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Sufficient conditions for the exponential stability of delay difference equations with linear parts defined by permutable matrices

M. Medved'[*], L. Škripková
Faculty of Mathematics, Physics and Informatics,
Comenius University, Mlynská dolina,
842 48 Bratislava, Slovakia,
e-mail: milan.medved@fmph.uniba.sk,
lucia.skripkova@fmph.uniba.sk

Abstract

This paper deals with the stability problem of nonlinear delay difference equations with linear parts defined by permutable matrices. Several criteria for exponential stability of systems with different types of nonlinearities are proved. Finally, a stability result for a model of population dynamics is proved by applying one of them.

1 Introduction

This paper is concerned with the stability of the nonlinear delay difference equations. Throughout the paper we use for zero matrix notation Θ, I represents identity matrix with $\|I\| = 1$. For given integers s, q, such that $s < q$

[*]Corresponding author

we denote $\mathbb{Z}_s^q := \{s, s+1, \dots, q\}$ the set of integers. First let us recall the result from the paper [2] concerning the representation of a solution of a linear delay difference system which will be starting point in our further study of the stability problems for nonlinear perturbation of this system.

Theorem 1.1. *Let $\varphi : \mathbb{Z}_{-m}^0 \to \mathbb{R}^n$ be a given function, A, B are $n \times n$ constant permutable matrices, i.e. $AB = BA$ with $\det A \neq 0$. Then the trivial solution of the initial-value problem*

$$x(k+1) = Ax(k) + Bx(k-m) + f(k), \quad k \in \mathbb{Z}_0^\infty,$$

$$x(k) = \varphi(k), \quad k \in \mathbb{Z}_{-m}^0$$

has the form

$$x(k) = A^{k+m} \mathrm{e}_m^{B_1 k} \varphi(-m) + \sum_{j=-m+1}^{0} A^{k-j} \mathrm{e}_m^{B_1(k-m-j)} \left[\varphi(j) - A\varphi(j-1)\right]$$
$$+ \sum_{j=1}^{k} A^{k-j} \mathrm{e}_m^{B_1(k-m-j)} f(j-1), \quad k \in \mathbb{Z}_{-m}^\infty,$$

where $B_1 = A^{-1} B A^{-m}$.

This theorem is a discrete version of [6, Th. 2.1]. Function e_m^{Bk} is called the discrete delayed-matrix exponential [3] and is given by

$$\mathrm{e}_m^{Bk} = \begin{cases} \Theta, & k < -m, \\ I + \sum_{j=1}^{l} B^j \binom{k-(j-1)m}{j}, & k \in \mathbb{Z}_{(l-1)(m+1)+1}^{l(m+1)}, l \in \mathbb{Z}_0^\infty. \end{cases}$$

This matrix function was used to construct the general solution of planar linear discrete systems with weak delay in [5]. Problems of controllability of linear discrete systems with constant coefficients and pure delay are considered in [4].

Applying Theorem 1.1 to the initial-value problem

$$x(k+1) = Ax(k) + Bx(k-m) + f(x(k), x(k-m)), \quad k \in \mathbb{Z}_0^\infty, \qquad (1)$$

$$x(k) = \varphi(k), \quad k \in \mathbb{Z}_{-m}^{0}, \tag{2}$$

where $m \geq 1$ is a constant delay, $\varphi : \mathbb{Z}_{-m}^{0} \to \mathbb{R}^n$ a given initial function, $f(x(k), x(k-m))$ a given function and constant $n \times n$ matrices A, B are permutable, we obtain the following representation of its solution:

$$\begin{aligned} x(k) = A^{k+m} e_m^{B_1 k} \varphi(-m) + \sum_{j=-m+1}^{0} A^{k-j} e_m^{B_1(k-m-j)} \left[\varphi(j) - A\varphi(j-1)\right] \\ + \sum_{j=1}^{k} A^{k-j} e_m^{B_1(k-m-j)} f(x(j-1), x(j-m-1)), \quad k \in \mathbb{Z}_{-m}^{\infty}. \end{aligned} \tag{3}$$

Our aim is to find some sufficient conditions for the exponential stability of the trivial solution of a nonlinear delay difference equation with different types of nonlinearities in the sense of the following definition.

