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On the Growth of Solutions of Some Higher Order Linear Differential Equations With Entire Coefficients

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Abstract. In this paper, we investigate the order and the hyper-order of solutions of the linear differential equation

$$f^{(k)} + \left(D_{k-1} + B_{k-1}e^{b_{k-1}z}\right) f^{(k-1)} + ... + \left(D_1 + B_1 e^{b_1 z}\right) f'$$

$$+ \left(D_0 + A_1 e^{a_1 z} + A_2 e^{a_2 z}\right) f = 0,$$

where $A_j(z)$ $(\not\equiv 0)$ $(j = 1, 2)$, $B_l(z)$ $(\not\equiv 0)$ $(l = 1, ..., k-1)$, D_m $(m = 0, ..., k-1)$ are entire functions with $\max\{\sigma(A_j), \sigma(B_l), \sigma(D_m)\} < 1$, a_1, a_2, b_l $(l = 1, ..., k-1)$ are complex numbers. Under some conditions, we prove that every solution $f(z) \not\equiv 0$ of the above equation is of infinite order and with hyper-order 1.

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1 Introduction and statement of results

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory (see [9], [14]). Let $\sigma(f)$ denote the order of growth of an entire function f and the hyper-order $\sigma_2(f)$ of f is defined by (see [10], [14])

$$\sigma_2(f) = \lim_{r\to+\infty}\sup \frac{\log\log T(r,f)}{\log r} = \lim_{r\to+\infty}\sup \frac{\log\log\log M(r,f)}{\log r},$$

where $T(r,f)$ is the Nevanlinna characteristic function of f and $M(r,f) = \max_{|z|=r}|f(z)|$.

For the second order linear differential equation

$$f'' + e^{-z}f' + B(z)f = 0, \tag{1.1}$$

where $B(z)$ is an entire function, it is well-known that each solution f of the equation (1.1) is an entire function, and that if f_1, f_2 are two linearly independent solutions of (1.1), then by [4], there is at least one of f_1, f_2 of infinite order. Hence, "most" solutions of (1.1) will have infinite order. But the equation (1.1) with $B(z) = -(1+e^{-z})$ possesses a solution $f(z) = e^z$ of finite order.

A natural question arises: What conditions on $B(z)$ will guarantee that every solution $f \not\equiv 0$ of (1.1) has infinite order? Many authors, Frei [5], Ozawa [12], Amemiya-Ozawa [1] and Gundersen [6], Langley [11] have studied this problem. They proved that when $B(z)$ is a nonconstant polynomial or $B(z)$ is a transcendental entire function with order $\rho(B) \neq 1$, then every solution $f \not\equiv 0$ of (1.1) has infinite order. In [3], Chen has considered equation (1.1) and obtained different results concerning the growth of its solutions when $\rho(B) = 1$.

Recently in [13], Peng and Chen have investigated the order and the hyper-order of solutions of some second order linear differential equations and have proved the following result.

Theorem A ([13]) *Let $A_j(z)$ $(\not\equiv 0)$ $(j = 1, 2)$ be entire functions with $\sigma(A_j) < 1$, a_1, a_2 be complex numbers such that $a_1a_2 \neq 0$, $a_1 \neq a_2$ (suppose that $|a_1| \leqslant |a_2|$). If $\arg a_1 \neq \pi$ or $a_1 < -1$, then every solution $f \not\equiv 0$ of the equation*

$$f'' + e^{-z}f' + (A_1e^{a_1z} + A_2e^{a_2z})f = 0$$

has infinite order and $\sigma_2(f) = 1$.

In this paper, we continue the research in this type of problems, the main purpose of this paper is to extend and improve the results of Theorem A to some higher order linear differential equations. In fact we will prove the following results.

Theorem 1.1 *Let* $A_j(z)$ $(\not\equiv 0)$ $(j = 1, 2)$, $B_l(z)$ $(\not\equiv 0)$ $(l = 1, ..., k-1)$, D_m $(m = 0, ..., k-1)$ *be entire functions with* $\max\{\sigma(A_j), \sigma(B_l), \sigma(D_m)\} < 1$, b_l $(l = 1, ..., k-1)$ *be complex constants such that* (i) $\arg b_l = \arg a_1$ *and* $b_l = c_l a_1$ $(0 < c_l < 1)$ $(l \in I_1)$ *and* (ii) b_l *is a real constant such that* $b_l \leqslant 0$ $(l \in I_2)$, *where* $I_1 \neq \varnothing$, $I_2 \neq \varnothing$, $I_1 \cap I_2 = \varnothing$, $I_1 \cup I_2 = \{1, 2, ..., k-1\}$, *and* a_1, a_2 *are complex numbers such that* $a_1 a_2 \neq 0$, $a_1 \neq a_2$ *(suppose that* $|a_1| \leqslant |a_2|$*). If* $\arg a_1 \neq \pi$ *or* a_1 *is a real number such that* $a_1 < \frac{b}{1-c}$, *where* $c = \max\{c_l : l \in I_1\}$ *and* $b = \min\{b_l : l \in I_2\}$, *then every solution* $f \not\equiv 0$ *of the equation*

$$f^{(k)} + \left(D_{k-1} + B_{k-1}e^{b_{k-1}z}\right) f^{(k-1)} + ... + \left(D_1 + B_1 e^{b_1 z}\right) f'$$

$$+ \left(D_0 + A_1 e^{a_1 z} + A_2 e^{a_2 z}\right) f = 0 \tag{1.2}$$

satisfies $\sigma(f) = +\infty$ *and* $\sigma_2(f) = 1$.

