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On the p-biharmonic equation involving concave-convex nonlinearities and sign-changing weight function

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Abstract

In this paper, we study the combined effect of concave and convex nonlinearities on the number of nontrivial solutions for the p-biharmonic equation of the form

$$\begin{cases} \Delta_p^2 u = |u|^{q-2}u + \lambda f(x)|u|^{r-2}u & \text{in } \Omega, \\ u = \nabla u = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.1}$$

where Ω is a bounded domain in R^N, $f \in C(\overline{\Omega})$ be a sign-changing weight function. By means of the Nehari manifold, we prove that there are at least two nontrivial solutions for the problem.

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1 Introduction

In this paper, we are concerned with the multiple solutions of the following p-biharmonic equation:

$$\begin{cases} \Delta_p^2 u = |u|^{q-2}u + \lambda f(x)|u|^{r-2}u & \text{in } \Omega, \\ u = \nabla u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in R^N, $1 < r < p < q < p_2^*(p_2^* = \frac{Np}{N-2p}$ if $p < \frac{N}{2}$, $p_2^* = \infty$ if $p \geq \frac{N}{2})$, $\lambda > 0$ and $f : \overline{\Omega} \to R$ is a continuous function which changes sign in $\overline{\Omega}$.

During the last ten years, several authors used the Nehari manifold and fibering maps to solve the problems involving sign-changing weight function, we refer the

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reader to [1, 2] for the semilinear elliptic equations, to [3, 4] for the elliptic problems with nonlinear boundary condition, to [5] for the problems in R^N, to [6] for the Kirchhoff type problems, and to [3, 4, 7] for the elliptic systems. Meanwhile, the positive solutions of semilinear biharmonic equations with Navier boundary on bounded domain in R^N are extensive studied, for example [8, 9], and so on. Although there are a lot of papers about the nontrivial solutions of biharmonic or p-biharmonic equations [10, 11, 12, 13] and references therein, there are less results about existence and multiplicity of solutions of p-biharmonic equations with Dirichlet boundary conditions on bounded domains. In [14], apart from the Kirchhoff function which can be taken identically 1, has been proved the existence of infinitely many solutions for an equation governed by the $p(x)$-ployharmonic operator, under Dirichlet boundary conditions, via variational methods.The main purpose of this paper is concerned with multiple solutions of the p-biharmonic equation involving concave-convex nonlinearities and sign-changing weight function and the combined effect of concave and convex nonlinearities on the number of nontrivial solutions.

We know that the corresponding energy functional of problem (0.1) is

$$J_\lambda(u) = \frac{1}{p}\int_\Omega |\Delta u|^p dx - \frac{1}{q}\int_\Omega |u|^q dx - \frac{\lambda}{r}\int_\Omega f(x)|u|^r dx,$$

where $u \in W_0^{2,p}(\Omega)$ with the norm $\|u\| = (\int_\Omega |\Delta u|^p dx)^{\frac{1}{p}}$, and J_λ is a C^1 functional and the critical points of J_λ are the weak solutions of problem (0.1).

The following is the main result of this paper.

Theorem 1. *There exists $\lambda_0 > 0$ such that for each $\lambda \in (0, \lambda_0)$, problem (0.1) has at least two nontrivial solutions.*

The paper is organized as follows. In Section 2, we give some preliminary lemmas. In Section 3, we give the proof of Theorem 1.

2 Preliminaries

Throughout this section, we denote by S the best Sobolev constant for the embedding of $W_0^{2,p}(\Omega)$ in $L^q(\Omega)$. We consider the Nehari minimization problem: for $\lambda > 0$,

$$\alpha_\lambda(\Omega) = \inf\left\{J_\lambda(u) \mid u \in M_\lambda(\Omega)\right\},$$

where $M_\lambda(\Omega) = \left\{u \in W_0^{2,p}(\Omega) \setminus \{0\} \mid \langle J'_\lambda(u), u\rangle = 0\right\}$. Define

$$\psi_\lambda(u) = \langle J'_\lambda(u), u\rangle = \|u\|^p - \int_\Omega |u|^q dx - \lambda\int_\Omega f(x)|u|^r dx.$$

Then for $u \in M_\lambda(\Omega)$,

$$\langle \psi'_\lambda(u), u\rangle = p\|u\|^p - q\int_\Omega |u|^q dx - \lambda r\int_\Omega f(x)|u|^r dx.$$

We may split $M_\lambda(\Omega)$ into three parts:

$$M_\lambda^+(\Omega) = \{u \in M_\lambda(\Omega) \mid \langle \psi'_\lambda(u), u\rangle > 0\},$$

$$M_\lambda^0(\Omega) = \{u \in M_\lambda(\Omega) \mid \langle \psi'_\lambda(u), u\rangle = 0\},$$

$$M_\lambda^-(\Omega) = \{u \in M_\lambda(\Omega) \mid \langle \psi'_\lambda(u), u\rangle < 0\}.$$

Now, we give the following lemmas.

Lemma 2.1. *There exists $\lambda_1 > 0$ such that for each $\lambda \in (0, \lambda_1)$, $M_\lambda^0(\Omega) = \emptyset$.*

Proof. We consider the following two cases.

Case (I). $u \in M_\lambda(\Omega)$ and $\int_\Omega f(x)|u|^r dx = 0$. We have

$$\|u\|^p - \int_\Omega |u|^q dx = 0.$$

Thus,

$$\langle \psi'_\lambda(u), u\rangle = p\|u\|^p - q\int_\Omega |u|^q dx = (p-q)\|u\|^p < 0$$

and so $u \notin M_\lambda^0(\Omega)$.