Definition 1.1. *Let $m > 1$, and $\varphi : \mathbb{Z}_{-m}^{0} \to \mathbb{R}^n$ be a given function. The solution $x_\varphi(k)$ of equation* (1) *satisfying initial condition* (2) *is called exponentially stable if there exist positive constants c_1, c_2, δ depending on A, B_1 and $\|\varphi\| = \max_{k \in \mathbb{Z}_{-m}^{0}} \|\varphi(k)\|$, such that*

$$\|x_\varphi(k) - x_\psi(k)\| < c_1 e^{-c_2 k}, \qquad k \geq 0$$

for any solution $x_\psi(k)$ of equation (1) *satisfying initial condition*

$$x_\psi(k) = \psi(k), \quad k \in \mathbb{Z}_{-m}^{0}$$

with $\|\varphi - \psi\| < \delta$.

The following lemma will be helpful in our estimations.

Lemma 1.1. *Let $m \geq 1$ be a constant delay. Then for any $k \in \mathbb{Z}$ the following inequality holds true*

$$\left\| e_m^{Bk} \right\| \leq e^{\|B\|(k+m)}. \tag{4}$$

Proof. Using the definition of delayed matrix exponential one can easily prove the statement. □

2 Systems with a nonlinearity independent of delay

Now we state sufficient conditions for the exponential stability of the trivial solution of the nonlinear equation

$$x(k+1) = Ax(k) + Bx(k-m) + f(x(k)), \quad k \in \mathbb{Z}_0^\infty. \tag{5}$$

Some analogical results for the delay differential equations are proved in the paper [7].

Definition 2.1. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ and $l > 0$. We say that $f(x) = o(\|x\|^l)$ if*

$$\lim_{\|x\|\to 0} \frac{\|f(x)\|}{\|x\|^l} = 0.$$

Theorem 2.1. *Let A, B be $n \times n$ permutable matrices, i.e. $AB = BA$, $B_1 = A^{-1}BA^{-m}$ and $\|A\| \mathrm{e}^{\|B_1\|} < 1$. If $f(x) = o(\|x\|)$ then the trivial solution of the equation (5) is exponentially stable.*

Proof. From Lemma 1.1 we can estimate the solution of the equation (5) as follows

$$\|x(k)\| \le \|A\|^{k+m} \mathrm{e}^{\|B_1\|(k+m)} \|\varphi(-m)\| + \sum_{j=-m+1}^{0} \|A\|^{k-j} \mathrm{e}^{\|B_1\|(k-j)} \|\varphi(j) - A\varphi(j-1)\| + \sum_{j=1}^{k} \|A\|^{k-j} \mathrm{e}^{\|B_1\|(k-j)} \|f(x(j-1))\|.$$

If $\|x(j)\| < \delta$ for each $j \in \mathbb{Z}_0^{k-1}$ then we get

$$\|x(k)\| \le \|A\|^{k+m} \mathrm{e}^{\|B_1\|(k+m)} \|\varphi(-m)\| + \sum_{j=-m+1}^{0} \|A\|^{k-j} \mathrm{e}^{\|B_1\|(k-j)} \|\varphi(j) - A\varphi(j-1)\| + P \sum_{j=1}^{k} \|A\|^{k-j} \mathrm{e}^{\|B_1\|(k-j)} \|x(j-1)\|.$$

Denoting

$$C := \|A\| \, \mathrm{e}^{\|B_1\|}, \quad u(k) := \|x(k)\| \, C^{-k}, \tag{6}$$

$$M := \|A\|^m \, \mathrm{e}^{\|B_1\| m} \, \|\varphi(-m)\| + \sum_{j=-m+1}^{0} \|A\|^{-j} \, \mathrm{e}^{-\|B_1\| j} \, \|\varphi(j) - A\varphi(j-1)\|, \tag{7}$$

we obtain

$$u(k) \leq M + P \sum_{j=1}^{k} C^{-1} u(j-1).$$

Now by applying the discrete version of the Gronwall's inequality (cf. [1]) we obtain

$$u(k) \leq M \mathrm{e}^{P \sum_{j=1}^{k} C^{-1}} = M \mathrm{e}^{P C^{-1} k}$$

and this yields the inequality

$$\|x(k)\| \leq M \mathrm{e}^{P C^{-1} k} \, \|A\|^k \, \mathrm{e}^{\|B_1\| k} = M \mathrm{e}^{(P C^{-1} + \|B_1\| + \ln \|A\|) k}.$$

