

## POSITIVE SOLUTIONS OF NONLINEAR FRACTIONAL BOUNDARY VALUE PROBLEMS WITH DIRICHLET BOUNDARY CONDITIONS

QINGKAI KONG<sup>†</sup> AND MIN WANG<sup>‡</sup>

ABSTRACT. In this paper, we study the existence and multiplicity of positive solutions of a class of nonlinear fractional boundary value problems with Dirichlet boundary conditions. By applying the fixed point theory on cones we establish a series of criteria for the existence of one, two, any arbitrary finite number, and an infinite number of positive solutions. A criterion for the nonexistence of positive solutions is also derived. Several examples are given for demonstration.

### 1. INTRODUCTION

In this paper, we study the boundary value problem (BVP) consisting of the fractional differential equation

$$-D_{0+}^{\alpha} u = w(t)f(u), \quad 0 < t < 1, \quad (1.1)$$

and the Dirichlet boundary condition (BC)

$$u(0) = 0, \quad u(1) = 0, \quad (1.2)$$

where  $1 < \alpha < 2$ ,  $w \in L[0, 1]$  is bounded with  $w(t) \geq 0$  on  $[0, 1]$  and  $w(t) > 0$  on  $[1/4, 3/4]$ ,  $f \in C(\mathbb{R}_+, \mathbb{R}_+)$  with  $\mathbb{R}_+ = [0, \infty)$ , and  $D_{0+}^{\alpha}$  is the  $\alpha$ th Riemann-Liouville fractional derivative of  $h : [0, 1] \rightarrow \mathbb{R}$  defined as

$$D_{0+}^{\alpha} h(t) = \frac{1}{\Gamma(2-\alpha)} \frac{d^2}{dt^2} \int_0^t (t-s)^{1-\alpha} h(s) ds,$$

whenever the right-hand side is defined.

Fractional differential equations have attracted extensive attention as they can be applied in various fields of science and engineering. Many phenomena in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc., can be modeled as fractional differential equations. We refer the reader to [10, 19] and references therein for the detail.

Due to the needs in applications, people have special interests in the existence of positive solutions of BVPs. In the study on integer order nonlinear BVPs, the fixed point theory on cones is a powerful tool in dealing with the existence of positive solutions. The main idea is to construct a cone in a Banach space and a

---

1991 *Mathematics Subject Classification.* primary 34B15; secondary 34B18.

*Key words and phrases.* Positive solutions, fractional calculus, fixed point index.

<sup>†</sup> This author is supported by the NNSF of China (No. 10971231).

<sup>‡</sup> Corresponding author, min-wang@utc.edu.

completely continuous operator defined on this cone based on the corresponding Green's function and then find fixed points of the operator. Many results have been obtained by this approach, see, for example, [5, 7, 8, 11, 14, 15, 16, 17, 18, 20, 21, 22]. This idea has also been used to the study of fractional BVPs, see [1, 2, 3, 6, 12, 13, 23, 25] and references therein for recent development.

In particular, Bai and Lü [3] proved that function  $G : [0, 1] \times [0, 1] \rightarrow \mathbb{R}_+$  defined by

$$G(t, s) = \begin{cases} \frac{[t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \\ \frac{[t(1-s)]^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1, \end{cases} \quad (1.3)$$

is the Green's function for the BVP consisting of the fractional differential equation

$$D_{0+}^{\alpha} u = 0$$

and BC (1.2). Based on it, they studied the BVP consisting of

$$-D_{0+}^{\alpha} u = g(t, u), \quad 0 < t < 1, \quad (1.4)$$

and BC (1.2) and obtained results on the existence of one or three positive solutions using the Krasnosel'skii fixed point theorem and the Leggett-Williams fixed point theorem. By further revealing the properties of the Green's function  $G(t, s)$ , Jiang and Yuan [12] obtained different conditions from those in [3] for BVP (1.4), (1.2) to have one positive solution and derived conditions for it to have two positive solutions.

To the best knowledge of the authors, there are no results yet on the existence of an arbitrary number of positive solutions of BVP (1.4), (1.2). This is due to the fact that although the Green's function  $G(t, s)$  is positive in the interior of its domain  $\{(t, s) : 0 \leq t, s \leq 1\}$ , it becomes zero on the boundary, and does not satisfy the following condition

$$\gamma \Phi(s) \leq G(t, s) \leq \Phi(s), \quad t \in [a, b] \subset [0, 1], \quad s \in [0, 1], \quad (1.5)$$

with a positive  $\Phi$  and  $\gamma \in (0, 1)$ . The lack of (1.5) prevents us from constructing a needed cone for the natural application of the fixed point theory as with positive Green's functions.

In this paper, by carefully manipulating the Green's function  $G(t, s)$ , we obtain a weaker condition similar to (1.5) so that we are able to apply the fixed point theory on cones to BVP (1.1), (1.2), where the functions  $w(t)$  and  $f(x)$  satisfy certain conditions. A series of new criteria for BVP (1.1), (1.2) to have one, two, an arbitrary number, and even an infinite number of positive solutions are obtained. Our results reveal the fact that under our conditions, the existence of one or more positive solutions is determined by the behavior

of  $f$  on certain intervals. Moreover, a theorem on the nonexistence of positive solutions is also derived.

