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Regularity of Weak Solutions for Nonlinear Parabolic Problem with $p(x)$-Growth

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Abstract

In this paper, we study the nonlinear parabolic problem with $p(x)$-growth conditions in the space $W^{1,x}L^{p(x)}(Q)$, and give a regularity theorem of weak solutions for the following equation

$$\frac{\partial u}{\partial t} + A(u) = 0$$

where $A(u) = -\mathrm{div}a(x,t,u,\nabla u) + a_0(x,t,u,\nabla u)$, $a(x,t,u,\nabla u)$ and $a_0(x,t,u,\nabla u)$ satisfy $p(x)$-growth conditions with respect to u and ∇u.

Keywords: nonlinear parabolic problem, regularity, $W^{1,x}L^{p(x)}(Q)$ space, $p(x)$-growth condition.
Mathematics Subject Classification: 35K15, 35K20.

1 Introduction

In recent years, the research of variational problems with nonstandard growth conditions is an interesting topic. $p(x)$-growth problems can be regarded as a kind of nonstandard growth problems and they appear in nonlinear elastic, electrorheological fluids and other physics phenomena. Many results have been obtained on this kind of problems, for examples [1-9].

In this paper, we will qualitatively study the properties of weak solutions. For more information about qualitative analysis, we refer to [10-11]. Let Q be $\Omega \times (0,T)$ where $T > 0$ is given. In [8], the authors studied the following equation in the space $W^{1,p(x,t)}_{loc}(Q) \cap C(0,T; L^2_{loc}(\Omega))$,

$$u_t - \mathrm{div}(|Du|^{p(x,t)-2}Du) = 0,$$

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where $\max\{1; \frac{2N}{N+2}\} < p_1 = \inf_{(x,t)\in Q} p(x,t) \le p(x,t) \le \sup_{(x,t)\in Q} p(x,t) = p_2 < \infty$, $p(x,t)$ is dependent on the space variable x and the time variable t, and satisfies the following Logarithmic Hölder condition

$$|p(x,t) - p(y,s)| \le \frac{C_1}{-\ln(|x-y| + C_2|t-s|^{p_2})}$$

for all $(x,t),(y,s) \in Q$, $|x-y| < \frac{1}{2}, |t-s| < \frac{1}{2}$, where $C_1, C_1 > 0$ are constants. The authors proved the Hölder continuity of the local weak solution with the scale transformation method.In this paper, we will study the following more general problem

$$\frac{\partial u}{\partial t} + A(u) = 0, \quad \text{in} \quad Q, \tag{1.1}$$

$$u(x,t) = 0, \quad \text{on} \quad \partial\Omega \times (0,T), \tag{1.2}$$

$$u(x,0) = \psi(x), \quad \text{in} \quad \Omega, \tag{1.3}$$

where $\psi(x)$ is a given function in $L^2(\Omega)$ and $A : W_0^{1,x}L^{p(x)}(Q) \to W^{-1,x}L^{q(x)}(Q)$ is an elliptic operator of the form $A(u) = -\mathrm{div}a(x,t,u,\nabla u) + a_0(x,t,u,\nabla u)$ with the coefficients a and a_0 satisfying the classical Leray-Lions conditions. In [12-13] we have proved the existence and the local boundedness of the solutions of (1.1)-(1.3) and have obtained $u \in W^{1,x}L^{p(x)}(Q) \cap L^\infty(0,T;L^2(\Omega))$. In this paper we will give the regularity theorem of the weak solutions in the framework space $W^{1,x}L^{p(x)}(Q)$, which can be considered as a special case of the space $W^{1,p(x,t)}(Q)$.

The space $W^{1,x}L^{p(x)}(Q)$ provides a suitable framework to discuss some physical problems. In [14], the authors studied a functional with variable exponent, $1 \le p(x) \le 2$, which provided a model for image denoising, enhancement, and restoration. Because in [14] the direction and speed of diffusion at each location depended on the local behavior, $p(x)$ only depended on the location x in the image. Consider that the space $W^{1,x}L^{p(x)}(Q)$ was introduced and discussed in [12] and [15], we think that the space $W^{1,x}L^{p(x)}(Q)$ is a reasonable framework to discuss the $p(x)$-growth problem (1.1)-(1.3), where $p(x)$ only depends on the space variable x similar to [14].

In this paper, let $a : Q \times R \times R^N \to R^N$ and $a_0 : Q \times R \times R^N \to R$ be the operators such that for any $s \in R$ and $\xi \in R^N$, $a(x,t,s,\xi)$ and $a_0(x,t,s,\xi)$ are both continuous in (t,s,ξ) for a.e. $x \in \Omega$ and measurable in x for all $(t,s,\xi) \in (0,T) \times R \times R^n$. They also satisfy that for a.e. $(x,t) \in Q$, any $s \in R$ and $\xi \ne \xi^* \in R^N$:

$$|a(x,t,s,\xi)| \le \alpha(|s|^{p(x)-1} + |\xi|^{p(x)-1}), \tag{1.4}$$

$$|a_0(x,t,s,\xi)| \le \alpha(|s|^{p(x)-1} + |\xi|^{p(x)-1}), \tag{1.5}$$

$$[a(x,t,s,\xi) - a(x,t,s,\xi^*)](\xi - \xi^*) > 0, \tag{1.6}$$

$$a(x,t,s,\xi)\xi \ge \beta(|\xi|^{p(x)} + |s|^{p(x)}), \tag{1.7}$$

$$a_0(x,t,s,\xi)s \ge \beta(|\xi|^{p(x)} + |s|^{p(x)}), \tag{1.8}$$

where $\alpha, \beta > 0$ are constants.

Throughout this paper, unless special statement, we always suppose that $p(x)$ is Lipschitz continuous on $\overline{\Omega}$, and satisfies

$$1 < p^- = \inf_{\overline{\Omega}} p(x) \leq p(x) \leq \sup_{\overline{\Omega}} p(x) = p^+ < \infty. \tag{1.9}$$

Because $p(x)$ is Lipschitz continuous, there exists a constant $C > 0$ such that

$$\rho^{-(p_\rho^+ - p_\rho^-)} \leq C, \qquad \forall Q_\rho \subset Q, \tag{1.10}$$

where $Q_\rho = K_\rho \times (-\rho^{p_\rho^+}, 0)$, $0 < \rho < 1$, $K_\rho = \{x \in \Omega | \max_{1 \leq i \leq N} |x_i| < \rho\}$, $p_\rho^+ = \sup_{K_\rho} p(x)$, $p_\rho^- = \inf_{K_\rho} p(x)$.

Definition 1.1 *A function $u \in W^{1,x}L^{p(x)}(Q) \cap L^\infty(0,T;L^2(\Omega))$ is called a weak solution of (1.1)-(1.3) if*

$$-\int_Q u \frac{\partial \varphi}{\partial t} dxdt + \int_\Omega u\varphi dx|_0^T$$

$$+ \int_Q [a(x,t,u,\nabla u)\nabla\varphi + a_0(x,t,u,\nabla u)\varphi] dxdt = 0$$

for all $\varphi \in C^1(0,T;C_0^\infty(\Omega))$.

Definition 1.2 *The functions $u_n \in C(0,T;C_0^\infty(\Omega))$ are called the Galerkin solutions of (1.1)-(1.3) if*

$$\int_{Q^t} \frac{\partial u_n}{\partial \tau} \varphi dxd\tau + \int_{Q^t} a(x,\tau,u_n,\nabla u_n)\varphi dxd\tau + \int_{Q^t} a_0(x,\tau,u_n,\nabla u_n)\varphi dxd\tau = 0 \tag{1.11}$$

for all $\varphi \in C^1(0,T;C_0^\infty(\Omega))$ and $Q^t = \Omega \times (0,t), t \in (0,T]$.

We will prove the following regularity theorem:

Theorem 1 *Let $p^- > 2$. If $u \in W^{1,x}L^{p(x)}(Q) \cap L^\infty(0,T;L^2(\Omega))$ is a local weak solution of (1.1)-(1.3), then u is local Hölder continuous in Q.*

2 Preliminaries

We first recall some facts on spaces $L^{p(x)}(\Omega)$, $W^{m,p(x)}(\Omega)$, $W^{m,x}L^{p(x)}(Q)$ and parabolic space. For the details see [15-18].

Although we assume (1.9) holds in this paper, in this section we introduce the general spaces $L^{p(x)}(\Omega)$, $W^{m,p(x)}(\Omega)$ and $W^{m,x}L^{p(x)}(Q)$.

Denote

$$E = \{\omega : \omega \text{ is a measurable function on } \Omega\},$$

where $\Omega \subset R^N$ is an open subset.

Let $p(x) : \Omega \to [1, \infty]$ be an element in E. Denote $\Omega_\infty = \{x \in \Omega : p(x) = \infty\}$. For $u \in E$, we define

$$\rho(u) = \int_{\Omega \setminus \Omega_\infty} |u(x)|^{p(x)} dx + \text{ess} \sup_{x \in \Omega_\infty} |u(x)|.$$

The space $L^{p(x)}(\Omega)$ is

$$L^{p(x)}(\Omega) = \{u \in E : \exists \lambda > 0, \rho(\lambda u) < \infty\}$$

endowed with the norm

$$\|u\|_{L^{p(x)}(\Omega)} = \inf\{\lambda > 0 : \rho(\frac{u}{\lambda}) \leq 1\}.$$

We define the conjugate function $q(x)$ of $p(x)$ by

$$q(x) = \begin{cases} \infty, & \text{if } p(x) = 1; \\ 1, & \text{if } p(x) = \infty; \\ \frac{p(x)}{p(x)-1}, & \text{if } 1 < p(x) < \infty. \end{cases}$$

Lemma 2.1 (see [18]) *(1) The dual space of $L^{p(x)}(\Omega)$ is $L^{q(x)}(\Omega)$, if $1 \leq p(x) < \infty$.*
(2)The space $L^{p(x)}(\Omega)$ is reflexive if and only if (1.9) is satisfied.
Lemma 2.2 (see [18]) *If $1 \leq p(x) < \infty$, $C_0^\infty(\Omega)$ is dense in the space $L^{p(x)}(\Omega)$ and $L^{p(x)}(\Omega)$ is separable.*
Lemma 2.3 (see [18]) *Let $1 \leq p(x) \leq \infty$, for every $u(x) \in L^{p(x)}(\Omega)$ and $v(x) \in L^{q(x)}(\Omega)$, we have*

$$\int_\Omega |u(x)v(x)| dx \leq C\|u(x)\|_{L^{p(x)}(\Omega)} \|v(x)\|_{L^{q(x)}(\Omega)},$$

where C is only dependent on $p(x)$ and Ω, not dependent on $u(x), v(x)$.

Next let $m > 0$ be an integer. For each $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, α_i are nonnegative integers and $|\alpha| = \Sigma_{i=1}^n \alpha_i$, and denote by D^α the distributional derivative of order α with respect to the variable x.

We now introduce the generalized Lebesgue-Sobolev space $W^{m,p(x)}(\Omega)$ which is defined as

$$W^{m,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : D^\alpha u \in L^{p(x)}(\Omega), |\alpha| \leq m\}.$$

$W^{m,p(x)}(\Omega)$ is a Banach space endowed with the norm

$$\|u\| = \sum_{|\alpha| \leq m} \|D^\alpha u\|_{L^{p(x)}(\Omega)}.$$

The space $W_0^{m,p(x)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $W^{m,p(x)}(\Omega)$. The dual space $(W_0^{m,p(x)}(\Omega))^*$ is denoted by $W^{-m,q(x)}(\Omega)$ equipped with the norm

$$\|f\|_{W^{-m,q(x)}(\Omega)} = \inf \Sigma_{|\alpha| \leq m} \|f_\alpha\|_{L^{q(x)}(\Omega)},$$

where infimum is taken on all possible decompositions

$$f = \Sigma_{|\alpha| \le m} (-1)^{|\alpha|} D^{\alpha} f_{\alpha}, \qquad f_{\alpha} \in L^{q(x)}(\Omega).$$

Lemma 2.4(see [18]) *(1) $W^{m,p(x)}(\Omega)$ and $W_0^{m,p(x)}(\Omega)$ are separable if $1 \le p(x) < \infty$.*
(2) $W^{m,p(x)}(\Omega)$ and $W_0^{m,p(x)}(\Omega)$ are reflexive if (1.9) holds.

We define the space $W^{m,x}L^{p(x)}(Q)$ as the following:

$$W^{m,x}L^{p(x)}(Q) = \{u \in L^{p(x)}(Q) : D^{\alpha} u \in L^{p(x)}(Q), |\alpha| \le m\}.$$

$W^{m,x}L^{p(x)}(Q)$ is a Banach space with the norm $\|u\| = \sum\limits_{|\alpha| \le m} \|D^{\alpha} u\|_{L^{p(x)}(Q)}$, where $p(x)$ is independent of t.