Case (II). $u \in M_\lambda(\Omega)$ and $\int_\Omega f(x)|u|^r dx \neq 0$.
Suppose that $M_\lambda^0(\Omega) \neq \emptyset$ for all $\lambda > 0$. If $u \in M_\lambda^0(\Omega)$, then we have

$$\begin{aligned} 0 = \langle \psi'_\lambda(u), u\rangle &= p\|u\|^p - q\int_\Omega |u|^q dx - \lambda r\int_\Omega f(x)|u|^r dx \\ &= (p-r)\|u\|^p - (q-r)\int_\Omega |u|^q dx. \end{aligned}$$

Thus,

$$\|u\|^p = \frac{q-r}{p-r}\int_\Omega |u|^q dx \tag{2.1}$$

and

$$\lambda\int_\Omega f(x)|u|^r dx = \|u\|^p - \int_\Omega |u|^q dx = \frac{q-p}{p-r}\int_\Omega |u|^q dx. \tag{2.2}$$

Moreover,

$$\begin{aligned} \frac{q-p}{q-r}\|u\|^p &= \|u\|^p - \int_\Omega |u|^q dx = \lambda\int_\Omega f(x)|u|^r dx \\ &\leq \lambda\|f\|_{L^{q^*}}\|u\|_{L^q}^r \leq \lambda\|f\|_{L^{q^*}} S^r\|u\|^r, \end{aligned}$$

where $q^* = \frac{q}{q-r}$. This implies

$$\|u\| \leq \Big(\lambda(\frac{q-r}{q-p})\|f\|_{L^{q^*}} S^r\Big)^{\frac{1}{p-r}}. \tag{2.3}$$

Let $I_\lambda : M_\lambda(\Omega) \to R$ be given by

$$I_\lambda(u) = K(q,r)\Big(\frac{\|u\|^q}{\int_\Omega |u|^q dx}\Big)^{\frac{p}{q-p}} - \lambda \int_\Omega f(x)|u|^r dx,$$

where $K(q,r) = (\frac{q-p}{q-r})(\frac{p-r}{q-r})^{\frac{p}{q-p}}$. Then $I_\lambda(u) = 0$ for all $u \in M_\lambda^0(\Omega)$. Indeed, from (2.1) and (2.2) it follows that for $u \in M_\lambda^0(\Omega)$, we have

$$\begin{aligned} I_\lambda(u) &= K(q,r)\Big(\frac{\|u\|^q}{\int_\Omega |u|^q dx}\Big)^{\frac{p}{q-p}} - \lambda \int_\Omega f(x)|u|^r dx \\ &= \Big(K(q,r)(\frac{q-r}{p-r})^{\frac{q}{q-p}} - \frac{q-p}{p-r}\Big)\int_\Omega |u|^q dx \\ &= 0. \end{aligned} \tag{2.4}$$

However, by (2.3), the Hölder and Sobolev inequality, for $u \in M_\lambda^0(\Omega)$,

$$\begin{aligned} I_\lambda(u) &\geq K(q,r)\Big(\frac{\|u\|^q}{\int_\Omega |u|^q dx}\Big)^{\frac{p}{q-p}} - \lambda\|f\|_{L^{q^*}}\|u\|^r_{L^q} \\ &\geq \|u\|^r_{L^q}\Big(K(q,r)\big(\frac{\|u\|^q}{S^{\frac{r(q-p)+pq}{p}}\|u\|^{\frac{r(q-p)+pq}{p}}}\big)^{\frac{p}{q-p}} - \lambda\|f\|_{L^{q^*}}\Big) \\ &= \|u\|^r_{L^q}\Big(K(q,r)\frac{1}{S^{\frac{r(q-p)+pq}{q-p}}}\|u\|^{-r} - \lambda\|f\|_{L^{q^*}}\Big) \\ &\geq \|u\|^r_{L^q}\Big\{K(q,r)\frac{1}{S^{\frac{r(q-p)+pq}{q-p}}}\lambda^{\frac{-r}{p-r}}\big[(\frac{q-r}{q-p})\|f\|_{L^{q^*}}S^r\big]^{\frac{-r}{p-r}} - \lambda\|f\|_{L^{q^*}}\Big\}. \end{aligned}$$

This implies that for λ sufficiently small we have $I_\lambda(u) > 0$ for all $u \in M_\lambda^0(\Omega)$, this contradicts (2.4). Thus, we can conclude that there exists $\lambda_1 > 0$ such that for $\lambda \in (0, \lambda_1)$, $M_\lambda^0(\Omega) = \emptyset$. □

Lemma 2.2. *If $u \in M_\lambda^+(\Omega)$, then $\int_\Omega f(x)|u|^r dx > 0$.*

Proof. For $u \in M_\lambda^+(\Omega)$, we have

$$\|u\|^p - \int_\Omega |u|^q dx - \lambda\int_\Omega f(x)|u|^r dx = 0$$

and

$$\|u\|^p > \frac{q-r}{p-r}\int_\Omega |u|^q dx.$$

Thus,

$$\lambda\int_\Omega f(x)|u|^r dx = \|u\|^p - \int_\Omega |u|^q dx > \frac{q-p}{p-r}\int_\Omega |u|^q dx > 0.$$

This completes the proof. □

By Lemma 2.1, for $\lambda \in (0, \lambda_1)$, we write $M_\lambda(\Omega) = M_\lambda^+(\Omega)\bigcup M_\lambda^-(\Omega)$ and define

$$\alpha_\lambda^+(\Omega) = \inf_{u\in M_\lambda^+(\Omega)} J_\lambda(u), \quad \alpha_\lambda^-(\Omega) = \inf_{u\in M_\lambda^-(\Omega)} J_\lambda(u).$$

The following lemma shows that the minimizers on $M_\lambda(\Omega)$ are the critical points for J_λ. We write $(W_0^{2,p}(\Omega))^*$ is the dual space of $W_0^{2,p}(\Omega)$.

Lemma 2.3. *For $\lambda \in (0, \lambda_1)$, if u_0 is a local minimizer for J_λ* on *$M_\lambda(\Omega)$,* then $J'_\lambda(u_0) = 0$ in $(W_0^{2,p}(\Omega))^*$.

Proof. If u_0 is a local minimizer for J_λ on $M_\lambda(\Omega)$, then u_0 is a solution of the optimization problem

$$\textit{minimize} \qquad J_\lambda(u) \quad \textit{subject to} \quad \psi_\lambda(u) = 0.$$

Hence, by the theory of Lagrange multipliers, there exists $\theta \in R$ such that

$$J'_\lambda(u_0) = \theta\psi'_\lambda(u_0) \quad in \quad (W_0^{2,p}(\Omega))^*.$$

Thus,

$$\langle J'_\lambda(u_0), u_0\rangle = \theta\langle\psi'_\lambda(u_0), u_0\rangle. \tag{2.5}$$

Since $u_0 \in M_\lambda(\Omega)$, so $\langle J'_\lambda(u_0), u_0\rangle = 0$. Moreover, since $M_\lambda^0(\Omega) = \emptyset$, so $\langle\psi'_\lambda(u_0), u_0\rangle \neq 0$ and by (2.5) $\theta = 0$. This completes the proof. □

For $u \in W_0^{2,p}(\Omega)$, we write

$$t_{\max} = \Big(\frac{(p-r)\|u\|^p}{(q-r)\int_\Omega |u|^q dx}\Big)^{\frac{1}{q-p}}.$$

Then we have the following lemma.