This paper is organized as follows: After this introduction, our main results are stated in Section 2. Several examples are given in Section 3. All the proofs of the main results are given in the last section.

## 2. MAIN RESULTS

The following assumptions are needed in the presentations of our main results.

(H1)  $f(x) > 0$  on  $\mathbb{R}_+$  and there exists a function  $B : (0, 1) \rightarrow (0, \infty)$  such that for any  $r > 0$  and  $k \in (0, 1)$

$$\frac{\max_{x \in [0, r]} f(x)}{\min_{x \in [kr, r]} f(x)} \leq B(k).$$

(H2) there exists  $k^* \in (0, 1)$  such that

$$\psi(k^*) := k^*(1 + B(k^*)M) \leq \underline{\gamma}; \tag{2.1}$$

where  $M = \max\{M_1, M_2\}$  with

$$M_1 := \max_{s \in [1/4, 1/2]} \frac{G(s - 1/4, s - 1/4)w(s - 1/4)}{G(s, s)w(s)} \quad \text{and}$$

$$M_2 := \max_{s \in [1/2, 3/4]} \frac{G(s + 1/4, s + 1/4)w(s + 1/4)}{G(s, s)w(s)},$$

here  $G$  is defined by (1.3); and  $\underline{\gamma} = \min_{s \in [1/4, 3/4]} \gamma(s)$  with

$$\gamma(s) = \begin{cases} \frac{[\frac{3}{4}(1-s)]^{\alpha-1} - (\frac{3}{4}-s)^{\alpha-1}}{[s(1-s)]^{\alpha-1}}, & 0 < s \leq s_1, \\ \frac{1}{(4s)^{\alpha-1}}, & s_1 \leq s < 1, \end{cases} \tag{2.2}$$

here  $s_1$  is the unique solution of the equation

$$[\frac{3}{4}(1-s)]^{\alpha-1} - (\frac{3}{4}-s)^{\alpha-1} = \frac{1}{4^{\alpha-1}}(1-s)^{\alpha-1}.$$

From the proof of [3, Lemma 2.4] we know that  $1/4 < s_1 < 3/4$ .

**Remark 2.1.** The Assumptions (H1) and (H2) are satisfied by a variety of functions of  $f$  and  $w$ .

(i) We claim that Assumption (H1) is satisfied by the following functions

$$f_1(x) = bx^\theta, \quad b > 0, \quad \theta > 0,$$

$$f_2(x) = b_1x^{\theta_1} + b_2x^{\theta_2}, \quad b_1, b_2 > 0, \quad 0 < \theta_1 < 1 < \theta_2, \quad \text{and}$$

$$f_3(x) = bx(\sin(x) + \mu), \quad b > 0, \quad \mu > 1.$$

In fact, for any  $r > 0$  and  $k \in (0, 1)$ , since

$$\frac{\max_{x \in [0, r]} f_1(x)}{\min_{x \in [kr, r]} f_1(x)} = \frac{br^\theta}{b(kr)^\theta} = k^{-\theta},$$

$f_1$  satisfies (H1) with  $B(k) = k^{-\theta}$ ; since

$$\frac{\max_{x \in [0, r]} f_2(x)}{\min_{x \in [kr, r]} f_2(x)} = \frac{r^{\theta_2}(b_1 r^{\theta_1 - \theta_2} + b_2)}{(kr)^{\theta_2}(b_1 (kr)^{\theta_1 - \theta_2} + b_2)} < k^{-\theta_2},$$

$f_2$  satisfies (H1) with  $B(k) = k^{-\theta_2}$ ; and since

$$\frac{\max_{x \in [0, r]} f_3(x)}{\min_{x \in [kr, r]} f_3(x)} \leq \frac{(\mu + 1)br}{(\mu - 1)bkr} = \frac{\mu + 1}{(\mu - 1)k},$$

$f_3$  satisfies (H1) with  $B(k) = (\mu + 1)(\mu - 1)^{-1}k^{-1}$ .

- (ii) It is easy to see that for any function  $f$  satisfying Assumption (H1), Assumption (H2) holds when the function  $w$  is relatively small on  $[0, 1/4] \cup [3/4, 1]$  compared with its values on  $[1/4, 3/4]$ . In this case, the number  $M$  in (H2) can be sufficiently small and hence inequality (2.1) has a solution  $k^* \in (0, 1)$ .

However, the choice of the cut-off points  $1/4$  and  $3/4$  in the definitions of  $M_1$  and  $M_2$  is only for convenience in computations. Actually, all results in this paper can be extended to the case when the function  $w(t)$  is sufficiently small near the endpoints  $0$  and  $1$ .