The space $W_0^{m,x}L^{p(x)}(Q)$ is defined as the closure of $C_0^{\infty}(Q)$ in $W^{m,x}L^{p(x)}(Q)$ and $W_0^{m,x}L^{p(x)}(Q) \hookrightarrow L^{p(x)}(Q)$ is continuous embedding. Let $\bar{M}$ be the number of multiindexes α which satisfies $0 \le |\alpha| \le m$, then the space $W_0^{m,x}L^{p(x)}(Q)$ can be considered as a close subspace of the product space $\Pi_{i=1}^{\bar{M}} L^{p(x)}(Q)$. So if $1 < p(x) < \infty$, $\Pi_{i=1}^{\bar{M}} L^{p(x)}(Q)$ is reflexive and further we can get that the space $W_0^{m,x}L^{p(x)}(Q)$ is reflexive. The dual space $(W_0^{m,x}L^{p(x)}(Q))^*$ is denoted by $W^{-m,x}L^{q(x)}(Q)$ equipped with the norm

$$\|f\|_{W^{-m,x}L^{q(x)}(Q)} = \sup_{\|u\|_{W_0^{m,x}L^{p(x)}(Q)} \le 1} |< f, u >| = \inf \Sigma_{|\alpha| \le m} \|f_{\alpha}\|_{L^{q(x)}(Q)},$$

where infimum is taken on all possible decompositions

$$f = \Sigma_{|\alpha| \le m} (-1)^{|\alpha|} D_x^{\alpha} f_{\alpha}, \qquad f_{\alpha} \in L^{q(x)}(Q).$$

Next, we will introduce the parabolic space and some results in [16]:

Definition 2.5 *Let $p, r \ge 1$. A function f defined in Q belongs to the space $L^r(0,T; L^p(\Omega))$, if*

$$\|f\|_{p,r,Q} = (\int_0^T (\int_{\Omega} |f|^p dx)^{\frac{r}{p}} dt)^{\frac{1}{r}} < \infty.$$

Definition 2.6 *Let $p, r \ge 1$. We define the function spaces*

$$V^{r,p}(Q) = L^{\infty}(0,T; L^r(\Omega)) \cap L^p(0,T; W^{1,p}(\Omega)),$$

$$V_0^{r,p}(Q) = L^{\infty}(0,T; L^r(\Omega)) \cap L^p(0,T; W_0^{1,p}(\Omega)),$$

which are both equipped with the norm

$$\|v\|_{V^{r,p}(Q)} = \operatorname*{ess\,sup}_{0<t<T} \|v(x,t)\|_{L^r(\Omega)} + \|\nabla v\|_{L^p(Q)}.$$

Lemma 2.7 *let $\{Y_n\}, n = 0, 1, 2, \cdots$, be a sequence of positive numbers, satisfying the inequalities $Y_{n+1} \leq Cb^n Y_n^{1+\alpha}$, where $C, b > 1$ and $\alpha > 0$ are given numbers. If $Y_0 \leq C^{-\frac{1}{\alpha}} b^{-\frac{1}{\alpha^2}}$, then $\{Y_n\}$ converges to 0 as $n \to \infty$.*

Lemma 2.8 *Let $r > 1$, there exists a constant C depending only on N, r, such that for every $v \in L^\infty(0, T; L^r(\Omega)) \cap L^r(0, T; W_0^{1,r}(\Omega))$,*

$$\|v\|_{L^r(Q)}^r \leq C||v| > 0|^{\frac{r}{r+N}} \|v\|_{V^{r,r}(Q)}^r$$

where $||v| > 0| = \text{meas}\{(x, t) : |v| > 0\}$.

Lemma 2.9 *Let $v \in W^{1,1}(K_\rho(x_0)) \cap C(K_\rho(x_0))$ for some $\rho > 0$ and some $x_0 \in R^N$, and let k and h be any pair of real numbers such that $k < h$, then there exists a constant C depending only upon N, such that*

$$(h - k)|A(h)| \leq C \frac{\rho^{N+1}}{|K_\rho(x_0) \backslash A(k)|} \int_{A(k) \backslash A(h)} |\nabla v| dx$$

where $A(k) = \{x \in K_\rho(x_0) : v(x) > k\}, |A(k)| = \text{meas}A(k)$.

Let $u \in L^1(Q)$. For any $0 < h < T$, we introduce the Steklov average function

$$u_h(x, t) = \begin{cases} \frac{1}{h} \int_t^{t+h} u(x, \tau) d\tau, & t \in (0, T - h], \\ 0, & t > T - h. \end{cases}$$

Lemma 2.10 *Let $u \in L^r(0, T; L^p(\Omega))$, then as $h \to 0$, $u_h \to u$ in $L^r(0, T - \varepsilon; L^p(\Omega))$ for every $\varepsilon \in (0, T)$. If $u \in C(0, T; L^2(\Omega))$, then as $h \to 0$, $u_h \to u$ in $L^2(\Omega)$ for every $t \in (0, T - \varepsilon)$.*

Similarly, we can get the following lemma in variable exponent space.

Lemma 2.11 *If $u \in L^{p(x)}(Q)$, then as $h \to 0$, $u_h \to u$ in $L^{p(x)}(Q)$.*

Proof: Because $p(x)$ is bounded and independent of t. We only need to notice that there exist $u_k \in C_0^1(Q)$ such that $u_k \to u$ in $L^{p(x)}(Q)$, and by the uniform continuity of u_k, we can conclude the lemma.□

3 Regularity of Weak Solutions

In [12-13] , we have obtained that for the Galerkin solution $u_n \in C^1(0, T; C_0^\infty(\Omega))$, $u_n \to u$ strongly in $L^2(Q)$ and $L^{p(x)}(Q)$, $u_n \rightharpoonup u$ weakly in $W_0^{1,x} L^{p(x)}(Q)$, $a(x, t, u_n, \nabla u_n) \rightharpoonup a(x, t, u, \nabla u)$ and $a_0(x, t, u_n, \nabla u_n) \rightharpoonup a_0(x, t, u, \nabla u)$ weakly in $L^{q(x)}(Q)$, $u_n \to u$ a.e. in Q and $\nabla u_n \to \nabla u$ a.e. in Q.

For (1.11), integrating by parts, we can get

$$\int_{Q^t} \frac{\partial u_n}{\partial \tau} \varphi dx d\tau = \int_\Omega u_n(x, t) \varphi(x, t) dx - \int_{Q^t} u_n \frac{\partial \varphi}{\partial \tau} dx d\tau,$$

therefore

$$\lim_{n \to \infty} \int_{Q^t} \frac{\partial u_n}{\partial \tau} \varphi dx d\tau = \int_\Omega u(x, t) \varphi(x, t) dx - \int_{Q^t} u \frac{\partial \varphi}{\partial \tau} dx d\tau.$$

As $a(x,t,u_n,\nabla u_n) \rightharpoonup a(x,t,u,\nabla u)$ weakly in $L^{q(x)}(Q)$ and $a_0(x,t,u_n,\nabla u_n) \rightharpoonup a_0(x,t,u,\nabla u)$ weakly in $L^{q(x)}(Q)$, we have

$$\begin{aligned} & \lim_{n\to\infty}(\textstyle\int_{Q^t} a(x,\tau,u_n,\nabla u_n)\varphi + \int_{Q^t} a_0(x,\tau,u_n,\nabla u_n)\varphi dxd\tau) \\ = & \textstyle\int_{Q^t} a(x,\tau,u,\nabla u)\varphi + a_0(x,\tau,u,\nabla u)\varphi dxd\tau, \end{aligned}$$

then (1.11) can be written as

$$\int_\Omega u(x,t)\varphi(x,t)dx - \int_{Q^t} u\frac{\partial\varphi}{\partial\tau}dxd\tau + \int_{Q^t} a(x,\tau,u,\nabla u)\varphi dxd\tau$$
$$+\int_{Q^t} a_0(x,\tau,u,\nabla u)\varphi dxd\tau = 0. \quad (3.1)$$

In (3.1), let φ be independent of t and $t = t+h$, then we get

$$\int_\Omega \frac{\partial u_h(x,\tau)}{\partial\tau}\varphi dx + \int_{Q_t}[a(x,\tau,u,\nabla u)]_h\varphi dxd\tau + \int_{Q_t}[a_0(x,\tau,u,\nabla u)]_h\varphi dxd\tau = 0, \quad (3.2)$$

where $\varphi \in C_0^\infty(\Omega)$.

Lemma 3.1 *If u is a weak solution of (1.1)-(1.3), then $u \in C(0,T;L^2(\Omega))$.*

Proof: Because $u_n \rightharpoonup u$ weakly in $W_0^{1,x}L^{p(x)}(Q)$, there exists convex combination of u_n, denoted by v_n, such that $v_n \to u$ strongly in $W_0^{1,x}L^{p(x)}(Q)$ and $v_n(x,0) \to \psi(x)$ strongly in $L^2(\Omega)$. Take $\varphi = u_n - v_m$ as the testing function in (1.11),

$$\int_{Q^t}\frac{\partial u_n}{\partial\tau}(u_n - v_m)dxd\tau + \int_{Q^t} a(x,\tau,u_n,\nabla u_n)\nabla(u_n - v_m)dxd\tau$$
$$+\int_{Q^t} a_0(x,\tau,u_n,\nabla u_n)(u_n - v_m)dxd\tau = 0,$$

then for the sufficient large m, we have

$$\begin{aligned} & \overline{\lim_{n\to\infty}} \textstyle\int_{Q^t}\frac{\partial u_n}{\partial\tau}(u_n - v_m)dxd\tau \\ \le & \textstyle\int_{Q^t} a(x,\tau,u,\nabla u)\nabla(u - v_m)dxd\tau + \int_{Q^t} a_0(x,\tau,u,\nabla u)\nabla(u - v_m)dxd\tau \\ \le & 2(\|a\|_{L^{q(x)}(Q)} + \|a_0\|_{L^{q(x)}(Q)})\|\nabla(u - v_m)\|_{L^{p(x)}(Q)} \le \varepsilon(m) \end{aligned}$$

and

$$\lim_{n\to\infty}\int_{Q^t}\frac{\partial v_m}{\partial\tau}(v_m - u_n)dxd\tau \le \varepsilon(m)$$

where $\varepsilon(m) \to 0$ as $m \to 0$.

In short,

$$\overline{\lim_{n\to\infty}}\int_{Q^t}\frac{\partial(u_n - v_m)}{\partial\tau}(u_n - v_m)dxd\tau \le \varepsilon(m),$$

i.e.

$$\overline{\lim_{n\to\infty}} \int_\Omega |u_n - v_m|^2 dx \le \varepsilon(m).$$

therefore, for $k > m$, we get

$$\max_{0<t<T} \|v_k - v_m\| \le \max_{0<t<T} [\overline{\lim_{n\to\infty}} (\|v_k - u_n\| + \|u_n - v_m\|)] \le \varepsilon(k) + \varepsilon(m),$$

namely $\{v_n\}$ is a Cauchy sequence in $C(0, T; L^2(\Omega))$, so we get the result.□

Next, we will prove the main theorem.

By [13], we know that there exists a constant $M > 0$, such that $\|u\|_{L^\infty_{loc}(Q)} \le M$. Fix a point (x_0, t_0) in Q, let $\rho \in (0, 1)$ be small enough such that

$$Q(\rho^{p^+_\rho - \varepsilon}, 2\rho) = K_{2\rho}(x_0) \times (t_0 - \rho^{p^+_\rho - \varepsilon}, t_0) \subset Q,$$

where $K_{2\rho}(x_0) = \{x \in \Omega | \max_{1\le i\le N} |x_i - x_{0,i}| < 2\rho\}$, $p^+_\rho = \sup_{K_{2\rho}(x_0)} p(x)$, $p^-_\rho = \inf_{K_{2\rho}(x_0)} p(x)$.

Denote $\mu^+ = \operatorname*{ess\,sup}_{Q(\rho^{p^+_\rho - \varepsilon}, 2\rho)} u$, $\mu^- = \operatorname*{ess\,inf}_{Q(\rho^{p^+_\rho - \varepsilon}, 2\rho)} u$, $\omega = \operatorname*{ess\,osc}_{Q(\rho^{p^+_\rho - \varepsilon}, 2\rho)} u = \mu^+ - \mu^-$. Consider the cylinder $Q(a\rho^{p^+_\rho}, \rho)$, $\frac{1}{a} = (\frac{\omega}{A})^{p^-_\rho - 2}$, where $A > 2$ is a constant to be determined later. We assume that

$$(\frac{\omega}{A})^{p^-_\rho - 2} > \rho^\varepsilon, \tag{3.3}$$

where $\varepsilon \in (0, 1)$ will be determined later. This implies the inclusion

$$Q(a\rho^{p^+_\rho}, \rho) \subset Q(\rho^{p^+_\rho - \varepsilon}, 2\rho)$$

and

$$\operatorname*{ess\,osc}_{Q(a\rho^{p^+_\rho}, \rho)} u \le \omega.$$

If (3.3) is not hold, $\omega \le A\rho^{\overline{\frac{\varepsilon}{p^-_\rho - 2}}}$. Take $C = A$, then the first iterative of proposition 3.4 is hold, so the proposition 3.4 is right. therefore we also assume that (3.3) is hold in the following proof.

