ON SINGULAR SOLUTIONS FOR SECOND ORDER DELAYED DIFFERENTIAL EQUATIONS

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Abstract. Asymptotic properties and estimate of singular solutions (either defined on a finite interval only or trivial in a neighbourhood of \(\infty\)) of the second order delay differential equation with \(p\)-Laplacian are investigated.

1. Introduction

In this paper, we consider the second order nonlinear delay differential equation

\[
(1) \quad (a(t)|y'|^{p-1}y')' + r(t)|y(\varphi(t))|^{\lambda} \sgn y(\varphi(t)) = 0
\]

where \(p > 0\), \(\lambda > 0\), \(a \in C^0(\mathbb{R}_+)\), \(r \in C^0(\mathbb{R}_+)\), \(a(t) > 0\), \(r(t) > 0\), \(\varphi(t) \leq t\) on \(\mathbb{R}_+\) and \(\lim_{t \to \infty} \varphi(t) = \infty\).

If \(p = \lambda\), it is known as the half-linear equation, while if \(\lambda > p\), we say that equation (1) is of the super-half-linear type, and if \(\lambda < p\), we will say that it is of the sub-half-linear type.

We begin by defining what is meant by a solution of equation (1) as well as some basic properties of solutions.

**Definition 1.** Let \(T \in (0, \infty]\), \(\varphi_0 = \inf_{t \in \mathbb{R}_+} \varphi(t)\), \(\phi \in C^0[\varphi_0, 0]\), and \(y'_0 \in \mathbb{R}\). We say that a function \(y\) is a solution of (1) on \([0, T)\) (with the initial conditions \((\varphi, y'_0)\)) if \(y \in C^1[\varphi_0, T), y \in C^1[0, T], a|y'|^{p-1}y' \in C^1[0, T], (1)\) holds on \([0, T), y(t) = \phi(t)\) on \([\varphi_0, 0)\), and \(y'_0 = y'_0\).

We assume that solutions are defined on their maximal interval of existence to the right.

Equation (1) can be written as the equivalent system

\[
(2) \quad \begin{align*}
y'_1 &= a^{-\frac{1}{p}}(t)|y_2|^{\frac{1}{p}} \sgn y_2, \\
y'_2 &= -r(t)|y(\varphi(t))|^{\lambda} \sgn y(\varphi(t)).
\end{align*}
\]

The relationship between a solution \(y\) of (1) and a solution \((y_1, y_2)\) of the system (2) is

\[
(3) \quad y_1(t) = y(t) \quad \text{and} \quad y_2(t) = a(t)|y'(t)|^{p-1}y'(t),
\]

and when discussing a solution \(y\) of (1), we will often use (3) without mention.

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Definition 2. Let $y$ be a solution of (1) defined on $[0,T)$, $T \leq \infty$. It is called singular of the 1st kind if $T = \infty$, $\tau \in (0, \infty)$ exists such that $y \equiv 0$ on $[\tau, \infty)$ and $y$ is nontrivial in any left neighbourhood of $\tau$. Solution $y$ is called singular of the 2nd kind if $T < \infty$ and put $\tau = T$. It is called proper if $T = \infty$ and it is nontrivial in any neighbourhood of $\infty$. Singular solutions of either 1st or 2nd kind are called singular.

Note, that a solution of (1) is either proper, or singular or trivial on $(\varphi_0, \infty)$.

Remark 1. If $y$ is a singular solution of (1) of the 2nd kind, then it is defined on $[0, \tau)$, $\tau < \infty$ and it cannot be defined at $t = \tau$; so, $\limsup_{t \to \tau} (|y_1(t)| + |y_2(t)|) = \infty$.

Definition 3. Let $y$ be a singular solution of (1) of the 1st kind (of the 2nd kind). Then it is called oscillatory if there exists a sequence of its zeros tending to $\tau$ and it is called nonoscillatory otherwise.

Singular solutions of (1) without delay, i.e. of
\begin{equation}
(a(t)|y'|^{p-1}y')' + r(t)|y|^{\lambda} \operatorname{sgn} y = 0,
\end{equation}
have been studied by many authors, see e.g. [1, 5], [9]–[16] and the references therein. Note, that the first existence results are obtained in [12] for $p = 1$, $a = 1$ and $r \leq 0$.

In the monography of Kiguradze and Chanturia [13] it is a good overview of results for $p = 1$ and $a = 1$.