- (iii) For some functions  $f$  satisfying (H1), (H2) holds for any  $w \in C([0, 1], \mathbb{R}_+)$ . For example, consider  $f_1(x) = bx^\theta$  with  $\theta \in (0, 1)$ . Since  $B(k) = k^{-\theta}$ ,  $\psi(k) = k + k^{1-\theta}M \rightarrow 0$  as  $k \rightarrow 0$ . Hence there exists  $k^* \in (0, 1)$  such that (2.1) holds.

In the following, we assume that the Assumptions (H1) and (H2) hold. Let

$$L = \underline{\gamma} \int_{1/4}^{3/4} G(s, s)w(s)ds, \quad (2.3)$$

$$U = (1 + B(k^*)M) \int_{1/4}^{3/4} G(s, s)w(s)ds. \quad (2.4)$$

First, we state our basic result on the existence of a positive solution.

**Theorem 2.1.** *Assume there exist  $r^*, r_* > 0$  with  $[k^*r^*, r^*] \cap [k^*r_*, r_*] = \emptyset$ , such that*

$$f(x) \leq U^{-1}r^* \text{ on } [k^*r^*, r^*], \quad (2.5)$$

and

$$f(x) \geq L^{-1}r_* \text{ on } [k^*r_*, r_*]. \quad (2.6)$$

Then BVP (1.1), (1.2) has at least one positive solution  $u$  with  $\min\{r^*, r_*\} \leq \|u\| \leq \max\{r^*, r_*\}$ .

In the sequel, we will use the following notation:

$$\begin{aligned} f_0 &= \liminf_{x \rightarrow 0} f(x)/x, & f_\infty &= \liminf_{x \rightarrow \infty} f(x)/x; \\ f^0 &= \limsup_{x \rightarrow 0} f(x)/x, & f^\infty &= \limsup_{x \rightarrow \infty} f(x)/x. \end{aligned}$$

The next three theorems are derived from Theorem 2.1 using  $f_0, f_\infty, f^0$ , and  $f^\infty$ .

**Theorem 2.2.** *BVP (1.1), (1.2) has at least one positive solution if either*

- (a)  $f^0 < U^{-1}$  and  $f_\infty > (k^*L)^{-1}$ ; or
- (b)  $f_0 > (k^*L)^{-1}$  and  $f^\infty < U^{-1}$ .

**Theorem 2.3.** *Assume there exists  $r^* > 0$  such that (2.5) holds.*

- (a) *If  $f_0 > (k^*L)^{-1}$ , then BVP (1.1), (1.2) has at least one positive solution  $u$  with  $\|u\| \leq r^*$ ;*
- (b) *if  $f_\infty > (k^*L)^{-1}$ , then BVP (1.1), (1.2) has at least one positive solution  $u$  with  $\|u\| \geq r^*$ .*

**Theorem 2.4.** *Assume there exists  $r_* > 0$  such that (2.6) holds.*

- (a) *If  $f^0 < U^{-1}$ , then BVP (1.1), (1.2) has at least one positive solution  $u$  with  $\|u\| \leq r_*$ ;*
- (b) *if  $f^\infty < U^{-1}$ , then BVP (1.1), (1.2) has at least one positive solution  $u$  with  $\|u\| \geq r_*$ .*

Combining Theorems 2.3 and 2.4 we obtain results on the existence of at least two positive solutions.

**Theorem 2.5.** *Assume either*

- (a)  $f_0 > (k^*L)^{-1}$  and  $f_\infty > (k^*L)^{-1}$ , and there exists  $r > 0$  such that

$$f(x) < U^{-1}r \text{ on } [k^*r, r]; \text{ or} \tag{2.7}$$

- (b)  $f^0 < U^{-1}$  and  $f^\infty < U^{-1}$ , and there exists  $r > 0$  such that

$$f(x) > L^{-1}r \text{ on } [k^*r, r]. \tag{2.8}$$

Then BVP (1.1), (1.2) has at least two positive solutions  $u_1$  and  $u_2$  with  $\|u_1\| < r < \|u_2\|$ .

Note that in Theorem 2.5, the inequalities in (2.7) and (2.8) are strict and hence are different from those in (2.5) and (2.6) in Theorem 2.1. This is to guarantee that the two solutions  $u_1$  and  $u_2$  are different. By applying Theorem 2.1 repeatedly, we can generalize the conclusion to obtain criteria for the existence of multiple positive solutions.

**Theorem 2.6.** Let  $\{r_i\}_{i=1}^N \subset \mathbb{R}$  such that  $0 < r_1 < r_2 < r_3 < \dots < r_N$  and  $[k^*r_i, r_i]$ ,  $i = 1, \dots, N$ , are mutually disjoint. Assume either

- (a)  $f$  satisfies (2.7) with  $r = r_i$  when  $i$  is odd, and satisfies (2.8) with  $r = r_i$  when  $i$  is even; or
- (b)  $f$  satisfies (2.7) with  $r = r_i$  when  $i$  is even, and satisfies (2.8) with  $r = r_i$  when  $i$  is odd.