Corollary 1.1 *Let* $A_j(z)$ $(\not\equiv 0)$ $(j = 1, 2)$, $B_l(z)$ $(\not\equiv 0)$ $(l = 1, ..., k-1)$, D_m $(m = 0, ..., k-1)$ *be entire functions with* $\max\{\sigma(A_j), \sigma(B_l), \sigma(D_m)\} < 1$, b_l $(l = 1, ..., k-1)$ *be complex constants such that* $\arg b_l = \arg a_1$ *and* $b_l = c_l a_1$ $(0 < c_l < 1)$ $(l = 1, ..., k-1)$, *and* a_1, a_2 *be complex numbers such that* $a_1 a_2 \neq 0$, $a_1 \neq a_2$ *(suppose that* $|a_1| \leqslant |a_2|$*). If* $\arg a_1 \neq \pi$ *or* a_1 *is a real number such that* $a_1 < 0$, *then every solution* $f \not\equiv 0$ *of equation* (1.2) *satisfies* $\sigma(f) = +\infty$ *and* $\sigma_2(f) = 1$.

Corollary 1.2 *Let* $A_j(z)$ $(\not\equiv 0)$ $(j = 1, 2)$, $B_l(z)$ $(\not\equiv 0)$ $(l = 1, ..., k-1)$, D_m $(m = 0, ..., k-1)$ *be entire functions with* $\max\{\sigma(A_j), \sigma(B_l), \sigma(D_m)\} < 1$, b_l $(l = 1, ..., k-1)$ *be real constants such that* $b_l \leqslant 0$, *and* a_1, a_2 *be complex numbers such that* $a_1 a_2 \neq 0$, $a_1 \neq a_2$ *(suppose that* $|a_1| \leqslant |a_2|$*). If* $\arg a_1 \neq \pi$ *or* a_1 *is a real number such that* $a_1 < b$, *where* $b = \min\{b_l : l = 1, ..., k-1\}$, *then every solution* $f \not\equiv 0$ *of equation* (1.2) *satisfies* $\sigma(f) = +\infty$ *and* $\sigma_2(f) = 1$.

2 Preliminary lemmas

To prove our theorem, we need the following lemmas.

Lemma 2.1 ([7]) *Let f be a transcendental meromorphic function with $\sigma(f) = \sigma < +\infty$, $H = \{(k_1, j_1), (k_2, j_2), ..., (k_q, j_q)\}$ be a finite set of distinct pairs of integers satisfying $k_i > j_i \geqslant 0$ $(i = 1, ..., q)$ and let $\varepsilon > 0$ be a given constant. Then,*
(i) *there exists a set $E_1 \subset \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right)$ with linear measure zero, such that, if $\psi \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right) \setminus E_1$, then there is a constant $R_0 = R_0(\psi) > 1$, such that for all z satisfying $\arg z = \psi$ and $|z| \geqslant R_0$ and for all $(k, j) \in H$, we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leqslant |z|^{(k-j)(\sigma-1+\varepsilon)}, \tag{2.1}$$

(ii) *there exists a set $E_2 \subset (1, +\infty)$ with finite logarithmic measure, such that for all z satisfying $|z| \notin E_2 \cup [0, 1]$ and for all $(k, j) \in H$, we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leqslant |z|^{(k-j)(\sigma-1+\varepsilon)}, \tag{2.2}$$

(iii) *there exists a set $E_3 \subset (0, \infty)$ with finite linear measure, such that for all z satisfying $|z| \notin E_3$ and for all $(k, j) \in H$, we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leqslant |z|^{(k-j)(\sigma+\varepsilon)}. \tag{2.3}$$

Lemma 2.2 ([3]) *Suppose that $P(z) = (\alpha + i\beta) z^n + ...$ (α, β are real numbers, $|\alpha| + |\beta| \neq 0$) is a polynomial with degree $n \geqslant 1$, that $A(z)$ ($\not\equiv 0$) is an entire function with $\sigma(A) < n$. Set $g(z) = A(z) e^{P(z)}$, $z = re^{i\theta}$, $\delta(P, \theta) = \alpha \cos n\theta - \beta \sin n\theta$. Then for any given $\varepsilon > 0$, there is a set $E_4 \subset [0, 2\pi)$ that has linear measure zero, such that for any $\theta \in [0, 2\pi) \setminus (E_4 \cup E_5)$, there is $R > 0$, such that for $|z| = r > R$, we have*
(i) *if $\delta(P, \theta) > 0$, then*

$$\exp\{(1-\varepsilon)\delta(P,\theta) r^n\} \leqslant \left|g\left(re^{i\theta}\right)\right| \leqslant \exp\{(1+\varepsilon)\delta(P,\theta) r^n\}; \tag{2.4}$$