Now if we take $\max\{\|\varphi(0)\|, M\} < \delta$ and $PC^{-1} < -\|B_1\| - \ln \|A\|$, then $\|x(k)\| \leq M \mathrm{e}^{-\eta k}$, where $\eta := -PC^{-1} - \|B_1\| - \ln \|A\| > 0$. Thus the trivial solution of (1) is exponentially stable. □

Theorem 2.2. *Let the matrices A, B and B_1 be as in Theorem 2.1. If $f(x) = o(\|x\|^\alpha)$ for $\alpha > 1$ then the trivial solution of (5) is exponentially stable.*

Proof. Similarly to the proof of the previous theorem we derive the following estimate

$$u(k) \leq M + P \sum_{j=1}^{k} C^{-\alpha} u^\alpha(j-1)$$

for $k \in \mathbb{Z}_0^\infty$, where we have used the notation of (6),(7) and the assumption $\|x(j)\| < \delta$ for each $j \in \mathbb{Z}_0^{k-1}$. Now let

$$c := \max\{M, \|\varphi(0)\|\}, \quad \lambda(j) := P C^{-\alpha} C^{(\alpha-1)j}, \quad \omega := u^\alpha.$$

Then

$$u(k) \le c + \sum_{j=1}^{k} \lambda(j)\omega(u(j-1)).$$

It is easy to see that

$$\|\lambda\| = \sum_{j=1}^{\infty} \lambda(j) = \sum_{j=1}^{\infty} PC^{-\alpha}C^{(\alpha-1)j} = PC^{-\alpha}\frac{C^{\alpha-1}}{1-C^{\alpha-1}}.$$

Consequently, applying the discrete version of the Bihari's theorem (cf. [9]) we obtain

$$u(k) \le W^{-1}\left[W(c) + \sum_{j=1}^{k}\lambda(j)\right] \le W^{-1}\left[W(c) + \|\lambda\|\right], \tag{8}$$

where

$$W(\widetilde{u}) = \int_c^{\widetilde{u}} \frac{\mathrm{d}\sigma}{\omega(\sigma)}, \quad \widetilde{u} > 0.$$

Note that the expression $W^{-1}\left[W(c) + \|\lambda\|\right]$ is surely less than infinity. If K denotes the constant on the right-hand side of (8) and $\max\{K, c\} < \delta$, we get

$$\|x(k)\| \le K\mathrm{e}^{(\|B_1\| + \ln\|A\|)k}$$

for P sufficiently small. Since $\|B_1\| < -\ln\|A\|$, the trivial solution of the equation (5) is exponentially stable. □

3 Systems with a nonlinearity depending on delay

In this section, we consider the system

$$x(k+1) = Ax(k) + Bx(k-m) + f(x(k), x(k-m)), \quad k \in \mathbb{Z}_0^{\infty} \tag{9}$$

$$x(k) = \varphi(k), \quad k \in \mathbb{Z}_{-m}^{0}, \tag{10}$$

where $m \geq 1$ is a constant delay, $\varphi : \mathbb{Z}_{-m}^{0} \to \mathbb{R}^n$ an initial function and matrices A, B are permutable. Here we derive sufficient conditions for the exponential stability of the trivial solution of equation (9) with different types of a given function f.

Definition 3.1. *Let $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $l_1, \dots, l_k, m_1, \dots, m_r > 0$ for $k, r \in \mathbb{N}$. We say that $f(x, y) = o(\|x\|^{l_1} + \cdots + \|x\|^{l_k} + \|y\|^{m_1} + \cdots + \|y\|^{m_r})$ if*

$$\lim_{\substack{\|x\| \to 0 \\ \|y\| \to 0}} \frac{\|f(x, y)\|}{\|x\|^{l_1} + \cdots + \|x\|^{l_k} + \|y\|^{m_1} + \cdots + \|y\|^{m_r}} = 0.$$

Theorem 3.1. *Let the constant $n \times n$ matrices A, B be permutable, $B_1 = A^{-1}BA^{-m}$ and $\|A\| \, \mathrm{e}^{\|B_1\|} < 1$. If $f(x, y) = o(\|x\| + \|y\|)$ then the trivial solution of the equation (9) is exponentially stable.*

Proof. Suppose that $\|x(k)\| < \delta$ for $k \in \mathbb{Z}_{-m}^{\infty}$. Then using the notation (6) and (7) we obtain the estimate for the solution $x(k)$ of (3)

$$u(k) \leq M + P \sum_{j=1}^{k} C^{-1} u(j-1) + P \sum_{j=1}^{k} C^{-(m+1)} u(j-m-1),$$