Then BVP (1.1), (1.2) has at least  $N - 1$  positive solutions  $u_i$  with  $r_i < \|u_i\| < r_{i+1}$ ,  $i = 1, 2, \dots, N - 1$ .

**Theorem 2.7.** Let  $\{r_i\}_{i=1}^\infty \subset \mathbb{R}$  such that  $0 < r_1 < r_2 < r_3 < \dots$  and  $[k^*r_i, r_i]$ ,  $i = 1, \dots$ , are mutually disjoint. Assume either

- (a)  $f$  satisfies (2.5) with  $r_* = r_i$  when  $i$  is odd, and satisfies (2.6) with  $r^* = r_i$  when  $i$  is even; or
- (b)  $f$  satisfies (2.5) with  $r_* = r_i$  when  $i$  is even, and satisfies (2.6) with  $r^* = r_i$  when  $i$  is odd.

Then BVP (1.1), (1.2) has an infinite number of positive solutions.

The following is an immediate consequence of Theorem 2.7.

**Corollary 2.1.** Let  $\{r_i\}_{i=1}^\infty \subset \mathbb{R}$  such that  $0 < r_1 < r_2 < r_3 < \dots$  and  $[k^*r_i, r_i]$ ,  $i = 1, \dots$ , are mutually disjoint. Let  $E_1 = \cup_{i=1}^\infty [k^*r_{2i-1}, r_{2i-1}]$  and  $E_2 = \cup_{i=1}^\infty [k^*r_{2i}, r_{2i}]$ . Assume

$$\limsup_{E_1 \ni x \rightarrow \infty} \frac{f(x)}{x} < U^{-1} \quad \text{and} \quad \liminf_{E_2 \ni x \rightarrow \infty} \frac{f(x)}{x} > (k^*L)^{-1}.$$

Then BVP (1.1), (1.2) has an infinite number of positive solutions.

Our last theorem is about the nonexistence of positive solutions of BVP (1.1), (1.2).

**Theorem 2.8.** BVP (1.1), (1.2) has no positive solutions if  $f(x)/x < U^{-1}$  on  $(0, \infty)$ .

### 3. EXAMPLES

**Example 1.** Let  $f(x) = x^\theta$  with  $\theta \in (0, 1)$ . By Remark 2.1, Assumptions (H1) and (H2) are satisfied for any  $w \in C([0, 1], \mathbb{R}_+)$ . It is easy to see that  $f_0 = \infty$  and  $f^\infty = 0$ . By Theorem 2.2 (b), BVP (1.1), (1.2) has at least one positive solution.

For easy computations, in Examples 2-4, we let  $\alpha = 3/2$  and

$$w(t) = \begin{cases} c, & 0 \leq t \leq 1/4, \\ 1/G(t, t), & 1/4 < t < 3/4, \\ c, & 3/4 \leq t \leq 1, \end{cases}$$

where  $G(t, s)$  is given in (1.3) and  $c > 0$  is a constant. By some computations, we know that  $\underline{\gamma} = \sqrt{3} - 2\sqrt{6}/3$  and  $M = \sqrt{3}c/(4\Gamma(3/2))$ .

**Example 2.** Let  $f(x) = x^\theta$  with  $\theta > 1$ . By a simple calculation we see that Assumptions (H1) and (H2) are satisfied when

$$c \leq (4 - 8\sqrt{2}/3)\Gamma(3/2)((\theta - 1)^{1/\theta} + (\theta - 1)^{\frac{1-\theta}{\theta}})^{-\theta}.$$

In this case, since  $f^0 = 0$  and  $f_\infty = \infty$ , by Theorem 2.2 (a), BVP (1.1), (1.2) has at least one positive solution.

**Example 3.** Let  $f(x) = \lambda(x^{\theta_1} + x^{\theta_2})$ , where  $0 < \theta_1 < 1 < \theta_2 < \infty$ . By a simple calculation we see that Assumptions (H1) and (H2) are satisfied when

$$c \leq (4 - 8\sqrt{2}/3)\Gamma(3/2)((\theta_2 - 1)^{1/\theta_2} + (\theta_2 - 1)^{\frac{1-\theta_2}{\theta_2}})^{-\theta_2}.$$

Furthermore,  $L = \sqrt{3}/2 - \sqrt{6}/3$  and  $U = 1/2 + \sqrt{3}c/(8\Gamma(3/2)k^{\theta_2})$ . In this case, we let  $r_1 = ((1 - \theta_1)/(\theta_2 - 1))^{1/(\theta_2 - \theta_1)}$ . We claim that BVP (1.1), (1.2) has

- (a) at least one positive solution if  $\lambda = (r_1^{\theta_1} + r_1^{\theta_2})^{-1} U^{-1}$ ;
- (b) at least two positive solutions if  $\lambda \in (0, (r_1^{\theta_1} + r_1^{\theta_2})^{-1} U^{-1})$ .