Eq. (5) may have singular solutions. Heidel [11] (Coffman, Ulrych [9]) proved the existence of an equation of type (5), $a \equiv 1$, $p = 1$ with singular solutions of the 1st kind (of the 2nd kind) in case $\lambda < p$ ($\lambda > p$); in this case $r$ is continuous but not of locally bounded variation. If $a$ and $r$ are smooth enough, then singular solutions of (5) do not exist (see Theorem A below). As concerns to Eq. (1), the existence of singular solutions of the second kind are investigated in [4] in case $r \leq 0$. The existence and properties of singular solutions of either the first kind or of the second kind in case $r \geq 0$ seem not to be studied at all.

The following theorem sums up results concerning to Eq. (5).

Theorem A. Let $r \in C^0(\mathbb{R}_+)$ and $r(t) > 0$ on $\mathbb{R}_+$.

(i) If $\lambda \geq p$, then there exists no singular solution of (5) of the 1st kind.

(ii) If $\lambda \leq p$, then there exists no singular solution of (5) of the 2nd kind.

(iii) If $a \neq 0$, then all solutions of (5) are proper.

Proof. (i), (ii): See Theorems 1.1 and 1.2 in [15]. (iii): It follows from Theorem 2 in [5].

Note that estimates of such kind of solutions are proved by Kvinikadze, see references in [13]. In [1] (for $p = 1$, $a = 1$, $r \leq 0$) precise asymptotic formulas of all
solutions are obtained for differential equations of the third and fourth orders, see also [3]. About uniform estimates of solutions of quasi-linear ordinary differential equations see [2]. In [16] estimates of singular solutions of the second kind of a system of second order differential equations (of the form (5)) are derived.

Theorem B ([16], Theorem 2). Let \( r \in C^0(\mathbb{R}_+) \) and \( r(t) > 0 \) on \( \mathbb{R}_+ \). Let \( \lambda > p, y \) be a singular solution of (5) of the second kind, \( T \in [0, \tau), \tau - T \leq 1, r_0 = \max_{T \leq s \leq \tau} r(s), C_0 = 2^{\lambda+2} \) in case \( p > 1 \) and \( C_0 = 2^{2\lambda+1} \) in case \( p \leq 1 \). Then a positive constant \( C = C(p, \lambda, r_0) \) exists such that

\[
|y_2(t)| + C_0 r_0 |y(t)|^\lambda \geq C(\tau - t)^{-\frac{\lambda+1}{p^2-1}}, \quad t \in [T, \tau).
\]

It is important to study the existence of proper/singular solutions. When studying solutions of (1) and (5), some authors sometimes investigate properties of solutions that are defined on \( \mathbb{R}_+ \) only without proving the existence of them. Moreover, sometimes, proper solutions have crucial role in a definition of some problems, see e.g. the limit-point/limit-circle problem in [6], [8]. Furthermore, noncontinuable solutions appear e.g. in water flow problems (flood waves, a flow in sewerage systems), see e.g. [4].

Our goal is to study properties of singular solutions and to extend Theorems A and B to (1).

For convenience, we define the constants and the function

\[
\delta = \frac{p+1}{p}, \quad \gamma = \frac{p+1}{p(\lambda+1)}, \quad R(t) = a^{\frac{1}{p}}(t) r(t), \quad t \in \mathbb{R}_+.
\]

If \( y \) is a solution of (1), then we set on its interval of existence

\[
F(t) = R^{-1}(t)|y_2(t)|^\delta + \gamma |y(t)|^{\lambda+1}.
\]

Notice that \( F(t) \geq 0 \) for every solution of (1) and

\[
F'(t) = -\frac{R'(t)}{R^2(t)}|y_2(t)|^\delta + \delta y'(t) e(t)
\]

with

\[
e(t) \overset{\text{def}}{=} |y(t)|^\lambda \text{sgn} y(t) - |y(\varphi(t))|^{\lambda} \text{sgn} y(\varphi(t)).
\]

From (6)

\[
|y(t)| \leq (\gamma^{-1} F(t))^{\frac{1}{\lambda+1}}, \quad |y_2(t)| \leq \left[R(t) F(t)\right]^{\frac{\lambda}{\lambda+1}},
\]

\[
|y'(t)| \leq a^{-\frac{1}{p}}(t) R^{\frac{1}{p+1}}(t) F^{\frac{1}{p+1}}(t).
\]

2. Singular solutions of the 2nd kind

The following theorem shows that such solutions do not exist in case \( \lambda \leq p \).

**Theorem 1.** If \( \lambda \leq p \), then all solutions of (1) are defined on \( \mathbb{R}_+ \).

**Proof.** It is proved in Lemma 7 in [6] for \( r < 0 \), for arbitrary \( r \) the proof is the same, it is necessary to replace \( r \) by \(|r|\). □

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The following theorem gives us basic properties.

