Regularity of Weak Solutions for Nonlinear Parabolic Problem with $p(x)$-Growth

Yongqiang Fu, Mingqi Xiang
Department of Mathematics, Harbin Institute of Technology, Harbin 150001, China

Ning Pan
Department of Mathematics, Northeast Forestry University, Harbin 150040, China

Abstract

In this paper, we study the nonlinear parabolic problem with $p(x)$-growth conditions in the space $W^{1,p}(x)(Q)$, and give a regularity theorem of weak solutions for the following equation

$$\frac{\partial u}{\partial t} + A(u) = 0$$

where $A(u) = -\text{div}(x, t, u, \nabla u) + a_0(x, t, u, \nabla u)$, $a(x, t, u, \nabla u)$ and $a_0(x, t, u, \nabla u)$ satisfy $p(x)$-growth conditions with respect to $u$ and $\nabla u$.

Keywords: nonlinear parabolic problem, regularity, $W^{1,p}(x)(Q)$ space, $p(x)$-growth condition.

Mathematics Subject Classification: 35K15, 35K20.

1 Introduction

In recent years, the research of variational problems with nonstandard growth conditions is an interesting topic. $p(x)$-growth problems can be regarded as a kind of nonstandard growth problems and they appear in nonlinear elastic, electrorheological fluids and other physics phenomena. Many results have been obtained on this kind of problems, for examples [1-9].

In this paper, we will qualitatively study the properties of weak solutions. For more information about qualitative analysis, we refer to [10-11]. Let $Q$ be $\Omega \times (0, T)$ where $T > 0$ is given. In [8], the authors studied the following equation in the space $W^{1, p(x,t)}_{loc}(Q) \cap C(0,T; L^2_{loc}(\Omega))$,

$$u_t - \text{div}(|Du|^{p(x,t)-2} Du) = 0, $$

*Corresponding author: Yongqiang Fu, fuyqhagd@yahoo.cn
where \( \max\{1, \frac{N}{N+2}\} < p_1 = \inf_{(x,t) \in Q} p(x,t) \leq p(x,t) \leq \sup_{(x,t) \in Q} p(x,t) = p_2 < \infty \), \( p(x,t) \) is dependent on the space variable \( x \) and the time variable \( t \), and satisfies the following Logarithmic Hölder condition

\[
|p(x,t) - p(y,s)| \leq \frac{C_1}{-\ln(|x-y| + C_2|t-s|^{p_2})}
\]

for all \((x,t), (y,s) \in Q, |x-y| < \frac{1}{2}, |t-s| < \frac{1}{2}\), where \( C_1, C_2 > 0 \) are constants. The authors proved the Hölder continuity of the local weak solution with the scale transformation method. In this paper, we will study the following more general problem

\[
\begin{align*}
\frac{\partial u}{\partial t} + A(u) &= 0, \quad \text{in} \quad Q, \\
u(x,t) &= 0, \quad \text{on} \quad \partial \Omega \times (0,T), \\
u(x,0) &= \psi(x), \quad \text{in} \quad \Omega,
\end{align*}
\]

where \( \psi(x) \) is a given function in \( L^2(\Omega) \) and \( A : W^{1,p(x)}_0(\Omega) \to W^{-1,q(x)}(\Omega) \) is an elliptic operator of the form \( A(u) = -\text{div}(a(x,t,u,\nabla u) + a_0(x,t,u,\nabla u)) \) with the coefficients \( a \) and \( a_0 \) satisfying the classical Leray-Lions conditions. In [12-13] we have proved the existence and the local boundedness of the solutions of (1.1)-(1.3) and have obtained \( u \in W^{1,p(x)}(Q) \cap L^\infty(0,T;L^2(\Omega)) \). In this paper we will give the regularity theorem of the weak solutions in the framework space \( W^{1,p(x)}(Q) \), which can be considered as a special case of the space \( W^{1,p(x)}(Q) \).

The space \( W^{1,p(x)}(Q) \) provides a suitable framework to discuss some physical problems. In [14], the authors studied a functional with variable exponent, \( 1 \leq p(x) \leq 2 \), which provided a model for image denoising, enhancement, and restoration. Because in [14] the direction and speed of diffusion at each location depended on the local behavior, \( p(x) \) only depended on the location \( x \) in the image. Consider that the space \( W^{1,p(x)}(Q) \) was introduced and discussed in [12] and [15], we think that the space \( W^{1,p(x)}(Q) \) is a reasonable framework to discuss the \( p(x) \)-growth problem (1.1)-(1.3), where \( p(x) \) only depends on the space variable \( x \) similar to [14].

In this paper, let \( a : Q \times R \times R^N \to R^N \) and \( a_0 : Q \times R \times R^N \to R \) be the operators such that for any \( s \in R \) and \( \xi \in R^N \), \( a(x,t,s,\xi) \) and \( a_0(x,t,s,\xi) \) are both continuous in \((t,s,\xi)\) for a.e. \( x \in \Omega \) and measurable in \( x \) for all \((t,s,\xi) \in (0,T) \times R^N \). They also satisfy that for a.e. \((x,t) \in Q\), any \( s \in R \) and \( \xi \neq \xi^* \in R^N \):

\[
\begin{align*}
|a(x,t,s,\xi)| &\leq \alpha(|s|^{p(x)-1} + |\xi|^{p(x)-1}), \\
|a_0(x,t,s,\xi)| &\leq \alpha(|s|^{p(x)-1} + |\xi|^{p(x)-1}), \\
|a(x,t,s,\xi) - a(x,t,s,\xi^*)|(|\xi - \xi^*|) &> 0, \\
a(x,t,s,\xi) &\geq \beta(|\xi|^{p(x)} + |s|^{p(x)}), \\
a_0(x,t,s,\xi) &\geq \beta(|\xi|^{p(x)} + |s|^{p(x)}),
\end{align*}
\]

EJQTDE, 2012 No. 4, p. 2
where $\alpha, \beta > 0$ are constants.

Throughout this paper, unless special statement, we always suppose that $p(x)$ is Lipschitz continuous on $\Omega$, and satisfies
\begin{equation}
1 < p^- = \inf_{\Omega} p(x) \leq p(x) \leq \sup_{\Omega} p(x) = p^+ < \infty. \tag{1.9}
\end{equation}
Because $p(x)$ is Lipschitz continuous, there exists a constant $C > 0$ such that
\begin{equation}
\rho^-(p^+ - \rho^-) \leq C, \quad \forall Q_\rho \subset Q, \tag{1.10}
\end{equation}
where $Q_\rho = K_\rho \times (-\rho^+, 0), 0 < \rho < 1, K_\rho = \{ x \in \Omega \mid \max_{1 \leq i \leq N} |x_i| < \rho \}, \quad p^+_\rho = \sup_{K_\rho} p(x), \quad p^-_\rho = \inf_{K_\rho} p(x)$.

**Definition 1.1** A function $u \in W^{1,x,L^p}(Q) \cap L^\infty(0,T; L^2(\Omega))$ is called a weak solution of (1.1)-(1.3) if
\begin{equation}
-\int_Q u \frac{\partial \varphi}{\partial t} dx dt + \int_\Omega u \varphi dx \bigg|_0^T + \int_Q [a(x,t,u,\nabla u)\nabla \varphi + a_0(x,t,u,\nabla u)\varphi] dx dt = 0
\end{equation}
for all $\varphi \in C^1(0,T; C_0^\infty(\Omega))$.

**Definition 1.2** The functions $u_n \in C(0,T; C_0^\infty(\Omega))$ are called the Galerkin solutions of (1.1)-(1.3) if
\begin{equation}
\int_{Q'} \frac{\partial u_n}{\partial \tau} \varphi dx d\tau + \int_{Q'} a(x,\tau,u_n,\nabla u_n)\varphi dx d\tau + \int_{Q'} a_0(x,\tau,u_n,\nabla u_n)\varphi dx d\tau = 0 \tag{1.11}
\end{equation}
for all $\varphi \in C^1(0,T; C_0^\infty(\Omega))$ and $Q' = \Omega \times (0,t), t \in (0,T]$.

We will prove the following regularity theorem:

**Theorem 1** Let $p^- > 2$. If $u \in W^{1,x,L^p}(Q) \cap L^\infty(0,T; L^2(\Omega))$ is a local weak solution of (1.1)-(1.3), then $u$ is local Hölder continuous in $Q$.

2 Preliminaries

We first recall some facts on spaces $L^p(\Omega), W^{m,p}(\Omega), W^{m,x,L^p}(Q)$ and parabolic space. For the details see [15-18].

Although we assume (1.9) holds in this paper, in this section we introduce the general spaces $L^p(\Omega), W^{m,p}(\Omega)$ and $W^{m,x,L^p}(Q)$.

Denote
\[ E = \{ \omega : \omega \text{ is a measurable function on } \Omega \}, \]
where $\Omega \subset \mathbb{R}^N$ is an open subset.
Let \( p(x) : \Omega \to [1, \infty] \) be an element in \( E \). Denote \( \Omega_\infty = \{ x \in \Omega : p(x) = \infty \} \). For \( u \in E \), we define

\[
\rho(u) = \int_{\Omega \setminus \Omega_\infty} |u(x)|^{p(x)} \, dx + \operatorname{ess \ sup} \frac{|u(x)|}{\lambda} \, \text{ for } x \in \Omega_\infty.
\]

The space \( L^{p(x)}(\Omega) \) is

\[
L^{p(x)}(\Omega) = \{ u \in E : \exists \lambda > 0, \rho(\lambda u) < \infty \}
\]

endowed with the norm

\[
\|u\|_{L^{p(x)}(\Omega)} = \inf \{ \lambda > 0 : \rho(\frac{u}{\lambda}) \leq 1 \}.
\]

We define the conjugate function \( q(x) \) of \( p(x) \) by

\[
q(x) = \begin{cases} 
\infty, & \text{if } p(x) = 1; \\
1, & \text{if } p(x) = \infty; \\
\frac{p(x)}{p(x) - 1}, & \text{if } 1 < p(x) < \infty.
\end{cases}
\]

Lemma 2.1 (see [18])

1. The dual space of \( L^{p(x)}(\Omega) \) is \( L^{q(x)}(\Omega) \), if \( 1 \leq p(x) < \infty \).
2. The space \( L^{p(x)}(\Omega) \) is reflexive if and only if (1.9) is satisfied.

Lemma 2.2 (see [18])

If \( 1 \leq p(x) < \infty \), \( C_\infty^0(\Omega) \) is dense in the space \( L^{p(x)}(\Omega) \) and \( L^{p(x)}(\Omega) \) is separable.

Lemma 2.3 (see [18])

Let \( 1 \leq p(x) \leq \infty \), for every \( u(x) \in L^{p(x)}(\Omega) \) and \( v(x) \in L^{q(x)}(\Omega) \), we have

\[
\int_{\Omega} |u(x)v(x)| \, dx \leq C \|u(x)\|_{L^{p(x)}(\Omega)} \|v(x)\|_{L^{q(x)}(\Omega)},
\]

where \( C \) is only dependent on \( p(x) \) and \( \Omega \), not dependent on \( u(x) \), \( v(x) \).

Next let \( m > 0 \) be an integer. For each \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \), \( \alpha_i \) are nonnegative integers and \( |\alpha| = \Sigma_{i=1}^n \alpha_i \), and denote by \( D^\alpha \) the distributional derivative of order \( \alpha \) with respect to the variable \( x \).