Lemma 2.4. *Let $q^* = \frac{q}{q-r}$ and $\lambda_2 = (\frac{p-r}{q-r})^{\frac{p-r}{q-p}}(\frac{q-p}{q-r})S^{\frac{p(r-q)}{q-p}}\|f\|_{L^{q^*}}^{-1}$. Then for each $u \in W_0^{2,p}(\Omega)\setminus\{0\}$ and $\lambda \in (0, \lambda_2)$, we have*

(i) *There is a unique $t^- = t^-(u) > t_{\max} > 0$ such that $t^-u \in M_\lambda^-(\Omega)$ and $J_\lambda(t^-u) = \max_{t\geq t_{\max}} J_\lambda(tu)$;*
(ii) *$t^-(u)$ is a continuous function for nonzero u;*
(iii) *$M_\lambda^-(\Omega) = \Big\{u \in W_0^{2,p}(\Omega)\setminus\{0\} \mid \frac{1}{\|u\|}t^-(\frac{u}{\|u\|}) = 1\Big\}$;*
(iv) *If $\int_\Omega f(x)|u|^r dx > 0$, then there is a unique $0 < t^+ = t^+(u) < t_{\max}$ such that $t^+u \in M_\lambda^+(\Omega)$ and $J_\lambda(t^+u) = \min_{0\leq t\leq t^-} J_\lambda(tu)$.*

Proof. **(i)** Fix $u \in W_0^{2,p}(\Omega)\setminus\{0\}$, let

$$s(t) = t^{p-r}\|u\|^p - t^{q-r}\int_\Omega |u|^q dx \quad \textit{for } t \geq 0.$$

We have $s(0) = 0$, $s(t) \to -\infty$ as $t \to +\infty$ and $s(t)$ achieves its maximum at $t_{\max}$. Moreover,

$$s(t_{\max}) \;=\; \Big(\frac{(p-r)\|u\|^p}{(q-r)\int_\Omega |u|^q dx}\Big)^{\frac{p-r}{q-p}}\|u\|^p$$

$$
\begin{aligned}
&- \Big(\frac{(p-r)\|u\|^p}{(q-r)\int_\Omega |u|^q dx}\Big)^{\frac{q-r}{q-p}} \int_\Omega |u|^q dx \\
&= \|u\|^r \Big[\big(\frac{(p-r)\|u\|^q}{(q-r)\int_\Omega |u|^q dx}\big)^{\frac{p-r}{q-p}} - \big(\frac{(p-r)\|u\|^{\frac{q(p-r)}{q-r}}}{(q-r)(\int_\Omega |u|^q dx)^{\frac{p-r}{q-r}}}\big)^{\frac{q-r}{q-p}}\Big] \\
&= \|u\|^r \Big[(\frac{p-r}{q-r})^{\frac{p-r}{q-p}} - (\frac{p-r}{q-r})^{\frac{q-r}{q-p}}\Big] \Big(\frac{\|u\|^q}{\int_\Omega |u|^q dx}\Big)^{\frac{p-r}{q-p}} \\
&\geq \|u\|^r (\frac{p-r}{q-r})^{\frac{p-r}{q-p}} (\frac{q-p}{q-r})(\frac{1}{S^q})^{\frac{p-r}{q-p}}. \qquad (2.6)
\end{aligned}
$$

Case (I). $\int_\Omega f(x)|u|^r dx \leq 0$.

There is a unique $t^- > t_{\max}$ such that $s(t^-) = \lambda \int_\Omega f(x)|u|^r dx$ and $s'(t^-) < 0$. Now

$$
\begin{aligned}
& (p-r)\|t^- u\|^p - (q-r)\int_\Omega |t^- u|^q dx \\
= \;& (t^-)^{r+1}\Big((p-r)(t^-)^{p-r-1}\|u\|^p - (q-r)(t^-)^{q-r-1}\int_\Omega |u|^q dx\Big) \\
= \;& (t^-)^{r+1} s'(t^-) < 0,
\end{aligned}
$$

and

$$
\begin{aligned}
& \big\langle J'_\lambda(t^- u), t^- u\big\rangle \\
= \;& (t^-)^p \|u\|^p - (t^-)^q \int_\Omega |u|^q dx - (t^-)^r \lambda \int_\Omega f(x)|u|^r dx \\
= \;& (t^-)^r \Big(s(t^-) - \lambda \int_\Omega f(x)|u|^r dx\Big) = 0.
\end{aligned}
$$

Thus, $t^- u \in M^-_\lambda(\Omega)$. Moreover, since for $t > t_{\max}$,

$$
\frac{d}{dt} J_\lambda(tu) = t^{p-1}\|u\|^p - t^{q-1}\int_\Omega |u|^q dx - t^{r-1}\lambda \int_\Omega f(x)|u|^r dx = 0 \quad \textit{for only } t = t^-,
$$

and

$$
\frac{d^2}{dt^2} J_\lambda(tu) < 0 \qquad \textit{for } t = t^-.
$$

Therefore, $J_\lambda(t^- u) = \max_{t \geq t_{\max}} J_\lambda(tu)$.

Case (II). $\int_\Omega f(x)|u|^r dx > 0$.

By (2.6) and

$$
\begin{aligned}
s(0) = 0 < \lambda \int_\Omega f(x)|u|^r dx \;&\leq\; \lambda \|f\|_{L^{q^*}} S^r \|u\|^r \\
&\leq\; \|u\|^r (\frac{p-r}{q-r})^{\frac{p-r}{q-p}} (\frac{q-p}{q-r})(\frac{1}{S^q})^{\frac{p-r}{q-p}}
\end{aligned}
$$

$$\leq \quad s(t_{\max}) \qquad for\, \lambda \in (0, \lambda_2),$$

there are unique t^+ and t^- such that $0 < t^+ < t_{\max} < t^-$,

$$s(t^+) = \lambda \int_\Omega f(x)|u|^r dx = s(t^-)$$

and

$$s'(t^+) > 0 > s'(t^-).$$

We have $t^+u \in M_\lambda^+(\Omega)$, $t^-u \in M_\lambda^-(\Omega)$, and $J_\lambda(t^-u) \geq J_\lambda(tu) \geq J_\lambda(t^+u)$ for each $t \in [t^+, t^-]$ and $J_\lambda(t^+u) \leq J_\lambda(tu)$ for each $t \in [0, t^+]$. Thus

$$J_\lambda(t^-u) = \max_{t \geq t_{\max}} J_\lambda(tu), \qquad J_\lambda(t^+u) = \min_{0 \leq t \leq t^-} J_\lambda(tu).$$

(ii) By the uniqueness of $t^-(u)$ and the external property of $t^-(u)$, we have that $t^-(u)$ is a continuous function of $u \neq 0$.