We note that  $f_0 = f_\infty = \infty$ ,  $f$  is strictly increasing, and  $r_1$  is the minimum point of  $f(x)/x$  on  $(0, \infty)$ . When  $\lambda = (r_1^{\theta_1} + r_1^{\theta_2})^{-1} U^{-1}$ ,  $f(x) \leq U^{-1}r_1$  on  $[k^*r_1, r_1]$ . By Theorem 2.3 (a), BVP (1.1), (1.2) has a positive solution  $u_1$  with  $\|u_1\| \leq r_1$ . Similarly, by Theorem 2.3 (b), BVP (1.1), (1.2) has a positive solution  $u_2$  with  $\|u_2\| \geq r_1$ . However,  $u_1$  and  $u_2$  may be the same when  $\|u_1\| = \|u_2\| = r_1$ .

Part (b) follows similarly from Theorem 2.5 (a).

**Example 4.** Let  $L$  and  $U$  be defined by (2.3) and (2.4), and

$$f(x) = \begin{cases} (2 + \underline{\gamma})x(\sin(m \ln x) + \mu)/(2\underline{\gamma}L), & x > 0, \\ 0, & x = 0, \end{cases}$$

where

$$1 < \mu < \min\{4/(2 + \underline{\gamma}), 1 + 2\underline{\gamma}L/((2 + \underline{\gamma})U)\}$$

and

$$0 < m < (\pi - 2 \sin^{-1} \delta)/\ln(2/\underline{\gamma})$$

with

$$\delta = \max\{|\delta_1|, |\delta_2|\},$$

here  $\delta_1 = (4 - 2\mu - \mu\underline{\gamma})/(2 + \underline{\gamma})$  and  $\delta_2 = 2\underline{\gamma}L/((2 + \underline{\gamma})U) - \mu$ . Note that  $\delta_2 < 0$ . By a simple calculation we see that Assumptions (H1) and (H2) are satisfied for  $k^* = \underline{\gamma}/2$  when  $c < \underline{\gamma}16^{\alpha-1}(\mu - 1)\Gamma(\alpha)3^{1-\alpha}/(2(\mu + 1))$ . In this case, we claim that BVP (1.1), (1.2) has an infinite number of positive solutions.

In fact, it is easy to verify that  $\delta \in (0, 1)$ . For  $i \in \mathbb{N}$ , let

$$\begin{aligned}\xi_i &= \exp(m^{-1}(\sin^{-1} \delta + (i - 1)\pi)), \\ \eta_i &= \exp(m^{-1}(i\pi - \sin^{-1} \delta)).\end{aligned}$$

Then

$$\frac{\eta_i}{\xi_i} = \exp(m^{-1}(\pi - 2 \sin^{-1} \delta)) > \exp(\ln(2/\underline{\gamma})) = k^{*-1}.$$

Hence  $\xi_i < k^*\eta_i$ . Then for any  $x \in [k^*\eta_i, \eta_i]$ ,  $x \in [\xi_i, \eta_i]$ .

When  $i$  is odd, for any  $x \in [k^*\eta_i, \eta_i]$

$$\sin(m \ln x) \geq \sin(m \ln \xi_i) = \sin(\sin^{-1} \delta) = \delta > \delta_1.$$

So

$$f(x) \geq (2 + \underline{\gamma})k^*\eta_i(\delta_1 + \mu)/(2\underline{\gamma}L) = L^{-1}\eta_i.$$

When  $i$  is even, for any  $x \in [k^*\eta_i, \eta_i]$

$$\sin(m \ln x) \leq \sin(m \ln \eta_i) = \sin(-\sin^{-1} \delta) = -\delta \leq \delta_2.$$

So

$$f(x) \leq (2 + \underline{\gamma})\eta_i(\delta_2 + \mu)/(2\underline{\gamma}L) = U^{-1}\eta_i.$$

Therefore, by Theorem 2.7 (b), BVP (1.1), (1.2) has an infinite number of positive solutions.

#### 4. PROOFS

Let  $X$  be a Banach space and  $K \subset X$  a cone in  $X$ . For  $r > 0$ , define

$$K_r = \{u \in K \mid \|u\| < r\} \quad \text{and} \quad \partial K_r = \{u \in K \mid \|u\| = r\}.$$

For an operator  $T : K_r \rightarrow K$ , let  $i(T, K_r, K)$  be the fixed point index of  $T$  on  $K_r$  with respect to  $K$ . We will use the following well-known lemmas on fixed point index to prove our main results. For the detail, see [4, 9] and [24, page 529, A2, A3].

**Lemma 4.1.** *Assume that for  $r > 0$ ,  $T : K_r \rightarrow K$  is a completely continuous operator such that  $Tu \neq u$  for  $u \in \partial K_r$ .*

(a) *If  $\|Tu\| \geq \|u\|$  for  $u \in \partial K_r$ , then  $i(T, K_r, K) = 0$ .*

(b) *If  $\|Tu\| \leq \|u\|$  for  $u \in \partial K_r$ , then  $i(T, K_r, K) = 1$ .*

**Lemma 4.2.** *Let  $0 < r_1 < r_2$  satisfy*

$$i(T, K_{r_1}, K) = 0 \quad \text{and} \quad i(T, K_{r_2}, K) = 1;$$

*or*

$$i(T, K_{r_1}, K) = 1 \quad \text{and} \quad i(T, K_{r_2}, K) = 0.$$

*Then  $T$  has a fixed point in  $K_{r_2} \setminus \overline{K_{r_1}}$ .*



We will also use the following property of the Green's function  $G(t, s)$  for BVP (1.1), (1.2), see [3, Lemma 2.4] for the proof.