We now introduce the generalized Lebesgue-Sobolev space \( W^{m,p(x)}(\Omega) \) which is defined as

\[
W^{m,p(x)}(\Omega) = \{ u \in L^{p(x)}(\Omega) : D^\alpha u \in L^{p(x)}(\Omega), |\alpha| \leq m \}.
\]

\( W^{m,p(x)}(\Omega) \) is a Banach space endowed with the norm

\[
\|u\| = \sum_{|\alpha| \leq m} \|D^\alpha u\|_{L^{p(x)}(\Omega)}.
\]

The space \( W_0^{m,p(x)}(\Omega) \) is defined as the closure of \( C_0^\infty(\Omega) \) in \( W^{m,p(x)}(\Omega) \). The dual space \( (W_0^{m,p(x)}(\Omega))^* \) is denoted by \( W^{-m,q(x)}(\Omega) \) equipped with the norm

\[
\|f\|_{W^{-m,q(x)}(\Omega)} = \inf \Sigma_{|\alpha| \leq m} \|f_\alpha\|_{L^{q(x)}(\Omega)},
\]

EJQTDE, 2012 No. 4, p. 4
where infimum is taken on all possible decompositions
\[
    f = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha f_\alpha, \quad f_\alpha \in L^q(x)(\Omega).
\]

**Lemma 2.4 (see [18])**  
(1) \( W^{m,p(x)}(\Omega) \) and \( W_0^{m,p(x)}(\Omega) \) are separable if \( 1 \leq p(x) < \infty \).

(2) \( W^{m,p(x)}(\Omega) \) and \( W_0^{m,p(x)}(\Omega) \) are reflexive if (1.9) holds.

We define the space \( W^{m,x}L^p(x)(Q) \) as the following:
\[
    W^{m,x}L^p(x)(Q) = \{ u \in L^p(x)(Q) : D^\alpha u \in L^p(x)(Q), |\alpha| \leq m \}.
\]

\( W^{m,x}L^p(x)(Q) \) is a Banach space with the norm \( \| u \| = \sum_{|\alpha| \leq m} \| D^\alpha u \|_{L^p(x)(Q)} \),
where \( p(x) \) is independent of \( t \).

The space \( W_0^{m,x}L^p(x)(Q) \) is defined as the closure of \( C_0^\infty(Q) \) in \( W^{m,x}L^p(x)(Q) \) and \( W_0^{m,x}L^p(x)(Q) \) is continuous embedding. Let \( M \) be the number of multiindexes \( \alpha \) which satisfies \( 0 \leq |\alpha| \leq m \), then the space \( W_0^{m,x}L^p(x)(Q) \) can be considered as a close subspace of the product space \( \prod_{i=1}^{M} L^p(x)(Q) \). So

if \( 1 < p(x) < \infty \), \( \prod_{i=1}^{M} L^p(x)(Q) \) is reflexive and further we can get that the space \( W_0^{m,x}L^p(x)(Q) \) is reflexive. The dual space \( (W_0^{m,x}L^p(x)(Q))^* \) is denoted by \( W^{-m,x}L^q(x)(Q) \) equipped with the norm
\[
    \| f \|_{W^{-m,x}L^q(x)(Q)} = \sup_{\| u \|_{W_0^{m,x}L^p(x)(Q)} \leq 1} | < f, u > | = \inf \sum_{|\alpha| \leq m} \| f_\alpha \|_{L^q(x)(Q)},
\]
where infimum is taken on all possible decompositions
\[
    f = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha f_\alpha, \quad f_\alpha \in L^q(x)(Q).
\]

Next, we will introduce the parabolic space and some results in [16]:

**Definition 2.5**  
Let \( p, r \geq 1 \). A function \( f \) defined in \( Q \) belongs to the space \( L^r(0,T; L^p(\Omega)) \), if
\[
    \| f \|_{p,r,Q} = (\int_0^T (\int_{\Omega} |f|^p dx)^{\frac{r}{p}} dt)^{\frac{1}{r}} < \infty.
\]

**Definition 2.6**  
Let \( p, r \geq 1 \). We define the function spaces
\[
    V^{r,p}(Q) = L^\infty(0,T; L^r(\Omega)) \cap L^p(0,T; W^{1,p}(\Omega)),
\]
\[
    V_0^{r,p}(Q) = L^\infty(0,T; L^r(\Omega)) \cap L^p(0,T; W_0^{1,p}(\Omega)),
\]
which are both equipped with the norm
\[
    \| v \|_{V^{r,p}(Q)} = \text{ess sup}_{0 < t < T} \| v(x,t) \|_{L^r(\Omega)} + \| \nabla v \|_{L^p(Q)}.
\]
Lemma 2.7  let \( \{Y_n\}, n = 0,1,2, \cdots, \) be a sequence of positive numbers, satisfying the inequalities \( Y_{n+1} \leq Cb^n Y_1^{-\alpha} \), where \( C,b > 1 \) and \( \alpha > 0 \) are given numbers. If \( Y_0 \leq C^{-\frac{1}{\alpha}}b^{-\frac{\alpha}{2}} \), then \( \{Y_n\} \) converges to 0 as \( n \to \infty \).

Lemma 2.8  Let \( r > 1 \), there exists a constant \( C \) depending only on \( N,r, \) such that for every \( v \in L^\infty(0,T;L^r(\Omega)) \cap L^r(0,T;W_0^{1,r}(\Omega)) \),

\[ \|v\|^r_{L^r(\Omega)} \leq C\|v\| > 0 \|

where \( \|v\| > 0 \) is \( \{x,t): |v| > 0 \} \).

Lemma 2.9  Let \( v \in W^{1,1}(K_p(x_0)) \cap C(K_p(x_0)) \) for some \( p > 0 \) and some \( x_0 \in R^N \), and let \( k \) and \( h \) be any pair of real numbers such that \( k < h \), then there exists a constant \( C \) depending only upon \( N, \) such that

\[ (h-k)|A(h)| \leq C \rho^{N+1} \int_{[A(k)\setminus A(h)]} |\nabla v|dx \]

where \( A(k) = \{x \in K_p(x_0) : v(x) > k\}, |A(k)| = \text{meas}(A(k)) \).

Let \( u \in L^1(Q) \). For any \( 0 < h < T, \) we introduce the Steklov average function

\[ u_h(x,t) = \begin{cases} \int_0^{t+h} u(x,t)dt, & t \in (0,T-h], \\ 0, & t > T-h. \end{cases} \]

Lemma 2.10  Let \( u \in L^r(0,T;L^p(\Omega)), \) then as \( h \to 0, u_h \to u \) in \( L^r(0,T-\varepsilon;L^p(\Omega)) \) for every \( \varepsilon \in (0,T) \). If \( u \in C(0,T;L^2(\Omega)), \) then as \( h \to 0, u_h \to u \) in \( L^2(\Omega) \) for every \( t \in (0,T-\varepsilon) \).

Similarly, we can get the following lemma in variable exponent space.

Lemma 2.11  If \( u \in L^{p(x)}(Q) \), then as \( h \to 0, u_h \to u \) in \( L^{p(x)}(Q) \).

Proof: Because \( p(x) \) is bounded and independent of \( t \). We only need to notice that there exist \( u_k \in C^1_0(Q) \) such that \( u_k \to u \) in \( L^{p(x)}(Q) \), and by the uniform continuity of \( u_k \), we can conclude the lemma.\( \square \)

3  Regularity of Weak Solutions

In [12-13], we have obtained that for the Galerkin solution \( u_n \in C^1(0,T;C^\infty(\Omega)), \) \( u_n \to u \) strongly in \( L^2(Q) \) and \( L^{p(x)}(Q) \), \( u_n \to u \) weakly in \( W_0^{1,2}(Q) \), \( a(x,t,u_n,\nabla u_n) \to a(x,t,u,\nabla u) \) and \( a_0(x,t,u_n,\nabla u_n) \to a_0(x,t,u,\nabla u) \) weakly in \( L^{q(x)}(Q), \) \( u_n \to u \) a.e. in \( Q \) and \( \nabla u_n \to \nabla u \) a.e. in \( Q \).

For (1.11), integrating by parts, we can get

\[ \int_{Q^r} \frac{\partial u_n}{\partial t} \varphi dxdt = \int_{Q^r} u_n(x,t)\varphi(x,t)dx - \int_{Q^r} u_n \frac{\partial \varphi}{\partial t} dxdt, \]

therefore

\[ \lim_{n \to \infty} \int_{Q^r} \frac{\partial u_n}{\partial t} \varphi dxdt = \int_{Q^r} u(x,t)\varphi(x,t)dx - \int_{Q^r} u \frac{\partial \varphi}{\partial t} dxdt. \]

EJQTDE, 2012 No. 4, p. 6
As \(a(x, t, u_n, \nabla u_n) \to a(x, t, u, \nabla u)\) weakly in \(L^{p(x)}(Q)\) and \(a_0(x, t, u_n, \nabla u_n) \to a_0(x, t, u, \nabla u)\) weakly in \(L^{p(x)}(Q)\), we have

\[
\lim_{n \to \infty} \int_{Q^t} a(x, \tau, u_n, \nabla u_n) \varphi \, dx \, d\tau = \int_{Q^t} a(x, \tau, u, \nabla u) \varphi \, dx \, d\tau + \int_{Q^t} a_0(x, \tau, u, \nabla u) \varphi \, dx \, d\tau,
\]

then (1.11) can be written as

\[
\int_{Q^t} u(x, t) \varphi(x, t) \, dx = \int_{Q^t} u \frac{\partial \varphi}{\partial \tau} \, dx \, d\tau + \int_{Q^t} a(x, \tau, u, \nabla u) \varphi \, dx \, d\tau
\]

\[
+ \int_{Q^t} a_0(x, \tau, u, \nabla u) \varphi \, dx \, d\tau = 0. \tag{3.1}
\]

In (3.1), let \(\varphi\) be independent of \(t\) and \(t = t + h\), then we get

\[
\int_{\Omega} \frac{\partial u_n(x, \tau)}{\partial \tau} \varphi(x) \, dx + \int_{Q^t} [a(x, \tau, u, \nabla u)]_h \varphi \, dx \, d\tau + \int_{Q^t} [a_0(x, \tau, u, \nabla u)]_h \varphi \, dx \, d\tau = 0, \tag{3.2}
\]

where \(\varphi \in C^\infty_c(\Omega)\).