(ii) *if $\delta(P, \theta) < 0$, then*

$$\exp\{(1+\varepsilon)\delta(P,\theta) r^n\} \leqslant \left|g\left(re^{i\theta}\right)\right| \leqslant \exp\{(1-\varepsilon)\delta(P,\theta) r^n\}, \tag{2.5}$$

where $E_5 = \{\theta \in [0, 2\pi) : \delta(P, \theta) = 0\}$ *is a finite set.*

Lemma 2.3 ([13]) *Suppose that* $n \geqslant 1$ *is a positive entire number. Let* $P_j(z) = a_{jn}z^n + ...$ $(j = 1, 2)$ *be nonconstant polynomials, where* a_{jq} $(q = 1, ..., n)$ *are complex numbers and* $a_{1n}a_{2n} \neq 0$*. Set* $z = re^{i\theta}$*,* $a_{jn} = |a_{jn}| e^{i\theta_j}$*,* $\theta_j \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right)$*,* $\delta(P_j, \theta) = |a_{jn}| \cos(\theta_j + n\theta)$*, then there is a set* $E_6 \subset \left[-\frac{\pi}{2n}, \frac{3\pi}{2n}\right)$ *that has linear measure zero. If* $\theta_1 \neq \theta_2$*, then there exists a ray* $\arg z = \theta$*,* $\theta \in \left(-\frac{\pi}{2n}, \frac{\pi}{2n}\right) \setminus (E_6 \cup E_7)$*, such that*

$$\delta(P_1, \theta) > 0 \ , \ \delta(P_2, \theta) < 0 \tag{2.6}$$

or

$$\delta(P_1, \theta) < 0 \ , \ \delta(P_2, \theta) > 0, \tag{2.7}$$

where $E_7 = \left\{\theta \in \left[-\frac{\pi}{2n}, \frac{3\pi}{2n}\right) : \delta(P_j, \theta) = 0\right\}$ *is a finite set, which has linear measure zero.*

Remark 2.1 ([13]) In Lemma 2.3, if $\theta \in \left(-\frac{\pi}{2n}, \frac{\pi}{2n}\right) \setminus (E_6 \cup E_7)$ is replaced by $\theta \in \left(\frac{\pi}{2n}, \frac{3\pi}{2n}\right) \setminus (E_6 \cup E_7)$, then we obtain the same result.

Lemma 2.4 ([2]) *Suppose that* $k \geqslant 2$ *and* B_0*,* $B_1, ...,$ B_{k-1} *are entire functions of finite order and let* $\sigma = \max\{\sigma(B_j) : j = 0, ..., k-1\}$*. Then every solution* f *of the equation*

$$f^{(k)} + B_{k-1}f^{(k-1)} + ... + B_1f' + B_0f = 0 \tag{2.8}$$

satisfies $\sigma_2(f) \leqslant \sigma$*.*

Lemma 2.5 ([7]) *Let* $f(z)$ *be a transcendental meromorphic function, and let* $\alpha > 1$ *be a given constant. Then there exist a set* $E_8 \subset (1, \infty)$ *with finite logarithmic measure and a constant* $B > 0$ *that depends only on* α *and* i, j $(0 \leqslant i < j \leqslant k)$*, such that for all* z *satisfying* $|z| = r \notin [0, 1] \cup E_8$*, we have*

$$\left|\frac{f^{(j)}(z)}{f^{(i)}(z)}\right| \leqslant B\left\{\frac{T(\alpha r, f)}{r}(\log^{\alpha} r)\log T(\alpha r, f)\right\}^{j-i}. \tag{2.9}$$

Lemma 2.6 ([8]) *Let* $\varphi : [0, +\infty) \to \mathbb{R}$ *and* $\psi : [0, +\infty) \to \mathbb{R}$ *be monotone non-decreasing functions such that* $\varphi(r) \leqslant \psi(r)$ *for all* $r \notin E_9 \cup [0, 1]$*, where* $E_9 \subset (1, +\infty)$ *is a set of finite logarithmic measure. Let* $\gamma > 1$ *be a given constant. Then there exists an* $r_1 = r_1(\gamma) > 0$ *such that* $\varphi(r) \leqslant \psi(\gamma r)$ *for all* $r > r_1$*.*

3 Proof of Theorem 1.1

Assume that $f\ (\not\equiv 0)$ is a solution of equation (1.2).