**Lemma 4.3.** *Let  $G(t, s)$  be defined by (1.3) and  $\underline{\gamma}$  defined in (H2). Then*

- (a)  $G(t, s) > 0$  on  $(0, 1) \times (0, 1)$ ,
- (b)  $\max_{t \in [0, 1]} G(t, s) = G(s, s)$  for  $0 < s < 1$  and  $\min_{1/4 \leq t \leq 3/4} G(t, s) \geq \underline{\gamma} G(s, s)$  for  $1/4 \leq s \leq 3/4$ .

Let  $X = C[0, 1]$  and  $\|u\| = \max_{t \in [0, 1]} |u(t)|$  for any  $u \in X$ . Then  $(X, \|\cdot\|)$  is a Banach space. Define

$$K = \{u \in X \mid u(t) \geq 0 \text{ on } [0, 1] \text{ and } \min_{t \in [1/4, 3/4]} u(t) \geq k^* \|u\|\}, \quad (4.1)$$

where  $k^*$  is defined in (H2), and  $T : K \rightarrow X$  as

$$(Tu)(t) = \int_0^1 G(t, s)w(s)f(u(s))ds. \quad (4.2)$$

Clearly,  $u(t)$  is a solution of BVP (1.1), (1.2) if and only if  $u \in K$  is a fixed point of  $T$ , and  $\|Tu\| \leq \int_0^1 G(s, s)w(s)f(u(s))ds$ .

**Lemma 4.4.** *Assume (H1), (H2) hold. Then  $TK \subset K$  and  $T$  is a completely continuous operator.*

*Proof.* For any  $u \in K$  with  $\|u\| = r$ ,  $k^*r \leq u(t) \leq r$  on  $[1/4, 3/4]$ . By (4.2) and Lemma 4.3, for  $t \in [1/4, 3/4]$

$$\begin{aligned} (Tu)(t) &= \int_0^1 G(t, s)w(s)f(u(s))ds \\ &\geq \underline{\gamma} \int_{1/4}^{3/4} G(s, s)w(s)f(u(s))ds. \end{aligned} \quad (4.3)$$

By (H1),  $\max_{u \in [0, r]} f(u) \leq B(k^*) \min_{u \in [k^*r, r]} f(u)$ . Then by (H2)

$$\begin{aligned} &\int_0^{1/4} G(s, s)w(s)f(u(s))ds \leq \max_{u \in [0, r]} f(u) \int_0^{1/4} G(s, s)w(s)ds \\ &\leq B(k^*) \min_{u \in [k^*r, r]} f(u) \int_0^{1/4} G(s, s)w(s)ds \\ &\leq B(k^*)M_1 \min_{u \in [k^*r, r]} f(u) \int_{1/4}^{1/2} G(s, s)w(s)ds \\ &\leq B(k^*)M \int_{1/4}^{1/2} G(s, s)w(s)f(u(s))ds. \end{aligned} \quad (4.4)$$

Similarly,

$$\int_{3/4}^1 G(s, s)w(s)f(u(s))ds \leq B(k^*)M \int_{1/2}^{3/4} G(s, s)w(s)f(u(s))ds. \quad (4.5)$$

Therefore, by (4.4) and (4.5)

$$\begin{aligned} & \int_0^1 G(s, s)w(s)f(u(s))ds \\ &= \left( \int_0^{1/4} + \int_{1/4}^{1/2} + \int_{1/2}^{3/4} + \int_{3/4}^1 \right) G(s, s)w(s)f(u(s))ds \\ &\leq (1 + B(k^*)M) \int_{1/4}^{3/4} G(s, s)w(s)f(u(s))ds, \end{aligned} \quad (4.6)$$

which follows that

$$\int_{1/4}^{3/4} G(s, s)w(s)f(u(s))ds \geq \frac{1}{1 + B(k^*)M} \int_0^1 G(s, s)w(s)f(u(s))ds.$$

Hence by (H2) and (4.3)

$$\begin{aligned} (Tu)(t) &\geq \frac{\underline{\gamma}}{1 + B(k^*)M} \int_0^1 G(s, s)w(s)f(u(s))ds \\ &\geq \frac{\underline{\gamma}}{1 + B(k^*)M} \|Tu\| \geq k^* \|Tu\| \text{ on } [1/4, 3/4]. \end{aligned}$$

It is clear that  $(Tu)(t) \geq 0$  on  $[0, 1]$ . Therefore,  $TK \subset K$ . By Arzela-Ascoli Theorem,  $T$  is completely continuous.  $\square$