**Theorem 2.** Let $y$ be a singular solution of (1) of the second kind. Then it is oscillatory and $\varphi(\tau) = \tau$. If, moreover, $R \in C^1(\mathbb{R}_+)$, then $\varphi(t) \not\equiv t$ in any left neighbourhood of $\tau$.

**Proof.** Suppose, contrarily, that $\varphi(\tau) < \tau$. Then an interval $I = [\tau_1, \tau)$ exists such that $\tau_1 < \tau$ and $\sup_{t \in I} \varphi(t) < \tau$. From this and from (1) we have $|y'(t)| = r(t)|y(\varphi(t))|^\lambda \leq \sup_{t \in I} r(t)|y(\varphi(t))|^\lambda < \infty$. Hence, $y_2$ is bounded on $I$ that contradicts (4). Hence, $\varphi(\tau) = \tau$.

Let $y$ be nonoscillatory. Suppose, for the simplicity, that $y$ is positive in a left neighbourhood of $\tau$. Then, with respect to $\varphi(\tau) = \tau$, $\tau_1 < \tau$ exists such that
\[(10)\quad y(\varphi(t)) > 0 \quad \text{on} \quad I \overset{\text{def}}{=} [\tau_1, \tau).
\]
As according to (2) and (10), $y_2$ is decreasing on $I$ and (4) implies
\[(11)\quad \lim_{t \to \tau^-} y_2(t) = -\infty.
\]
From this $\tau_2 \in I$ exists such that
\[(12)\quad y'(t) < 0 \quad \text{on} \quad [\tau_2, \tau)
\]
and the integration of (1) and (11)
\[\int_{\tau_2}^\tau r(t)y^\lambda(\varphi(t))\, dt = y_2(\tau_2) - \lim_{t \to \tau^-} y_2(t) = \infty.
\]
Hence, $\limsup_{t \to \tau^-} y(t) = \infty$ that contradicts (12) and $y$ is oscillatory.

Let $y$ be a singular solution of (1) and $\varphi(t) \equiv t$ on a left neighbourhood $J$ on $\tau$. Then $y$ is a singular solution of (5) on $J$. A contradiction with Theorem A(iii) proves that $\varphi(t) \not\equiv t$ in any left neighbourhood of $\tau$. $\Box$

**Remark 2.** According to Theorem 1 there exists no singular solution of (1) of the second kind in case $\varphi(t) < t$ on $\mathbb{R}_+$; all solutions are defined on $\mathbb{R}_+$. This fact was used by many authors for special types of (1), see e.g. [10], [4] ($r < 0$).

The following two lemmas serve us for estimate of solutions.

**Lemma 1.** Let $\omega > 1$, $t_0 \in \mathbb{R}_+$, $K > 0$, $Q$ be a continuous nonnegative function on $[t_0, \infty)$ and $u$ be continuous and nonnegative on $[t_0, \infty)$ satisfying
\[(13)\quad u(t) \leq K + \int_{t_0}^t Q(s)u^\omega(s)\, ds \quad \text{on} \quad [t_0, T), T \leq \infty.
\]
If
\[(14)\quad (\omega - 1)K^{\omega-1}\int_{t_0}^\infty Q(s)\, ds < 1
\]
then
\[(15)\quad u(t) \leq K[1 - (\omega - 1)K^{\omega-1}\int_{t_0}^t Q(s)\, ds]^{1/(1-\omega)}, \quad t \in [t_0, T).
\]

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Proof. It is proved in Lemma 2.1 in [14] for \( m = \omega \) and \( p = 1 \). \( \square \)

**Lemma 2.** Let \( \lambda > p \), \( \int_0^\infty r(s)\left( \int_0^s a^{-\frac{p}{\lambda}}(\sigma)\,d\sigma \right)^\lambda \,ds < \infty \), \( y \) be a solution of (1) defined on \([0,T]\), \( T < \infty \) and let \( t_0 \in [0,T) \). If \( y_* = \max_{\varphi(t_0) \leq s \leq t_0} |y(s)| \) and

\[
|y_2(t_0)| + 2^\lambda y_*^\lambda \int_{t_0}^\infty r(s)\,ds \leq \lambda \int_{t_0}^\infty r(s)\left( \int_{t_0}^s a^{-\frac{p}{\lambda}}(\sigma)\,d\sigma \right)^\lambda \,ds < 2^{-\lambda} \frac{p}{\lambda - p} - \lambda.
\]

Then \( T = \infty \) and \( y \) is defined on \( \mathbb{R}^+ \).