(iii) For $u \in M_\lambda^-(\Omega)$, let $v = \frac{u}{\|u\|}$. By part (i), there is unique $t^-(v) > 0$ such that $t^-(v)v \in M_\lambda^-(\Omega)$, that is $t^-(\frac{u}{\|u\|})\frac{1}{\|u\|}u \in M_\lambda^-(\Omega)$. Since $u \in M_\lambda^-(\Omega)$, we have $t^-(\frac{u}{\|u\|})\frac{1}{\|u\|} = 1$, which implies

$$M_\lambda^-(\Omega) \subset \Big\{u \in W_0^{2,p}(\Omega) \setminus \{0\} \mid t^-(\frac{u}{\|u\|})\frac{1}{\|u\|} = 1\Big\}.$$

Conversely, let $u \in W_0^{2,p}(\Omega) \setminus \{0\}$ such that $t^-(\frac{u}{\|u\|})\frac{1}{\|u\|} = 1$, then

$$t^-(\frac{u}{\|u\|})\frac{u}{\|u\|} \in M_\lambda^-(\Omega).$$

Thus,

$$M_\lambda^-(\Omega) = \Big\{u \in W_0^{2,p}(\Omega) \setminus \{0\} \mid t^-(\frac{u}{\|u\|})\frac{1}{\|u\|} = 1\Big\}.$$

(iv) By Case (II) of part (i). □

By $f : \overline{\Omega} \to R$ is continuous function which changes sign in Ω, we have $\Theta = \{x \in \Omega \mid f(x) > 0\}$ is a open set in R^N. Consider the following p-biharmonic equation:

$$\begin{cases} \Delta_p^2 u = |u|^{q-2}u & \text{in } \Theta, \\ u = \nabla u = 0 & \text{on } \partial\Theta. \end{cases} \tag{2.7}$$

Associated with (2.7), we consider the energy functional

$$K(u) = \frac{1}{p}\int_\Omega |\Delta u|^p dx - \frac{1}{q}\int_\Omega |u|^q dx$$

and the minimization problem

$$\beta(\Theta) = \inf \Big\{K(u) \mid u \in N(\Theta)\Big\},$$

where $N(\Theta) = \Big\{ u \in W_0^{2,p}(\Theta) \setminus \{0\} \mid \langle K'(u), u \rangle = 0 \Big\}$. Now we prove that problem (2.7) has a nontrivial solution ω_0 such that $K(\omega_0) = \beta(\Theta) > 0$.

Lemma 2.5. *For any $u \in W_0^{2,p}(\Theta) \setminus \{0\}$ there exists a unique $t(u) > 0$ such that $t(u)u \in N(\Theta)$. The maximum of $K(tu)$ for $t \geq 0$ is achieved at $t = t(u)$, The function*

$$W_0^{2,p}(\Theta) \setminus \{0\} \to (0, +\infty) : u \to t(u)$$

is continuous and the map $u \to t(u)u$ defines a homeomorphism of the unit sphere of $W_0^{2,p}(\Theta)$ with $N(\Theta)$.

Proof. Let $u \in W_0^{2,p}(\Theta) \setminus \{0\}$ be fixed and define the function $g(t) := K(tu)$ on $[0, \infty)$. Clearly we have

$$\begin{aligned} g'(t) = 0 \quad &\Leftrightarrow \quad tu \in N(\Theta) \\ &\Leftrightarrow \quad \|u\|^p = t^{q-p} \int_\Omega |u|^q dx. \end{aligned} \tag{2.8}$$

It is easy to verify that $g(0) = 0$, $g(t) > 0$ for $t > 0$ small and $g(t) < 0$ for $t > 0$ large. Therefore $\max_{[0,\infty)} g(t)$ is achieved at a unique $t = t(u)$ such that $g'(t(u)) = 0$ and $t(u)u \in N(\Theta)$. To prove the continuity of $t(u)$, assume that $u_n \to u$ in $W_0^{2,p}(\Theta) \setminus \{0\}$. It is easy to verify that $\{t(u_n)\}$ is bounded. If a subsequence of $\{t(u_n)\}$ converges to t_0, it follows from (2.8) that $t_0 = t(u)$, But then $t(u_n) \to t(u)$. Finally the continuous map from the unit sphere of $W_0^{2,p}(\Theta)$ to $N(\Theta)$, $u \to t(u)u$, is inverse to the retraction $u \to \frac{u}{\|u\|}$. □

Define

$$\begin{aligned} c_1 &:= \inf_{u \in W_0^{2,p}(\Theta) \setminus \{0\}} \max_{t \geq 0} K(tu), \\ c &:= \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} K\big(\gamma(tu)\big), \end{aligned}$$

where $\Gamma := \Big\{ \gamma \in C\big([0,1], W_0^{2,p}(\Theta)\big) : \gamma(0) = 0, K(\gamma(1)) < 0 \Big\}$.

Lemma 2.6. *$\beta(\Theta) = c_1 = c > 0$ and c is a critical value of K.*

Proof. The lemma 2.5 implies that $\beta(\Theta) = c_1$. Since $K(tu) < 0$ for $u \in W_0^{2,p}(\Theta) \setminus \{0\}$ and t large, we obtain $c \leq c_1$. The manifold $N(\Theta)$ separates $W_0^{2,p}(\Theta)$ into two components. The component containing the origin also contains a small ball around the origin. Moreover $K(u) \geq 0$ for all u in this component, because $\langle K'(tu), u \rangle \geq 0$ for all $0 \leq t \leq t(u)$. Thus every $\gamma \in \Gamma$ has to cross $N(\Theta)$ and $\beta(\Theta) \leq c$. Since the embedding $W_0^{2,p}(\Theta) \hookrightarrow L^q(\Theta)$ is compact, it is easy to prove that $c > 0$ is a critical value of K and ω_0 a nontrivial solution corresponding to c. □

With the help of Lemma 2.6, we have the following result.