*Proof of Theorem 2.1.* Without loss of the generality, we assume  $r_* < r^*$ . For any  $u \in \partial K_{r^*}$ ,  $\|u\| = r^*$  and  $k^*r^* \leq u(t) \leq r^*$  on  $[1/4, 3/4]$ . By (2.4), (2.5), (4.2), (4.6), and Lemma 4.3 (a)

$$\begin{aligned} \|Tu\| &= \max_{t \in [0, 1]} \int_0^1 G(t, s)w(s)f(u(s))ds \\ &\leq \int_0^1 G(s, s)w(s)f(u(s))ds \\ &\leq U^{-1}r^* (1 + B(k^*)M) \int_{1/4}^{3/4} G(s, s)w(s)ds = r^*. \end{aligned}$$

By Lemma 4.1 (a),  $i(T, K_{r^*}, K) = 1$ .

For any  $u \in \partial K_{r_*}$ ,  $\|u\| = r_*$  and  $k^*r_* \leq u(t) \leq r_*$  on  $[1/4, 3/4]$ . By (2.3), (2.6), (4.2), and Lemma 4.3 (b)

$$\begin{aligned} \|Tu\| &\geq (Tu)(1/2) = \int_0^1 G(1/2, s)w(s)f(u(s))ds \\ &\geq \underline{\gamma} \int_{1/4}^{3/4} G(s, s)w(s)f(u(s))ds \\ &\geq L^{-1}r_*\underline{\gamma} \int_{1/4}^{3/4} G(s, s)w(s)ds = r_*. \end{aligned}$$

By Lemma 4.1 (b),  $i(T, K_{r_*}, K) = 0$ .

By Lemma 4.2,  $T$  has a fixed point  $u$  in  $K$  with  $r_* \leq \|u\| \leq r^*$ . Hence BVP (1.1), (1.2) has at least one positive solution  $u(t)$ .  $\square$

*Proof of Theorem 2.2.* (a) If  $f^0 < U^{-1}$ , there exists a sufficiently small  $r^* > 0$  such that

$$f(x) < U^{-1}x \leq U^{-1}r^* \text{ on } [k^*r^*, r^*],$$

i.e., (2.5) holds.

If  $f_\infty > (k^*L)^{-1}$ , there exists  $\hat{r} > r^*$  such that

$$f(x) > (k^*L)^{-1}x \text{ on } [\hat{r}, \infty).$$

Then for any  $r_*$  with  $k^*r_* \geq \hat{r}$

$$f(x) > (k^*L)^{-1}x \geq L^{-1}r_* \text{ for all } x \in [k^*r_*, r_*],$$

i.e., (2.6) holds. Then the conclusion follows from Theorem 2.1.

(b) The proof is similar to Part (a) and hence is omitted.  $\square$

The proofs of Theorems 2.3 and 2.4 are in the same way and hence are omitted.

*Proof of Theorem 2.5.* (a) If there exists  $r > 0$  such that (2.7) holds, then by the continuity of  $f(x)/x$  on  $(0, \infty)$ , there exist  $r_1, r_2 > 0$  such that  $r_1 < r < r_2$  and  $f(x) < U^{-1}r_i$  on  $[k^*r_i, r_i]$ ,  $i = 1, 2$ . By Theorem 2.3 (a) and (b), BVP (1.1), (1.2) has two positive solutions  $u_1$  and  $u_2$  satisfying  $\|u_1\| \leq r_1$  and  $\|u_2\| \geq r_2$ .

Similarly, Case (b) follows from Theorem 2.4.  $\square$

The proofs of Theorems 2.6 and 2.7 are in the same way and are hence omitted.

*Proof of Corollary 2.1.* From the assumption we see that for sufficiently large  $i$

$$\frac{f(x)}{x} < U^{-1} \text{ on } [k^*r_{2i-1}, r_{2i-1}]$$

and

$$\frac{f(x)}{x} > (k^*L)^{-1} \text{ on } [k^*r_{2i}, r_{2i}].$$

This shows that for sufficiently large  $i$

$$f(x) < U^{-1}x \leq U^{-1}r_{2i-1} \text{ on } [k^*r_{2i-1}, r_{2i-1}]$$

and

$$f(x) > (k^*L)^{-1}x \geq L^{-1}r_{2i} \text{ on } [k^*r_{2i}, r_{2i}].$$

Therefore, the conclusion follows from Theorem 2.7.  $\square$

*Proof of Theorem 2.8.* Assume BVP (1.1), (1.2) has a positive solution  $u$  with  $\|u\| = r$  for some  $r > 0$ . Then  $u$  is a fixed point of the operator  $T$  defined by (4.2). For any  $t \in [0, 1]$ , by (4.6)

$$\begin{aligned} u(t) &= (Tu)(t) = \int_0^1 G(t, s)w(s)f(u(s))ds \\ &\leq (1 + B(k^*)M) \int_{1/4}^{3/4} G(s, s)w(s)f(u(s))ds \\ &< U^{-1}r(1 + B(k^*)M) \int_{1/4}^{3/4} G(s, s)w(s)ds = r, \end{aligned}$$

which contradicts  $\|u\| = r$ . Therefore, BVP (1.1), (1.2) has no positive solutions.  $\square$

#### ACKNOWLEDGMENTS

The authors are grateful to the referee for his/her careful checking and valuable suggestions for changes which help us to improve the quality of the paper.