First step: We prove that $\sigma(f) = +\infty$. Suppose that $\sigma(f) = \sigma < +\infty$. Set $\max\{\sigma(A_j), \sigma(B_l), \sigma(D_m)\} = \beta < 1$ where $(j = 1, 2)$, $(l = 1, ..., k-1)$, $(m = 0, ..., k-1)$. Then, for any given ε $(0 < \varepsilon < 1-\beta)$ and for sufficiently large r, we have

$$|A_j(z)| \leqslant \exp\left\{r^{\beta+\varepsilon}\right\},\ |B_l(z)| \leqslant \exp\left\{r^{\beta+\varepsilon}\right\},\ |D_m(z)| \leqslant \exp\left\{r^{\beta+\varepsilon}\right\}. \tag{3.1}$$

By Lemma 2.1 (i), for the above ε, there exists a set $E_1 \subset \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right)$ of linear measure zero, such that if $\theta \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right) \setminus E_1$, then there is a constant $R_0 = R_0(\theta) > 1$, such that for all z satisfying $\arg z = \theta$ and $|z| = r \geqslant R_0$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leqslant r^{j(\sigma-1+\varepsilon)}\ \ (j = 1, ..., k). \tag{3.2}$$

Let $z = re^{i\theta}$, $a_1 = |a_1| e^{i\theta_1}$, $a_2 = |a_2| e^{i\theta_2}$, $\theta_1, \theta_2 \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right)$. We know that $\delta(b_l z, \theta) = \delta(c_l a_1 z, \theta) = c_l \delta(a_1 z, \theta)$ $(l \in I_1)$.

Case 1: $\arg a_1 \neq \pi$, which is $\theta_1 \neq \pi$.
(i) Assume that $\theta_1 \neq \theta_2$. By Lemma 2.3, for any given ε $(0 < \varepsilon < \min\{\frac{|a_2|-|a_1|}{|a_2|+|a_1|}, 1-\beta, \frac{1-c}{2(1+c)}\})$, there is a ray $\arg z = \theta$ such that $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus (E_1 \cup E_6 \cup E_7)$ (where E_6 and E_7 are defined as in Lemma 2.3, $E_1 \cup E_6 \cup E_7$ is of the linear measure zero), and satisfying

$$\delta(a_1 z, \theta) > 0,\ \delta(a_2 z, \theta) < 0 \text{ or } \delta(a_1 z, \theta) < 0,\ \delta(a_2 z, \theta) > 0.$$

a) When $\delta(a_1 z, \theta) > 0$, $\delta(a_2 z, \theta) < 0$, for sufficiently large r, we get by Lemma 2.2

$$|A_1 e^{a_1 z}| \geqslant \exp\{(1-\varepsilon)\delta(a_1 z, \theta) r\}, \tag{3.3}$$

$$|A_2 e^{a_2 z}| \leqslant \exp\{(1-\varepsilon)\delta(a_2 z, \theta) r\} < 1. \tag{3.4}$$

By (3.3) and (3.4), we have

$$|A_1 e^{a_1 z} + A_2 e^{a_2 z}| \geqslant |A_1 e^{a_1 z}| - |A_2 e^{a_2 z}|$$

$$\geqslant \exp\{(1-\varepsilon)\delta(a_1 z, \theta) r\} - 1$$

$$\geqslant (1-o(1))\exp\{(1-\varepsilon)\,\delta(a_1z,\theta)\,r\}. \tag{3.5}$$

By (1.2), we get

$$|A_1e^{a_1z}+A_2e^{a_2z}| \leqslant \left|\frac{f^{(k)}(z)}{f(z)}\right| + \left(|D_{k-1}|+\left|B_{k-1}(z)\,e^{b_{k-1}z}\right|\right)\left|\frac{f^{(k-1)}(z)}{f(z)}\right|$$

$$+\ldots+\left(|D_1|+\left|B_1(z)\,e^{b_1z}\right|\right)\left|\frac{f'(z)}{f(z)}\right|+|D_0(z)|. \tag{3.6}$$

For $l\in I_1$, we have

$$\left|B_l(z)\,e^{b_lz}\right| \leqslant \exp\{(1+\varepsilon)\,c_l\delta(a_1z,\theta)\,r\} \leqslant \exp\{(1+\varepsilon)\,c\delta(a_1z,\theta)\,r\}. \tag{3.7}$$

For $l\in I_2$, we have

$$\left|B_l(z)\,e^{b_lz}\right| = |B_l(z)|\left|e^{b_lz}\right| \leqslant \exp\left\{r^{\beta+\varepsilon}\right\}e^{b_lr\cos\theta} \leqslant \exp\left\{r^{\beta+\varepsilon}\right\} \tag{3.8}$$

because $b_l\leqslant 0$ and $\cos\theta>0$. Substituting (3.1) , (3.2) , (3.5), (3.7) and (3.8) into (3.6), we obtain

$$\begin{aligned}&(1-o(1))\exp\{(1-\varepsilon)\,\delta(a_1z,\theta)\,r\}\\ &\leqslant r^{k(\sigma-1+\varepsilon)}+\left(\exp\left\{r^{\beta+\varepsilon}\right\}+\left|B_{k-1}(z)\,e^{b_{k-1}z}\right|\right)r^{(k-1)(\sigma-1+\varepsilon)}\\ &+\ldots+\left(\exp\left\{r^{\beta+\varepsilon}\right\}+\left|B_1(z)\,e^{b_1z}\right|\right)r^{\sigma-1+\varepsilon}+\exp\left\{r^{\beta+\varepsilon}\right\}\\ &\leqslant M_0r^{k(\sigma-1+\varepsilon)}\exp\left\{r^{\beta+\varepsilon}\right\}\exp\{(1+\varepsilon)\,c\delta(a_1z,\theta)\,r\},\end{aligned} \tag{3.9}$$