Proof. Suppose, contrarily, that \( y \) is singular of the 2nd kind. Then \( T = \tau < \infty \) and denote by

\[
v(t) = \sup_{t_0 \leq s \leq t} |y_2(s)| \quad \text{for} \quad t \in I \overset{\text{def}}{=} [t_0, T).
\]

It follows from (2) that

\[
|y_2(t)| \leq |y_2(t_0)| + \int_{t_0}^t r(s) |y_2^\varphi(s)| ds
\]

and

\[
|y(t)| \leq |y(t_0)| + \int_{t_0}^t a^{-\frac{p}{\lambda}}(s)|y_2(s)| \,ds, \quad t \in I.
\]

Hence, for \( t_0 \leq s \leq t < T \) we have

\[
|y_2(s)| \leq |y_2(t_0)| + \int_{t_0}^s r(z) \left[ y_* + v^\varphi(z) \int_{t_0}^z a^{-\frac{p}{\lambda}}(\sigma)\,d\sigma \right]^\lambda \,dz
\]

\[
\leq |y_2(t)| + 2^\lambda y_*^\lambda \int_{t_0}^\infty r(\sigma)\,d\sigma + 2^\lambda \int_{t_0}^t r(z) \left( \int_{t_0}^z a^{-\frac{p}{\lambda}}(\sigma)\,d\sigma \right)^\lambda \,v^\varphi(z) \,dz.
\]

From this

\[
v(t) \leq |y_2(t_0)| + 2^\lambda y_*^\lambda \int_{t_0}^\infty r(\sigma)\,d\sigma + 2^\lambda \int_{t_0}^t r(z) \left( \int_{t_0}^z a^{-\frac{p}{\lambda}}(\sigma)\,d\sigma \right)^\lambda \,v^\varphi(z) \,dz.
\]

Put \( \omega = \frac{\lambda}{p} > 1 \), \( u = v \), \( K = |y_2(t_0)| + 2^\lambda y_*^\lambda \int_{t_0}^\infty r(s)\,ds \)

and \( Q(t) = 2^\lambda r(t) \left( \int_{t_0}^t a^{-\frac{p}{\lambda}}(\sigma)\,d\sigma \right)^\lambda \).

Then (16) and (17) imply (13) and (14), and according to Lemma 1, (15) is valid. As \( T < \infty \), \( y_* \) is bounded on \( J \). A contradiction with (4) proves the statement. \( \square \)

**Remark 3.** Note that Lemma 2 is valid even if we suppose \( r \geq 0 \) instead of \( r > 0 \) on \( \mathbb{R}^+ \).

**Remark 4.** The idea of the proof is due to Medvedě and Pekářková [14] (with \( \varphi(t) \equiv t \)); it is used also in [7] for (1) with \( t - \varphi(t) \leq \text{const} \) on \( \mathbb{R}^+ \).

The next theorem derives an estimate from below of a singular solution of the second kind.
Theorem 3. Let $\lambda > p$ and let $y$ be a singular solution of (1) of the 2nd kind. Let $T \in [0, \tau)$, $a_* = \min_{T \leq s \leq \tau} a(s)$, $r_* = \max_{T \leq s \leq \tau} r(s)$ and $y_*(t) = \max_{\varphi(t) \leq s \leq \tau} |y(s)|$ on $[T, \tau)$.

Then
\[ (18) \quad |y_2(t)| + 2^{\lambda+1} y_*(t) r_*(\tau - t) \geq K(\tau - t)^{-\frac{\varphi(\lambda+1)}{\lambda-p}} \]
on $[T, \tau)$ with $K = \left( 2^{-2\lambda-1} \frac{(\lambda+1)p}{\lambda-p} a_1^2 r_*^{-1} \right)^{\frac{1}{\lambda-p}}$. Especially, a left neighbourhood $I$ of $\tau$ exists such that

\[ (19) \quad a(\tau)|y'(t)|^p + 2^{\lambda+1} y_*(t) r_*(\tau - t) \geq K_1(\tau - t)^{-\frac{\varphi(\lambda+1)}{\lambda-p}} \]
on $I$ with $K_1 = \left( 2^{-2\lambda-3-\frac{\varphi(\lambda+1)}{\lambda-p}} a_t(\tau) r_*^{-1}(\tau) \right)^{\frac{1}{\lambda-p}}$.