**Lemma 3.1** If \(u\) is a weak solution of (1.1)-(1.3), then \(u \in C(0, T; L^2(\Omega))\).

Proof: Because \(u_n \rightharpoonup u\) weakly in \(W^{1,p}_0(L^{p(x)}(Q))\), there exists convex combination of \(u_n\), denoted by \(v_n\), such that \(v_n \to u\) strongly in \(W^{1,p}_0(L^{p(x)}(Q))\) and \(v_n(x, 0) \to \psi(x)\) strongly in \(L^2(\Omega)\). Take \(\varphi = u_n - v_m\) as the testing function in (1.11),

\[
\int_{Q^t} \frac{\partial u_n(x, \tau)}{\partial \tau} (u_n - v_m) \, dx \, d\tau + \int_{Q^t} a(x, \tau, u_n, \nabla u_n) \nabla (u_n - v_m) \, dx \, d\tau
\]

\[
+ \int_{Q^t} a_0(x, \tau, u_n, \nabla u_n) (u_n - v_m) \, dx \, d\tau = 0,
\]

then for the sufficient large \(m\), we have

\[
\lim_{n \to \infty} \int_{Q^t} \frac{\partial u_n}{\partial \tau} (u_n - v_m) \, dx \, d\tau \leq \int_{Q^t} a(x, \tau, u, \nabla u) \nabla (u - v_m) \, dx \, d\tau + \int_{Q^t} a_0(x, \tau, u, \nabla u) \nabla (u - v_m) \, dx \, d\tau
\]

\[
\leq 2(\|a\|_{L^{p(x)}(Q)} + \|a_0\|_{L^{p(x)}(Q)}) \|\nabla (u - v_m)\|_{L^{p(x)}(Q)} \leq \varepsilon(m)
\]

and

\[
\lim_{n \to \infty} \int_{Q^t} \frac{\partial v_m}{\partial \tau} (v_m - u_n) \, dx \, d\tau \leq \varepsilon(m)
\]

where \(\varepsilon(m) \to 0\) as \(m \to 0\).

In short,

\[
\lim_{n \to \infty} \int_{Q^t} \frac{\partial (u_n - v_m)}{\partial \tau} (u_n - v_m) \, dx \, d\tau \leq \varepsilon(m),
\]
Consider the cylinder $K$ where $K \subseteq M$ and be determined later. We assume that (3.3) is hold in the following proof.

By [13], we know that there exists a constant $M > 0$, such that $\|u\|_{L^\infty(Q)} \leq M$. Fix a point $(x_0, t_0)$ in $Q$, let $\rho \in (0, 1)$ be small enough such that

$$Q(\rho^{p^*_\rho - \varepsilon}, 2\rho) = K_{2\rho}(x_0) \times (t_0 - \rho^{p^*_\rho - \varepsilon}, t_0) \subseteq Q,$$

where $K_{2\rho}(x_0) = \{x \in \Omega \mid \max_{1 \leq i \leq N} |x_i - x_{0,i}| < 2\rho\}$, $p^*_\rho = \sup_{K_{2\rho}(x_0)} p(x)$, $p^-_\rho = \inf_{K_{2\rho}(x_0)} p(x)$.

Denote $\mu^+ = \operatorname{ess sup} u_\rho \mu^-$, $\omega = \operatorname{ess osc} u_\rho \mu^+ - \mu^-$. Consider the cylinder $Q(a^{p^*_\rho}, \rho)$, $\frac{1}{A} = (\frac{\mu^+}{A})^{p^*_\rho - 2} > \rho^\varepsilon$, where $A > 2$ is a constant to be determined later. We assume that

$$\left(\frac{\omega}{A}\right)^{p^*_\rho - 2} > \rho^\varepsilon,$$

where $\varepsilon \in (0, 1)$ will be determined later. This implies the inclusion

$$Q(a^{p^*_\rho}, \rho) \subseteq Q(\rho^{p^*_\rho - \varepsilon}, 2\rho)$$

and

$$\operatorname{ess osc} u_\rho \leq \omega.$$  

If (3.3) is not hold, $\omega \leq A^{p^*_\rho - 2}$. Take $C = A$, then the first iterative of proposition 3.4 is hold, so the proposition 3.4 is right. Therefore we also assume that (3.3) is hold in the following proof.

Let $[(0, t^*) + Q(l \rho^{p^*_\rho}, \rho)] = \{x \in \Omega \mid \max_{1 \leq i \leq N} |x_i| < \rho\} \times [t^* - l \rho^{p^*_\rho}, t^*]$, $\frac{1}{t} = (\frac{\omega}{2})^{p^*_\rho - 2}$. For $[(0, t^*) + Q(l \rho^{p^*_\rho}, \rho)] \subseteq Q(a^{p^*_\rho}, \rho), -(A^{p^*_\rho - 2} - 2^{p^*_\rho - 2}) \rho^{p^*_\rho - 2} \omega^2 \rho^\varepsilon < t^* < 0$. We assume $(x_0, t_0) = (0, 0)$ and define $(u - k)_\pm = \max\{-u + (u - k), 0\}$.

**Lemma 3.2** There exists a number $\sigma \in (0, 1)$ independent of $\omega, \rho$ such that if (3.3) and

$$|(x, t) \in [(0, t^*) + Q(l \rho^{p^*_\rho}, \rho)] : u < \mu^- + \frac{\omega}{2} \leq \sigma |Q(l \rho^{p^*_\rho}, \rho)|$$

hold, then $u > \mu^- + \frac{\omega}{2}$, a.e. $(x, t) \in [(0, t^*) + Q(l \rho^{p^*_\rho}, \rho)]$. 

EJQTDE, 2012 No. 4, p. 8
Proof: Up to a translation we may assume that $(0,t^*)=(0,0)$. Let $\rho_m = \frac{\xi}{2} + \frac{\mu_0}{2\mu_0 + \xi}, k_m = \mu_0 + \frac{\xi}{2} + \frac{\mu_0}{2\mu_0 + \xi}, Q_{\rho_m} = K_{\rho_m} \times (-l_{\rho_m}^+, 0), m = 0, 1, 2,...$ We choose smooth cutoff function $\eta_m = \xi_1(x)\xi_2(t)$, where $0 \leq \xi_1 \leq 1, 0 \leq \xi_2 \leq 1$ and

$$\xi_1 = 1, \text{ if } x \in K_{\rho_{m+1}}; \quad \xi_1 = 0, \text{ if } x \in K_{\rho_m}; \quad \text{and } |\nabla \xi_1| \leq \frac{1}{\rho_m - \rho_{m+1}}.$$

$$\xi_2 = 1, \text{ if } t \geq -l_{\rho_m}^+; \quad \xi_2 = 0, \text{ if } t \leq -l_{\rho_m}^+; \quad \text{and } 0 \leq \frac{\partial \xi_2}{\partial t} \leq \frac{1}{l(\rho_m^+ - \rho_m^+)}.$$

Take $\varphi = -(u_n - k_m^+) - \eta_m^+$ as the testing function in (1.11), then

$$\int_{Q_m^t} \frac{\partial u_n}{\partial t}[-(u_n - k_m^+) - \eta_m^+] \, dx \, dt + \int_{Q_m^t} a(x, \tau, u_n, \nabla u_n)[\nabla ((u_n - k_m^+) - \eta_m^+)] \, dx \, dt$$

$$= \int_{Q_m^t} a_0(x, \tau, u_n, \nabla u_n)[-\nabla(\rho_m^+)] \, dx \, dt = 0,$$

where $Q_m^t = K_{\rho_m} \times (-l_{\rho_m}^+, t), t \in (-l_{\rho_m}^+, 0)$.

First, integrating by parts,

$$\int_{Q_m^t} \frac{\partial u_n}{\partial t}[-(u_n - k_m^+) - \eta_m^+] \, dx \, dt$$

$$= \frac{1}{2} \int_{Q_m^t} \frac{\partial}{\partial \tau}((u_n - k_m^+) - \eta_m^+) \, dx - \frac{l_{\rho_m}^+}{2} \int_{Q_m^t} (u_n - k_m^+)^2 \eta_m^+ \, dx \, dt$$

$$= \frac{1}{2} \int_{K_{\rho_m}} (u_n - k_m^+) - \eta_m^+ \, dx + \frac{1}{2} \int_{K_{\rho_m}} (u_n - k_m^+) - \eta_m^+ \, dx$$

$$- \frac{l_{\rho_m}^+}{2} \int_{Q_m^t} [(u_n - k_m^+)^2 \eta_m^+ - \eta_m^+ \frac{\partial \eta_m}{\partial \tau}] \, dx \, dt.$$

Since $u_n \rightarrow u$ in $L^2(Q)$ and $u \in C(0, T; L^2(\Omega)), u_n \rightarrow u$ in $L^2(\Omega)$ for $\forall t \in (0, T)$, therefore we can get

$$\lim_{n \rightarrow \infty} \int_{Q_m^t} \frac{\partial u_n}{\partial t}[-(u_n - k_m^+) - \eta_m^+] \, dx \, dt$$

$$= \frac{1}{2} \int_{K_{\rho_m}} (u - k_m^+) - \eta_m^+ \, dx - \frac{1}{2} \int_{K_{\rho_m}} (u - k_m^+) - \eta_m^+ \, dx$$

$$- \frac{l_{\rho_m}^+}{2} \int_{Q_m^t} [(u - k_m^+)^2 \eta_m^+ - \eta_m^+ \frac{\partial \eta_m}{\partial \tau}] \, dx \, dt.$$

Since $\nabla (u_n - k_m^+) \rightarrow \nabla (u - k_m^+)$ and $a(x, \tau, u_n, \nabla u_n) \rightarrow a(x, \tau, u, \nabla u)$ a.e. in $Q_m^t$, by Fatou lemma,
By the fact that $u_n \rightharpoonup u$ strongly in $L^{p(x)}(Q)$, $a(x,t,u_n,\nabla u_n) \rightharpoonup a(x,t,u,\nabla u)$ weakly and $a_0(x,t,u_n,\nabla u_n) \rightharpoonup a_0(x,t,u,\nabla u)$ weakly in $L^{q(x)}(Q)$, we have

$$
\lim_{n \to \infty} \int_{Q_m} \left( a(x,\tau,u_n,\nabla u_n)[-(u_n - k_m) - \eta_m^{p^+}] \right) dx d\tau
\geq \int_{Q_m} \left( a(x,\tau,u,\nabla u)[-(u - k_m) - \eta_m^{p^+}] \right) dx d\tau.
$$

By (1.4)-(1.5), (1.7)-(1.8), $\|u\|_{L^{p(x)}(Q_m)} \leq M$ and $\|(u - k_m)\|_{L^{p(x)}(Q_m)} \leq \frac{\|k_m - u\|}{\frac{2}{p^+}}$, we have

$$
I \geq \beta \int_{Q_m} (|\nabla (u - k_m)|^{p(x)} + |u|^{p(x)}) \eta_m^{p^+} dx d\tau
- \alpha p^+ \int_{Q_m} (|\nabla (u - k_m)|^{p(x)-1} + |\nabla (u - k_m)|^{p(x)-1}) (u - k_m - \eta_m^{p^+}) |\nabla \eta_m| dx d\tau
- \alpha \int_{Q_m} (|\nabla (u - k_m)|^{p(x)-1} + |u|^{p(x)-1}) (u - k_m - \eta_m^{p^+}) dx d\tau
\geq \frac{\beta}{2} \int_{Q_m} |\nabla (u - k_m)|^{p(x)} \eta_m^{p^+} dx d\tau - C2^{mp^+} \rho^{-p^+} |A_m|,
$$

where $A_m = \{(x,t) \in Q_{\rho m} : u(x,t) < k_m\}$, $C = C(M, p^+)$. 

EJQTDE, 2012 No. 4, p. 10
So we can get the following inequality
\[
\sup_{-l_0^m < t < 0} \int_{K_{pm}} (u - k_m)^{\bar{p}_m} \eta_m^{p_m} \frac{dx}{\rho_m} + \int_{Q_{pm}} |\nabla (u - k_m)|^{p(x)} \eta_m^{p_m} \frac{dx dt}{\rho_m} 
\leq C 2^{mp_r} \rho^{-p_r} |A_m|, \tag{3.5}
\]
where \( C = C(M, p^+) \).