where $M_0>0$ is a some constant. From (3.9) and $0<\varepsilon<\frac{1-c}{2(1+c)}$, we get

$$(1-o(1))\exp\left\{\frac{1-c}{2}\delta(a_1z,\theta)\,r\right\} \leqslant M_0r^{k(\sigma-1+\varepsilon)}\exp\left\{r^{\beta+\varepsilon}\right\}. \tag{3.10}$$

By $\delta(a_1z,\theta)>0$ and $\beta+\varepsilon<1$ we know that (3.10) is a contradiction.

b) When $\delta(a_1z,\theta)<0$, $\delta(a_2z,\theta)>0$, for sufficiently large r, we get by Lemma 2.2

$$|A_1e^{a_1z}| \leqslant \exp\{(1-\varepsilon)\,\delta(a_1z,\theta)\,r\}<1, \tag{3.11}$$

$$|A_2e^{a_2z}| \geqslant \exp\{(1-\varepsilon)\,\delta(a_2z,\theta)\,r\}. \tag{3.12}$$

By (3.11) and (3.12), we have

$$|A_1e^{a_1z}+A_2e^{a_2z}| \geqslant (1-o(1))\exp\{(1-\varepsilon)\,\delta(a_2z,\theta)\,r\}. \tag{3.13}$$

For $l \in I_1$, we have

$$\left|B_l(z) e^{b_l z}\right| \leqslant \exp\left\{(1+\varepsilon) c_l \delta(a_1 z, \theta) r\right\} < 1. \tag{3.14}$$

Substituting $(3.1), (3.2), (3.8), (3.13)$ and (3.14) into (3.6), we obtain

$$(1 - o(1)) \exp\left\{(1-\varepsilon) \delta(a_2 z, \theta) r\right\} \leqslant M_0 r^{k(\sigma-1+\varepsilon)} \exp\left\{r^{\beta+\varepsilon}\right\}. \tag{3.15}$$

By $\delta(a_2 z, \theta) > 0$ and $\beta + \varepsilon < 1$ we know that (3.15) is a contradiction.

(ii) Assume that $\theta_1 = \theta_2$. By Lemma 2.3, for the above ε, there is a ray $\arg z = \theta$ such that $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus (E_1 \cup E_6 \cup E_7)$ and $\delta(a_1 z, \theta) > 0$. Since $|a_1| \leqslant |a_2|$, $a_1 \neq a_2$ and $\theta_1 = \theta_2$, then $|a_1| < |a_2|$, thus $\delta(a_2 z, \theta) > \delta(a_1 z, \theta) > 0$. For sufficiently large r, we have by Lemma 2.2

$$|A_1 e^{a_1 z}| \leqslant \exp\left\{(1+\varepsilon) \delta(a_1 z, \theta) r\right\}, \tag{3.16}$$

$$|A_2 e^{a_2 z}| \geqslant \exp\left\{(1-\varepsilon) \delta(a_2 z, \theta) r\right\} \tag{3.17}$$

and $(3.7), (3.8)$ hold. By (3.16) and (3.17), we get

$$|A_1 e^{a_1 z} + A_2 e^{a_2 z}| \geqslant |A_2 e^{a_2 z}| - |A_1 e^{a_1 z}|$$

$$\geqslant \exp\left\{(1-\varepsilon) \delta(a_2 z, \theta) r\right\} - \exp\left\{(1+\varepsilon) \delta(a_1 z, \theta) r\right\}$$
$$= \exp\left\{(1+\varepsilon) \delta(a_1 z, \theta) r\right\} \left[\exp\left\{\alpha r\right\} - 1\right], \tag{3.18}$$

where

$$\alpha = (1-\varepsilon) \delta(a_2 z, \theta) - (1+\varepsilon) \delta(a_1 z, \theta).$$

Since $0 < \varepsilon < \frac{|a_2| - |a_1|}{|a_2| + |a_1|}$, then

$$\alpha = (1-\varepsilon) |a_2| \cos(\theta_2 + \theta) - (1+\varepsilon) |a_1| \cos(\theta_1 + \theta)$$

$$= \cos(\theta_1 + \theta) \left[(1-\varepsilon) |a_2| - (1+\varepsilon) |a_1|\right]$$
$$= \cos(\theta_1 + \theta) \left[|a_2| - |a_1| - \varepsilon(|a_2| + |a_1|)\right] > 0.$$

Then, by $\alpha > 0$ and from (3.18), we get

$$|A_1 e^{a_1 z} + A_2 e^{a_2 z}| \geqslant (1 - o(1)) \exp\left\{(1+\varepsilon) \delta(a_1 z, \theta) r\right\} \exp\left\{\alpha r\right\}. \tag{3.19}$$