Let $[(0, t^*) + Q(l\rho^{p^+_\rho}, \rho)] = \{x \in \Omega | \max_{1\le i\le N} |x_i| < \rho\} \times [t^* - l\rho^{p^+_\rho}, t^*]$, $\frac{1}{l} = (\frac{\omega}{2})^{p^-_\rho - 2}$. For $[(0, t^*) + Q(l\rho^{p^+_\rho}, \rho)] \subset Q(a\rho^{p^+_\rho}, \rho)$, $-(A^{p^-_\rho - 2} - 2^{p^-_\rho - 2})\rho^{p^+_\rho}\omega^{2 - p^-_\rho} < t^* < 0$. We assume $(x_0, t_0) = (0, 0)$ and define $(u - k)_\pm = \max\{\pm(u - k), 0\}$.

Lemma 3.2 *There exists a number $\sigma \in (0, 1)$ independent of ω, ρ such that if (3.3) and*

$$|(x, t) \in [(0, t^*) + Q(l\rho^{p^+_\rho}, \rho)] : u < \mu^- + \frac{\omega}{2}| \le \sigma |Q(l\rho^{p^+_\rho}, \rho)| \tag{3.4}$$

hold, then $u > \mu^- + \frac{\omega}{4}$, a.e.$(x, t) \in [(0, t^*) + Q(l(\frac{\rho}{2})^{p^+_\rho}, \frac{\rho}{2})]$.

Proof: Up to a translation we may assume that $(0,t^*) = (0,0)$. Let $\rho_m = \frac{\rho}{2} + \frac{\rho}{2^{m+1}}, k_m = \mu^- + \frac{\omega}{4} + \frac{\omega}{2^{m+2}}$, $Q_{\rho_m} = K_{\rho_m} \times (-l\rho_m^{p_\rho^+}, 0), m = 0,1,2....$ We choose smooth cutoff function $\eta_m = \xi_1(x)\xi_2(t)$, where $0 \le \xi_1 \le 1, 0 \le \xi_2 \le 1$ and

$$\xi_1 = 1, \text{ if } x \in K_{\rho_{m+1}}; \quad \xi_1 = 0, \text{ if } x \overline{\in} K_{\rho_{m+1}}; \quad \text{and } |\nabla \xi_1| \le \frac{1}{\rho_m - \rho_{m+1}}.$$

$$\xi_2 = 1, \text{ if } t \ge -l\rho_{m+1}^{p_\rho^+}; \quad \xi_2 = 0, \text{ if } t \le -l\rho_m^{p_\rho^+}; \text{ and } 0 \le \frac{\partial \xi_2}{\partial t} \le \frac{1}{l(\rho_m^{p_\rho^+} - \rho_{m+1}^{p_\rho^+})}.$$

Take $\varphi = -(u_n - k_m)_- \eta_m^{p_\rho^+}$ as the testing function in (1.11), then

$$\int_{Q_m^t} \frac{\partial u_n}{\partial \tau}[-(u_n - k_m)_- \eta_m^{p_\rho^+}] dx d\tau + \int_{Q_m^t} a(x,\tau,u_n,\nabla u_n)[-\nabla((u_n - k_m)_- \eta_m^{p_\rho^+})] dx d\tau$$

$$+ \int_{Q_m^t} a_0(x,\tau,u_n,\nabla u_n)[-(u_n - k_m)_- \eta_m^{p_\rho^+}] dx d\tau = 0,$$

where $Q_m^t = K_{\rho_m} \times (-l\rho_m^{p_\rho^+}, t)$, $t \in (-l\rho_m^{p_\rho^+}, 0)$.

First, integrating by parts,

$$\begin{aligned}
&\int_{Q_m^t} \frac{\partial u_n}{\partial \tau}[-(u_n - k_m)_- \eta_m^{p_\rho^+}] dx d\tau \\
=&\frac{1}{2}\int_{Q_m^t} \frac{\partial[(u_n - k_m)_- \eta_m^{p_\rho^+}]}{\partial \tau} dx - \frac{p_\rho^+}{2}\int_{Q_m^t} (u_n - k_m)_-^2 \eta_m^{p_\rho^+ - 1}\frac{\partial \eta_m}{\partial \tau} dx d\tau \\
=&\frac{1}{2}\int_{K_{\rho_m}} (u_n - k_m)_- \eta_m^{p_\rho^+}(x,t) dx - \frac{1}{2}\int_{K_{\rho_m}} (u_n - k_m)_- \eta_m^{p_\rho^+}(x, -l\rho_m^{p_\rho^+}) dx \\
&- \frac{p_\rho^+}{2}\int_{Q_m^t} [(u_n - k_m)_-^2 \eta_m^{p_\rho^+ - 1}\frac{\partial \eta_m}{\partial \tau}] dx d\tau.
\end{aligned}$$

Since $u_n \to u$ in $L^2(Q)$ and $u \in C(0,T;L^2(\Omega))$, $u_n \to u$ in $L^2(\Omega)$ for $\forall t \in (0,T)$, therefore we can get

$$\begin{aligned}
&\lim_{n\to\infty}\int_{Q_m^t} \frac{\partial u_n}{\partial \tau}[-(u_n - k_m)_- \eta_m^{p_\rho^+}] dx d\tau \\
=&\frac{1}{2}\int_{K_{\rho_m}} (u - k_m)_- \eta_m^{p_\rho^+}(x,t) dx - \frac{1}{2}\int_{K_{\rho_m}} (u - k_m)_- \eta_m^{p_\rho^+}(x, -l\rho_m^{p_\rho^+}) dx \\
&- \frac{p_\rho^+}{2}\int_{Q_m^t} [(u - k_m)_-^2 \eta_m^{p_\rho^+ - 1}\frac{\partial \eta_m}{\partial \tau}] dx d\tau.
\end{aligned}$$

Since $\nabla(u_n - k_m)_- \to \nabla(u - k_m)_-$ and $a(x,t,u_n,\nabla u_n) \to a(x,t,u,\nabla u)$ a.e. in Q_m^t, by Fatou lemma,

$$
\begin{aligned}
&\varliminf_{n\to\infty}\int_{Q_m^t} a(x,\tau,u_n,\nabla u_n)[-\nabla(u_n-k_m)_-\eta_m^{p_\rho^+}]dxd\tau\\
\geq&\int_{Q_m^t} a(x,\tau,u,\nabla u)[-\nabla(u-k_m)_-\eta_m^{p_\rho^+}]dxd\tau.
\end{aligned}
$$

By the fact that $u_n \to u$ strongly in $L^{p(x)}(Q)$, $a(x,t,u_n,\nabla u_n) \rightharpoonup a(x,t,u,\nabla u)$ weakly and $a_0(x,t,u_n,\nabla u_n) \rightharpoonup a_0(x,t,u,\nabla u)$ weakly in $L^{q(x)}(Q)$, we have

$$
\begin{aligned}
&\lim_{n\to\infty}\int_{Q_m^t} a(x,\tau,u_n,\nabla u_n)[-(u_n-k_m)_-\eta_m^{p_\rho^+-1}\nabla\eta_m]dxd\tau\\
=&\int_{Q_m^t} a(x,\tau,u,\nabla u)[-(u-k_m)_-\eta_m^{p_\rho^+-1}\nabla\eta_m]dxd\tau,
\end{aligned}
$$

and

$$
\begin{aligned}
&\lim_{n\to\infty}\int_{Q_m^t} a_0(x,\tau,u_n,\nabla u_n)[-(u_n-k_m)_-\eta_m^{p_\rho^+}]dxd\tau\\
=&\int_{Q_m^t} a_0(x,\tau,u,\nabla u)[-(u-k_m)_-\eta_m^{p_\rho^+}]dxd\tau.
\end{aligned}
$$

Let $I=\varliminf_{n\to\infty}(\int_{Q_m^t} a(x,\tau,u_n,\nabla u_n)\nabla\varphi dxd\tau+\int_{Q_m^t} a_0(x,\tau,u_n,\nabla u_n)\varphi dxd\tau)$, so

$$
\begin{aligned}
I\geq&-\int_{Q_m^t} a(x,\tau,u,\nabla u)[\nabla(u-k_m)_-\eta_m^{p_\rho^+}]dxd\tau\\
&-p_\rho^+\int_{Q_m^t} a(x,\tau,u,\nabla u)[(u-k_m)_-\eta_m^{p_\rho^+-1}\nabla\eta_m]dxd\tau\\
&-\int_{Q_m^t} a_0(x,\tau,u,\nabla u)[(u-k_m)_-\eta_m^{p_\rho^+}]dxd\tau.
\end{aligned}
$$

By (1.4)-(1.5),(1.7)-(1.8), $\|u\|_{L^\infty_{loc}(Q_m)}\leq M$ and $\|(u-k_m)_-\|_{L^\infty_{loc}(Q_m)}\leq \|(k_m-u)\|_{L^\infty_{loc}(Q_m)}\leq\frac{\omega}{2}$, we have

$$
\begin{aligned}
I\geq&\beta\int_{Q_m^t}(|\nabla(u-k_m)|^{p(x)}+|u|^{p(x)})\eta_m^{p_\rho^+}dxd\tau\\
&-\alpha p^+\int_{Q_m^t}(|\nabla(u-k_m)|^{p(x)-1}+|\nabla(u-k_m)|^{p(x)-1})(u-k_m)_-\eta_m^{p_\rho^+-1}|\nabla\eta_m|dxd\tau\\
&-\alpha\int_{Q_m^t}(|\nabla(u-k_m)|^{p(x)-1}+|u|^{p(x)-1})(u-k_m)_-\eta_m^{p_\rho^+}dxd\tau\\
\geq&\frac{\beta}{2}\int_{Q_m^t}|\nabla(u-k_m)|^{p(x)}\eta_m^{p_\rho^+}dxd\tau-C2^{mp_\rho^+}\rho^{-p_\rho^+}|A_m|,
\end{aligned}
$$

where $A_m=\{(x,t)\in Q_{\rho_m}: u(x,t)<k_m\}$, $C=C(M,p^+)$.

So we can get the following inequality

$$\begin{aligned}&\sup_{-l\rho_m^{p_\rho^+}<t<0}\int_{K_{\rho_m}}(u-k_m)_-^2\eta_m^{p_\rho^+}\,dx+\int_{Q_{\rho_m}}|\nabla(u-k_m)_-|^{p(x)}\eta_m^{p_\rho^+}\,dxdt\\ \le\;& C2^{mp_\rho^+}\rho^{-p_\rho^+}|A_m|,\end{aligned}\tag{3.5}$$

where $C=C(M,p^+)$.