Let us denote $c := \max\{M, \|\varphi\|\}$. Then

$$u(k) \leq c + P \sum_{j=1}^{k} C^{-1} u(j-1) + P \sum_{j=1}^{k} C^{-(m+1)} u(j-m-1).$$

Now denote $g(k)$ the right-hand side of the above inequality. Note that it is a nondecreasing function. Apparently, $u(k) \leq g(k)$ and from the property of the maximum we have

$$\begin{aligned} u(j-m-1) &\leq \max_{s \in \mathbb{Z}_1^j} u(s-m-1) \\ &\leq \max_{s \in \mathbb{Z}_1^m} u(s-m-1) + \max_{s \in \mathbb{Z}_{m+1}^j} u(s-m-1) \\ &\leq \|\varphi\| + g(j-1) \leq 2g(j-1). \end{aligned} \tag{11}$$

Therefore

$$g(k) \leq c + P\sum_{j=1}^{k} C^{-1} g(j-1) + 2P\sum_{j=1}^{k} C^{-(m+1)} g(j-1)$$

$$\leq c + \left(PC^{-1} + 2PC^{-(m+1)}\right)\sum_{j=1}^{k} g(j-1) \leq c + 3PC^{-(m+1)}\sum_{j=1}^{k} g(j-1).$$

Using the Gronwall's inequality we obtain

$$g(k) \leq c\mathrm{e}^{3PC^{-(m+1)}k}.$$

Consequently, for the solution $x(k)$ we have the estimate

$$\|x(k)\| \leq c\mathrm{e}^{\left(\ln C + 3PC^{-(m+1)}\right)k}.$$

One can see that the trivial solution of the equation (9) is exponentially stable whenever $c < \delta$ and

$$P < \frac{-\ln C}{3C^{-(m+1)}}.$$

□

Theorem 3.2. *Let* $\alpha_1 > 1$, $\alpha_2 > 1$ *and matrices* A, B *and* B_1 *be as in the Theorem 3.1. Assume that* $\|A\|\,\mathrm{e}^{\|B_1\|} < 1$ *and* $f(x,y) = o(\|x\|^{\alpha_1} + \|y\|^{\alpha_2})$. *Then the trivial solution of the equation* (9) *is exponentially stable.*

Proof. Similarly to the proof of the previous theorem we estimate the solution of the equation (9). Supposing $\|x(k)\| < \delta$ for $k \in \mathbb{Z}_{-m}^{\infty}$ and using notation (6) and (7) we get

$$u(k) \leq c + \sum_{j=1}^{k} \lambda_1(j) u^{\alpha_1}(j-1) + \sum_{j=1}^{k} \widetilde{\lambda}_2(j) u^{\alpha_2}(j-m-1), \quad k \in \mathbb{Z}_{-m}^{\infty}, \quad (12)$$

where

$$\lambda_1(j) := PC^{-\alpha_1} C^{(\alpha_1 - 1)j}, \quad \widetilde{\lambda}_2(j) := PC^{-(m+1)\alpha_2} C^{(\alpha_2 - 1)j},$$
$$c := \max\{M, \|\varphi\|\}.$$

Without any loss of generality one can assume that $\alpha_1 \leq \alpha_2$. Then $\omega = \frac{\omega_2}{\omega_1}$ marks a nondecreasing function, where $\omega_i(u) := u^{\alpha_i}$ for $i = 1, 2$. If we denote the right-hand side of the inequality (12) by $g(k)$ we obtain

$$u(k) \leq g(k) \leq c + \sum_{j=1}^{k} \lambda_1(j)\omega_1(g(j-1)) + \sum_{j=1}^{k} \widetilde{\lambda_2}(j)\omega_2(g(j-m-1)).$$