Substituting (3.1), (3.2), (3.7), (3.8) and (3.19) into (3.6), we obtain

$$(1 - o(1)) \exp\left\{(1+\varepsilon) \delta(a_1 z, \theta) r\right\} \exp\left\{\alpha r\right\}$$

$$\leqslant M_1 r^{k(\sigma-1+\varepsilon)} \exp\left\{r^{\beta+\varepsilon}\right\} \exp\left\{(1+\varepsilon) c\delta(a_1 z,\theta) r\right\}, \tag{3.20}$$

where $M_1 > 0$ is a some constant. By (3.20), we have

$$(1-o(1)) \exp\left\{[(1+\varepsilon)(1-c)\delta(a_1 z,\theta)+\alpha]r\right\} \leqslant M_1 r^{k(\sigma-1+\varepsilon)} \exp\left\{r^{\beta+\varepsilon}\right\}. \tag{3.21}$$

By $\delta(a_1 z,\theta) > 0$, $\alpha > 0$ and $\beta+\varepsilon < 1$ we know that (3.21) is a contradiction.

Case 2: $a_1 < \frac{b}{1-c}$, which is $\theta_1 = \pi$.
(i) Assume that $\theta_1 \neq \theta_2$, then $\theta_2 \neq \pi$. By Lemma 2.3, for the above ε, there is a ray $\arg z = \theta$ such that $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus (E_1 \cup E_6 \cup E_7)$ and $\delta(a_2 z,\theta) > 0$. Because $\cos\theta > 0$, we have $\delta(a_1 z,\theta) = |a_1| \cos(\theta_1+\theta) = -|a_1|\cos\theta < 0$. For sufficiently large r, we obtain by Lemma 2.2

$$|A_1 e^{a_1 z}| \leqslant \exp\left\{(1-\varepsilon)\delta(a_1 z,\theta) r\right\} < 1, \tag{3.22}$$

$$|A_2 e^{a_2 z}| \geqslant \exp\left\{(1-\varepsilon)\delta(a_2 z,\theta) r\right\} \tag{3.23}$$

and (3.8), (3.14) hold. By (3.22) and (3.23), we obtain

$$\begin{aligned} |A_1 e^{a_1 z} + A_2 e^{a_2 z}| &\geqslant |A_2 e^{a_2 z}| - |A_1 e^{a_1 z}| \\ &\geqslant \exp\left\{(1-\varepsilon)\delta(a_2 z,\theta) r\right\} - 1 \\ &\geqslant (1-o(1)) \exp\left\{(1-\varepsilon)\delta(a_2 z,\theta) r\right\}. \end{aligned} \tag{3.24}$$

Using the same reasoning as in **Case 1**(i), we can get a contradiction.

(ii) Assume that $\theta_1 = \theta_2$, then $\theta_1 = \theta_2 = \pi$. By Lemma 2.3, for the above ε, there is a ray $\arg z = \theta$ such that $\theta \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \setminus (E_1 \cup E_6 \cup E_7)$, then $\cos\theta < 0$, $\delta(a_1 z,\theta) = |a_1|\cos(\theta_1+\theta) = -|a_1|\cos\theta > 0$, $\delta(a_2 z,\theta) = |a_2|\cos(\theta_2+\theta) = -|a_2|\cos\theta > 0$. Since $|a_1| \leqslant |a_2|$, $a_1 \neq a_2$ and $\theta_1 = \theta_2$, then $|a_1| < |a_2|$, thus $\delta(a_2 z,\theta) > \delta(a_1 z,\theta) > 0$. For sufficiently large r, we get (3.7), (3.16), (3.17) and (3.19) holds. For $l \in I_2$, we have

$$\begin{aligned} \left|B_l(z) e^{b_l z}\right| = |B_l(z)| \left|e^{b_l z}\right| &\leqslant \exp\left\{r^{\beta+\varepsilon}\right\} \exp\left\{b_l r\cos\theta\right\} \\ &\leqslant \exp\left\{r^{\beta+\varepsilon}\right\} \exp\left\{b r\cos\theta\right\} \end{aligned} \tag{3.25}$$

because $b_l \leqslant 0$, $b = \min\{b_l : l \in I_2\}$ and $\cos\theta < 0$. Substituting (3.1), (3.2), (3.7), (3.19) and (3.25) into (3.6), we obtain

$$(1-o(1)) \exp\left\{(1+\varepsilon)\delta(a_1 z,\theta) r\right\} \exp\left\{\alpha r\right\}$$

$$\leqslant M_2 r^{k(\sigma-1+\varepsilon)} \exp\left\{r^{\beta+\varepsilon}\right\} \exp\left\{(1+\varepsilon) c\delta(a_1 z, \theta) r\right\} \exp\left\{br\cos\theta\right\},$$