On the other hand, we have
\[
\int_{K_{pm}} (u - k_m)^{\bar{p}_m} \eta_m^{p_m} \frac{dx}{\rho_m} 
\leq \int_{K_{pm}} (u - k_m)^2 \eta_m^{p_m} \frac{dx}{\rho_m}
\]
and
\[
\int_{Q_{pm}} |\nabla (u - k_m)|^{p(x)} \eta_m^{p_m} \frac{dx dt}{\rho_m} 
\leq \int_{Q_{pm}} |\nabla (u - k_m)|^{p(x)} \eta_m^{p_m} \frac{dx dt}{\rho_m}
+ \int_{Q_{pm}} \chi_\eta [(u - k_m) - > 0] \frac{\rho_m}{\rho_m} \eta_m^{p_m} \frac{dx dt}{\rho_m},
\]
then by (3.5),
\[
\sup_{-l_0^m < t < 0} \int_{K_{pm}} (u - k_m)^{\bar{p}_m} \eta_m^{p_m} \frac{dx}{\rho_m} + \int_{Q_{pm}} |\nabla (u - k_m)|^{p(x)} \eta_m^{p_m} \frac{dx dt}{\rho_m} 
\leq C 2^{mp_r} \rho^{-p_r} \frac{1}{[A_m]}.
\]

Next, we introduce the change of time-variable \( z = l^{-1} t \) which transforms \( Q_{pm} \) into \( \tilde{Q}_{pm} = K_{pm} \times (-\rho_m, 0) \). Setting also \( v(x, t) = u(x, zl), \tilde{\eta}_m(x, z) = \eta_m(x, zl), |A_m| = \text{meas}\{ (x, z) \in \tilde{Q}_{pm} : v(x, z) < k_m \} \), then
\[
\sup_{-\rho_m^z < t < 0} \int_{K_{pm}} (v - k_m)^{\bar{p}_m} \tilde{\eta}_m^{p_m} \frac{dx}{\rho_m} \frac{dz}{z} + \int_{Q_{pm}} |\nabla (v - k_m)|^{p(x)} \tilde{\eta}_m^{p_m} \frac{dx dz}{\rho_m} 
\leq C 2^{mp_r} \rho^{-p_r} \frac{1}{[A_m]}.
\]

By lemma 2.8,
\[
\frac{1}{2p_r (m+1)} \frac{\rho_m}{(2)^{\bar{p}_m}} |A_m + 1| = |k_m - k_m + 1|^{p_r} \frac{1}{[A_m + 1]}
\leq \frac{1}{(v - k_m)^{p_r}} \frac{dx}{\rho_m (\tilde{Q}_{pm+1})} \leq \frac{1}{|\nabla (v - k_m)|^{p(x)} \tilde{\eta}_m^{p_m} \frac{dx dz}{\rho_m (\tilde{Q}_{pm})}}
\leq \frac{1}{|\nabla (v - k_m)|^{p(x)} \tilde{\eta}_m^{p_m} \frac{dx dz}{\rho_m (\tilde{Q}_{pm})}} \frac{1}{[A_m]^{p_r + \frac{1}{p_r} + \frac{1}{p_r + \infty}}} 
\leq C 2^{mp_r} \rho^{-p_r} |A_m|^{1 + \frac{1}{p_r + \infty}}.
\]

By (3.3), when \( A > 2 \), we choose \( \varepsilon \leq p_r - 2 \), then \( \left( \frac{\varepsilon}{\rho_m} \right)^{p_r} \leq \rho^{-p_r} \). Next, denote \( Y_m = \frac{|A_m|}{|Q_{pm}|} \), then by (1.10) we obtain

EJQTE, 2012 No. 4, p. 11
\[ Y_{m+1} \leq \frac{C4^{m}\rho_{m}^{+}\rho_{m}^{-}|A_{m}|^{\frac{1+\rho_{m}^{-}}{\rho_{m}^{-}+n}}}{|Q_{m+1}|} = \frac{C4^{m}\rho_{m}^{+}\rho_{m}^{-}|\tilde{Q}_{m}|^{\frac{1+\rho_{m}^{-}}{\rho_{m}^{-}+n}}}{|Q_{m+1}|} Y_{m}^{1+\frac{\rho_{m}^{-}}{\rho_{m}^{-}+n}} \leq C4^{m}\rho_{m}^{+}Y_{m}^{1+\frac{\rho_{m}^{-}}{\rho_{m}^{-}+n}}. \]

By lemma 2.7, when \( m \to \infty, Y_{m} \to 0 \) if \( Y_{0} \leq C \frac{\rho_{m}^{-}}{\rho_{m}^{-}+n} \left( \rho_{m}^{-}+n \right)^2 \equiv \sigma \) which just satisfies the condition of this lemma, i.e.

\[ Y_{0} = \frac{|\{(x,t) \in Q(l\rho^{+},\rho) : u < \mu^{-} + \frac{\omega}{4} \}|}{|Q(l\rho^{+},\rho)|} \leq \sigma. \]

By the fact that \( \rho_{m} \to \frac{\theta}{4}, \mu^{-} + \frac{\omega}{4} \) and \( |A_{m}| \to 0 \), we can get

\[ |\{(x,t) \in Q(l\rho^{+},\rho) : u(x,t) \leq \mu^{-} + \frac{\omega}{4} \}| = 0, \]

therefore \( u > \mu^{-} + \frac{\omega}{4}, \) a.e. \( (x,t) \in Q(l\rho^{+},\rho) \).

Let \( \theta = l\rho^{+}, \) by lemma 3.2 and \( u \in C(0,T;L^{2}(\Omega)) \), we obtain \( u(x,-\theta) > \mu^{-} + \frac{\omega}{4} \) a.e. \( x \in K_{\frac{\theta}{4}}. \)

**Lemma 3.3** Let (3.3)-(3.4) hold, then for every number \( \sigma_{1} \in (0,1) \), there exists a positive integer \( s \) such that

\[ |x \in K_{\frac{\theta}{4}} : u(x,t) < \mu^{-} + \frac{\omega}{4}^{2} \| \leq \sigma_{1}|K_{\frac{\theta}{4}}|, \quad \forall t \in (-\theta,0). \]

Proof: Set \( \rho^{+} = 2^{-1}\rho, \) we will consider the problem in \( Q(\theta^{+},\rho^{+}) = K_{\rho^{+}} \times (-\theta,0). \)

Let \( k = \mu^{-} + \frac{\omega}{4}, H_{k}^{-} = \text{ess sup}(u-k), \) thus \( H_{k}^{-} \leq \frac{\theta}{4} \). Then we take

\[ \Psi(u) = \max\{0, \ln \frac{H_{k}^{-}}{H_{k}^{-} - (u-k)} + \omega 2^{-(m+2)} \} = \ln^{+} \frac{H_{k}^{-}}{H_{k}^{-} - (u-k)} + \omega 2^{-(m+2)} . \]

By lemma 3.2, we know \( u(x,-\theta) > \mu^{-} + \frac{\omega}{4} \) a.e. \( x \in K_{\rho^{+}} \), so \( (u-k)_{-} = 0 \) a.e. in \( K_{\rho^{+}} \times \{-\theta\} \), moreover \( \Psi(u(x,-\theta)) = 0 \) a.e. \( x \in K_{\rho^{+}} \). Since \( \frac{\theta}{4} \geq H_{k}^{-} \geq (u-k)_{-}, \) we get \( \Psi(u) \leq \ln^{+} \frac{\theta}{2^{m+2}} = m \ln 2 \) and

\[ \frac{\partial \Psi(u)}{\partial u} = \begin{cases} \frac{1}{H_{k}^{-} - (u-k)_{-} + \omega 2^{-(m+2)}}, & u < k - \omega 2^{-(m+2)}, \\ 0, & u \geq k - \omega 2^{-(m+2)} , \end{cases} \]

therefore when \( u < k - \omega 2^{-(m+2)} \),

\[ \frac{\partial \Psi(u)}{\partial u} \leq \frac{\partial \Psi(u)}{\partial u} \leq \frac{2^{(m+2)}}{2}. \]
Take $\varphi = \frac{\partial}{\partial \tau}(|\Psi^2(d)|\eta^p^\tau)|_{d=u_h}$ as the testing function in (3.2), where $\eta$ is the cutoff function independent of $t$ and satisfies $0 < \eta < 1$ in $K^\rho$, $\eta = 1$ in $K_{2-1}^\rho$, and $|\nabla \eta| \leq 4\rho^{-1}$, then

$$
\int_{Q^\rho(\theta, \rho^*)} \frac{\partial}{\partial d} [\Psi^2(d)] \eta^p^\tau |_{d=u_h} \frac{\partial u_h}{\partial \tau} dxd\tau
$$

$$
+ \int_{Q^\rho(\theta, \rho^*)} [a(x, \tau, u, \nabla u)]_h \nabla \frac{\partial}{\partial d} [\Psi^2(d)] \eta^p^\tau |_{d=u_h} dxd\tau
$$

$$
+ \int_{Q^\rho(\theta, \rho^*)} [a_0(x, \tau, u, \nabla u)]_h \frac{\partial}{\partial d} [\Psi^2(d)] \eta^p^\tau |_{d=u_h} dxd\tau = 0,
$$

(3.6)

where $Q^\rho(\theta, \rho^*) = K^\rho \times (-\theta, \theta)$, $t \in (-\theta, 0)$.

Integrating by parts,

$$
\int_{Q^\rho(\theta, \rho^*)} \frac{\partial}{\partial \tau} [\Psi^2(d)] \eta^p^\tau |_{d=u_h} \frac{\partial u_h}{\partial \tau} dxd\tau = \int_{Q^\rho(\theta, \rho^*)} \frac{\partial}{\partial \tau} [\Psi^2(u_h)] \eta^p^\tau dxd\tau
$$

by $\Psi(u_h) \leq m \ln 2$, $\Psi(u) \leq m \ln 2$, $|\Psi^2(u_h) - \Psi^2(u)| \leq \frac{m^2}{2} \ln 2 |u_h - u|$, and $u_h \rightarrow u$ in $L^2(K^\rho)$ for $\forall t \in (-\theta, 0), so$

$$
\int_{K_{2-1}^\rho} \Psi^2(u_h(x, t)) \eta^p^\tau dx \rightarrow \int_{K_{2-1}^\rho} \Psi^2(u(x, t)) \eta^p^\tau dx,
$$

$$
\int_{K_{2-1}^\rho} \Psi^2(u_h(x, -\theta)) \eta^p^\tau dx \rightarrow \int_{K_{2-1}^\rho} \Psi^2(u(x, -\theta)) \eta^p^\tau dx,
$$

therefore we obtain

$$
\int_{Q^\rho(\theta, \rho^*)} \frac{\partial}{\partial \tau} [\Psi^2(d)] \eta^p^\tau |_{d=u_h} \frac{\partial u_h}{\partial \tau} dxd\tau
$$

$$
- \int_{K_{2-1}^\rho} \Psi^2(u_h(x, t)) \eta^p^\tau dx - \int_{K_{2-1}^\rho} \Psi^2(u(x, -\theta)) \eta^p^\tau dx,
$$

(3.7)