Proof. Let $y$ be a singular solution of (1) of the 2nd kind defined on $[0, \tau)$. Let $t \in [T, \tau)$ be fixed. Define

\[
\begin{align*}
\bar{r}(t) &= r(t) \\ \bar{a}(t) &= a(t) & \text{for } t \in [0, \tau) \\
\tilde{r}(t) &= \frac{r_*(\tau)}{\tau - t}(-t + 2\tau - \bar{t}) , \\
\tilde{a}(t) &= \frac{a(\tau)}{\tau - t}(-t + 2\tau - \bar{t}) & \text{for } t \in (\tau, 2\tau - \bar{t}) \\
\tilde{r}(t) = 0 , \quad \tilde{a}(t) &= 0 & \text{for } t > 2\tau - \bar{t} ;
\end{align*}
\]

note that $\tilde{r}$ and $\tilde{a}$ are continuous on $\mathbb{R}_+$ and are linear on $[\tau, 2\tau - \bar{t}]$. Furthermore, we have

\[ (20) \quad \int_{\bar{t}}^{\tau} \tilde{r}(s) \left( \int_{\bar{t}}^{s} \bar{a}^{-\frac{1}{p}}(\sigma) d\sigma \right)^{\lambda} ds \leq r_* a_*^{-\frac{1}{p}} \int_{\bar{t}}^{2\tau - l} (s - \bar{t})^\lambda ds \leq \frac{2^{\lambda+1}}{\lambda+1} r_* a_*^{-\frac{1}{p}} (\tau - \bar{t})^{\lambda+1} \]

and
\[ (21) \quad \int_{\bar{t}}^{\tau} r_*(s) ds \leq \int_{\bar{t}}^{2\tau - l} r_* ds = 2r_*(\tau - \bar{t}) . \]

Consider an auxiliary equation
\[ (22) \quad (\bar{a}(t)|z'|^{p-1}z')' + \tilde{r}(t)|z(\varphi)|^\lambda \sgn z(\varphi) = 0 . \]

Then $z = y$ is the singular solution of (22) of the second kind defined on $[0, \tau)$. Suppose that (18) is not valid for $t = \bar{t}$, i.e.

\[ (23) \quad |y_2(\bar{t})| + 2^{\lambda+1} y_*(\bar{t}) r_*(\tau - \bar{t}) \leq 2^{-2\lambda-1} \frac{(\lambda+1)p}{\lambda-p} \frac{\varphi(\lambda+1)}{\lambda-p} \]

holds. We apply Lemma 2 and Remark 3 with $T = \tau$ and $t_0 = \bar{t}$. Then it follows from (20), (21) and (23) that all assumptions of Lemma 2 are valid. Hence, $z$ is defined on $\mathbb{R}_+$ and the contradiction with $z$ to be singular proves that (18) is valid. Furthermore, a left neighbourhood $I$ of $t = \tau$ exists such that

\[ r_* \leq 2r(\tau) \quad \text{and} \quad \frac{a(\tau)}{2} \leq a_* \leq 2a(\tau) \]

and (20) follows from this and from (18). \[ \square \]

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Theorem 4. Let $\tau$, then a left neighbourhood oscillatory and $\phi$. Then

**Proof.** Suppose, contrarily, that $y$ is a singular solution of (1) of the second kind. Then according to Lemma 2 and Corollary 2 it is oscillatory and unbounded. Hence, an increasing sequence $\{t_k\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} t_k = \tau$ and

$$|y(t_k)| \geq M(\tau - t_k, \phi(t_k)) = 0, \quad k = 1, 2, \ldots$$

Proof. Let $y$ be a singular solution of the 2nd kind. Then according to Lemma 2 and Corollary 2 it is oscillatory and unbounded. Hence, an increasing sequence $\{t_k\}_{k=1}^{\infty}$ exists such that $\lim_{k\to\infty} t_k = \tau$, $y$ has the local extreme at $t_k$ and

$$|y(t_k)| \geq |y(t)| \quad \text{for } t \in [\phi, t_k], \quad k = 1, 2, \ldots$$

Then $y'(t_k) = 0$, $\max_{\phi(t_k) \leq s \leq t_k} |y(s)| = |y(t_k)|$, and the statement follows from (19). □

3. SINGULAR SOLUTION OF THE 1ST KIND

This paragraph begins with some basic properties

**Theorem 4.** Let $y$ be a singular solution of (1) of the first kind. Then it is oscillatory and $\phi(t) = \tau$. Moreover,

(i) if $R \in C^1(\mathbb{R}_+)$, then $\phi(t) \neq t$ in any left neighbourhood of $\tau$;
(ii) if $R \in C^1(\mathbb{R}_+)$, $\lambda \geq p$ and $\phi$ is nondecreasing in a left neighbourhood $J$ of $\tau$, then a left neighbourhood $J_1$ of $\tau$ exists such that $\phi(t) < t$ on $J_1$.