Lemma 2.7.
(i) *There exists* $\tilde{t} > 0$ *such that*

$$\alpha_\lambda(\Omega) \leq \alpha_\lambda^+(\Omega) < \frac{r-p}{r}\tilde{t}^p \beta(\Theta) < 0;$$

(ii) J_λ *is coercive and bounded below on* $M_\lambda(\Omega)$ *for all* $\lambda \in (0, \frac{q-p}{q-r}]$.

Proof. **(i)** Let ω_0 be a nontrivial solution of problem (2.7) such that $K(\omega_0) = \beta(\Theta) > 0$. Then

$$\int_\Omega f(x)|\omega_0|^r dx = \int_\Theta f(x)|\omega_0|^r dx > 0.$$

Set $\tilde{t} = t^+(\omega_0)$ as defined by Lemma 2.4(iv). Hence $\tilde{t}\omega_0 \in M_\lambda^+(\Omega)$ and

$$\begin{aligned} J_\lambda(\tilde{t}\omega_0) &= \frac{\tilde{t}^p}{p}\int_\Omega |\Delta\omega_0|^p dx - \frac{\tilde{t}^q}{q}\int_\Omega |\omega_0|^q dx - \frac{\lambda\tilde{t}^r}{r}\int_\Omega f(x)|\omega_0|^r dx \\ &= (\frac{1}{p} - \frac{1}{r})\tilde{t}^p \int_\Omega |\Delta\omega_0|^p dx + (\frac{1}{r} - \frac{1}{q})\tilde{t}^q \int_\Omega |\omega_0|^q dx \\ &< \frac{r-p}{r}\tilde{t}^p \beta(\Theta) < 0. \end{aligned}$$

This yields

$$\alpha_\lambda(\Omega) \leq \alpha_\lambda^+(\Omega) < \frac{r-p}{r}\tilde{t}^p \beta(\Theta) < 0.$$

(ii) For $u \in M_\lambda(\Omega)$, we have $\int_\Omega |\Delta u|^p dx = \int_\Omega |u|^q dx + \int_\Omega f(x)|u|^r dx$. Then by the Hölder and Young inequality

$$\begin{aligned} J_\lambda(u) &= \frac{q-p}{pq}\int_\Omega |\Delta u|^p dx - \lambda\frac{q-r}{qr}\int_\Omega f(x)|u|^r dx \\ &\geq \frac{q-p}{pq}\int_\Omega |\Delta u|^p dx - \lambda\frac{q-r}{qr}\|f\|_{L^{q^*}} S^r \|u\|^r \\ &\geq \frac{1}{qp}\Big[(q-p) - \lambda(q-r)\Big]\|u\|^p - \lambda\frac{(q-r)(p-r)}{qpr}(\|f\|_{L^{q^*}} S^r)^{\frac{p}{p-r}}. \end{aligned}$$

Thus J_λ is coercive on $M_\lambda(\Omega)$ and

$$J_\lambda(u) \geq -\lambda\frac{(q-r)(p-r)}{qpr}(\|f\|_{L^{q^*}} S^r)^{\frac{p}{p-r}}$$

for all $\lambda \in (0, \frac{q-p}{q-r}]$. □

3 Proof of Theorem 1

For the proof of theorem, we need the following lemmas.

Lemma 3.1. *For* $u \in M_\lambda(\Omega)$, *there exist* $\epsilon > 0$ *and a differentiable function*

$\xi : B(0;\epsilon) \subset W_0^{2,p}(\Omega) \to R^+$ *such that* $\xi(0) = 1$, *the function* $\xi(v)(u-v) \in M_\lambda(\Omega)$ *and*

$$\langle \xi'(0), v\rangle = \frac{p\int_\Omega |\Delta u|^{p-2}\Delta u \Delta v dx - q\int_\Omega |u|^{q-2}uvdx - r\lambda \int_\Omega f(x)|u|^{r-2}uvdx}{(p-r)\int_\Omega |\Delta u|^p dx - (q-r)\int_\Omega |u|^q dx} \tag{3.1}$$

for all $v \in W_0^{2,p}(\Omega)$.

Proof. For $u \in M_\lambda(\Omega)$, define a function $F : R \times W_0^{2,p}(\Omega) \to R$ by

$$\begin{aligned} F_u(\xi,\omega) &= \langle J'_\lambda(\xi(u-\omega)), \xi(u-\omega)\rangle \\ &= \xi^p \int_\Omega |\Delta(u-\omega)|^p dx - \xi^q \int_\Omega |u-\omega|^q dx - \xi^r \lambda \int_\Omega f(x)|u-\omega|^r dx. \end{aligned}$$

Then $F_u(1,0) = \langle J'_\lambda(u), u\rangle = 0$ and

$$\begin{aligned} \frac{d}{dt}F_u(1,0) &= p\int_\Omega |\Delta u|^p dx - q\int_\Omega |u|^q dx - r\lambda \int_\Omega f(x)|u|^r dx \\ &= (p-r)\int_\Omega |\Delta u|^p dx - (q-r)\int_\Omega |u|^q dx \neq 0. \end{aligned}$$

According to the implicit function theorem, there exist $\epsilon > 0$ and a differentiable function $\xi : B(0;\epsilon) \subset W_0^{2,p}(\Omega)) \to R^+$ such that $\xi(0) = 1$ and

$$\langle \xi'(0), v\rangle = \frac{p\int_\Omega |\Delta u|^{p-2}\Delta u \Delta v dx - q\int_\Omega |u|^{q-2}uvdx - r\lambda \int_\Omega f(x)|u|^{r-2}uvdx}{(p-r)\int_\Omega |\Delta u|^p dx - (q-r)\int_\Omega |u|^q dx}$$

and

$$F_u(\xi(v), v) = 0 \qquad \textit{for all } v \in B(0;\epsilon),$$

which is equivalent to

$$\Big\langle J'_\lambda(\xi(v)(u-v)), \xi(v)(u-v)\Big\rangle = 0 \qquad \textit{for all } v \in B(0;\epsilon),$$

that is $\xi(v)(u-v) \in M_\lambda(\Omega)$. □

Similarity, we have

Lemma 3.2. *For each* $u \in M_\lambda^-(\Omega)$, *there exist* $\epsilon > 0$ *and a differentiable function* $\xi^- : B(0;\epsilon) \subset W_0^{2,p}(\Omega) \to R^+$ *such that* $\xi^-(0) = 1$, *the function* $\xi^-(v)(u-v) \in M_\lambda^-(\Omega)$ *and*

$$\langle (\xi^-)'(0), v\rangle = \frac{p\int_\Omega |\Delta u|^{p-2}\Delta u \Delta v dx - q\int_\Omega |u|^{q-2}uvdx - r\lambda \int_\Omega f(x)|u|^{r-2}uvdx}{(p-r)\int_\Omega |\Delta u|^p dx - (q-r)\int_\Omega |u|^q dx} \tag{3.2}$$

for all $v \in W_0^{2,p}(\Omega)$.