On the other hand, we have

$$\int_{K_{\rho_m}}(u-k_m)_-^{p_\rho^-}\eta_m^{p_\rho^+}\,dx\le\left(\frac{\omega}{2}\right)^{p_\rho^--2}\int_{K_{\rho_m}}(u-k_m)_-^2\eta_m^{p_\rho^+}\,dx$$

and

$$\begin{aligned}\int_{Q_{\rho_m}}|\nabla(u-k_m)_-|^{p_\rho^-}\eta_m^{p_\rho^+}\,dxdt\le\;&\int_{Q_{\rho_m}}|\nabla(u-k_m)_-|^{p(x)}\eta_m^{p_\rho^+}\,dxdt\\ &+\int_{Q_{\rho_m}}\chi[(u-k_m)_->0]\eta_m^{p_\rho^+}\,dxdt,\end{aligned}$$

then by (3.5),

$$\begin{aligned}&\sup_{-l\rho_m^{p_\rho^+}<t<0}\int_{K_{\rho_m}}(u-k_m)_-^{p_\rho^-}\eta_m^{p_\rho^+}\,dx+\frac{1}{l}\int_{Q_{\rho_m}}|\nabla(u-k_m)_-|^{p_\rho^-}\eta_m^{p_\rho^+}\,dxdt\\ \le\;& C2^{mp_\rho^+}\rho^{-p_\rho^+}\frac{1}{l}|A_m|.\end{aligned}$$

Next, we introduce the change of time-variable $z=l^{-1}t$ which transforms Q_{ρ_m} into $\widetilde{Q}_{\rho_m}=K_{\rho_m}\times(-\rho_m^{p_\rho^+},0)$. Setting also $v(x,t)=u(x,zl)$, $\tilde\eta_m(x,z)=\eta_m(x,zl)$, $|\widetilde{A}_m|=\text{meas}\{(x,z)\in\widetilde{Q}_{\rho_m}:v(x,z)<k_m\}$, then

$$\begin{aligned}&\|(v-k_m)_-\tilde\eta_m^{p_\rho^+}\|^{p_\rho^-}_{V^{p_\rho^-,p_\rho^-}(\tilde Q_{\rho_m})}\\ \le\;& C\Big(\sup_{-\rho_m^{p_\rho^+}<z<0}\int_{K_{\rho_m}}(v-k_m)_-^{p_\rho^-}\tilde\eta_m^{p_\rho^+}\,dx+\int_{\tilde Q_{\rho_m}}|\nabla(v-k_m)_-|^{p_\rho^-}\tilde\eta_m^{p_\rho^+}\,dxdz\\ &+\int_{\tilde Q_{\rho_m}}|(v-k_m)_-\nabla\tilde\eta_m^{p_\rho^+}|^{p_\rho^-}\,dxdz\Big)\le C2^{mp_\rho^+}\rho^{-p_\rho^+}|\tilde A_m|.\end{aligned}$$

By lemma 2.8,

$$\begin{aligned}\frac{1}{2^{p_\rho^-(m+2)}}\left(\frac{\omega}{2}\right)^{p_\rho^-}|\tilde A_{m+1}|&=|k_m-k_{m+1}|^{p_\rho^-}|\tilde A_{m+1}|\\ &\le\|(v-k_m)_-\|^{p_\rho^-}_{L^{p_\rho^-}(\tilde Q_{\rho_{m+1}})}\le\|(v-k_m)_-\tilde\eta_m^{p_\rho^+}\|^{p_\rho^-}_{L^{p_\rho^-}(\tilde Q_{\rho_m})}\\ &\le\|(v-k_m)_-\tilde\eta_m^{p_\rho^+}\|^{p_\rho^-}_{V^{p_\rho^-,p_\rho^-}(\tilde Q_{\rho_m})}|\tilde A_m|^{\frac{p_\rho^-}{p_\rho^-+N}}\\ &\le C2^{mp^+}\rho^{-p_\rho^+}|\widetilde{A}_m|^{1+\frac{p_\rho^-}{p_\rho^-+N}}.\end{aligned}$$

By (3.3), when $A>2$, we choose $\varepsilon\le p_\rho^--2$, then $(\frac{\omega}{2})^{-p_\rho^-}\le\rho^{-p_\rho^-}$. Next, denote $Y_m=\frac{|\tilde A_m|}{|\tilde Q_{\rho_m}|}$, then by (1.10) we obtain

$$
\begin{aligned}
Y_{m+1} &\leq \frac{C4^{mp_\rho^+}\rho^{-p_\rho^+}|A_m|^{1+\frac{p_\rho^-}{p_\rho^-+N}}}{|\tilde{Q}_{\rho_{m+1}}|} \\
&= \frac{C4^{mp_\rho^+}\rho^{-p_\rho^+}|\tilde{Q}_{\rho_m}|^{1+\frac{p_\rho^-}{p_\rho^-+N}}}{|\tilde{Q}_{\rho_{m+1}}|} Y_m^{1+\frac{p_\rho^-}{p_\rho^-+N}} \\
&\leq C4^{mp^+} Y_m^{1+\frac{p_\rho^-}{p_\rho^-+N}}.
\end{aligned}
$$

By lemma 2.7, when $m \to \infty$, $Y_m \to 0$ if $Y_0 \leq C^{-\frac{N+p^-}{p^-}} 4^{-p^+(\frac{N+p^-}{p^-})^2} \equiv \sigma$ which just satisfies the condition of this lemma, i.e.

$$
Y_0 = \frac{|\{(x,t) \in Q(l\rho^{p_\rho^+}, \rho) : u < \mu^- + \frac{\omega}{2}\}|}{|Q(l\rho^{p_\rho^+}, \rho)|} \leq \sigma.
$$

By the fact that $\rho_m \searrow \frac{\rho}{2}$, $k_m \searrow \mu^- + \frac{\omega}{4}$ and $|A_m| \to 0$, we can get

$$
|\{(x,t) \in Q(l(\frac{\rho}{2})^{p_\rho^+}, \frac{\rho}{2}) : u(x,t) \leq \mu^- + \frac{\omega}{4}\}| = 0,
$$

therefore $u > \mu^- + \frac{\omega}{4}$, a.e. $(x,t) \in Q(l(\frac{\rho}{2})^{p_\rho^+}, \frac{\rho}{2})$.

Let $\theta = l(\frac{\rho}{2})^{p_\rho^+}$, by lemma 3.2 and $u \in C(0,T;L^2(\Omega))$, we obtain $u(x,-\theta) > \mu^- + \frac{\omega}{4}$ a.e. $x \in K_{\frac{\rho}{2}}$.□

Lemma 3.3 *Let (3.3)-(3.4) hold, then for every number $\sigma_1 \in (0,1)$, there exists a positive integer s such that*

$$
|x \in K_{\frac{\rho}{4}} : u(x,t) < \mu^- + \frac{\omega}{2^s}| \leq \sigma_1 |K_{\frac{\rho}{4}}|, \qquad \forall t \in (-\theta, 0).
$$

Proof: Set $\rho^* = 2^{-1}\rho$, we will consider the problem in $Q(\theta,\rho^*) = K_{\rho^*} \times (-\theta, 0)$. Let $k = \mu^- + \frac{\omega}{4}$, $H_k^- = \operatorname{ess\,sup}_{Q(\theta,\rho^*)} (u-k)_-$, thus $H_k^- \leq \frac{\omega}{4}$. Then we take

$$
\Psi(u) = \max\{0, \ln \frac{H_k^-}{H_k^- - (u-k)_- + \omega 2^{-(m+2)}}\} = \ln^+ \frac{H_k^-}{H_k^- - (u-k)_- + \omega 2^{-(m+2)}}.
$$

By lemma 3.2, we know $u(x,-\theta) > \mu^- + \frac{\omega}{4}$ a.e. $x \in K_{\rho^*}$, so $(u-k)_- = 0$ a.e. in $K_{\rho^*} \times \{-\theta\}$, moreover $\Psi(u(x,-\theta)) = 0$, a.e. $x \in K_{\rho^*}$. Since $\frac{\omega}{4} \geq H_k^- \geq (u-k)_-$, we get $\Psi(u) \leq \ln \frac{\frac{\omega}{4}}{\frac{\omega}{2^{m+2}}} = m \ln 2$ and

$$
|\frac{\partial \Psi(u)}{\partial u}| = \begin{cases} \frac{1}{H_k^- - (u-k)_- + \omega 2^{-(m+2)}}, & u < k - \omega 2^{-(m+2)}, \\ 0, & u \geq k - \omega 2^{-(m+2)}, \end{cases}
$$

therefore when $u < k - \omega 2^{-(m+2)}$, $\frac{2}{\omega} \leq |\frac{\partial \Psi(u)}{\partial u}| \leq \frac{2^{(m+2)}}{\omega}$.

Take $\varphi=\frac{\partial}{\partial d}[\Psi^2(d)]\eta^{p_\rho^+}|_{d=u_h}$ as the testing function in (3.2), where η is the cutoff function independent of t and satisfies $0<\eta<1$ in K_{ρ^*}, $\eta=1$ in $K_{2^{-1}\rho^*}$, and $|\nabla\eta|\le 4\rho^{-1}$, then

$$\begin{aligned}&\int_{Q^t(\theta,\rho^*)}\frac{\partial}{\partial d}[\Psi^2(d)]\eta^{p_\rho^+}|_{d=u_h}\frac{\partial u_h}{\partial\tau}dxd\tau\\&+\int_{Q^t(\theta,\rho^*)}[a(x,\tau,u,\nabla u)]_h\nabla[\frac{\partial}{\partial d}[\Psi^2(d)]\eta^{p_\rho^+}|_{d=u_h}]dxd\tau\\&+\int_{Q^t(\theta,\rho^*)}[a_0(x,\tau,u,\nabla u)]_h\frac{\partial}{\partial d}[\Psi^2(d)]\eta^{p_\rho^+}|_{d=u_h}dxd\tau=0,\end{aligned}\tag{3.6}$$

where $Q^t(\theta,\rho^*)=K_{\rho^*}\times(-\theta,t)$, $t\in(-\theta,0)$.

Integrating by parts,

$$\begin{aligned}&\int_{Q^t(\theta,\rho^*)}\frac{\partial}{\partial d}[\Psi^2(d)]\eta^{p_\rho^+}|_{d=u_h}\frac{\partial u_h}{\partial\tau}dxd\tau=\int_{Q^t(\theta,\rho^*)}\frac{\partial}{\partial t}[\Psi^2(u_h)]\eta^{p_\rho^+}dxd\tau\\=&\int_{K_{\rho^*}}\Psi^2(u_h(x,t))\eta^{p_\rho^+}dx-\int_{K_{\rho^*}}\Psi^2(u_h(x,-\theta))\eta^{p_\rho^+}dx,\end{aligned}$$

by $\Psi(u_h)\le m\ln 2$, $\Psi(u)\le m\ln 2$, $|\Psi^2(u_h)-\Psi^2(u)|\le\frac{m2^{m+3}\ln 2}{\omega}|u_h-u|$, and $u_h\to u$ in $L^2(K_{\rho^*})$ for $\forall t\in(-\theta,0)$, so

$$\int_{K_{\rho^*}}\Psi^2(u_h(x,t))\eta^{p_\rho^+}dx\to\int_{K_{\rho^*}}\Psi^2(u(x,t))\eta^{p_\rho^+}dx,$$

$$\int_{K_{\rho^*}}\Psi^2(u_h(x,-\theta))\eta^{p_\rho^+}dx\to\int_{K_{\rho^*}}\Psi^2(u(x,-\theta))\eta^{p_\rho^+}dx,$$

therefore we obtain

$$\begin{aligned}&\int_{Q^t(\theta,\rho^*)}\frac{\partial}{\partial d}[\Psi^2(d)]\eta^{p_\rho^+}|_{d=u_h}\frac{\partial u_h}{\partial\tau}dxd\tau\\\to&\int_{K_{\rho^*}}\Psi^2(u(x,t))\eta^{p_\rho^+}dx-\int_{K_{\rho^*}}\Psi^2(u(x,-\theta))\eta^{p_\rho^+}dx,\end{aligned}\tag{3.7}$$

Denote $\Psi'(u)=\frac{\partial(\Psi(u))}{\partial d}|_{d=u_h}$. Since $\frac{\partial^2}{\partial d^2}(\Psi^2(d))|_{d=u_h}=2(1+\Psi(u_h))\Psi'(u_h)^2$, for the other parts of (3.6),

$$\begin{aligned}I\equiv&\int_{Q^t(\theta,\rho^*)}[a(x,\tau,u,\nabla u)]_h\nabla[\frac{\partial}{\partial d}[\Psi^2(d)]\eta^{p_\rho^+}|_{d=u_h}]dxd\tau\\&+\int_{Q^t(\theta,\rho^*)}[a_0(x,\tau,u,\nabla u)]_h\frac{\partial}{\partial d}[\Psi^2(d)]\eta^{p_\rho^+}|_{d=u_h}dxd\tau\\=&2\int_{Q^t(\theta,\rho^*)}[a(x,\tau,u,\nabla u)]_h\nabla u_h(1+\Psi(u_h))\Psi'(u_h)^2\eta^{p_\rho^+}dxd\tau\\&+2\int_{Q^t(\theta,\rho^*)}[a(x,\tau,u,\nabla u)]_h\Psi'(u_h)\Psi(u_h)\nabla\eta^{p_\rho^+}dxd\tau\\&+2\int_{Q^t(\theta,\rho^*)}[a_0(x,\tau,u,\nabla u)]_h\Psi'(u_h)\Psi(u_h)\eta^{p_\rho^+}dxd\tau.\end{aligned}$$

Next, we consider the problem on the set $\{(x,t) \in K_{\rho^*} \times (-\theta, 0) : u(x,t) < k - \omega 2^{-(m+2)}\}$, thus $\frac{\omega}{2} \le |\Psi'(u)| \le \frac{2^{(m+2)}}{\omega}$. When $h \to 0$, $u_h \to u$ and $(u_h - k)_- \to (u-k)_-$ a.e. in $(x,t) \in Q(\theta, \rho^*)$, so $(1+\Psi(u_h))\Psi'(u_h)^2 \to (1+\Psi(u))\Psi'(u)^2$ a.e. in $(x,t) \in Q(\theta,\rho^*)$. Since

$$|(1+\Psi(u_h))\Psi'(u_h)^2 - (1+\Psi(u))\Psi'(u)^2|^{p(x)} \le [2(1+m\ln 2)(\frac{2^{m+2}}{\omega})^2]^{p_\rho^+}$$

and by Lebesgue's theorem, we get

$$(1+\Psi(u_h))\Psi'(u_h)^2 \nabla u \to (1+\Psi(u))\Psi'(u)^2 \nabla u$$

in $L^{p(x)}(Q^t(\theta,\rho^*))$ for a.e. $t \in (-\theta, 0)$. Because $[a(x,t,u,\nabla u)]_h \to a(x,t,u,\nabla u)$ in $L^{p(x)}(Q^t(\theta,\rho^*))$,

$$\begin{aligned} &\int_{Q^t(\theta,\rho^*)} [a(x,\tau,u,\nabla u)]_h \nabla u_h (1+\Psi(u_h))\Psi'(u_h)^2 \eta^{p_\rho^+} \, dx d\tau \\ &\to \int_{Q^t(\theta,\rho^*)} a(x,\tau,u,\nabla u) \nabla u (1+\Psi(u))\Psi'(u)^2 \eta^{p_\rho^+} \, dx d\tau. \end{aligned}$$

In the same way

$$\begin{aligned} &\int_{Q^t(\theta,\rho^*)} [a(x,\tau,u,\nabla u)]_h \Psi'(u_h)\Psi(u_h) \nabla \eta^{p_\rho^+} \, dx d\tau \\ &\to \int_{Q^t(\theta,\rho^*)} a(x,\tau,u,\nabla u) \Psi'(u)\Psi(u) \nabla \eta^{p_\rho^+} \, dx d\tau, \end{aligned}$$

and

$$\begin{aligned} &\int_{Q^t(\theta,\rho^*)} [a_0(x,\tau,u,\nabla u)]_h \Psi'(u_h)\Psi(u_h) \eta^{p_\rho^+} \, dx d\tau \\ &\to \int_{Q^t(\theta,\rho^*)} a_0(x,\tau,u,\nabla u) \Psi'(u)\Psi(u) \eta^{p_\rho^+} \, dx d\tau, \end{aligned}$$

are both valid.