Denote $\Psi'(u) = \frac{\partial (\Psi(u))}{\partial \tau}|_{d=u_h}$. Since $\frac{\partial^2}{\partial \tau^2} (\Psi^2(d))|_{d=u_h} = 2(1+\Psi(u_h))\Psi'(u_h)^2$, for the other parts of (3.6),

$$
I \equiv \int_{Q^\rho(\theta, \rho^*)} [a(x, \tau, u, \nabla u)]_h \nabla \frac{\partial}{\partial d} [\Psi^2(d)] \eta^p^\tau |_{d=u_h} dxd\tau
$$

$$
+ \int_{Q^\rho(\theta, \rho^*)} [a_0(x, \tau, u, \nabla u)]_h \frac{\partial}{\partial d} [\Psi^2(d)] \eta^p^\tau |_{d=u_h} dxd\tau
$$

$$
= 2 \int_{Q^\rho(\theta, \rho^*)} [a(x, \tau, u, \nabla u)]_h \nabla u_h (1 + \Psi(u_h)) \Psi'(u_h)^2 \eta^p^\tau dxd\tau
$$

$$
+ 2 \int_{Q^\rho(\theta, \rho^*)} [a_0(x, \tau, u, \nabla u)]_h \Psi(u_h) \nabla \eta^p^\tau dxd\tau
$$

$$
+ 2 \int_{Q^\rho(\theta, \rho^*)} [a_0(x, \tau, u, \nabla u)]_h \Psi'(u_h) \Psi(u_h) \eta^p^\tau dxd\tau.
$$

EJQTDE, 2012 No. 4, p. 13
Next, we consider the problem on the set \( \{(x, t) \in K_{\rho^*} \times (-\theta, 0) : u(x, t) < k - \omega 2^{-(m+2)} \} \), thus \( \frac{\omega}{2} \leq |\Psi'(u)| \leq 2^{m+2}. \) When \( h \to 0, \) \( u_h \to u \) and \( (u_h - k)_{-} \to (u - k)_{-} \) a.e. in \( (x, t) \in Q(\theta, \rho^*), \) so \( (1 + \Psi(u_h))\Psi'(u)^2 \to (1 + \Psi(u))\Psi'(u)^2 \) a.e. in \( (x, t) \in Q(\theta, \rho^*). \) Since
\[
|(1 + \Psi(u_h))\Psi'(u_h)^2 - (1 + \Psi(u))\Psi'(u)^2|^2 \leq |\Omega(1 + \Psi(u)\Psi'(u))|\leq 2(1 + m \ln 2)(\frac{2^{m+2}}{\omega})2\rho^*
\]
and by Lebesgue’s theorem, we get
\[
(1 + \Psi(u_h))\Psi'(u_h)^2 \to (1 + \Psi(u))\Psi'(u)^2 \nabla u
\]
in \( L^{p(x)}(Q^*(\theta, \rho^*)) \) for a.e. \( t \in (-\theta, 0). \) Because \( [a(x, t, u, \nabla u)]_h \to a(x, t, u, \nabla u) \) in \( L^{p(x)}(Q^*(\theta, \rho^*)), \)
\[
\int_{Q^*(\theta, \rho^*)} a(x, \tau, u, \nabla u) \nabla u \to \int_{Q^*(\theta, \rho^*)} a(x, \tau, u, \nabla u) \nabla u \\nabla (1 + \Psi(u))\Psi'(u)^2 \eta_\rho^* \mathrm{d}x \mathrm{d}t.
\]
In the same way
\[
\int_{Q^*(\theta, \rho^*)} a_0(x, \tau, u, \nabla u) \nabla u \to \int_{Q^*(\theta, \rho^*)} a_0(x, \tau, u, \nabla u) \nabla u \Psi'(u)^2 \eta_\rho^* \mathrm{d}x \mathrm{d}t,
\]
are both valid.

Combining these estimates, we have
\[
\lim_{h \to 0} I = 2 \int_{Q^*(\theta, \rho^*)} a(x, \tau, u, \nabla u) \nabla u (1 + \Psi(u))\Psi'(u)^2 \eta_\rho^* \mathrm{d}x \mathrm{d}t
\]

With (1.4)-(1.5), (1.7)-(1.8), we can get
\[
\lim_{h \to 0} I \geq 2\beta \int_{Q^*(\theta, \rho^*)} (|\nabla u|^{p(x)} + |u|^{p(x)}) (1 + \Psi(u))\Psi'(u)^2 \eta_\rho^* \mathrm{d}x \mathrm{d}t
\]

Since \( \frac{\rho^* - 1}{p(x) - 1} > \frac{p^*(p(x) - 1)}{p(x) - 1}, \) by Young’s inequality,
\[
\int_{Q^*(\theta, \rho^*)} |\nabla u|^{p(x) - 1}|\Psi'(u)|\Psi(u)\eta_\rho^* \mathrm{d}x \mathrm{d}t
\]

EJQTDE, 2012 No. 4, p. 14
In the same way, we have

\[
\int_{Q^t(\theta,\rho^*)} |u|^{p(x)-1}|\Psi'(u)|\Psi(u)\eta^{p^\ast} \leq \varepsilon \int_{Q^t(\theta,\rho^*)} |u|^{p(x)}|\Psi'(u)|^2(\Psi(u) + 1)\eta^{p^\ast} d\tau + C(\varepsilon) \int_{Q^t(\theta,\rho^*)} (\Psi'(u))^{2-p(x)}\Psi(u)|\nabla \eta|^{p(x)} d\tau,
\]

\[
\int_{Q^t(\theta,\rho^*)} |\nabla u|^{p(x)-1}|\Psi'(u)|\Psi(u)\eta^{p^\ast} d\tau 
\leq \varepsilon \int_{Q^t(\theta,\rho^*)} |\nabla u|^{p(x)}|\Psi'(u)|^2(\Psi(u) + 1)\eta^{p^\ast} d\tau + C(\varepsilon) \int_{Q^t(\theta,\rho^*)} (\Psi'(u))^{2-p(x)}\Psi(u)|\nabla \eta|^{p(x)} d\tau,
\]

and

\[
\int_{Q^t(\theta,\rho^*)} |\nabla u|^{p(x)-1}|\Psi'(u)|\Psi(u)\eta^{p^\ast} d\tau 
\leq \varepsilon \int_{Q^t(\theta,\rho^*)} |\nabla u|^{p(x)}|\Psi'(u)|^2(\Psi(u) + 1)\eta^{p^\ast} d\tau + C(\varepsilon) \int_{Q^t(\theta,\rho^*)} (\Psi'(u))^{2-p(x)}\Psi(u)(\eta^{p^\ast} + |\nabla \eta|^{p(x)}) d\tau.
\]

Combining (3.8)-(3.11),

\[
\lim_{h \to 0} I \geq (2\beta - 4\alpha p^\ast \varepsilon) \int_{Q^t(\theta,\rho^*)} (|\nabla u|^{p(x)} + |u|^{p(x)})(1 + \Psi(u))\Psi'(u)^2\eta^{p^\ast} d\tau - C(\varepsilon) \int_{Q^t(\theta,\rho^*)} (\Psi'(u))^{2-p(x)}\Psi(u)(\eta^{p^\ast} + |\nabla \eta|^{p(x)}) d\tau.
\]

Take \(4\alpha p^\ast \varepsilon = \beta\), then

\[
\lim_{h \to 0} I \geq \beta \int_{Q^t(\theta,\rho^*)} (|\nabla u|^{p(x)} + |u|^{p(x)})(1 + \Psi(u))\Psi'(u)^2\eta^{p^\ast} d\tau - C(p^\ast) \int_{Q^t(\theta,\rho^*)} (\Psi'(u))^{2-p(x)}\Psi(u)(\eta^{p^\ast} + |\nabla \eta|^{p(x)}) d\tau.
\]  

In view of (3.7) and (3.12),

\[
\int_{K_{\rho^*}} \Psi^2(u(x,t))\eta^{p^\ast} dx \leq C \int_{Q^t(\theta,\rho^*)} (\Psi'(u))^{2-p(x)}\Psi(u)(\eta^{p^\ast} + |\nabla \eta|^{p(x)}) d\tau.
\]

By \(\Psi(u) \leq m \ln 2, |\Psi'(u)|^{-1} \leq \frac{\omega}{\beta}, |\nabla \eta| \leq \frac{1}{\beta}, |\Psi(u)| \leq \frac{2m+2\omega}{\beta}\), we can get

\[
\int_{K_{\rho^*}} \Psi^2(u(x,t))\eta^{p^\ast} dx \leq C m |K_{\rho^*}|.
\]  

\(\forall t \in (-\theta,0),\) for such a set \(\{(x,t) \in K_{2\rho^*} : u(x,t) < \mu^- + \frac{m}{\ln 2}\}\) we have

\[
\Psi^2(u) \geq \ln^2 \frac{H^+_k}{H^-_k - \frac{\omega}{4} + \frac{\omega}{2m+2\omega}}.
\]

Since \(-\frac{\omega}{4} + \frac{\omega}{2m+2\omega} < 0\), we obtain \(\ln^2 \frac{H^+_k}{H^-_k - \frac{\omega}{4} + \frac{\omega}{2m+2\omega}} \) is decreasing about \(H^-_k\) and \(H^-_k \leq \frac{\omega}{4}\), thus

\[
\Psi^2(u) \geq \ln^2 \frac{H^+_k}{H^-_k - \frac{\omega}{4} + \frac{\omega}{2m+2\omega}} \geq \ln^2 \frac{\omega}{4} - \frac{\omega}{4} + \frac{\omega}{2m+2\omega} = [(m-1)\ln 2]^2.
\]
Because $\eta = 1$ in $K_{r_m}^+$, by (3.13)

$$|x \in K_{r_m}^+ : u(x,t) < \mu^{-} - \frac{\omega}{2m^2e^2}| \leq C \frac{m}{(m-1)^2}|K_{r_m}^+|,$$

where $C = C(M, p^+)$. To prove the lemma we have only to choose $m$ sufficiently large and $s = m + 2$.

**Lemma 3.4** Let (3.3)-(3.4) hold, then there exist $\sigma_1 \in (0, 1)$ and an integer $s > 1$ independent of $\omega$ and $\rho$, so that $u(x,t) > \mu^{-} + \frac{\omega}{2m^2e^2}$, a.e. $(x,t) \in Q(\theta, \frac{r_m}{4})$.

Proof: Let $\rho_m = \frac{\omega}{

Proof: Let $\rho_m = \frac{\omega}{2m^2e^2}$, $k_m = \mu^{-} + \frac{\omega}{2m^2e^2}$, $m = 0, 1, 2, \ldots$, and $s > 1$ is to be chosen later. By lemma 3.2, for a.e. $x \in K_{r_m}$, we have $u(x,\theta) > \mu^{-} + \frac{\omega}{2m^2e^2} \geq k_m$, thus $(u-k_m)_{-}(x, -\theta) = 0$. Let $\eta_m(x)$ be a smooth cutoff function in $K_{r_m}$ satisfying $\eta_m \equiv 1$ in $K_{r_m+1}$, $|\nabla \eta_m| \leq \frac{2m^2e^2}{\rho_m}$, and $\eta_m = 0$ outside $K_{r_m}$.