#### REFERENCES

- [1] R. P. Agarwal, D. O'Regan, and S. Staněk, Positive solutions for Dirichlet problems of singular nonlinear fractional differential equations, *J. Math. Anal. Appl.* **371** (2010), 57–68.
- [2] Z. Bai, On positive solutions of a nonlocal fractional boundary value problem, *Nonlinear Anal.* **72** (2010), 916–924.
- [3] Z. Bai and H. Lü, Positive solutions for boundary value problem of nonlinear fractional differential equation, *J. Math. Anal. Appl.* **311** (2005), 495–505.
- [4] K. Deimling, *Nonlinear Functional Analysis* Springer-Verlag, New York, 1985.
- [5] L. Erbe, Eigenvalue criteria for existence of positive solutions to nonlinear boundary value problems, *Math. Computer Modelling* **32** (2000), 529–539.
- [6] C. S. Goodrich, Existence of a positive solution to a class of fractional differential equations, *Appl. Math. Lett.* **23** (2010), 1050–1055.
- [7] J. R. Graef and L. Kong, Existence results for nonlinear periodic boundary value problems, *Proc. Edinburgh Math. Soc.* **52** (2009), 79–95.

- [8] J. R. Graef, L. Kong, and H. Wang, Existence, multiplicity, and dependence on a parameter for a periodic boundary value problem, *J. Differential Equations* **245** (2008), 1185–1197.
- [9] D. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, New York, 1988.
- [10] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.
- [11] D. Jiang, J. Chua, D. O'Regan, and R. P. Agarwal, Multiple positive solutions to superlinear periodic boundary value problems with repulsive singular forces, *J. Math. Anal. Appl.* **286** (2003), 563–576.
- [12] D. Jiang and C. Yuan, The positive properties of the Green function for Dirichlet-type boundary value problems of nonlinear fractional differential equations and its application, *Nonlinear Anal.* **72** (2010), 710–719.
- [13] E. R. Kaufmann and E. Mboumi, Positive solutions of a boundary value problem for a nonlinear fractional differential equation, *Electron. J. Qual. Theory Differ. Equ.* 2008, No. 3, 11 pp.
- [14] L. Kong and Q. Kong, Positive solutions of higher-order boundary value problems, *Proc. Edinburgh Math. Soc.* **48** (2005), 445–464.
- [15] Q. Kong and M. Wang, Positive solutions of even order periodic boundary value problems, *Rocky Mountain J. Math.* **41** (2011), 1907–1931.
- [16] Q. Kong and M. Wang, Positive solutions of even order system periodic boundary value problems, *Nonlinear Anal.* **72** (2010), 1778–1791.
- [17] K. Q. Lan and J. R. L. Webb, Positive solutions of semilinear differential equations with singularities, *J. Differential Equations* **148** (1998), 407–421.
- [18] D. O'Regan and H. Wang, Positive periodic solutions of systems of second order ordinary differential equations, *Positivity* **10** (2006), 285–298.
- [19] V. E. Tarasov, *Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles, Fields and Media*, Springer-Verlag, New York, 2011.
- [20] P. J. Torres, Existence of one-signed periodic solutions of some second-order differential equations via a Krasnoselskii fixed point theorem, *J. Differential Equations* **190** (2003), 643–662.
- [21] J. R. L. Webb and G. Infante, Positive solutions of nonlocal boundary value problems: a unified approach, *J. London Math. Soc. (2)* **74** (2006), 673–693.
- [22] J. R. L. Webb and G. Infante, Non-local boundary value problems of arbitrary order, *J. Lond. Math. Soc. (2)* **79** (2009), 238–258.
- [23] L. Yang and H. Chen, Unique positive solutions for fractional differential equation boundary value problems, *Appl. Math. Lett.* **23** (2010), 1095–1098.
- [24] E. Zeidler, *Nonlinear Functional Analysis and its Applications I: Fixed-Point Theorems*, Springer-Verlag, New York, 1986.
- [25] S. Zhang, Positive solutions to singular boundary value problem for nonlinear fractional differential equation, *Comput. Math. Appl.* **59** (2010), 1300–1309.

(Received September 27, 2011)

DEPARTMENT OF MATHEMATICAL SCIENCES, NORTHERN ILLINOIS UNIVERSITY, DEKALB, IL 60115, USA. EMAIL: KONG@MATH.NIU.EDU

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TENNESSEE AT CHATTANOOGA, CHATTANOOGA, TN 37403, USA. EMAIL: MIN-WANG@UTC.EDU