Combining these estimates, we have

$$\begin{aligned} \lim_{h\to 0} I = \;& 2\int_{Q^t(\theta,\rho^*)} a(x,\tau,u,\nabla u) \nabla u (1+\Psi(u))\Psi'(u)^2 \eta^{p_\rho^+} \, dx d\tau \\ &+2\int_{Q^t(\theta,\rho^*)} a(x,\tau,u,\nabla u) \Psi'(u)\Psi(u) \nabla \eta^{p_\rho^+} \, dx d\tau \\ &+2\int_{Q^t(\theta,\rho^*)} a_0(x,\tau,u,\nabla u) \Psi'(u)\Psi(u) \eta^{p_\rho^+} \, dx d\tau. \end{aligned} \tag{3.8}$$

With (1.4)-(1.5), (1.7)-(1.8), we can get

$$\begin{aligned} \lim_{h\to 0} I \ge \;& 2\beta \int_{Q^t(\theta,\rho^*)} (|\nabla u|^{p(x)} + |u|^{p(x)})(1+\Psi(u))|\Psi'(u)|^2 \eta^{p_\rho^+} \, dx d\tau \\ &-2\alpha \int_{Q^t(\theta,\rho^*)} (|\nabla u|^{p(x)-1} + |u|^{p(x)-1})|\Psi'(u)|\Psi(u) \nabla \eta^{p_\rho^+} \, dx d\tau \\ &-2\alpha \int_{Q^t(\theta,\rho^*)} (|\nabla u|^{p(x)-1} + |u|^{p(x)-1})|\Psi'(u)|\Psi(u) \eta^{p_\rho^+} \, dx d\tau. \end{aligned} \tag{3.9}$$

Since $\frac{(p_\rho^+ - 1)p(x)}{p(x)-1} > \frac{p_\rho^+(p(x)-1)}{p(x)-1}$, by Young's inequality,

$$\begin{aligned} &\int_{Q^t(\theta,\rho^*)} |\nabla u|^{p(x)-1} |\Psi'(u)|\Psi(u) \eta^{p_\rho^+ - 1} |\nabla \eta| dx d\tau \\ \le\;& \varepsilon \int_{Q^t(\theta,\rho^*)} |\nabla u|^{p(x)} (\Psi'(u))^2 (\Psi(u)+1) \eta^{p_\rho^+} \, dx d\tau \\ &+C(\varepsilon) \int_{Q^t(\theta,\rho^*)} (\Psi'(u))^{2-p(x)} \Psi(u) |\nabla \eta|^{p(x)} dx d\tau. \end{aligned} \tag{3.10}$$

In the same way, we have

$$\begin{array}{rl} & \int_{Q^t(\theta,\rho^*)} |u|^{p(x)-1}\Psi'(u)\Psi(u)\eta^{p_\rho^+-1}|\nabla\eta| dxd\tau \\ \leq & \varepsilon\int_{Q^t(\theta,\rho^*)} |u|^{p(x)}(\Psi'(u))^2(\Psi(u)+1)\eta^{p_\rho^+}dxd\tau \\ & +C(\varepsilon)\int_{Q^t(\theta,\rho^*)}(\Psi'(u))^{2-p(x)}\Psi(u)|\nabla\eta|^{p(x)}dxd\tau, \end{array}$$

$$\begin{array}{rl} & \int_{Q^t(\theta,\rho^*)} |\nabla u|^{p(x)-1}\Psi'(u)\Psi(u)\eta^{p_\rho^+}dxd\tau \\ \leq & \varepsilon\int_{Q^t(\theta,\rho^*)} |\nabla u|^{p(x)}(\Psi'(u))^2(\Psi(u)+1)\eta^{p_\rho^+}dxd\tau \\ & +C(\varepsilon)\int_{Q^t(\theta,\rho^*)}(\Psi'(u))^{2-p(x)}\Psi(u)\eta^{p_\rho^+}dxd\tau, \end{array} \tag{3.11}$$

and

$$\begin{array}{rl} & \int_{Q^t(\theta,\rho^*)} |u|^{p(x)-1}\Psi'(u)\Psi(u)\eta^{p_\rho^+}dxd\tau \\ \leq & \varepsilon\int_{Q^t(\theta,\rho^*)} |u|^{p(x)}(\Psi'(u))^2(\Psi(u)+1)\eta^{p_\rho^+}dxd\tau \\ & +C(\varepsilon)\int_{Q^t(\theta,\rho^*)}(\Psi'(u))^{2-p(x)}\Psi(u)\eta^{p_\rho^+}dxd\tau. \end{array}$$

Combining (3.8)-(3.11),

$$\begin{array}{rl} \varliminf\limits_{h\to 0} I \geq & (2\beta-4\alpha p^+\varepsilon)\int_{Q^t(\theta,\rho^*)}(|\nabla u|^{p(x)}+|u|^{p(x)})(1+\Psi(u))\Psi'(u)^2\eta^{p_\rho^+}dxd\tau \\ & -C(\varepsilon)\int_{Q^t(\theta,\rho^*)}(\Psi'(u))^{2-p(x)}\Psi(u)(\eta^{p_\rho^+}+|\nabla\eta|^{p(x)})dxd\tau. \end{array}$$

Take $4\alpha p^+\varepsilon=\beta$, then

$$\begin{array}{rl} \varliminf\limits_{h\to 0} I \geq & \beta\int_{Q^t(\theta,\rho^*)}(|\nabla u|^{p(x)}+|u|^{p(x)})(1+\Psi(u))\Psi'(u)^2\eta^{p_\rho^+}dxd\tau \\ & -C(p^+)\int_{Q^t(\theta,\rho^*)}(\Psi'(u))^{2-p(x)}\Psi(u)(\eta^{p_\rho^+}+|\nabla\eta|^{p(x)})dxd\tau. \end{array} \tag{3.12}$$

In view of (3.7) and (3.12),

$$\int_{K_{\rho^*}}\Psi^2(u(x,t))\eta^{p_\rho^+}dx \leq C\int_{Q^t(\theta,\rho^*)}(\Psi'(u))^{2-p(x)}\Psi(u)(\eta^{p_\rho^+}+|\nabla\eta|^{p(x)})dxd\tau.$$

By $\Psi(u)\leq m\ln 2$, $|\Psi'(u)|^{-1}\leq\frac{\omega}{2}$, $|\nabla\eta|\leq\frac{4}{\rho}$, $|\Psi'(u)|\leq\frac{2^{m+2}}{\omega}$, we can get

$$\int_{K_{\rho^*}}\Psi^2(u(x,t))\eta^{p_\rho^+}dx \leq Cm|K_{\rho^*}|. \tag{3.13}$$

$\forall t\in(-\theta,0)$, for such a set $\{(x,t)\in K_{2\rho^*}: u(x,t)<\mu^-+\frac{\omega}{2^{m+2}}\}$ we have

$$\Psi^2(u)\geq \ln^2\frac{H_k^-}{H_k^- -\frac{\omega}{4}+\frac{\omega}{2^{m+1}}}.$$

Since $-\frac{\omega}{4}+\frac{\omega}{2^{m+1}}<0$, we obtain $\ln^2\frac{H_k^-}{H_k^- -\frac{\omega}{4}+\frac{\omega}{2^{m+1}}}$ is decreasing about H_k^- and $H_k^-\leq\frac{\omega}{4}$, thus

$$\Psi^2(u)\geq \ln^2\frac{H_k^-}{H_k^- -\frac{\omega}{4}+\frac{\omega}{2^{m+1}}}\geq \ln^2\frac{\frac{\omega}{4}}{\frac{\omega}{4}-\frac{\omega}{4}+\frac{\omega}{2^{m+1}}}=[(m-1)\ln 2]^2.$$

Because $\eta = 1$ in $K_{\frac{\rho^*}{2}}$, by (3.13)

$$|x \in K_{\frac{\rho^*}{2}} : u(x,t) < \mu^- + \frac{\omega}{2^{m+2}}| \le C\frac{m}{(m-1)^2}|K_{\frac{\rho^*}{2}}|,$$

where $C = C(M, p^+)$. To prove the lemma we have only to choose m sufficiently large and $s = m + 2$.□

Lemma 3.4 *Let (3.3)-(3.4) hold, then there exist $\sigma_1 \in (0,1)$ and an integer $s > 1$ independent of ω and ρ, so that $u(x,t) > \mu^- + \frac{\omega}{2^{s+1}}$, a.e. $(x,t) \in Q(\theta, \frac{\rho^*}{4})$.*

Proof: Let $\rho_m^* = \frac{\rho^*}{4} + \frac{\rho^*}{2^{m+2}}$, $k_m = \mu^- + \frac{\omega}{2^{s+1}} + \frac{\omega}{2^{s+m+1}}$, $m = 0, 1, 2, ...,$ and $s > 1$ is to be chosen later. By lemma 3.2, for a.e. $x \in K_{\rho_m^*}$ we have $u(x, -\theta) > \mu^- + \frac{\omega}{4} \ge k_m$, thus $(u - k_m)_-(x, -\theta) = 0$. Let $\eta_m(x)$ be a smooth cutoff function in $K_{\rho_m^*}$ satisfying $\eta_m \equiv 1$ in $K_{\rho_{m+1}^*}$, $|\nabla\eta_m| \le \frac{2^{m+4}}{\rho}$, and $\eta_m = 0$ outside $K_{\rho_m^*}$. We take $\varphi = -(u_n - k_m)_- \eta_m^{p_\rho^+}$ as the testing function in (1.11), by the fact that

$$\|u\|_{L^\infty_{loc}(Q_{\rho_m^*})} \le M, \quad \|(u - k_m)_-\|_{L^\infty_{loc}(Q_{\rho_m^*})} \le \|(k_m - u)\|_{L^\infty_{loc}(Q_{\rho_m^*})} \le \frac{\omega}{2^s},$$

similar to lemma 3.2, we have

$$\begin{aligned} & \sup_{-\theta<t<0} \int_{K_{\rho_m^*}} (u-k_m)_-^2 \eta_m^{p_\rho^+} dx + \int_{Q(\theta,\rho_m^*)} |\nabla(u-k_m)_-|^{p(x)} \eta_m^{p_\rho^+} dxdt \\ \le \quad & C 2^{m p_\rho^+} \rho^{-p_\rho^+} \int_{Q(\theta,\rho_m^*)} \chi[(u-k_m)_- > 0] dxdt. \end{aligned} \tag{3.14}$$

On the other hand, we have

$$\int_{K_{\rho_m^*}} (u-k_m)_-^2 \eta_m^{p_\rho^+} dx \ge (\frac{\omega}{2^s})^{2-p_\rho^-} \int_{K_{\rho_m^*}} (u-k_m)_-^{p_\rho^-} \eta_m^{p_\rho^+} dx \ge \frac{\theta}{\rho^*} \int_{K_{\rho_m^*}} (u-k_m)_-^{p_\rho^-} \eta_m^{p_\rho^+} dx$$

and

$$\begin{aligned} & \int_{Q(\theta,\rho_m^*)} |\nabla(u-k_m)_-|^{p_\rho^-} \eta_m^{p_\rho^+} dxdt \\ \le \quad & \int_{Q(\theta,\rho_m^*)} |\nabla(u-k_m)_-|^{p(x)} \eta_m^{p_\rho^+} dxdt + \int_{Q(\theta,\rho_m^*)} \chi[(u-k_m)_- > 0] \eta_m^{p_\rho^+} dxdt, \end{aligned}$$

where s is chosen so large as to satisfy the conclusion of lemma 3.3.