We take $\varphi = -(u-k_m)_{-}\eta_m^+$ as the testing function in (1.11), by the fact that

$$\|u\|_{L^\infty(Q_{r_m})} \leq M, \quad \|(u-k_m)_{-}\|_{L^\infty(Q_{r_m})} \leq \|(k_m-u)\|_{L^\infty(Q_{r_m})} \leq \frac{\omega}{2m^2e^2},$$

similar to lemma 3.2, we have

$$\sup_{-\theta < t < 0} \int_{K_{r_m}} (u-k_m)^2 \eta_m^+ d\mathcal{L} dx + \int_{Q(\theta, r_m^+)} |\nabla (u-k_m)-|^{p^+}\eta_m^+ d\mathcal{L} dx dt \leq C 2^{m^2e^2} \rho^{-p^+}\int_{Q(\theta, r_m^+)} \chi\{(u-k_m)_{-} > 0\} dx dt. \tag{3.14}$$

On the other hand, we have

$$\int_{K_{r_m}} (u-k_m)^2 \eta_m^+ d\mathcal{L} dx \geq \left(\frac{\omega}{2m^2e^2}\right)^{2-p^+} \int_{K_{r_m}} (u-k_m)^{p^+}\eta_m^+ d\mathcal{L} dx \geq \frac{\theta}{\rho^+} \int_{K_{r_m}} (u-k_m)^{p^+}\eta_m^+ d\mathcal{L} dx$$

and

$$\int_{Q(\theta, r_m^+)} |\nabla (u-k_m)-|^{p^+}\eta_m^+ d\mathcal{L} dx dt \leq \int_{Q(\theta, r_m^+)} |\nabla (u-k_m)-|^{p^+}\eta_m^+ d\mathcal{L} dx dt + \int_{Q(\theta, r_m^+)} \chi\{(u-k_m)_{-} > 0\}\eta_m^+ d\mathcal{L} dx dt,$$

where $s$ is chosen so large as to satisfy the conclusion of lemma 3.3.

Combining the above two inequalities with (3.14), we get

$$\sup_{-\theta < t < 0} \int_{K_{r_m}} (u-k_m)^{p^+}\eta_m^+ d\mathcal{L} dx + \left(\frac{\theta}{\rho^+}\right)^{p^+\frac{p^+}{p^+-1}} \int_{Q(\theta, r_m^+)} |\nabla (u-k_m)-|^{p^+}\eta_m^+ d\mathcal{L} dx dt \leq C 2^{m^2e^2} \rho^{-p^+}\frac{(p^+)^{p^+\frac{p^+}{p^+-1}}}{\theta} \int_{Q(\theta, r_m^+)} \chi\{(u-k_m)_{-} > 0\} dx dt.$$

We introduce the change of variable $z = t(p^+)\eta_m^+ \theta^{-1}$, which maps $Q(\theta, r_m^+)$ into $Q_m = K_{r_m} \times (-\rho^+, 0)$. Let $\nu(x, t) = u(x, \theta z \rho^+ \eta_m^+, 0)$. Let $v(x, t) = u(x, \theta z \rho^+ \eta_m^+, 0)$, $\tilde{\eta}_m(x, z) = \chi(x \in K_{r_m}, z \in (-\rho^+, 0))$. Then

$$\int_{K_{r_m} \times (-\rho^+, 0)} \int_{\mathbb{R}} |\nabla \nu|^2 \tilde{\eta}_m^+ \eta_m^+ d\mathcal{L} dx dz \leq C 2^{m^2e^2} \rho^{-p^+}\frac{(p^+)^{p^+\frac{p^+}{p^+-1}}}{\theta} \int_{Q(\theta, r_m^+)} \chi\{(u-k_m)_{-} > 0\} dx dt.$$
by lemma 2.6 and (3.15),

$$Z = \text{meas}\{(x, z) \in Q : v(x, z) < k_m\},$$

then by lemma 2.6 it follows that

$$\|v - k_m\|_{L^p(\varOmega)} \leq C(\sup_{x, t} (v - k_m) - \eta_m)^{p^+} + \int_{Q_m} |\nabla[(v - k_m) - \eta_m]|^{p^+} dx dz$$

by lemma 2.6 and (3.15),

$$\frac{1}{2} \left( \frac{\omega}{2^+} \right)^{p^+} |A_{m+1}| = |k_m - k_{m+1}|^{p^+} |A_{m+1}|$$

We take $A > 2^+$, by (3.3), we get $(\frac{\omega}{2^+})^{p^- - 2} \geq p^+ \geq p^{p^--2}$, therefore $\frac{\omega}{2^+} \geq p^+$. Thus we obtain

$$\left( \frac{\omega}{2^+} \right)^{p^+} \leq p^+$$

Denote $Z_m = \frac{|A_m|}{|Q_m|}$. By (3.16) and (1.10),

$$Z_m \leq C 4^{m p^+} Z_m^{1+} \leq C 4^{m p^+} Z_m^{1+} \leq C 4^{m p^+} Z_m^{1+}$$

where $C = C(M, p^+)$. Since

$$Z_0 = \frac{|A_0|}{|Q_0|} = \frac{|\{(x, t) \in Q(\theta, \frac{\omega}{2^+}) : u(x, t) < \mu^- + \frac{\omega}{2^+} \}|}{|Q(\theta, \frac{\omega}{2^+})|},$$

by lemma 3.3 there exists $s$ such that $Z_0 < \sigma_1$ where $\sigma_1 \equiv C - \frac{\omega}{2^+} - 4^{-p^+(\frac{\omega}{2^+})^2}$. Then by lemma 2.6 it follows that $Z_m \to 0$ as $m \to \infty$. So we can get

$$u(x, t) > \mu^- + \frac{\omega}{2^+}, \quad \text{a.e.} \quad (x, t) \in Q(\theta, \frac{\rho^+}{4}).$$

**Proposition 3.1** There exist $\sigma \in (0, 1)$, $\nu_1 \in (0, 1)$ and $A_1 \gg 1$ independent of $\omega$ and $\rho$, such that if for some cylinder of the type $[(0, t^*) + Q(\rho, \rho^+), \rho]$, then

$$|(x, t) \in [(0, t^*) + Q(\rho, \rho^+), \rho) : u < \mu^- + \frac{\omega}{2^+} | \leq \sigma |Q(\rho, \rho^+), \rho)|,$$

EJQTDE, 2012 No. 4, p. 17
then either
\[ \omega \leq A_1 \rho^{s-2} \]  \hspace{1cm} (3.17)

or
\[ \text{ess osc } u \leq \nu_1 \omega. \] \hspace{1cm} (3.18)

Proof: Assume (3.17) is violated. By lemma 3.4, we can determine a positive integer number \( s \) such that
\[ \text{ess inf } u \geq \mu^- + \frac{\omega}{2s+1}, \]
this gives
\[ \text{ess inf } u \leq -\mu^- - \frac{\omega}{2s+1}, \] \hspace{1cm} (3.19)
and further
\[ \text{ess osc } u \leq (1 - \frac{1}{2s+1})\omega. \]

therefore the proposition follows with \( \nu_1 = \left( 1 - \frac{1}{2s+1} \right) \), since \( Q(l(\frac{\rho}{\theta}), \frac{\rho}{\theta}) \subset Q(\theta, \frac{\rho}{\theta}). \]

Next assume that the condition of proposition 3.1 is violated, i.e. for every cylinder \([0, t^*) + Q(l(\rho^p), \rho)] \subset Q(\rho^p, \rho), \)
where \( \frac{1}{2} = (\frac{\rho}{\theta})^{p-2} \),
\[ |(x, t) \in [(0, t^*) + Q(l(\rho^p), \rho) : u < \mu^- + \frac{\omega}{2}] > \sigma Q(l(\rho^p), \rho)|. \]

Since \( \mu^- + \frac{\omega}{2} \leq \mu^+ - \frac{\omega}{2}, \) we can get
\[ |(x, t) \in [(0, t^*) + Q(l(\rho^p), \rho) : u > \mu^+ - \frac{\omega}{2}] \leq (1 - \sigma)Q(l(\rho^p), \rho)|. \] \hspace{1cm} (3.20)

Lemma 3.5 \hspace{1cm} Let (3.20) hold, then there exists a \( \bar{t} \in [t^* - l\rho^p, t^* - \frac{s}{2}\rho^p] \) such that
\[ \{|x \in \rho^p : u(x, \bar{t}) > \mu^+ - \frac{\omega}{2}\} \leq \frac{1 - \sigma}{1 - \frac{\sigma}{2}}|K_\rho|. \]

Proof: If not, for all \( t \in [t^* - l\rho^p, t^* - \frac{s}{2}\rho^p] \),
\[ \{|x \in \rho^p : u(x, t) > \mu^+ - \frac{\omega}{2}\} > \frac{1 - \sigma}{1 - \frac{\sigma}{2}}|K_\rho| \]
and
\[ |(x, t) \in [(0, t^*) + Q(l(\rho^p), \rho) : u > \mu^+ - \frac{\omega}{2}] | \geq \int_{t^* - l\rho^p}^{t^* - \frac{s}{2}\rho^p} \{|x \in \rho^p : u(x, t) > \mu^+ - \frac{\omega}{2}\}dt \]
\[ > (1 - \frac{s}{2})l(\rho^p(1 - \sigma)(1 - \frac{\sigma}{2})|K_\rho| = (1 - \sigma)Q(l(\rho^p), \rho), \]
contradicting (3.20). \( \square \)