**Proof.** Let $y$ be a singular solution of (1) of the first kind. Then

$$y(t) = 0 \quad \text{for } t \geq \tau$$

and

$$y(t) \neq 0 \quad \text{in any left neighbourhood of } \tau.$$

Suppose, contrarily, that $\phi(t) < \tau$. Then $\lim_{t\to\infty} \phi(t) = \infty$ implies the existence of $\tau_1$ such that $\tau_1 > \tau$ and $\phi(t) > \tau$ for $t \geq \tau_1$. Denote $I = [\tau, \tau_1]$. Then according to (1) and (24)

$$y(\phi(t)) = -r^{-\frac{\lambda}{p-1}}(t)[a(t)|y'(t)|^{p-1}y'(t)]^{1/\lambda} \sgn(a(t)|y'(t)|^{p-1}y'(t))' = 0$$

for $t \in I$. As $\phi(\tau_1) > \tau$ we have

$$[\phi(\tau), \tau] \subset [\phi(\tau), \phi(\tau_1)] \subset \{\phi(t): t \in I\}.$$

From this and from (26), $y(t) = 0$ on $[\phi(\tau), \tau]$ that contradicts (25). Hence, $\phi(\tau) = \tau.$

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We prove that \( y \) is oscillatory. Suppose, contrarily, that \( y(t) > 0 \) in a left neighbourhood of \( \tau \); case \( y(t) < 0 \) can be studied similarly. From this and from \( \varphi(\tau) = \tau \) an interval \( I_1 = [\tau_2, \tau], \tau_2 < \tau \) exists such

\[
y(\varphi(t)) > 0 \quad \text{for} \quad t \in I_1.
\]

As, according to (2), \( y_2 \) is decreasing on \( I_1 \) and (24) implies \( y_2(\tau) = 0 \) we have \( y_2 > 0 \) on \( I_1 \); hence, \( y' > 0 \) on \( I_1 \). The contradiction with (27) and (24) proves that \( y \) is oscillatory.

Case (i). The proof follows from Theorem A(iii) by the same way as in the proof of Theorem 1.

Case (ii). Let \( \lambda \geq p \) and \( R \in C^1(\mathbb{R}_+) \). Then (i) implies \( \varphi \) is nontrivial in any left neighbourhood of \( \tau \). Suppose that an increasing sequence \( \{\tau_k\}_{k=1}^\infty \) exists such that \( \lim_{k \to \infty} \tau_k = \tau \) and \( \varphi(\tau_k) = \tau_k \). As \( \varphi \) is nondecreasing in \( J \), \( \{\tau_k\} \) may be chosen such that

\[
\varphi(t) \in [\tau_k, \tau] \quad \text{for} \quad t \in [\tau_k, \tau].
\]

It follows from (24) and (25) that \( y_2(\tau) = 0 \) and \( F(\tau) = 0 \). Denote \( \bar{F}_k = \max_{\tau_k \leq s \leq \tau} F(s) \). Then (28), (7) and (9) imply

\[
F(s) = -\int_s^\tau F'(\sigma) \, d\sigma \leq \bar{F}_k \int_{\tau_k}^\tau \left| \frac{R'(\sigma)}{R(\sigma)} \right| \, d\sigma + 2\delta \gamma^{-\lambda} \bar{F}_k \int_{\tau_k}^\tau \frac{a^{-\frac{1}{p}}(\sigma)R^\frac{1}{p+1}(\sigma)}{R(\sigma)} \, d\sigma
\]

for \( s \in [\tau_k, \tau] \) where \( \omega = \frac{1}{p+1} + \frac{\lambda}{\lambda+1} \geq 1 \) due to \( \lambda \geq p \). Hence,

\[
\bar{F}_k \leq \bar{F}_k \int_{\tau_k}^\tau \left| \frac{R'(\sigma)}{R(\sigma)} \right| \, d\sigma + 2\delta \gamma^{-\lambda} \bar{F}_k \int_{\tau_k}^\tau \frac{a^{-\frac{1}{p}}(\sigma)R^\frac{1}{p+1}(\sigma)}{R(\sigma)} \, d\sigma
\]

\( k = 1, 2, \ldots \). As \( \lim_{k \to \infty} \bar{F}_k = F(\tau) = 0 \) and

\[
\lim_{k \to \infty} \int_{\tau_k}^\tau \left| \frac{R'(\sigma)}{R(\sigma)} \right| \, d\sigma = 0, \quad \lim_{k \to \infty} \int_{\tau_k}^\tau a^{-\frac{1}{p}}(\sigma)R^\frac{1}{p+1}(\sigma) \, d\sigma = 0
\]

we obtain the contradiction in (29) for large \( k \). Hence, \( \{\tau_k\} \) does not exists and the statement holds in this case. \( \square \)

The following result is a consequence of Theorem 2 and Theorem 4.