Combining the above two inequalities with (3.14), we get

$$\begin{aligned} & \sup_{-\theta<t<0} \int_{K_{\rho_m^*}} (u-k_m)_-^{p_\rho^-} \eta_m^{p_\rho^+} dx + \frac{(\rho^*)^{p_\rho^+}}{\theta} \int_{Q(\theta,\rho_m^*)} |\nabla(u-k_m)_-|^{p_\rho^-} \eta_m^{p_\rho^+} dxdt \\ \le \quad & C 2^{m p_\rho^+} \rho^{-p_\rho^+} \frac{(\rho^*)^{p_\rho^+}}{\theta} \int_{Q(\theta,\rho_m^*)} \chi[(u-k_m)_- > 0] dxdt. \end{aligned}$$

We introduce the change of variable $z = t(\rho^*)^{p_\rho^+}\theta^{-1}$, which maps $Q(\theta, \rho_m^*)$ into $Q_m = K_{\rho_m^*} \times (-(\rho^*)^{p_\rho^+}, 0)$. Let $v(x,t) = u(x, \theta z(\rho^*)^{-p_\rho^+})$, $\tilde{\eta}_m(x,z) =$

$\eta_m(x,\theta z(\rho^*)^{-p_\rho^+})$, and denote $|A_m| = \text{meas}\{(x,z)\in Q : v(x,z) < k_m\}$, then

$$\begin{aligned}
& \|(v-k_m)_-\tilde{\eta}_m^{p_\rho^+}\|^{p_\rho^-}_{V^{p_\rho^-,p_\rho^-}(Q_m)} \\
= & \operatorname*{ess\,sup}_{-(\rho^*)^{p_\rho^+}<z<0} \int_{K_{\rho_m^*}} (v-k_m)_-^{p_\rho^-}(\tilde{\eta}_m^{p_\rho^+})^{p_\rho^-}\,dx + \int_{Q_m} |\nabla[(v-k_m)_-\tilde{\eta}_m^{p_\rho^+}]|^{p_\rho^-}\,dxdz \\
\le & C(\sup_{-(\rho^*)^{p_\rho^+}<z<0} \int_{K_{\rho_m^*}} (v-k_m)_-^{p_\rho^-}\tilde{\eta}_m^{p_\rho^+}\,dx \\
& + \int_{Q_m} |\nabla(v-k_m)_-|^{p_\rho^-}\tilde{\eta}_m^{p_\rho^+}\,dxdz + \int_{Q_m} |(v-k_m)_-\nabla\tilde{\eta}_m^{p_\rho^+}|^{p_\rho^-}\,dxdz) \\
\le & C2^{mp_\rho^+}\rho^{-p_\rho^+}|A_m|,
\end{aligned} \tag{3.15}$$

by lemma 2.6 and (3.15),

$$\begin{aligned}
\frac{1}{2^{p_\rho^-(m+2)}}\left(\frac{\omega}{2^s}\right)^{p_\rho^-}|A_{m+1}| &= |k_m-k_{m+1}|^{p_\rho^-}|A_{m+1}| \\
&\le \|(v-k_m)_-\|^{p_\rho^-}_{L^{p_\rho^-}(Q_{m+1})} \le \|(v-k_m)_-\tilde{\eta}_m^{p_\rho^+}\|^{p_\rho^-}_{L^{p_\rho^-}(Q_m)} \\
&\le \|(v-k_m)_-\tilde{\eta}_m^{p_\rho^+}\|^{p_\rho^-}_{V^{p_\rho^-,p_\rho^-}(Q_m)}|A_m|^{\frac{p_\rho^-}{p_\rho^-+N}} \\
&\le C2^{mp_\rho^+}\rho^{-p_\rho^+}|A_m|^{1+\frac{p_\rho^-}{p_\rho^-+N}}.
\end{aligned} \tag{3.16}$$

We take $A > 2^s$, by (3.3), we get $(\frac{\omega}{2^s})^{p_\rho^- -2} \ge \rho^\varepsilon \ge \rho^{p_\rho^- -2}$, therefore $\frac{\omega}{2^s} \ge \rho$. Thus we obtain

$$\left(\frac{\omega}{2^s}\right)^{-p_\rho^+} \le \rho^{-p_\rho^+}.$$

Denote $Z_m = \frac{|A_m|}{|Q_m|}$. By (3.16) and (1.10),

$$Z_{m+1} \le C4^{mp_\rho^+} Z_m^{1+\frac{p_\rho^-}{p_\rho^-+N}} \le C4^{mp^+} Z_m^{1+\frac{p^-}{p^-+N}},$$

where $C = C(M,p^+)$. Since

$$Z_0 = \frac{|A_0|}{|Q_0|} = \frac{|\{(x,t)\in Q(\theta,\frac{\rho^*}{2}) : u(x,t) < \mu^- + \frac{\omega}{2^s}\}|}{|Q(\theta,\frac{\rho^*}{2})|},$$

by lemma 3.3 there exists s such that $Z_0 < \sigma_1$ where $\sigma_1 \equiv C^{-\frac{N+p^-}{p^-}}4^{-p^+(\frac{N+p^-}{p^-})^2}$. Then by lemma 2.6 it follows that $Z_m \to 0$ as $m\to\infty$. So we can get

$$u(x,t) > \mu^- + \frac{\omega}{2^{s+1}}, \qquad a.e. \quad (x,t)\in Q(\theta,\frac{\rho^*}{4}).$$

Proposition 3.1 *There exist $\sigma\in(0,1)$, $\nu_1\in(0,1)$ and $A_1 \gg 1$ independent of ω and ρ, such that if for some cylinder of the type $[(0,t^*)+Q(l\rho^{p_\rho^+},\rho)]$,*

$$|(x,t)\in[(0,t^*)+Q(l\rho^{p_\rho^+},\rho)] : u < \mu^- + \frac{\omega}{2}| \le \sigma|Q(l\rho^{p_\rho^+},\rho)|,$$

then either

$$\omega \leq A_1 \rho^{\frac{\varepsilon}{p_\rho^- - 2}} \tag{3.17}$$

or

$$\operatorname*{ess\,osc}_{Q(l(\frac{\rho}{8})^{p_\rho^+},\frac{\rho}{8})} u \leq \nu_1 \omega. \tag{3.18}$$

Proof: Assume (3.17) is violated. By lemma 3.4, we can determine a positive integer number s such that $\operatorname*{ess\,inf}_{Q(l(\frac{\rho}{8})^{p_\rho^+},\frac{\rho}{8})} u \geq \mu^- + \frac{\omega}{2^{s+1}}$, this gives

$$-\operatorname*{ess\,inf}_{Q(l(\frac{\rho}{8})^{p_\rho^+},\frac{\rho}{8})} u \leq -\mu^- - \frac{\omega}{2^{s+1}}, \tag{3.19}$$

and further

$$\operatorname*{ess\,osc}_{Q(\theta,\frac{\rho}{8})} u \leq (1 - \frac{1}{2^{s+1}})\omega.$$

therefore the proposition follows with $\nu_1 = (1 - \frac{1}{2^{s+1}})$, since $Q(l(\frac{\rho}{8})^{p_\rho^+}, \frac{\rho}{8}) \subset Q(\theta, \frac{\rho}{8})$.□

Next assume that the condition of proposition 3.1 is violated, i.e. for every cylinder $[(0,t^*) + Q(l\rho^{p_\rho^+}, \rho)] \subset Q(a\rho^{p_\rho^+}, \rho)$, where $\frac{1}{a} = (\frac{\omega}{A})^{p_\rho^- - 2}$,

$$|(x,t) \in [(0,t^*) + Q(l\rho^{p_\rho^+}, \rho)] : u < \mu^- + \frac{\omega}{2}| > \sigma |Q(l\rho^{p_\rho^+}, \rho)|.$$

Since $\mu^- + \frac{\omega}{2} \leq \mu^+ - \frac{\omega}{2}$, we can get

$$|(x,t) \in [(0,t^*) + Q(l\rho^{p_\rho^+}, \rho)] : u > \mu^+ - \frac{\omega}{2}| \leq (1-\sigma)|Q(l\rho^{p_\rho^+}, \rho)|. \tag{3.20}$$

Lemma 3.5 *Let (3.20) hold, then there exists a* $\bar{t} \in [t^* - l\rho^{p_\rho^+}, t^* - \frac{\sigma}{2} l\rho^{p_\rho^+}]$ *such that*

$$|\{x \in K_\rho : u(x,\bar{t}) > \mu^+ - \frac{\omega}{2}\}| \leq \frac{1-\sigma}{1-\frac{\sigma}{2}} |K_\rho|.$$

Proof: If not, for all $t \in [t^* - l\rho^{p_\rho^+}, t^* - \frac{\sigma}{2} l\rho^{p_\rho^+}]$,

$$|\{x \in K_\rho : u(x,t) > \mu^+ - \frac{\omega}{2}\}| > \frac{1-\sigma}{1-\frac{\omega}{2}} |K_\rho|$$

and

$$\begin{aligned} & |(x,t) \in [(0,t^*) + Q(l\rho^{p_\rho^+}, \rho)] : u > \mu^+ - \tfrac{\omega}{2}| \\ \geq \ & \int_{t^* - lp_\rho^+}^{t^* - \frac{\sigma}{2} lp_\rho^+} |\{x \in K_\rho : u(x,t) > \mu^+ - \tfrac{\omega}{2}\}| dt \\ > \ & (1 - \tfrac{\sigma}{2}) l\rho^{p_\rho^+} (1-\sigma)(1-\tfrac{\sigma}{2})^{-1} |K_\rho| = (1-\sigma)|Q(l\rho^{p_\rho^+}, \rho)|, \end{aligned}$$

contradicting (3.20).□

Lemma 3.6 *Let (3.20) hold, then there exists a positive integer* $\bar{s} > 2$*, such that*

$$|\{x \in K_\rho : u(x,t) > \mu^+ - \frac{\omega}{2^{\bar{s}}}\}| \leq (1 - (\frac{\sigma}{2})^2)|K_\rho|, \quad \forall t \in [t^* - \frac{\sigma}{2} l\rho^{p_\rho^+}, t^*].$$

Proof: Let $k=\mu^+-\frac{\omega}{2}$, $Q_\rho=K_\rho\times(\bar{t},t^*)$. Similar to lemma 3.3, we take $\varphi=\frac{\partial}{\partial d}[\Psi^2(d)]\eta^{p_\rho^+}|_{d=u_h}$ as the testing function in (3.2), where the cutoff function η independent of t is taken so that $\eta\equiv 1$ in the cube $K_{(1-\alpha)\rho}$, $\alpha\in(0,1)$, and $|\nabla\eta|\le\frac{1}{\alpha\rho}$, $0<\alpha<1$. We take $H_k^+=\operatorname*{ess\,sup}_{[(0,t^*)+Q(\theta,\rho^*)]}(u-k)_+$, and consider

$$\Psi(u)=\max\{0,\ln\frac{H_k^+}{H_k^+-(u-k)_++\omega 2^{-(m+2)}}\}=\ln^+\frac{H_k^+}{H_k^+-(u-k)_++\omega 2^{-(m+2)}},$$

then

$$\begin{aligned}\int_{K_{(1-\alpha)\rho}}\Psi^2(u(x,t))dx &\le \int_{K_\rho}\Psi^2(u(x,t))\eta^{p_\rho^+}dx\le\int_{K_\rho}\Psi^2(u(x,\bar{t}))dx\\ &+C\int_{\bar{t}}^{t^*}\int_{K_\rho}(\Psi'(u))^{2-p(x)}\Psi(u)(\eta^{p_\rho^+}+|\nabla\eta|^{p(x)})dx,\end{aligned}$$

where $|t^*-\bar{t}|\le l\rho^{p_\rho^+}$, $l=(\frac{\omega}{2})^{2-p_\rho^-}$, $C=C(M,p^+)$.