Proof. Similar to the proof in Lemma 3.1, there exist $\epsilon > 0$ and a differentiable function $\xi^- : B(0;\epsilon) \subset W_0^{2,p}(\Omega) \to R^+$ such that $\xi^-(0) = 1$ and $\xi^-(v)(u-v) \in M_\lambda(\Omega)$ for all $v \in B(0;\epsilon)$. Since

$$\langle \psi'_\lambda(u), u\rangle = (p-r)\|u\|^p - (q-r)\int_\Omega |u|^q dx < 0.$$

Thus, by the continuity of the function ψ'_λ and ξ^-, we have

$$\begin{aligned}&\left\langle \psi'_\lambda(\xi^-(v)(u-v)), \xi^-(v)(u-v)\right\rangle\\ = \;& (p-r)\|\xi^-(v)(u-v)\|^p - (q-r)\int_\Omega |\xi^-(v)(u-v)|^q dx < 0.\end{aligned}$$

If ϵ sufficiently small, this implies that $\xi^-(v)(u-v) \in M^-_\lambda(\Omega)$. □

Proposition 3.1. *Let* $\lambda_0 = \inf\{\lambda_1, \lambda_2, \frac{q-p}{q-r}\}$*, for* $\lambda \in (0, \lambda_0)$.
(i) *There exists a minimizing sequence* $\{u_n\} \subset M_\lambda(\Omega)$ *such that*

$$\begin{aligned}J_\lambda(u_n) &= \alpha_\lambda(\Omega) + o(1),\\ J'_\lambda(u_n) &= o(1), \qquad \textit{for } (W_0^{2,p}(\Omega))^*;\end{aligned}$$

(ii) *There exists a minimizing sequence* $\{u_n\} \subset M^-_\lambda(\Omega)$ *such that*

$$\begin{aligned}J_\lambda(u_n) &= \alpha^-_\lambda(\Omega) + o(1),\\ J'_\lambda(u_n) &= o(1), \qquad \textit{for } (W_0^{2,p}(\Omega))^*.\end{aligned}$$

Proof. (i) By Lemma 2.7(ii) and the Ekeland variational principle[15], there exists a minimizing sequence $\{u_n\} \subset M_\lambda(\Omega)$ such that

$$J_\lambda(u_n) < \alpha_\lambda(\Omega) + \frac{1}{n}, \tag{3.3}$$

and

$$J_\lambda(u_n) < J_\lambda(\omega) + \frac{1}{n}\|\omega - u_n\| \textit{ for each } \omega \in M_\lambda(\Omega). \tag{3.4}$$

By taking n enough large, from Lemma 2.7(i), we have

$$\begin{aligned}J_\lambda(u_n) &= (\frac{1}{p}-\frac{1}{q})\|u_n\|^p - (\frac{1}{r}-\frac{1}{q})\lambda\int_\Omega f(x)|u_n|^r dx\\ &< \alpha_\lambda(\Omega) + \frac{1}{n} < \frac{r-p}{r}\tilde{t}^p\beta(\Theta) < 0.\end{aligned} \tag{3.5}$$

This implies

$$\|f\|_{L^{q^*}} S^r\|u_n\|^r \geq \int_\Omega f(x)|u_n|^r dx > \frac{q(p-r)}{\lambda(q-r)}\tilde{t}^p\beta(\Theta). \tag{3.6}$$

Consequently $u_n \neq 0$ and putting together (3.5), (3.6) and the Hölder inequality, we obtain

$$\|u_n\| \geq \Big[\frac{q(p-r)}{\lambda(q-r)}\frac{\tilde{t}^p}{\|f\|_{L^{q^*}} S^r}\beta(\Theta)\Big]^{\frac{1}{r}}, \tag{3.7}$$

and

$$\|u_n\| \leq \Big[\frac{\lambda p(q-r)}{r(q-p)}\|f\|_{L^{q^*}} S^r\Big]^{\frac{1}{p-r}}. \tag{3.8}$$

Now we show that

$$\|J'_\lambda(u_n)\|_{(W_0^{2,p}(\Omega))^*} \to 0 \ as \ n \to \infty.$$

Applying Lemma 3.1 with u_n to obtain the function $\xi_n : B(0;\epsilon_n) \subset W_0^{2,p}(\Omega) \to R^+$ for some $\epsilon_n > 0$, such that $\xi_n(\omega)(u_n - \omega) \in M_\lambda(\Omega)$. Choose $0 < \rho < \epsilon_n$. Let $u \in W_0^{2,p}(\Omega)$ with $u \not\equiv 0$ and let $\omega_\rho = \frac{\rho u}{\|u\|}$. We set $\eta_\rho = \xi_n(\omega_\rho)(u_n - \omega_\rho)$. Since $\eta_\rho \in M_\lambda(\Omega)$, we deduce from (3.4) that

$$J_\lambda(\eta_\rho) - J_\lambda(u_n) \geq -\frac{1}{n}\|\eta_\rho - u_n\|,$$

and by the mean value theorem, we have

$$\langle J'_\lambda(u_n), \eta_\rho - u_n\rangle + o(\|\eta_\rho - u_n\|) \geq -\frac{1}{n}\|\eta_\rho - u_n\|.$$

Thus,

$$\begin{aligned} &\langle J'_\lambda(u_n), -\omega_\rho\rangle + (\xi_n(\omega_\rho) - 1)\langle J'_\lambda(u_n), (u_n - \omega_\rho)\rangle \\ \geq\ & -\frac{1}{n}\|\eta_\rho - u_n\| + o(\|\eta_\rho - u_n\|). \end{aligned} \tag{3.9}$$

From $\xi_n(\omega_\rho)(u_n - \omega_\rho) \in M_\lambda(\Omega)$ and (3.9) it follows that

$$\begin{aligned} &-\rho\langle J'_\lambda(u_n), \frac{u}{\|u\|}\rangle + (\xi_n(\omega_\rho) - 1)\langle J'_\lambda(u_n) - J'_\lambda(\eta_\rho), (u_n - \omega_\rho)\rangle \\ \geq\ & -\frac{1}{n}\|\eta_\rho - u_n\| + o(\|\eta_\rho - u_n\|). \end{aligned}$$