**Theorem 5.** If \( \varphi(t) < t \) on \( \mathbb{R}_+ \), then all solutions of (1) are proper.

**Lemma 3.** Let \( y \) be a singular solution of the 1st kind, let \( T \in [0, \tau) \) be such that

\[
\int_T^\tau R^{-1}(t)|R'(t)| \, dt \leq \frac{1}{2},
\]

\( I = [T, \tau], K > 0, \omega \geq 0 \) and \( |e(t)| \leq K(\tau - t)^\omega \) on \( I \). Then

\[
F(t) \leq K_1(\tau - t)^{(\omega + 1)} \quad \text{in} \quad I
\]

where \( K_1 = [2\delta (\omega + 1)^{-1} K \max_{0 \leq \sigma \leq \tau} a^{-\frac{1}{p}}(\sigma)R^\frac{1}{p+1}(\sigma)]^\delta \).

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Proof. Let $y$ be a singular solution of the 1st kind. Then (9) implies
\[ R^{-1}(t)|y_2(t)|^\delta \leq F(t), \quad |y'(t)| \leq C F^{1/(\omega + 1)}(t) \]
on $I$ with $C = \max_{t \in I} a^{-\frac{1}{\omega}}(t)R^{1/(\omega + 1)}(t) > 0$. Define $F(t) = \max F(s)$ for $t \in I$. From this and from (7), (8) and (30)
\[
F(s) = -\int_t^\tau F'(\sigma) d\sigma \leq \int_t^\tau R^{-1}(\sigma)|R'(\sigma)|F(\sigma) d\sigma + \delta \int_t^\tau |y'(\sigma)e(\sigma)| d\sigma
\]
\[
\leq \bar{F}(t) \int_t^\tau R^{-1}(\sigma)|R'(\sigma)|d\sigma + C_1 \int_t^\tau \bar{F}^{1/(\omega + 1)}(\tau - \sigma)^{\omega} d\sigma
\]
\[
\leq \bar{F}(t) + \frac{C_1}{\omega + 1} \bar{F}^{1/(\omega + 1)}(\tau - t)^{\omega + 1}
\]
for $t \in I$ and $t \leq s \leq \tau$ where $C_1 = \delta KC$. Hence,
\[
\bar{F}(t) \leq \bar{F}(t) + \frac{C_1}{\omega + 1} \bar{F}^{1/(\omega + 1)}(\tau - t)^{\omega + 1}
\]
or
\[
F(t) \leq \bar{F}(t) \leq K_1(\tau - t)^{\omega(\omega + 1)} \quad \text{on } I.
\]

The following theorem gives us an estimate from above of singular solutions of the 1st kind.

**Theorem 6.** Let $y$ be a singular solution of (1) of the 1st kind and $M > 0$ be such that $\varphi'(t) \leq M$ in a left neighbourhood $S$ of $\tau$.

(i) Let $\lambda \geq p$ and $m > 0$. Then a positive constant $K$ and a left neighbourhood $J$ of $\tau$ exist such that
\[
|y(t)| \leq K(\tau - t)^m, \quad |y_2(t)| \leq K(\tau - t)^{\frac{(\lambda + 1)m}{p + 1}} \quad \text{on } J.
\]

(ii) Let $\lambda < p$ and $\varepsilon > 0$. Then a positive constant $K$ and a left neighbourhood $J$ of $\tau$ exist such that
\[
|y(t)| \leq K(\tau - t)^{\frac{m + 1}{p + 1} - \varepsilon}, \quad |y_2(t)| \leq K(\tau - t)^{\frac{m + 1}{p + 1} - \varepsilon} \quad \text{on } J.
\]

Proof. Let $y$ be a singular solution of the 1st kind. According to Theorem 4 $\varphi(\tau) = \tau$. Moreover, $\lim_{t \to \tau^-} y(t) = \lim_{t \to \tau^-} y_2(t) = 0$ and an interval $I = [T, \tau] \subset S$, $0 \leq T < T$ exists such that (30) and
\[
|y(t)|^\lambda \leq \frac{1}{2}, \quad |y(\varphi(t))|^\lambda \leq \frac{1}{2} \quad \text{for } t \in I.
\]
Hence, (8) implies $|e(t)| \leq 1$ on $I$ and it follows from Lemma 3 (with $I = I, K = 1$, $\omega = 0$)
\[
F(t) \leq K(T - t)^\delta, \quad t \in I
\]
with
\[
K = \left[2\delta \max_{0 \leq \sigma \leq T} a^{-\frac{1}{\omega}}(\sigma)R^{1/(\omega + 1)}(\sigma)\right]^\delta
\]
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Let \( \{I_n\}_{n=1}^{\infty} \) be such that \( I_1 = I, I_n = [T_n, \tau], T_n < T_{n+1} < \tau \) and \( \varphi(t) \in I_n \) for \( t \in I_{n+1}, n = 1, 2, \ldots; \) this sequence exists due to \( \varphi(t) \leq t \) and \( \varphi(\tau) = \tau. \)