Using the property of the maximum (11) we have

$$g(k) \leq c + \sum_{j=1}^{k} \lambda_1(j)\omega_1(g(j-1)) + \sum_{j=1}^{k} \lambda_2(j)\omega_2(g(j-1)),$$

where $\lambda_2(j) := 2^{\alpha_2}\widetilde{\lambda_2}(j)$. Now we apply the estimation proposed by Pinto, Medina (cf. [8]). If

$$W_i(u) = \int_{u_i}^{u} \frac{\mathrm{d}\sigma}{\omega_i(\sigma)}, \quad u_i, u > 0, \quad i = 1, 2,$$

$$\|\lambda_i\| = \sum_{j=1}^{\infty} \lambda_i(j) \leq \int_{c_{i-1}}^{\infty} \frac{\mathrm{d}\sigma}{\omega_i(\sigma)}, \quad i = 1, 2,$$

$$c_0 := c \quad \text{and} \quad c_1 := W_1^{-1}\left(W_1(c_0) + \|\lambda_1\|\right),$$

we obtain

$$g(k) \leq W_2^{-1}\left(W_2(c_1) + \sum_{j=1}^{k} \lambda_2(j)\right) \leq W_2^{-1}\left(W_2(c_1) + \|\lambda_2\|\right) < \infty.$$

Thus one can estimate the function $g(k)$ with a constant and denote it K. So we get

$$\|x(k)\| \leq K\mathrm{e}^{(\|B_1\| + \ln\|A\|)k}, \quad k \in \mathbb{Z}_0^{\infty}.$$

Since $\frac{1}{P}\lambda_{1,2} \in l_1$, one can find constant $P > 0$ such small that the following conditions

$$\|\lambda_i\| < \int_{c_{i-1}}^{\infty} \frac{\mathrm{d}z}{z^{\alpha_i}}, \quad i = 1, 2$$

are fulfilled. Apparently, if $\|B_1\| < -\ln\|A\|$ the trivial solution of the equation (9) is exponentially stable whenever $\max\{K, \|\varphi\|\} < \delta$. □

Note that the following theorem is a generalization of the previous one.

Theorem 3.3. *Let* $\alpha_1, \dots, \alpha_a, \beta_1, \dots, \beta_b > 1$ *be given constants such that* $\alpha_i \neq \alpha_j$ *and* $\beta_k \neq \beta_l$ *for* $i \neq j, k \neq l$, $a, b \in \mathbb{N}$, *the matrices* A, B *and* B_1 *be as in the Theorem 3.1. Assume that* $\|A\| \mathrm{e}^{\|B_1\|} < 1$ *and*

$$f(x,y) = o\left(\|x\|^{\alpha_1} + \dots + \|x\|^{\alpha_a} + \|y\|^{\beta_1} + \dots + \|y\|^{\beta_b}\right).$$

Then the trivial solution of the equation (9) *is exponentially stable.*

Proof. Using the notation (6),(7) we get the inequality

$$u(k) \leq M + \sum_{j=1}^{k} \sum_{i=1}^{a} \lambda_i(j) u^{\alpha_i}(j-1) + \sum_{j=1}^{k} \sum_{i=1}^{b} \eta_i(j) u^{\beta_i}(j-m-1), \qquad (13)$$

where

$$\lambda_i(j) = P C^{-\alpha_i} C^{(\lambda_i - 1)j}, \quad i = 1, \dots, a,$$
$$\eta_i(j) = 2^{\beta_i} P C^{-\beta_i(m+1)} C^{(\beta_i - 1)j}, \quad i = 1, \dots, b.$$

If we denote the right-hand side of the inequality (13) by $g(k)$ and apply the inequality (11) we obtain

$$u(k) \leq g(k) \leq c_0 + \sum_{j=1}^{k} \sum_{i=1}^{p} \nu_i(j) g^{\gamma_i}(j-1),$$

where

$$c_0 := \max\{M, \|\varphi\|\}, \quad \max\{a, b\} \leq p \leq a + b,$$
$$\{\gamma_1, \dots, \gamma_p\} = \{\alpha_1, \dots, \alpha_a\} \cup \{\beta_1, \dots, \beta_b\}$$

is an increasing sequence of exponents $0 < \gamma_1 < \dots < \gamma_p$ and for each $l \in \{1, \dots, p\}$, coefficient ν_l is given by one the following possible formulas:

1. $\nu_l = P(C^{-\alpha_i} C^{(\alpha_i - 1)j} + 2^{\beta_k} C^{-\beta_k(m+1)} C^{(\beta_k - 1)j}) \, (= \lambda_i(j) + \eta_k(j))$ if $\alpha_i = \beta_k = (\gamma_l)$ for some $i \in \{1, \dots, a\}$ and $k \in \{1, \dots, b\}$,
2. $\nu_l = P C^{-\alpha_i} C^{(\alpha_i - 1)j} \, (= \lambda_i(j))$ for $\gamma_l = \alpha_i$, $i = \{1, \dots, a\}$ if $\alpha_i \neq \beta_k$ for all $k \in \{1, \dots, b\}$,

3. $\nu_l = 2^{\beta_k} P C^{-\beta_k(m+1)} C^{(\beta_k-1)j}\ (= \eta_k(j))$ for $\gamma_l = \beta_k$, if $\alpha_i \neq \beta_k$ for all $i \in \{1, \ldots, a\}$.