Thus,

$$\begin{aligned} \langle J'_\lambda(u_n), \frac{u}{\|u\|}\rangle \ \leq\ & \frac{(\xi_n(\omega_\rho) - 1)}{\rho}\langle J'_\lambda(u_n) - J'_\lambda(\eta_\rho), (u_n - \omega_\rho)\rangle \\ & + \frac{1}{n\rho}\|\eta_\rho - u_n\| + \frac{o(\|\eta_\rho - u_n\|)}{\rho}. \end{aligned} \tag{3.10}$$

Since

$$\|\eta_\rho - u_n\| \leq |\xi_n(\omega_\rho) - 1|\|u_n\| + \rho|\xi_n(\omega_\rho)|$$

and

$$\lim_{\rho\to 0}\frac{|\xi_n(\omega_\rho) - 1|}{\rho} \leq \|\xi'_n(0)\|.$$

If we let $\rho \to 0$ in (3.10) for a fixed n, then by (3.8) we can find a constant $C > 0$, independent of ρ, such that

$$\langle J'_\lambda(u_n), \frac{u}{\|u\|}\rangle \leq \frac{C}{n}(1 + \|\xi'_n(0)\|).$$

We are done once we show that $\|\xi'_n(0)\|$ is uniformly bounded in n. By (3.1), (3.8) and Hölder inequality, we have

$$\langle \xi'_n(0), v\rangle \leq \frac{b\|v\|}{|(p-r)\int_\Omega |\Delta u_n|^p dx - (q-r)\int_\Omega |u_n|^q dx|} \ for\ some\ b > 0.$$

We only need to show that

$$\Big|(p-r)\int_\Omega |\Delta u_n|^p dx-(q-r)\int_\Omega |u_n|^q dx\Big|>c \tag{3.11}$$

for some $c>0$ and n large enough. We argue by contradiction. Assume that there exists a subsequence $\{u_n\}$ such that

$$(p-r)\int_\Omega |\Delta u_n|^p dx-(q-r)\int_\Omega |u_n|^q dx=o(1). \tag{3.12}$$

Combining (3.12) with (3.7), we can find a suitable constant $d>0$ such that

$$\int_\Omega |u_n|^q dx\geq d \qquad \textit{for } n \textit{ sufficiently large.} \tag{3.13}$$

In addition (3.12), and the fact $\{u_n\}\subset M_\lambda(\Omega)$ also give

$$\lambda\int_\Omega f(x)|u_n|^r dx=\|u_n\|^p-\int_\Omega |u_n|^q dx>\|u_n\|^p>\frac{q-p}{p-r}\int_\Omega |u_n|^q dx>0$$

and

$$\|u_n\|\leq\Big(\lambda(\frac{q-r}{q-p})\|f\|_{L^{q^*}}S^r\Big)^{\frac{1}{p-r}}+o(1). \tag{3.14}$$

This implies

$$\begin{aligned}
I_\lambda(u_n) &= K(q,r)\Big(\frac{\|u_n\|^q}{\int_\Omega |u_n|^q dx}\Big)^{\frac{p}{q-p}}-\lambda\int_\Omega f(x)|u_n|^r dx\\
&= \Big(K(q,r)(\frac{q-r}{p-r})^{\frac{q}{q-p}}-\frac{q-p}{p-r}\Big)\int_\Omega |u_n|^q dx+o(1)\\
&= o(1).
\end{aligned} \tag{3.15}$$

However, by (3.13), (3.14) and $\lambda\in(0,\lambda_0)$,

$$\begin{aligned}
I_\lambda(u_n) &\geq K(q,r)\Big(\frac{\|u_n\|^q}{\int_\Omega |u_n|^q dx}\Big)^{\frac{p}{q-p}}-\lambda\|f\|_{L^{q^*}}\|u_n\|^r_{L^q}\\
&\geq \|u_n\|^r_{L^q}\Big(K(q,r)(\frac{\|u_n\|^q}{S^{\frac{r(q-p)+pq}{p}}\|u_n\|^{\frac{r(q-p)+pq}{p}}})^{\frac{p}{q-p}}-\lambda\|f\|_{L^{q^*}}\Big)\\
&= \|u_n\|^r_{L^q}\Big(K(q,r)\frac{1}{S^{\frac{r(q-p)+pq}{q-p}}}\|u_n\|^{-r}-\lambda\|f\|_{L^{q^*}}\Big)\\
&\geq \|u_n\|^r_{L^q}\Big\{K(q,r)\frac{1}{S^{\frac{r(q-p)+pq}{q-p}}}\lambda^{\frac{-r}{p-r}}\big[(\frac{q-r}{q-p})\|f\|_{L^{q^*}}S^r\big]^{\frac{-r}{p-r}}-\lambda\|f\|_{L^{q^*}}\Big\},
\end{aligned}$$

This contradicts (3.15). We get

$$\langle J'_\lambda(u_n),\frac{u}{\|u\|}\rangle\leq\frac{C}{n}.$$

The proof is complete.

(ii) Similar to the proof of (i), we may prove (ii). □

Now, we establish the existence of a local minimum for J_λ on $M_\lambda^+(\Omega)$.

Theorem 3.1. *Let λ_0 as in Proposition 3.1, then for $\lambda \in (0, \lambda_0)$, the functional J_λ has a minimizer $u_0^+ \in M_\lambda^+(\Omega)$ and it satisfies*
(i) $J_\lambda(u_0^+) = \alpha_\lambda(\Omega) = \alpha_\lambda^+(\Omega)$ *;*
(ii) u_0^+ *is a nontrivial solution of problem (0.1);*
(iii) $J_\lambda(u_0^+) \to 0$ *as* $\lambda \to 0$.