When $u(x,\bar{t})>k+\frac{\omega}{2^{m+1}}>\mu^+-\frac{\omega}{2}$, $\Psi^2(u(x,\bar{t}))\neq 0$, by lemma 3.5,

$$\begin{aligned}&\int_{K_\rho}\Psi^2(u(x,\bar{t}))dx=\int_{\{x\in K_\rho:u(x,\bar{t})>\mu^+-\frac{\omega}{2}+\frac{\omega}{2^{m+1}}\}}\Psi^2(u(x,\bar{t}))dx\\ \le\ &\int_{\{x\in K_\rho:u(x,\bar{t})>\mu^+-\frac{\omega}{2}\}}\Psi^2(u(x,\bar{t}))dx\le(m\ln 2)^2(1-\sigma)(1-\frac{\sigma}{2})^{-1}|K_\rho|,\end{aligned}$$

so we have

$$\int_{K_{(1-\alpha)\rho}}\Psi^2(u(x,t))dx\le C[m^2(1-\sigma)(1-\frac{\sigma}{2})^{-1}+m\alpha^{-p_\rho^+}]|K_\rho|.$$

$\forall t\in(\bar{t},t^*)$, in $\{x\in K_{(1-\alpha)\rho}:u(x,t)>\mu^+-\frac{\omega}{2^{m+2}}\}$ we can get

$$\Psi^2(u)\ge\ln^2\frac{H_k^+}{H_k^+-\frac{\omega}{4}+\omega 2^{-(m+1)}}\ge\ln^2\frac{\omega 2^{-2}}{\omega 2^{-(m+1)}}=(m-1)^2\ln^2 2,$$

so $\forall t\in(\bar{t},t^*)$,

$$\begin{aligned}&|x\in K_{(1-\alpha)\rho}:u(x,t)>\mu^+-\omega 2^{-(m+2)}|\\ \le\ & C[(\tfrac{m}{m+1})^2(1-\sigma)(1-\tfrac{\sigma}{2})^{-1}+\tfrac{1}{m}\alpha^{-p_\rho^+}]|K_\rho|.\end{aligned}$$

On the other hand, $\forall t\in(\bar{t},t^*)$,

$$|x\in K_\rho:u(x,t)>\mu^+-\omega 2^{-(m+2)}|\le|x\in K_{(1-\alpha)\rho}:u(x,t)>\mu^+-\omega 2^{-(m+1)}|$$

$$+|K_\rho\setminus K_{(1-\alpha)\rho}|\le|x\in K_{(1-\alpha)\rho}:u(x,t)>\mu^+-\omega 2^{-(m+1)}|+\alpha N|K_\rho|,$$

so $\forall t\in(\bar{t},t^*)$,

$$\begin{aligned}&|x\in K_{(1-\alpha)\rho}:u(x,t)>\mu^+-\omega 2^{-(m+1)}|\\ \le\ & C(\tfrac{m}{m-1})^2[(1-\sigma)(1-\tfrac{\sigma}{2})^{-1}+\tfrac{1}{m}\alpha^{-p_\rho^+}+N\alpha]|K_\rho|.\end{aligned}$$

Choose α so small and then m so large that $C(\frac{m}{m-1})^2 \le (1+\sigma)(1-\frac{\sigma}{2})$, $\frac{C}{m}\alpha^{-p_\rho^+} \le \frac{3}{8}\sigma^2$ and $C\alpha N \le \frac{3}{8}\sigma^2$. Then for such a choice of m the lemma follows with $\bar{s} = m+1.\Box$

Since (3.20) holds for all $[(0,t^*) + Q(l\rho^{p_\rho^+},\rho)]$, the conclusion of lemma 3.6 holds for all time levels satisfying $t \ge -(a-l)\rho^{p_\rho^+} = -(1-(\frac{2}{A})^{p_\rho^- -2})a\rho^{p_\rho^+}$. If the number A is chosen sufficiently large such that $1-(\frac{2}{A})^{p_\rho^- -2} > \frac{1}{2}$, we deduce the following corollary.

Corollary 3.1 *Let (3.20) hold, then for all* $t \in (-\frac{a}{2}\rho^{p_\rho^+}, 0)$,

$$|\{x \in K_\rho : u(x,t) > \mu^+ - \omega 2^{-\bar{s}}\}| \le (1-(\frac{\sigma}{2})^2)|K_\rho|.$$

Lemma 3.7 *Let (3.20) hold, then for every* $\bar{\sigma} \in (0,1)$, *there exists positive integer* $s^* > \bar{s}$, *such that*

$$|\{x \in K_\rho : u(x,t) > \mu^+ - \frac{\omega}{2^{s^*}}\}| \le \bar{\sigma}|Q(2^{-1}a\rho^{p_\rho^+},\rho)|, \quad \forall t \in (-\frac{a}{2}\rho^{p_\rho^+}, 0).$$

Proof: Consider the problem in $Q(a\rho^{p_\rho^+}, 2\rho)$. Let $k = \mu^+ - \frac{\omega}{2^s}$, where $\bar{s} \le s \le s^*$. Take $\varphi = (u_n - k)_+ \zeta^{p_\rho^+}$ as the testing function in (1.9), where ζ is a cutoff function that equals one on $Q(\frac{a}{2}\rho^{p_\rho^+},\rho)$, vanishes on the parabolic boundary of $Q(a\rho^{p_\rho^+}, 2\rho)$ and such that $|\nabla\zeta| \le \frac{1}{\rho}$, $0 \le \zeta_t \le \frac{2}{a\rho^{p_\rho^+}}$. Similar to lemma 3.2, we get

$$\begin{aligned} &\int_{A_s} |\nabla u|^{p_\rho^-} dxdt \le \int_{Q(\frac{a}{2}\rho^{p_\rho^+},\rho)} |\nabla (u-k)_+|^{p(x)} dxdt + |A_s| \\ \le\ & C\rho^{-p_\rho^+}|Q(\tfrac{a}{2}\rho^{p_\rho^+},\rho)|, \end{aligned}$$

where $C = C(p^+)$ and

$$A_s = \{(x,t) \in Q(\frac{a}{2}\rho^{p_\rho^+},\rho) : u(x,t) > \mu^+ - \frac{\omega}{2^s}\},$$

$$A_s(t) = \{x \in K_\rho : u(x,t) > \mu^+ - \frac{\omega}{2^s}\}.$$

By corollary 3.1, $\forall t \in (-\frac{a}{2}\rho^{p_\rho^+}, 0)$,

$$|\{x \in K_\rho : u(x,t) < \mu^+ - \frac{\omega}{2^s}\}| = |K_\rho| - |A_s(t)| \ge (\frac{\sigma}{2})^2|K_\rho|. \tag{3.21}$$

In lemma 2.8, take $k = \mu^+ - \frac{\omega}{2^s}$, $h = \mu^+ - \frac{\omega}{2^{s+1}}$, then $\forall t \in [-\frac{a}{2}\rho^{p_\rho^+}, 0]$, by (3.21), we get

$$\frac{\omega}{2^{s+1}}|A_{s+1}(t)| \le \frac{C}{\sigma^2}\frac{\rho^{N+1}}{|K_\rho|}\int_{A_s(t)\setminus A_{s+1}(t)} |\nabla u| dx. \tag{3.22}$$

Take $A > 2^s$, there exists $C = C(M, p^+, p^-)$ such that $(\frac{\omega}{2})^{p_\rho^+ - p_\rho^-} \le C$ and $(\frac{\omega}{2^s})^{-p_\rho^-} \le \rho^{-p_\rho^-}$ hold. Integrating on $(-a\rho^{p_\rho^+}, 0)$, from (3.22) we get

$$\begin{aligned} & (\tfrac{\omega}{2^s})^{-p_\rho^-} \tfrac{\omega}{2^{s+1}} |A_{s+1}| \le (\tfrac{\omega}{2^s})^{-p_\rho^-} \tfrac{C\rho}{\sigma^2} \textstyle\int_{A_s \setminus A_{s+1}} |\nabla u| dxdt \\ \le \; & (\tfrac{\omega}{2^s})^{-p_\rho^-} \tfrac{C\rho}{\sigma^2} (\textstyle\int_{A_s} |\nabla u|^{p_\rho^-} dxdt)^{\frac{1}{p_\rho^-}} |A_s \setminus A_{s+1}|^{\frac{p_\rho^- - 1}{p_\rho^-}} \\ \le \; & \tfrac{C}{\sigma^2} |Q(\tfrac{a}{2}\rho^{p_\rho^+}, \rho)|^{\frac{1}{p_\rho^-}} |A_s \setminus A_{s+1}|^{\frac{p_\rho^- - 1}{p_\rho^-}}, \end{aligned} \tag{3.23}$$

If s is large enough so that $(\frac{\omega}{2^s})^{p_\rho^-} \frac{2^{s+1}}{\omega} < 1$, from (3.23) we get

$$|A_{s+1}|^{\frac{p_\rho^-}{p_\rho^- - 1}} \le C\sigma^{-2\frac{p_\rho^-}{p_\rho^- - 1}} |Q(\frac{a}{2}\rho^{p_\rho^+}, \rho)|^{\frac{1}{p_\rho^- - 1}} |A_s \setminus A_{s+1}|, \tag{3.24}$$

for all $\bar{s} \le s \le s^*$. We add them for $s = \bar{s}, \bar{s}+1, \bar{s}+2, ..., s^*-1$, then

$$(s^* - \bar{s}) |A_{s^*}|^{\frac{p_\rho^-}{p_\rho^- - 1}} \le C\sigma^{-2\frac{p_\rho^-}{p_\rho^- - 1}} |Q(\frac{a}{2}\rho^{p_\rho^+}, \rho)|^{\frac{p_\rho^-}{p_\rho^- - 1}}.$$

After taking s^* so large that $C(s^* - \bar{s})^{\frac{1-p^-}{p^-}} \le \sigma^2 \bar{\sigma}$, we conclude the lemma.□

Lemma 3.8 *Let (3.20) hold, then there exists $\bar{\sigma} \in (0,1)$ so that*

$$u(x,t) \le \mu^+ - \frac{\omega}{2^{s^*+1}}, \qquad a.e. \quad Q(\frac{a}{2}(\frac{\rho}{2})^{p_\rho^+}, \frac{\rho}{2}),$$

where $\frac{1}{a} = (\frac{\omega}{A})^{p_\rho^- - 2}$, $A = 2^{s^*}$.

Proof: We will consider the problem over the boxes $Q(\frac{a}{2}\rho_m^{p_\rho^+}, \rho_m)$. Let $\rho_m = \frac{\rho}{2} + \frac{\rho}{2^{m+1}}, k_m = \mu^+ - \frac{\omega}{2^{s^*+1}} - \frac{\omega}{2^{s^*+m+1}}$. ζ_m is a cutoff function with $0 \le \zeta_m \le 1$ in $Q(\frac{a}{2}\rho_m^{p_\rho^+}, \rho_m)$, $\zeta_m \equiv 1$ in $Q(\frac{a}{2}\rho_{m+1}^{p_\rho^+}, \rho_{m+1})$, $\zeta_m \equiv 0$ on the parabolic boundary of $Q(\frac{a}{2}\rho_m^{p_\rho^+}, \rho_m)$, $|\nabla \zeta_m| \le \frac{2^{m+2}}{\rho}$, $0 \le \frac{\partial \zeta_m}{\partial t} \le \frac{2}{a}(\frac{2^{m+2}}{\rho})^{p_\rho^+}$. Take $(u_m - k_m)_+ \zeta_m^{p_\rho^+}$ as the testing function in (1.11), by $\|u\|_{L^\infty_{loc}(Q(\frac{a}{2}\rho_m^{p_\rho^+}, \rho_m))} \le M$ and $\|(u - k_m)_+\|_{L^\infty_{loc}(Q(\frac{a}{2}\rho_m^{p_\rho^+}, \rho_m))} \le \|(u - k_m)\|_{L^\infty_{loc}(Q(\frac{a}{2}\rho_m^{p_\rho^+}, \rho_m))} \le \frac{\omega}{2^{s^*}}$, similar to lemma 3.2, we obtain

$$\begin{aligned} & \sup_{-\frac{a}{2}\rho_m^{p_\rho^+} < t < 0} \textstyle\int_{K_{\rho_m}} (u - k_m)_+^2 \zeta_m^{p_\rho^+} dx + \int_{Q(\frac{a}{2}\rho_m^{p_\rho^+}, \rho_m)} |\nabla (u - k_m)_+|^{p(x)} \zeta_m^{p_\rho^+} dxdt \\ \le \; & C 2^{m p_\rho^+} \rho^{-p_\rho^+} \textstyle\int_{Q(\frac{a}{2}\rho_m^{p_\rho^+}, \rho_m)} \chi[(u - k_m)_+ > 0] dxdt. \end{aligned} \tag{3.25}$$