Suppose that P is such small that

$$\|\nu_i\| = \sum_{i=1}^{\infty} \nu_i(j) < \int_{c_{i-1}}^{\infty} \frac{\mathrm{d}\sigma}{\omega_i(\sigma)}, \quad i = 1, \ldots, p, \tag{14}$$

where

$$\omega_i(u) = u^{\gamma_i}, \quad i = 1, \ldots, p, \quad c_i = W_i^{-1}\left[W_i(c_{i-1}) + \|\nu_i\|\right], \quad i = 1, \ldots, p-1$$
$$W_i(u) = \int_{u_i}^{u} \frac{\mathrm{d}\sigma}{\omega_i(\sigma)}, u_i, u > 0, \quad i = 1, \ldots, p.$$

Now by applying the Pinto, Medina inequality (cf. [8]) we get

$$u(k) \le g(k) \le W_p^{-1}\left[W_p(c_{p-1}) + \sum_{i=1}^{p} \nu_i(j)\right] \le W_p^{-1}\left[W_p(c_{p-1}) + \|\nu_p\|\right].$$

It is easy to see that the right-hand side of the latter inequality is a positive constant. If we denote it by K, for the trivial solution of the equation (9) we obtain

$$\|x(k)\| \le K \mathrm{e}^{(\ln\|A\| + \|B_1\|)k}, \quad k \ge 0.$$

Since $\|B_1\| < -\ln\|A\|$, if P is such small that (14) holds and $\max\{K, \|\varphi\|\} < \delta$, the trivial solution of the equation (9) is exponentially stable. □

4 Application to a biomathematical model

Consider a model of population dynamics with delayed birthrates of the form:

$$\begin{aligned} x_{n+1}(k) &= x_n(k)\left[1 - \alpha - \gamma_1 y_n(k) - r\left(x_n(k-m) + y_n(k-m)\right)\right] + \epsilon_1 x_n(k-m) \\ y_{n+1}(k) &= y_n(k)\left[1 - \beta + \gamma_2 x_n(k) - r\left(x_n(k-m) + y_n(k-m)\right)\right] + \epsilon_2 y_n(k-m), \end{aligned} \tag{15}$$

where $\alpha, \beta, \gamma_1, \gamma_2 > 0$ represent coefficients of the mortality rate and interaction between populations. Parameter $r > 0$ is the logistic coefficient and $\epsilon_1, \epsilon_2 > 0$ are the delayed growth coefficients.

Theorem 4.1. *Assume that* $0 < \alpha < \beta < 1$. *If*

$$\epsilon < \frac{|1-\beta|^{m+1} \ln(\frac{1}{\sqrt{2}|1-\alpha|})}{\sqrt{2}},$$

where $\epsilon := \max\{\epsilon_1, \epsilon_2\}$, *then the trivial solution of* (15) *is exponentially stable.*

Proof. It is easy to see that matrices $A = \left(\begin{smallmatrix} 1-\alpha & 0 \\ 0 & 1-\beta \end{smallmatrix}\right)$, $B = \left(\begin{smallmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{smallmatrix}\right)$ representing the linear parts of system (15) are permutable. Function $f : \mathbb{R}^4 \to \mathbb{R}^2$, $f(x,y) = (-\gamma_1 x_1 x_2 - r x_1 y_1 - r x_1 y_2, \gamma_2 x_1 x_2 - r x_2 y_1 - r x_2 y_2)$ can be estimated as follows $\|f(x,y)\| \leq P\left(\|x\|^2 + \|y\|^2\right)$. Since $\|A\| \leq \sqrt{2}\,|1-\alpha|$, $\|B_1\| \leq \frac{\sqrt{2}\epsilon}{|1-\beta|^{m+1}}$ for $\alpha, \beta \in (0,1)$, the statement follows from Theorem 3.3. □

This result means that if birthrates are low, both populations in (15) tend to zero exponentially, assuming initial states of populations to be sufficiently small.

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