where $M_2 > 0$ is a some constant. Thus

$$(1-o(1)) \exp\{\gamma r\} \leqslant M_2 r^{k(\sigma-1+\varepsilon)} \exp\left\{r^{\beta+\varepsilon}\right\}, \tag{3.26}$$

where $\gamma = (1+\varepsilon)(1-c)\delta(a_1 z, \theta) + \alpha - b\cos\theta$. Since $\alpha > 0$, $\cos\theta < 0$, $\delta(a_1 z, \theta) = -|a_1|\cos\theta$, $a_1 < \frac{b}{1-c}$ and $b \leqslant 0$, then

$$\begin{aligned} \gamma &= -(1+\varepsilon)(1-c)|a_1|\cos\theta - b\cos\theta + \alpha \\ &= -\left[(1+\varepsilon)(1-c)|a_1| + b\right]\cos\theta + \alpha \\ &> -\left[(1+\varepsilon)(1-c)\frac{|b|}{1-c} + b\right]\cos\theta + \alpha \\ &= -\left[-(1+\varepsilon)b + b\right]\cos\theta + \alpha = \alpha + b\varepsilon\cos\theta > 0. \end{aligned}$$

By $\beta + \varepsilon < 1$ and $\gamma > 0$, we know that (3.26) is a contradiction. Concluding the above proof, we obtain $\sigma(f) = +\infty$.

Second step: We prove that $\sigma_2(f) = 1$. By

$$\max\left\{\sigma\left(D_l + B_l e^{b_l z}\right)\ \ (l = 1, ..., k-1),\ \sigma\left(D_0 + A_1 e^{a_1 z} + A_2 e^{a_2 z}\right)\right\} = 1$$

and Lemma 2.4, we obtain $\sigma_2(f) \leqslant 1$. By Lemma 2.5, we know that there exists a set $E_8 \subset (1, +\infty)$ with finite logarithmic measure and a constant $B > 0$, such that for all z satisfying $|z| = r \notin [0,1] \cup E_8$, we get

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leqslant B\left[T(2r, f)\right]^{j+1}\ \ (j = 1, ..., k). \tag{3.27}$$

Case 1: $\arg a_1 \neq \pi$.
(i) $(\theta_1 \neq \theta_2)$. In first step, we have proved that there is a ray $\arg z = \theta$ where $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus (E_1 \cup E_6 \cup E_7)$, satisfying

$$\delta(a_1 z, \theta) > 0,\ \delta(a_2 z, \theta) < 0 \text{ or } \delta(a_1 z, \theta) < 0,\ \delta(a_2 z, \theta) > 0.$$

a) When $\delta(a_1 z, \theta) > 0$, $\delta(a_2 z, \theta) < 0$, for sufficiently large r, we get (3.5) holds. Substituting (3.1), (3.5), (3.7), (3.8) and (3.27) into (3.6), we obtain for all $z = re^{i\theta}$ satisfying $|z| = r \notin [0,1] \cup E_8$, $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus (E_1 \cup E_6 \cup E_7)$

$$(1-o(1)) \exp\left\{(1-\varepsilon)\delta(a_1 z, \theta) r\right\}$$

$$\leqslant B\left[T(2r,f)\right]^{k+1}+B\left[\exp\left\{r^{\beta+\varepsilon}\right\}+\left|B_{k-1}\left(z\right)e^{b_{k-1}z}\right|\right]\left[T(2r,f)\right]^{k}$$
$$+...+B\left[\exp\left\{r^{\beta+\varepsilon}\right\}+\left|B_{1}\left(z\right)e^{b_{1}z}\right|\right]\left[T(2r,f)\right]^{2}+\exp\left\{r^{\beta+\varepsilon}\right\}$$
$$\leqslant M_{0}\exp\left\{r^{\beta+\varepsilon}\right\}\exp\left\{(1+\varepsilon)\,c\delta\left(a_{1}z,\theta\right)r\right\}\left[T(2r,f)\right]^{k+1}, \tag{3.28}$$

where $M_0 > 0$ is a some constant. From (3.28) and $0 < \varepsilon < \frac{1-c}{2(1+c)}$, we get

$$(1-o\left(1\right))\exp\left\{\frac{1-c}{2}\delta\left(a_{1}z,\theta\right)r\right\}\leqslant M_{0}\exp\left\{r^{\beta+\varepsilon}\right\}\left[T(2r,f)\right]^{k+1}. \tag{3.29}$$