Lemma 3.6 \hspace{1cm} Let (3.20) hold, then there exists a positive integer \( \bar{s} > 2, \) such that
\[ \{|x \in \rho^p : u(x, t) > \mu^+ - \frac{\omega}{2}\} \leq (1 - (\frac{\sigma}{2})^2)|K_\rho|, \hspace{0.5cm} \forall t \in [t^* - \frac{s}{2}\rho^p, t^*]. \]
Proof: Let \( k = \mu^+ - \frac{\sigma}{2} \), \( Q_\rho = K_\rho \times (\bar{t}, t^*) \). Similar to lemma 3.3, we take
\( \varphi = \frac{\partial}{\partial t} [\Psi^2(x)] \eta^p \mid_{x=0} \) as the testing function in (3.2), where the cutoff function \( \eta \) independent of \( t \) is taken so that \( \eta \equiv 1 \) in the cube \( K_{(1-\alpha)\rho}, \alpha \in (0, 1), \) and \( |\nabla \eta| \leq \frac{1}{\alpha \rho}, 0 < \alpha < 1 \). We take \( H_k^+ = \text{ess sup}_{(0, t^*) \times Q(x_\rho, \rho^*)} \frac{(u - k)^+}{2} \), and consider
\[
\Psi(u) = \max\{0, \ln \frac{H_k^+}{H_k^+ - (u - k)^+ + \omega 2^{-(m+2)}}\} = \ln^+ \frac{H_k^+}{H_k^+ - (u - k)^+ + \omega 2^{-(m+2)}},
\]
then
\[
\int_{K_{(1-\alpha)\rho}} \Psi^2(u(x, t))dx \leq \int_{K_\rho} \Psi^2(u(x, t))\eta^p dx \leq \int_{K_\rho} \Psi^2(u(x, t))dx + C \int_{K_\rho} \Psi'(|u|)2^{-p(x)}\Psi(u)\eta^{2(p^*)} + |\nabla \eta|^{p^*}dx,
\]
where \( |\bar{t} - \bar{t}| \leq l \rho^2, l = (\frac{\sigma}{2})^2 \rho^2, C = C(M, \rho^*) \).
When \( u(x, t) > k + \frac{\omega 2^{-(m+1)}}{\rho}, \Psi^2(u(x, \bar{t})) \neq 0 \), by lemma 3.5,
\[
\int_{x \in K_{(1-\alpha)\rho}, u(x, t) > \mu^+ - \frac{\omega 2^{-(m+1)}}{\rho}} \Psi^2(u(x, \bar{t}))dx \leq \int_{x \in K_{(1-\alpha)\rho}, u(x, t) > \mu^+ - \frac{\omega 2^{-(m+1)}}{\rho}} \Psi^2(u(x, \bar{t}))dx \leq (m \ln 2)^2 (1 - \sigma)(1 - \frac{\sigma}{2})^{-1}|K_\rho|,
\]
so we have
\[
\int_{K_{(1-\alpha)\rho}} \Psi^2(u(x, t))dx \leq C[m^2 (1 - \sigma)(1 - \frac{\sigma}{2})^{-1} + m \alpha^{-\rho^*_2}]|K_\rho|.
\]
\forall t \in (\bar{t}, t^*), \in \{x \in K_{(1-\alpha)\rho} : u(x, t) > \mu^+ - \frac{\omega 2^{-(m+1)}}{\rho}\} \text{ we can get }
\[
\Psi^2(u) \geq \ln^2 \frac{H_k^+}{H_k^+ - \frac{\omega 2^{-(m+1)}}{\rho} + \omega 2^{-(m+1)}} \geq \ln^2 \frac{\omega 2^{-2}}{\omega 2^{-(m+1)}} = (m - 1)^2 \ln^2 2,
\]
so \( \forall t \in (\bar{t}, t^*) \),
\[
|x \in K_{(1-\alpha)\rho} : u(x, t) > \mu^+ - \omega 2^{-(m+2)}| \leq C[(\frac{\omega 2^{-(m+1)}}{\rho})^2 (1 - \sigma)(1 - \frac{\sigma}{2})^{-1} + m \alpha^{-\rho^*_2}]|K_\rho|.
\]
On the other hand, \( \forall t \in (\bar{t}, t^*) \),
\[
|x \in K_\rho \setminus K_{(1-\alpha)\rho} : u(x, t) > \mu^+ - \omega 2^{-(m+1)}| \leq |K_\rho \setminus K_{(1-\alpha)\rho}| \leq |x \in K_{(1-\alpha)\rho} : u(x, t) > \mu^+ - \omega 2^{-(m+1)}| + \alpha N|K_\rho|,
\]
so \( \forall t \in (\bar{t}, t^*) \),
\[
|x \in K_{(1-\alpha)\rho} : u(x, t) > \mu^+ - \omega 2^{-(m+1)}| \leq C(\frac{\omega 2^{-(m+1)}}{\rho})^2 [1 - \sigma^{-1} + \sum \alpha^{-\rho^*_2} + N\alpha]|K_\rho|.
\]

EJQTDE, 2012 No. 4, p. 19
Choose \( \alpha \) so small and then \( m \) so large that \( C(\frac{m}{m-1})^2 \leq (1+\sigma)(1-\frac{s}{2}) \), \( \frac{C}{m}\alpha^{-p_0^+} \leq \frac{2}{3}\sigma^2 \) and \( C\alpha N \leq \frac{4}{3}\sigma^2 \). Then for such a choice of \( m \) the lemma follows with \( \bar{s} = m + 1 \). \( \Box \)

Since (3.20) holds for all \( [0, t^*] + Q(l\rho^{p_0^+}, \rho) \), the conclusion of lemma 3.6 holds for all time levels satisfying \( t \geq -(a - t)\rho^{p_0^+} = -(1 - (\frac{2}{a})\rho^{p_0^+})a\rho^{p_0^+} \). If the number \( A \) is chosen sufficiently large such that \( 1 - (\frac{2}{a})\rho^{p_0^+} - 2 > \frac{2}{3} \), we deduce the following corollary.

**Corollary 3.1** Let (3.20) hold, then for all \( t \in (-\frac{2}{a}\rho^{p_0^+}, 0) \),

\[
|\{x \in K_\rho : u(x, t) > \mu^+ - \omega 2^{-s^*}\}| \leq (1 - \frac{\sigma^2}{2})|K_\rho|.
\]

**Lemma 3.7** Let (3.20) hold, then for every \( \bar{s} \in (0, 1) \), there exists positive integer \( s^* > \bar{s} \), such that

\[
|\{x \in K_\rho : u(x, t) > \mu^+ - \omega \frac{2^s}{2^s}\}| \leq \bar{s}|Q(2^{-1}a\rho^{p_0^+}, \rho)|, \quad \forall t \in (-\frac{2}{a}\rho^{p_0^+}, 0).
\]

Proof: Consider the problem in \( Q(a\rho^{p_0^+}, 2\rho) \). Let \( k = \mu^+ - \omega \frac{2^s}{2^s} \), where \( \bar{s} \leq s \leq s^* \). Take \( \varphi = (u_n - k)_+ \zeta^{p_0^+} \) as the testing function in (1.9), where \( \zeta \) is a cutoff function that equals one on \( Q(\frac{\rho}{2}\rho^{p_0^+}, \rho) \), vanishes on the parabolic boundary of \( Q(a\rho^{p_0^+}, 2\rho) \) and such that \( |\nabla \zeta| \leq \frac{1}{2}, 0 \leq \zeta t \leq \frac{2}{a\rho^{p_0^+}} \). Similar to lemma 3.2, we get

\[
\int_{A_\ast} |\nabla u|^{p_0^+} |dxdt| \leq \int_{Q(\frac{\rho}{2}\rho^{p_0^+}, \rho)} |\nabla (u - k)_+ |^{p(x)} |dxdt| + |A_\ast|
\]

where \( C = C(p^+) \) and

\[
A_\ast = \{(x, t) \in Q(\frac{\rho}{2}\rho^{p_0^+}, \rho) : u(x, t) > \mu^+ - \omega \frac{2^s}{2^s}\},
\]

\[
A_\ast(t) = \{x \in K_\rho : u(x, t) > \mu^+ - \omega \frac{2^s}{2^s}\}.
\]

By corollary 3.1, \( \forall t \in (-\frac{2}{a}\rho^{p_0^+}, 0) \),

\[
|\{x \in K_\rho : u(x, t) < \mu^+ - \omega \frac{2^s}{2^s}\}| = |K_\rho| - |A_\ast(t)| \geq (\frac{\sigma^2}{2})|K_\rho|.
\] (3.21)

In lemma 2.8, take \( k = \mu^+ - \omega \frac{2^s}{2^s}, h = \mu^+ - \omega \frac{2^{s+1}}{2^{s+1}} \), \( \forall t \in [-\frac{a}{2}\rho^{p_0^+}, 0] \), by (3.21), we get

\[
\frac{\omega \frac{2^{s+1}}{2^{s+1}}}{2^{s+1}}|A_{s+1}(t)| \leq \frac{C}{\sigma^2} |\rho^{p_0^+}| \int_{A_\ast(t) \setminus A_{s+1}(t)} |\nabla u| |dx|.
\] (3.22)
Take $A > 2^s$, there exists $C = C(M, p^+, p^-)$ such that $(\frac{\omega}{A})^{p^+_\nu} - p^-_\nu \leq C$ and $(\frac{\omega}{A})^{p^-_\nu} \leq \rho^{p^-_\nu}$ hold. Integrating on $(-\alpha \rho^{p^+_\nu}, 0)$, from (3.22) we get

\[
(\frac{\omega}{A})^{p^-_\nu} \frac{\omega}{A^{p^+_\nu - 1}} |A_{s+1}| \leq (\frac{\omega}{A})^{p^-_\nu} \frac{C\rho}{\sigma} \int_{A_s \setminus A_{s+1}} |\nabla u| dx dt
\]

\[
\leq (\frac{\omega}{A})^{p^-_\nu} \frac{C\rho}{\sigma} (\int_{A_s} |\nabla u|^{p^-_\nu} dx dt)^{\frac{1}{p^-_\nu}} |A_s \setminus A_{s+1}|^{\frac{1}{p^-_\nu}}
\]

\[
\leq \frac{C\rho}{\sigma} |Q(\frac{\omega}{A}\rho^{p^+_\nu}, \rho)|^{\frac{1}{p^-_\nu}} |A_s \setminus A_{s+1}|^{\frac{1}{p^-_\nu}}.
\]

(3.23)

If $s$ is large enough so that $(\frac{\omega}{A})^{p^-_\nu} \frac{2^{s+1}}{2^s} < 1$, from (3.23) we get

\[
|A_{s+1}|^{\frac{1}{p^-_\nu}} \leq C_\sigma \frac{2^{-s} \sigma_\nu}{\rho^{p^-_\nu - 1}} |Q(\frac{\omega}{A}\rho^{p^+_\nu}, \rho)|^{\frac{1}{p^-_\nu}} |A_s \setminus A_{s+1}|,
\]

(3.24)

for all $s \leq s \leq s^*$. We add them for $s = \bar{s}, \bar{s} + 1, \bar{s} + 2, ..., s^* - 1$, then

\[
(s^* - \bar{s}) |A_{s^*}|^{\frac{1}{p^-_\nu}} \leq C_\sigma \frac{2^{-s} \sigma_\nu}{\rho^{p^-_\nu - 1}} |Q(\frac{\omega}{A}\rho^{p^+_\nu}, \rho)|^{\frac{1}{p^-_\nu}}.
\]

After taking $s^*$ so large that $C(s^* - \bar{s}) \frac{2^{-s^*}}{2^{s^*}} \leq \sigma^2 \sigma_\nu$, we conclude the lemma.□

**Lemma 3.8** Let (3.20) hold, then there exists $\bar{\sigma} \in (0, 1)$ so that

\[
u(x, t) \leq \frac{\omega}{2^{s+1}}, \quad a.e. \quad Q(\frac{w}{2}\rho^{p^+_\nu}, \rho),
\]

where $\frac{1}{a} = (\frac{\omega}{A})^{p^-_\nu - 2}$, $A = 2^s$.