Proof. Let $\{u_n\} \subset M_\lambda(\Omega)$ is a minimizing sequence for J_λ on $M_\lambda(\Omega)$ such that

$$\begin{aligned} &J_\lambda(u_n) = \alpha_\lambda(\Omega) + o(1), \\ &J'_\lambda(u_n) = o(1), \qquad \textit{for } (W_0^{2,p}(\Omega))^*. \end{aligned}$$

Then by Lemma 2.7 and the compact imbedding theorem, there exists a subsequence $\{u_n\}$ and $u_0^+ \in W_0^{2,p}(\Omega)$ such that

$$\begin{aligned} u_n &\rightharpoonup u_0^+ \qquad \textit{weakly in} \quad W_0^{2,p}(\Omega) \\ u_n &\to u_0^+ \qquad \textit{strongly in} \quad L^q(\Omega) \end{aligned}$$

and

$$u_n \to u_0^+ \qquad \textit{strongly in} \quad L^r(\Omega). \tag{3.16}$$

We firstly show that $\int_\Omega f(x)|u_0^+|^r dx \neq 0$. If not, by (3.16) we can conclude that

$$\int_\Omega f(x)|u_0^+|^r dx = 0$$

and

$$\int_\Omega f(x)|u_n|^r dx \to 0 \qquad as \qquad n \to \infty.$$

Thus,

$$\int_\Omega |\Delta u_n|^p dx = \int_\Omega |u_n|^q dx + o(1)$$

$$\begin{aligned} J_\lambda(u_n) &= \frac{1}{p}\int_\Omega |\Delta u_n|^p dx - \frac{1}{q}\int_\Omega |u_n|^q dx - \frac{\lambda}{r}\int_\Omega f(x)|u_n|^r dx \\ &= (\frac{1}{p} - \frac{1}{q})\int_\Omega |u_n|^q dx + o(1) \\ &= (\frac{1}{p} - \frac{1}{q})\int_\Omega |u_0^+|^q dx \qquad as \quad n \to \infty, \end{aligned}$$

this contradicts $J_\lambda(u_n) \to \alpha_\lambda(\Omega) < 0$ as $n \to \infty$. In particular, $u_0^+ \in M_\lambda^+(\Omega)$ is a nontrivial solution of problem (1.1) and $J_\lambda(u_0^+) \geq \alpha_\lambda(\Omega)$. We now prove that $u_n \rightharpoonup u_0^+$ strongly in $W_0^{2,p}(\Omega)$. Supposing the contrary, then $\|u_0^+\| < \liminf_{n\to\infty} \|u_n\|$ and so

$$\|u_0^+\|^p - \int_\Omega |u_0^+|^q dx - \lambda \int_\Omega f(x)|u_0^+|^r dx$$

$$< \liminf_{n\to\infty}\Big(\|u_n\|^p - \int_\Omega |u_n|^q dx - \lambda\int_\Omega f(x)|u_n|^r dx\Big) = 0,$$

this contradicts $u_0^+ \in M_\lambda(\Omega)$. In fact, if $u_0^+ \in M_\lambda^-(\Omega)$, by Lemma 2.4, there are unique t_0^+ *and* t_0^- such that $t_0^+u_0^+ \in M_\lambda^+(\Omega)$ *and* $t_0^-u_0^+ \in M_\lambda^-(\Omega)$, we have $t_0^+ < t_0^- = 1$. Since

$$\frac{d}{dt}J_\lambda(t_0^+u_0^+) = 0 \quad \textit{and} \quad \frac{d^2}{dt^2}J_\lambda(t_0^+u_0^+) > 0,$$

there exists $t_0^+ < \bar{t} \le t_0^-$ such that $J_\lambda(t_0^+u_0^+) < J_\lambda(\bar{t}u_0^+)$. By

$$J_\lambda(t_0^+u_0^+) < J_\lambda(\bar{t}u_0^+) \le J_\lambda(t_0^-u_0^+) = J_\lambda(u_0^+),$$

which is a contradiction. By Lemma 2.3, we know that u_0^+ is a nontrivial solution. Moreover, by Lemma 2.7,

$$0 > J_\lambda(u_0^+) \ge -\lambda\frac{(q-r)(p-r)}{qpr}(\|f\|_{L^{q^*}}S^r)^{\frac{p}{p-r}},$$

it is clear that $J_\lambda(u_0^+) \to 0$ as $\lambda \to 0$. □

Next, we establish the existence of a local minimum for J_λ on $M_\lambda^-(\Omega)$.

Theorem 3.2. *Let λ_0 as in Proposition 3.1, then for $\lambda \in (0, \lambda_0)$, the functional J_λ has a minimizer $u_0^- \in M_\lambda^-(\Omega)$ and it satisfies*
(i) $J_\lambda(u_0^-) = \alpha_\lambda^-(\Omega)$*;*
(ii) u_0^- *is a nontrivial solution of problem (0.1).*

Proof. Let $\{u_n\}$ is a minimizing sequence for J_λ on $M_\lambda^-(\Omega)$ such that

$$\begin{aligned} &J_\lambda(u_n) = \alpha_\lambda^-(\Omega) + o(1),\\ &J'_\lambda(u_n) = o(1), \qquad \textit{for } (W_0^{2,p}(\Omega))^*. \end{aligned}$$

Then by Proposition 3.1(ii) and the compact imbedding theorem, there exists a subsequence $\{u_n\}$ and $u_0^- \in M_\lambda^-(\Omega)$ such that

$$\begin{aligned} u_n &\rightharpoonup u_0^- \qquad \textit{weakly in} \quad W_0^{2,p}(\Omega)\\ u_n &\to u_0^- \qquad \textit{strongly in} \quad L^q(\Omega) \end{aligned}$$

and

$$u_n \to u_0^- \qquad \textit{strongly in} \quad L^r(\Omega). \tag{3.17}$$

We now prove that $u_n \to u_0^-$ strongly in $W_0^{2,p}(\Omega)$. Supposing the contrary, then $\|u_0^-\| < \liminf_{n\to\infty}\|u_n\|$ and so

$$\begin{aligned} &\|u_0^-\|^p - \int_\Omega |u_0^-|^q dx - \lambda\int_\Omega f(x)|u_0^-|^r dx\\ < &\liminf_{n\to\infty}\Big(\|u_n\|^p - \int_\Omega |u_n|^q dx - \lambda\int_\Omega f(x)|u_n|^r dx\Big) = 0, \end{aligned}$$

this contradicts $u_0^- \in M_\lambda^-(\Omega)$. Hence $u_n \to u_0^-$ strongly in $W_0^{2,p}(\Omega)$. This implies

$$J_\lambda(u_n) \to J_\lambda(u_0^-) = \alpha_\lambda^-(\Omega) \qquad as \quad as\ n \to \infty.$$

By Lemma 2.3, we know that u_0^- is a nontrivial solution. $\square$

Combing with Theorem 3.1 and Theorem 3.2, for problem (0.1) there exist two nontrivial solution u_0^+ and u_0^- such that $u_0^+ \in M_\lambda^+(\Omega)$, $u_0^- \in M_\lambda^-(\Omega)$. Since $M_\lambda^+(\Omega) \bigcap M_\lambda^+(\Omega) = \emptyset$, this shows that u_0^+ and u_0^- are different.

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