We prove the estimate
\[
F(t) \leq K_n(\tau - t)^{\omega_n} \quad \text{on} \quad I_n
\]
by the mathematical induction, where
\[
\omega_1 = \delta, \quad \omega_{n+1} = \delta \left[ \frac{\lambda}{\lambda + 1} \omega_n + 1 \right], \quad n = 1, 2, \ldots
\]
and
\[
K_1 = K, \quad K_{n+1} = K \left[ \gamma^{-\frac{1}{\lambda + 1}} \left( 1 + \frac{\lambda}{\lambda + 1} \omega_n \right)^{-1} (1 + M^{\omega_n}) \right]^{\delta}, \quad n = 1, 2, \ldots
\]
For \( n = 1 \) (33) follows from (31) and (32). Suppose the validity of (33) for \( n. \) Then (6) and (33) imply
\[
|y(t)|^\lambda \leq (\gamma^{-1} F(t))^{\frac{1}{\lambda + 1}} \leq \gamma^{-\frac{1}{\lambda + 1}} K_n^{1/\lambda} (\tau - t)^{\frac{1}{\lambda + 1} \omega_n}, \quad t \in I_n
\]
and
\[
|y(\varphi(t))|^\lambda \leq \gamma^{-\frac{1}{\lambda + 1}} K_n^{1/\lambda} M^{\frac{1}{\lambda + 1} \omega_n} (\tau - t)^{\frac{1}{\lambda + 1} \omega_n}, \quad t \in I_{n+1}
\]
as
\[
0 \leq \tau - \varphi(t) = \varphi(\tau) - \varphi(t) = \varphi'(\xi)(\tau - t) \leq M(\tau - t), \quad \xi \in [t, \tau].
\]
From this and from (8)
\[
|\varepsilon(t)| \leq \gamma^{-\frac{1}{\lambda + 1}} K_n^{1/\lambda} [1 + M^{\frac{1}{\lambda + 1} \omega_n}] (\tau - t)^{\frac{1}{\lambda + 1} \omega_n} = L_n(\tau - t)^{\omega_n},
\]
where
\[
w_n = \frac{\lambda}{\lambda + 1} \omega_n \quad \text{and} \quad L_n = \gamma^{-\frac{1}{\lambda + 1}} K_n^{1/\lambda} [1 + M^{\omega_n}].
\]
Now, we use Lemma 3 with \( I = I_{n+1}, K = L_n \) and \( \omega = w_n \) and we obtain
\[
F(t) \leq K_{n+1}(\tau - t)^{\omega_{n+1}}. \quad \text{Hence, (33) holds for all} \ n = 1, 2, \ldots \text{ Denote by}
\]
\[
z = \frac{\lambda(p+1)}{(\lambda+1)p}.
\]
We prove that
\[
\omega_n \leq \delta \frac{1 - z^n}{1 - z}, \quad n = 1, 2, \ldots \quad \text{for} \ z \neq 1
\]
\[
\omega_n = \delta n \quad \text{for} \ z = 1.
\]
If \( v_n = \frac{\omega_n}{\delta}, \) then (34) implies \( v_1 = 1, v_{n+1} = z v_n + 1, n = 1, \ldots \) Hence, \( v_n = 1 + z + z^2 + \ldots + z^{n-1} = \frac{1 - z^n}{1 - z} \) in case \( z \neq 1 \) and \( v_n = n \) in case \( z = 1. \) Now, (36) follows from this.

We have from (35) that
\[
z > 1 \Leftrightarrow \lambda > p, \quad z = 1 \Leftrightarrow \lambda = p, \quad z < 1 \Leftrightarrow \lambda < p.
\]
Furthermore, from this and from (36) \( \lim_{n \to \infty} \omega_n = \infty \) in case \( \lambda \geq p \) and \( \lim_{n \to \infty} \omega_n = \frac{\delta}{1 - p} = \frac{(p+1)(\lambda+1)}{p - \lambda} \) in case \( \lambda < p. \) Hence, the statement follows from (33) and (6). \( \square \)

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