**Proof:** We will consider the problem over the boxes $Q(\frac{\omega}{A}\rho^{p^+_\nu}, \rho_m)$. Let $\rho_m = \frac{\omega}{2^{s+1}}$, $k_m = \frac{\omega}{2^{s+1}} - \frac{\omega}{A}$. $\zeta_m$ is a cutoff function with $0 \leq \zeta_m \leq 1$ in $Q(\frac{\omega}{A}\rho^{p^+_\nu}, \rho_m)$, $\zeta_m \equiv 1$ in $Q(\frac{\omega}{A}\rho^{p^+_\nu}, \rho_m)$, $\zeta_m \equiv 0$ on the parabolic boundary of $Q(\frac{\omega}{A}\rho^{p^+_\nu}, \rho_m)$, $|\nabla \zeta_m| \leq \frac{\omega}{A}$, $0 \leq \frac{\partial \zeta_m}{\partial \rho} \leq \frac{\omega}{A \rho^{p^+_\nu}}$

Take $(u_m - k_m)\zeta_m^{p^+_\nu}$ as the testing function in (1.11), by $\|u\|_{L^\infty(\rho^{p^+_\nu}, \rho_m)} \leq M$ and $\|(u_k - k_m)\|_{L^\infty(\rho^{p^+_\nu}, \rho_m)} \leq \|(u_k - k_m)\|_{L^\infty(\rho^{p^+_\nu}, \rho_m)} \leq 2^{-s}$, similar to lemma 3.2, we obtain

\[
K_{e^{2\nu}(u - k_m)^2} \zeta_m dx + \int_{Q(\frac{\omega}{A}\rho^{p^+_\nu}, \rho_m)} |\nabla u| dx dt
\]

\[
\leq C^2 k_m^2 \int_{Q(\frac{\omega}{A}\rho^{p^+_\nu}, \rho_m)} \chi[(u - k_m)_+] > 0] dx dt.
\]

(3.25)

On the other hand, we have

\[
\int_{K_{e^{2\nu}(u_k - k_m)^2}} \zeta_m dx \leq \frac{\omega}{2^{s+1}} \int_{K_{e^{2\nu}(u_k - k_m)^2}} \zeta_m dx
\]

EJQTDE, 2012 No. 4, p. 21
and
\[
\int_{Q(\tilde{\Omega}_{\rho}^+, \rho_m)} |\nabla (u - k_m) + \rho \zeta_m^2| \, dx dt \leq \int_{Q(\tilde{\Omega}_{\rho}^+, \rho_m)} |\nabla (u - k_m) + \rho (x) \zeta_m^2| \, dx dt \\
+ \int_{Q(\tilde{\Omega}_{\rho}^+, \rho_m)} \chi[(u - k_m) > 0] \zeta_m^2 \, dx dt
\]
then by (3.25),
\[
\sup_{-\rho_m < t < 0} \int_{K_{\rho m}} (u - k_m)_{+}^{\rho \zeta_m^2} \, dx + \frac{1}{a} \int_{Q(\tilde{\Omega}_{\rho}^+, \rho_m)} |\nabla (u - k_m) + \rho \zeta_m^2| \, dx dt \\
\leq C \sup_{-\rho_m < t < 0} \int_{K_{\rho m}} (v - k_m)_{+}^{\rho \zeta_m^2} \, dx + \int_{Q_m} |\nabla (v - k_m) + \rho \zeta_m^2| \, dx dz \\
+ \int_{Q_m} |(v - k_m) + \nabla \epsilon_m \rho | \, dx dz \leq C 2^{m + p} \rho^{1 - p} |A_m|.
\]

By lemma 2.8 and (3.26),
\[
\frac{1}{2^p + 1} \tau^{p_r} |A_{m+1}| = |k_m - k_{m+1}|^{p_r} |A_{m+1}| \\
\leq \| (v - k_m)_{+}^{\rho \zeta_m^2} \|_{L^{p_r}(Q_{m+1})} \\
\leq \| (v - k_m)_{+}^{\rho \zeta_m^2} \|_{V^{p_r}(Q_{m+1})} \\
\leq C 2^{m + p} \rho^{1 - p} |A_m|^{1 + \frac{p_r}{p_r + N}}.
\]

Take \( A = 2^s \), then \( \frac{1}{2^s} |A_m| \leq \rho^{1 - p} \).

Next, we obtain
\[
Z_{m+1} \leq C 4^{m + p} 2^s + N.
\]

By lemma 2.7, when \( m \to \infty \), \( Z_m \to 0 \) where \( Z_0 \leq C^{-\frac{N + p}{p - 1}} 4^{-p} (\frac{N + p}{p - 1})^2 \equiv \delta. \)

Thus as \( m \to \infty \),
\[
\int_{Q_m} \chi[(v - k_m) > 0] \, dx dz \to 0,
\]
i.e. \( u(x, t) \leq \mu^+ - \frac{1}{\rho^{1 - p}} \) a.e. in \( Q(\tilde{\Omega}(\tilde{\Omega})^+, \tilde{\Omega}) \).

EJQTDE, 2012 No. 4, p. 22
Proposition 3.2 There exist \( \sigma \in (0,1) \), \( \nu_2 \in (0,1) \) and \( A_2 \gg 1 \) independent of \( \omega \) and \( \rho \), such that if for all cylinders of the type \([0, t^*] + Q(l\rho^8, \rho)\),
\[
(x, t) \in [(0, t^*) + Q(l\rho^8, \rho)] : u > \mu^+ - \frac{\omega}{2} \leq (1 - \sigma)|Q(l\rho^8, \rho)|,
\]
then either
\[
\omega \leq A_2\rho^{n_0 - 2}
\] (3.27)
or
\[
\text{ess osc } u \leq \nu_2 \omega.
\] (3.28)

Proof: Assume (3.27) is violated. By lemma 3.8, we can determine a positive integer number \( s^* \) such that
\[
\text{ess inf } Q(l(\frac{p}{\omega})^{p\xi}, \frac{p}{\omega}) u \leq \mu^+ + \frac{\omega}{2^{s^*+1}},
\] (3.29)
and further
\[
\text{ess osc } Q(l(\frac{p}{\omega})^{p\xi}, \frac{p}{\omega}) u \leq (1 - \frac{1}{2^{s^*+1}})\omega,
\]
therefore (3.28) holds with \( \nu_2 = (1 - \frac{1}{2^{s^*+1}}) \). We get the conclusion. \( \square \)

Combine proposition 1 and proposition 2, we can get

Proposition 3.3 There exist \( \nu = \max\{\nu_1, \nu_2\} \) and \( A = \{A_1, A_2\} \), such that either \( \omega \leq A \rho ^{n_0 - 2} \) or \( \text{ess osc } u \leq \nu \omega \), where \( \nu_1, \nu_2, A_1, A_2 \) are determined by proposition 1 and proposition 2.

Next we assume \( \omega_1 = \max\{\nu \omega, A \rho ^{n_0 - 2}\} \) and \( \frac{1}{\omega_1} = (\frac{p}{\omega})^{p\xi - 2} \). Since
\[
\nu(\frac{\rho}{\omega})^{p\xi - 2}(\frac{\rho}{\omega})^{p\xi} \geq 2^{3-3p_\nu \rho^{p_\xi - 2}}(\frac{A}{\omega_1})^{p\xi - 2}(\frac{A}{\omega_1})^{p\xi - 2}(\frac{A}{\omega_1})^{p\xi - 2}(\frac{A}{\omega_1})^{p\xi - 2} = a_1 \rho_1^{p\xi},
\]
where \( \rho_1 = C^{-1} \rho \) and \( C = 8(\frac{1}{\omega_1})^{p\xi - 2} \), so \( Q(a_1 \rho_1^{p\xi}, \rho_1) \subset Q(l(\frac{p}{\omega})^{p\xi}, \frac{p}{\omega}) \).
Then we can get \( \text{ess osc } u \leq \omega \) and \( (\frac{p}{\omega})^{p\xi - 2} > a_1 \rho_1^{p\xi} > \rho^{p\xi} \). So for \( Q(a_1 \rho_1^{p\xi}, \rho_1) \)
\( Q(a_1 \rho_1^{p\xi}, \rho_1) \), repeating the process above, we can get the similar result, and moreover the following proposition 3.4 can be obtained:

Proposition 3.4 There exist \( 0 < \varepsilon_0 < 1 \), \( \nu \in (0,1) \), \( C = C(N, M, p^+, p^-) > 1 \) and \( A \geq 1 \) satisfy \( \rho_0 = \rho \), \( \omega_0 = \omega \), \( \rho_n = C^{-n} \rho \) and \( \omega_n = \max\{\nu \omega_n, C \rho_n^{p_\nu}\} \), \( n = 1, 2, ... \), such that for all boxes \( Q^{(n)} = Q(a_n \rho_n^{p_\nu}, \rho_n) \), \( \frac{1}{\omega_n} = (\frac{p}{\omega})^{p\xi - 2} \), \( n = 1, 2, ... \), we have
\[
Q^{(n+1)} \subset Q^{(n)}, \quad \text{ess osc } u \leq \omega_n.
\]

In view of proposition 3.4, we get
Proposition 3.5  There exist \( \lambda \in (0,1) \), \( C = C(N,M,p^+,p^-) \) and \( 0 < \tilde{\rho} \leq \rho \) such that for all boxes \( Q(a\tilde{\rho}^+,\rho) \), \( \frac{1}{\alpha} = \left(\frac{\tilde{\rho}}{\rho}\right)^{p^- - 2} \), we have

\[
\text{ess osc } u \leq C(\omega + \rho^\alpha)(\frac{\tilde{\rho}}{\rho})^\lambda.
\]

Proof: From the iterative construction of \( \omega_n \), it follows that \( \omega_{n+1} \leq \nu \omega_n + C\rho_0^\alpha \) and by iteration

\[
\omega_n \leq \nu^n \omega + C(\sum_{i=0}^{n-1} \nu^i C^{-\varepsilon_0(n-i)})\rho^\alpha.
\]

We may assume without loss of generality that \( \varepsilon_0 \) is so small that \( \nu \leq C^{-\varepsilon_0} \), then \( \omega_n \leq \nu^n \omega \). Let \( 0 < \tilde{\rho} \leq \rho \) be fixed, then there exists a nonnegative integer \( n \) such that

\[
C^{-(n+1)} \rho \leq \tilde{\rho} \leq C^{-n} \rho,
\]

which implies the inequalities

\[
(n+1) \geq \ln\left(\frac{\tilde{\rho}}{\rho}\right)^{-\frac{1}{\alpha}},
\]

\[
\nu^n \leq \nu^{-1}\left(\frac{\tilde{\rho}}{\rho}\right)^{\lambda_1}, \quad \lambda_1 = \left\lfloor \frac{\ln \nu}{\ln C} \right\rfloor,
\]

\[
Cu\left(\frac{\rho}{C^n}\right)^\varepsilon_0 \leq C^{1+\varepsilon_0} \ln\left(\frac{\tilde{\rho}}{\rho}\right)^{-\frac{\varepsilon_0}{\lambda_1}} \tilde{\rho}^\varepsilon_0 \leq C(\varepsilon_0)\rho^\alpha \tilde{\rho}^\frac{\varepsilon_0}{\lambda_1}.
\]

Therefore

\[
\omega_n \leq C(\omega + \rho^\alpha)(\frac{\tilde{\rho}}{\rho})^\lambda, \quad \lambda = \min\{\lambda_1, \frac{\varepsilon_0}{2}\}.
\]

On the other hand, by (3.3) we get \( \omega > C\rho^\alpha \). Thus by the definition of \( \omega_n \), \( \omega_1 = \max\{\nu \omega, C\rho^\alpha\} \leq \omega \) and \( \omega_2 = \max\{\nu \omega_1, C(C^{-1}\rho)^\alpha\} \leq \omega \),..., so \( \omega_n \leq \omega \).

Since \( Q(a\tilde{\rho}^+, \tilde{\rho}) \subset Q(\tilde{\omega}^+, \tilde{\rho}) \), by proposition 3.4, we obtain \( \text{ess osc } u \leq \omega_n \), so we conclude proposition 3.5. \( \square \)

By proposition 3.5, we know \( u \) is Hölder continuity in \( Q(a\tilde{\rho}^+, \tilde{\rho}) \), so for every point in \( Q \) we can obtain such a cylinder as \( Q(a\tilde{\rho}^+, \tilde{\rho}) \), then by limited coverage theorem, \( u \) is local Hölder continuity in \( Q \), thus we get theorem 1.

Acknowledgments:
This work is supported by the Fundamental Research Funds for the Central Universities (DL11BB40).
References


(Received August 13, 2011)