Since $\delta\left(a_1 z,\theta\right) > 0$, $\beta+\varepsilon < 1$, then by using Lemma 2.6 and (3.29), we obtain $\sigma_2\left(f\right) \geqslant 1$, hence $\sigma_2\left(f\right) = 1$.
b) When $\delta\left(a_1 z,\theta\right) < 0$, $\delta\left(a_2 z,\theta\right) > 0$, for sufficiently large r, we get (3.13) holds. Substituting (3.1), (3.8) , (3.13), (3.14) and (3.27) into (3.6), we obtain for all $z = re^{i\theta}$ satisfying $|z| = r \notin [0,1] \cup E_8$, $\theta \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right) \setminus \left(E_1 \cup E_6 \cup E_7\right)$

$$(1-o\left(1\right))\exp\left\{(1-\varepsilon)\,\delta\left(a_{2}z,\theta\right)r\right\}\leqslant M_{0}\exp\left\{r^{\beta+\varepsilon}\right\}\left[T(2r,f)\right]^{k+1}, \tag{3.30}$$

where $M_0 > 0$ is a some constant. By $\delta\left(a_2 z,\theta\right) > 0$, $\beta+\varepsilon < 1$ and (3.30), we have $\sigma_2\left(f\right) \geqslant 1$, then $\sigma_2\left(f\right) = 1$.
(ii) $(\theta_1 = \theta_2)$. In first step, we have proved that there is a ray $\arg z = \theta$ where $\theta \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right) \setminus \left(E_1 \cup E_6 \cup E_7\right)$, satisfying $\delta\left(a_2 z,\theta\right) > \delta\left(a_1 z,\theta\right) > 0$ and for sufficiently large r, we get (3.19) holds. Substituting (3.1), (3.7) , (3.8), (3.19) and (3.27) into (3.6), we obtain for all $z = re^{i\theta}$ satisfying $|z| = r \notin [0,1] \cup E_8$, $\theta \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right) \setminus \left(E_1 \cup E_6 \cup E_7\right)$

$$(1-o\left(1\right))\exp\left\{(1+\varepsilon)\,\delta\left(a_{1}z,\theta\right)r\right\}\exp\left\{\alpha r\right\}$$
$$\leqslant M_{1}\exp\left\{r^{\beta+\varepsilon}\right\}\exp\left\{(1+\varepsilon)\,c\delta\left(a_{1}z,\theta\right)r\right\}\left[T(2r,f)\right]^{k+1}, \tag{3.31}$$

where $M_1 > 0$ is a some constant. By (3.31), we have

$$(1-o\left(1\right))\exp\left\{\left[(1+\varepsilon)\left(1-c\right)\delta\left(a_{1}z,\theta\right)+\alpha\right]r\right\}\leqslant M_{1}\exp\left\{r^{\beta+\varepsilon}\right\}\left[T(2r,f)\right]^{k+1}. \tag{3.32}$$

Since $\delta\left(a_1 z,\theta\right) > 0$, $\alpha > 0$, $\beta+\varepsilon < 1$, then by using Lemma 2.6 and (3.32), we obtain $\sigma_2\left(f\right) \geqslant 1$, hence $\sigma_2\left(f\right) = 1$.
Case 2: $a_1 < \frac{b}{1-c}$.
(i) $(\theta_1 \neq \theta_2)$. In first step, we have proved that there is a ray $\arg z = \theta$ where $\theta \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right) \setminus \left(E_1 \cup E_6 \cup E_7\right)$, satisfying $\delta\left(a_2 z,\theta\right) > 0$ and $\delta\left(a_1 z,\theta\right) < 0$ and for sufficiently large r, we get (3.24) holds. Using the same reasoning as in second step (**Case 1** (i)), we can get $\sigma_2\left(f\right) = 1$.

(*ii*) ($\theta_1 = \theta_2$) In first step, we have proved that there is a ray $\arg z = \theta$ where $\theta \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \setminus (E_1 \cup E_6 \cup E_7)$, satisfying $\delta(a_2 z, \theta) > \delta(a_1 z, \theta) > 0$ and for sufficiently large r, we get (3.19) holds. Substituting (3.1), (3.7), (3.19), (3.25) and (3.27) into (3.6), we obtain for all $z = re^{i\theta}$ satisfying $|z| = r \notin [0,1] \cup E_8$, $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus (E_1 \cup E_6 \cup E_7)$

$$(1 - o(1)) \exp\{(1+\varepsilon)\delta(a_1 z, \theta) r\} \exp\{\alpha r\}$$

$$\leqslant M_2 \exp\left\{r^{\beta+\varepsilon}\right\} \exp\{(1+\varepsilon) c\delta(a_1 z, \theta) r\} \exp\{br\cos\theta\} [T(2r, f)]^{k+1},$$

where $M_2 > 0$ is a some constant. Thus

$$(1 - o(1)) \exp\{\gamma r\} \leqslant M_2 \exp\left\{r^{\beta+\varepsilon}\right\} [T(2r, f)]^{k+1}, \tag{3.33}$$

where $\gamma = (1+\varepsilon)(1-c)\delta(a_1 z, \theta) + \alpha - b\cos\theta$. Since $\gamma > 0$, $\beta + \varepsilon < 1$, then by using Lemma 2.6 and (3.33), we have $\sigma_2(f) \geqslant 1$, hence $\sigma_2(f) = 1$. Concluding the above proof, we obtain that every solution $f \not\equiv 0$ of (1.2) satisfies $\sigma_2(f) = 1$. The proof of Theorem 1.1 is complete.

4 Proofs of Corollary 1.1 and Corollary 1.2

Using the same reasoning as in the proof of Theorem 1.1, we can obtain Corollary 1.1 and Corollary 1.2.

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