On the other hand, we have

$$\int_{K_{\rho_m}} (u - k_m)_+^{p_\rho^-} \zeta_m^{p_\rho^+} dx \le (\frac{\omega}{2^{s^*}})^{p_\rho^- - 2} \int_{K_{\rho_m}} (u - k_m)_+^2 \zeta_m^{p_\rho^+} dx$$

and

$$\int_{Q(\frac{a}{2}\rho_m^{p_\rho^+},\rho_m)}|\nabla(u-k_m)_+|^{p_\rho^-}\zeta_m^{p_\rho^+}dxdt \le \int_{Q(\frac{a}{2}\rho_m^{p_\rho^+},\rho_m)}|\nabla(u-k_m)_+|^{p(x)}\zeta_m^{p_\rho^+}dxdt + \int_{Q(\frac{a}{2}\rho_m^{p_\rho^+},\rho_m)}\chi[(u-k_m)_+>0]\zeta_m^{p_\rho^+}dxdt$$

then by (3.25),

$$\begin{aligned} & \sup_{-\frac{a}{2}\rho_m^{p_\rho^+}<t<0}\int_{K_{\rho_m}}(u-k_m)_+^{p_\rho^-}\zeta_m^{p_\rho^+}dx + \frac{1}{a}\int_{Q(\frac{a}{2}\rho_m^{p_\rho^+},\rho_m)}|\nabla(u-k_m)_+|^{p_\rho^-}\zeta_m^{p_\rho^+}dxdt \\ \le\ & C2^{mp_\rho^+}\rho^{-p_\rho^+}\frac{1}{a}\int_{Q(\frac{a}{2}\rho_m^{p_\rho^+},\rho_m)}\chi[(u-k_m)_+>0]dxdt. \end{aligned}$$

Next, we introduce the change of time-variable $z=2l^{-1}t$ which transforms $Q(2^{-1}a\rho_m^{p_\rho^+},\rho_m)$ into $Q_m=K_{\rho_m}\times(-\rho_m^{p_\rho^+},0)$. Setting $v(x,t)=u(x,2^{-1}az)$, $\tilde{\zeta}_m(x,z)=\zeta_m(x,2^{-1}az)$, $|A_m|=\text{meas}\{(x,z)\in Q_m: v(x,z)>k_m\}$, then

$$\begin{aligned} & \|(v-k_m)_+\tilde{\zeta}_m^{p_\rho^+}\|^{p_\rho^-}_{V^{p_\rho^-,p_\rho^-}(Q_m)} \\ \le\ & C(\sup_{-\rho_m^{p_\rho^+}<z<0}\int_{K_{\rho_m}}(v-k_m)_+^{p_\rho^-}\tilde{\zeta}_m^{p_\rho^+}dx + \int_{Q_m}|\nabla(v-k_m)_+|^{p_\rho^-}\tilde{\zeta}_m^{p_\rho^+}dxdz \\ & +\int_{Q_m}|(v-k_m)_+\nabla\tilde{\zeta}_m^{p_\rho^+}|^{p_\rho^-}dxdz) \le C2^{mp_\rho^+}\rho^{-p_\rho^+}|A_m|. \end{aligned} \tag{3.26}$$

By lemma 2.8 and (3.26),

$$\begin{aligned} \frac{1}{2^{p_\rho^-(m+2)}}\left(\frac{\omega}{2^{s^*}}\right)^{p_\rho^-}|A_{m+1}| &= |k_m-k_{m+1}|^{p_\rho^-}|A_{m+1}| \\ &\le \|(v-k_m)_+\|^{p_\rho^-}_{L^{p_\rho^-}(Q_{m+1})} \le \|(v-k_m)_+\tilde{\zeta}_m^{p_\rho^+}\|^{p_\rho^-}_{L^{p_\rho^-}(Q_m)} \\ &\le \|(v-k_m)_+\tilde{\zeta}_m^{p_\rho^+}\|^{p_\rho^-}_{V^{p_\rho^-,p_\rho^-}(Q_m)}|A_m|^{\frac{p_\rho^-}{p_\rho^-+N}} \\ &\le C2^{mp_\rho^+}\rho^{-p_\rho^+}|A_m|^{1+\frac{p_\rho^-}{p_\rho^-+N}}. \end{aligned}$$

Take $A=2^{s^*}$, then $(\frac{\omega}{2^{s^*}})^{-p_\rho^-}\le\rho^{-p_\rho^-}$.

Next, we obtain

$$Z_{m+1}\le C4^{mp^+}Z_m^{1+\frac{p_\rho^-}{p_\rho^-+N}}.$$

By lemma 2.7, when $m\to\infty$, $Z_m\to 0$ where $Z_0\le C^{-\frac{N+p^-}{p^-}}4^{-p^+(\frac{N+p^-}{p^-})^2}\equiv\bar{\sigma}$. Thus as $m\to\infty$,

$$\int_{Q_m}\chi[(v-k_m)_+>0]dxdz\to 0,$$

i.e. $u(x,t)\le\mu^+-\frac{\omega}{2^{s^*+1}}$ a.e. in $Q(\frac{a}{2}(\frac{\rho}{2})^{p_\rho^+},\frac{\rho}{2})$.□

Proposition 3.2 *There exist $\sigma \in (0,1)$, $\nu_2 \in (0,1)$ and $A_2 \gg 1$ independent of ω and ρ, such that if for all cylinders of the type $[(0,t^*)+Q(l\rho^{p_\rho^+},\rho)]$,*

$$|(x,t) \in [(0,t^*)+Q(l\rho^{p_\rho^+},\rho)] : u > \mu^+ - \frac{\omega}{2}| \le (1-\sigma)|Q(l\rho^{p_\rho^+},\rho)|,$$

then either

$$\omega \le A_2 \rho^{\frac{\varepsilon}{p_\rho^- -2}} \tag{3.27}$$

or

$$\operatorname*{ess\,osc}_{Q(\frac{a}{2}(\frac{\rho}{2})^{p_\rho^+},\frac{\rho}{2})} u \le \nu_2 \omega. \tag{3.28}$$

Proof: Assume (3.27) is violated. By lemma 3.8, we can determine a positive integer number s^* such that

$$\operatorname*{ess\,inf}_{Q(\frac{a}{2}(\frac{\rho}{2})^{p_\rho^+},\frac{\rho}{2})} u \le \mu^+ + \frac{\omega}{2^{s^*+1}}, \tag{3.29}$$

and further

$$\operatorname*{ess\,osc}_{Q(\frac{a}{2}(\frac{\rho}{2})^{p_\rho^+},\frac{\rho}{2})} u \le (1-\frac{1}{2^{s^*+1}})\omega,$$

therefore (3.28) holds with $\nu_2 = (1-\frac{1}{2^{s^*+1}})$. We get the conclusion.□

Combine proposition 1 and proposition 2, we can get

Proposition 3.3 *There exist $\nu = \max\{\nu_1,\nu_2\}$ and $\bar{A} = \{A_1, A_2\}$, such that either $\omega \le \bar{A}\rho^{\frac{\varepsilon}{p_\rho^- -2}}$ or $\operatorname*{ess\,osc}_{Q(l(\frac{\rho}{8})^{p_\rho^+},\frac{\rho}{8})} u \le \nu\omega$, where ν_1,ν_2,A_1,A_2 are determined by proposition 1 and proposition 2.*

Next we assume $\omega_1 = \max\{\nu\omega, \bar{A}\rho^{\frac{\varepsilon}{p_\rho^- -2}}\}$ and $\frac{1}{a_1} = (\frac{\omega_1}{A})^{p_\rho^- -2}$. Since

$$l(\frac{\rho}{8})^{p_\rho^+} = (\frac{2}{\omega})^{p_\rho^- -2}(\frac{\rho}{8})^{p_\rho^+} \ge 2^{-3p_\rho^+}\nu^{p_\rho^- -2}(\frac{2}{A})^{p_\rho^- -2}(\frac{A}{\omega_1})^{p_\rho^- -2}\rho^{p_\rho^+} = a_1\rho_1^{p_\rho^+},$$

where $\rho_1 = C^{-1}\rho$ and $C = 8(\frac{1}{\nu})^{\frac{p_\rho^- -2}{p_\rho^-}}(\frac{A}{2})^{\frac{p_\rho^- -2}{p_\rho^-}}$, so $Q(a_1(\rho_1)^{p_\rho^+},\rho_1) \subset Q(l(\frac{\rho}{8})^{p_\rho^+},\frac{\rho}{8})$. Then we can get $\operatorname*{ess\,osc}_{Q(a_1(\rho_1)^{p_\rho^+},\rho_1)} u \le \omega_1$ and $(\frac{\omega_1}{A})^{p_\rho^- -2} > 8^{p_\rho^+}(\frac{\omega}{2})^{p_\rho^- -2} > \rho^\varepsilon$. So for $Q(a_1(\rho_1)^{p_\rho^+},\rho_1)$, repeating the process above, we can get the similar result, and moreover the following proposition 3.4 can be obtained:

Proposition 3.4 *There exist $0 < \varepsilon_0 < 1$, $\nu \in (0,1)$, $C = C(N,M,p^+,p^-) > 1$ and $A > 1$ satisfy $\rho_0 = \rho$, $\omega_0 = \omega$, $\rho_n = C^{-n}\rho$ and $\omega_{n+1} = \max\{\nu\omega_n, C\rho_n^{\varepsilon_0}\}, n = 1,2,...,$ such that for all boxes $Q^{(n)} = Q(a_n\rho_n^{p_\rho^+},\rho_n)$, $\frac{1}{a_n} = (\frac{\omega_n}{A})^{p_\rho^- -2}, n = 1,2,...,$ we have*

$$Q^{(n+1)} \subset Q^{(n)}, \quad \operatorname*{ess\,osc}_{Q^{(n)}} u \le \omega_n.$$

In view of proposition 3.4, we get

Proposition 3.5 *There exist* $\lambda \in (0,1)$, $C = C(N, M, p^+, p^-)$ *and* $0 < \tilde{\rho} \le \rho$ *such that for all boxes* $Q(a\rho^{p_\rho^+}, \rho)$, $\frac{1}{a} = (\frac{\omega}{A})^{p_\rho^- - 2}$, *we have*

$$\operatorname*{ess\,osc}_{Q(a(\tilde{\rho})^{p_\rho^+}, \tilde{\rho})} u \le C(\omega + \rho^{\varepsilon_0})(\frac{\tilde{\rho}}{\rho})^{\lambda}.$$

Proof: From the iterative construction of ω_n, it follows that $\omega_{n+1} \le \nu\omega_n + C\rho_n^{\varepsilon_0}$ and by iteration

$$\omega_n \le \nu^n \omega + C(\Sigma_{i=0}^{n-1} \nu^i C^{-\varepsilon_0(n-i)})\rho^{\varepsilon_0}.$$

We may assume without loss of generality that ε_0 is so small that $\nu \le C^{-\varepsilon_0}$, then $\omega_n \le \nu^n \omega + Cn(\frac{\rho}{C^n})^{\varepsilon_0}$. Let $0 < \tilde{\rho} \le \rho$ be fixed, then there exists a nonnegative integer n such that

$$C^{-(n+1)}\rho \le \tilde{\rho} \le C^{-n}\rho,$$

which implies the inequalities

$$(n+1) \ge \ln(\frac{\tilde{\rho}}{\rho})^{-\frac{1}{\ln C}},$$

$$\nu^n \le \nu^{-1}(\frac{\tilde{\rho}}{\rho})^{\lambda_1}, \quad \lambda_1 = \frac{|\ln \nu|}{\ln C},$$

$$Cn(\frac{\rho}{C^n})^{\varepsilon_0} \le C^{1+\varepsilon_0} \ln(\frac{\tilde{\rho}}{\rho})^{-\frac{1}{\ln C}} \tilde{\rho}^{\varepsilon_0} \le C(\varepsilon_0)\rho^{\frac{\varepsilon_0}{2}} \tilde{\rho}^{\frac{\varepsilon_0}{2}}.$$

Therefore

$$\omega_n \le C(\omega + \rho^{\varepsilon_0})(\frac{\tilde{\rho}}{\rho})^{\lambda}, \qquad \lambda = \min\{\lambda_1, \frac{\varepsilon_0}{2}\}.$$

On the other hand, by (3.3) we get $\omega > C\rho^{\varepsilon_0}$. Thus by the definition of ω_n, $\omega_1 = \max\{\nu\omega, C\rho^{\varepsilon_0}\} \le \omega$ and $\omega_2 = \max\{\nu\omega_1, C(C^{-1}\rho)^{\varepsilon_0}\} \le \omega, ...,$ so $\omega_n \le \omega$. Since $Q(a\tilde{\rho}^{p_\rho^+}, \tilde{\rho}) \subset Q^{(n)}$, by proposition 3.4, we obtain $\operatorname*{ess\,osc}_{Q(a(\tilde{\rho})^{p_\rho^+}, \tilde{\rho})} u \le \omega_n$, so we conclude proposition 3.5.□

By proposition 3.5, we know u is Hölder continuity in $Q(a\tilde{\rho}^{p_\rho^+}, \tilde{\rho})$, so for every point in Q we can obtain such a cylinder as $Q(a\tilde{\rho}^{p_\rho^+}, \tilde{\rho})$, then by limited coverage theorem, u is local Hölder continuity in Q, thus we get theorem 1.

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