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Stability Analysis for Nonlinear Second Order Differential Equations with Impulses*

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Abstract

In this paper we investigate the impulsive equation

$$\begin{cases} (r(t)x')' + a(t)x + f(t, x, x') = p(t), t \geq t_0,\ t \neq t_k, \\ x(t_k) = c_k x(t_k - 0),\ x'(t_k) = d_k x'(t_k - 0),\ k = 1, 2, 3, \ldots, \end{cases}$$

and establish a couple of criteria to guarantee the equations of this type to possess the stability, including boundedness and asymptotic properties. Some examples are given to illustrate our results and the last one shows that, to some extent, our criteria have more comprehensive suitability than those given by G. Morosanu and C. Vladimirescu.

Keywords: Impulsive equations; Stability; Bernoulli type differential inequality; Bellman's inequality.

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1 Introduction

As the motion of damped oscillator of one degree of freedom can be described by second order equations, there have been quite a little literature to study

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the properties of the equations of this type with or without impulses, see, e.g., [1-8] and the references therein. We observe that Morosanu et al [8] investigated the stability of differential equation of the form

$$x'' + 2h(t)x' + x + g(t, x) = 0,\ t \geq 0 \tag{1}$$

and obtained an interesting and feasible criterion ([8, Theorem 2.1] as follows.

Theorem A: Suppose that $h \in C^1([0,\infty),(0,\infty))$ and $g \in C([0,\infty) \times (-\infty,\infty))$ with the local Lipschitzian condition for g in x. Suppose further that there exist constants $a \geq 0$, $K \in (0,1)$, $M > 0$ and $\alpha > 1$ such that

$$\begin{aligned} |h'(t) + h^2(t)| &\leq Kh(t) \ \text{ for all } \ t \geq a, \\ |g(t,x)| &\leq Mh(t)|x|^\alpha \ \text{ for all } \ (t,x) \in [0,\infty) \times (-\infty,\infty). \end{aligned}$$

Then the null solution of (1) is stable.

The questions posed here to answer are whether we can weaken the conditions in **Theorem A**, such as weakening the restrictions that $h(t) > 0$ and $\alpha > 1$, and the conclusion is also true. To these ends, in this paper we consider a more general form than (1) and study the impulsive second order nonlinear differential equation

$$\begin{cases} (r(t)x')' + a(t)x + f(t, x, x') = p(t), t \geq t_0,\ t \neq t_k, \\ x(t_k) = c_k x(t_k - 0),\ x'(t_k) = d_k x'(t_k - 0),\ k = 1, 2, 3, \cdots, \end{cases} \tag{2}$$

where

$$x(t_k) = x(t_k + 0) = \lim_{h\to 0^+} x(t_k + h), \quad x(t_k - 0) = \lim_{h\to 0^-} x(t_k + h)$$

and

$$x'(t_k) = x'(t_k + 0) = \lim_{h\to 0^+} \frac{x(t_k + h) - x_k}{h}, \quad x'(t_k - 0) = \lim_{h\to 0^-} \frac{x(t_k + h) - x_k}{h}.$$

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{R}$ be the real axis. Before proceeding our discussions, we give the blanket assumptions for (2) as follows:

(H1) $r, a \in C([t_0,\infty),\mathbb{R})$ with $r(t) \neq 0$, $p \in C([t_0,\infty),\mathbb{R})$ with $p(t)$ of constant sign and $f \in C([t_0,\infty) \times \mathbb{R} \times \mathbb{R},\mathbb{R})$;
(H2) $f(t, u, v)$ monotone decreasing in u provided $p(t)$ is not identically zero;
(H3) $b_i \in C([t_0,\infty),[0,\infty))$, $i = 1, 2$, and $\alpha \geq 1$ is some constant such that

$|f(t,u,v)| \le b_1(t)|u|^\alpha + b_2(t)|v|^\alpha$; and
(H4) $t_0 < t_1 < ... < t_k \to \infty$ as $k \to \infty$ and $\{c_k\}, \{d_k\}$ are positive sequences.

Let $x_1 = x$ and $x_2 = rx'$. Then (2) can be rewritten as

$$\begin{cases} X' = A(t)X + \begin{pmatrix} 0 \\ p(t) - f(t, x_1, x_2/r(t)) \end{pmatrix}, \ t \ge t_0, \ t \ne t_k, \\ X(t_k) = I_k X(t_k - 0), \ k \in \mathbb{N}, \end{cases} \tag{3}$$

where $X = (x_1, x_2)^T$ and

$$A(t) = \begin{bmatrix} 0 & \frac{1}{r(t)} \\ -a(t) & 0 \end{bmatrix}, \ I_k = \begin{bmatrix} c_k & 0 \\ 0 & d_k \end{bmatrix}, \quad k \in \mathbb{N}.$$

Set $t_0 < \zeta \le \infty$. As usual, a function $X : [t_0, \zeta) \to \mathbb{R}^2$ is said to be a solution of (3) if it satisfies

$$X' = A(t)X + \begin{pmatrix} 0 \\ p(t) - f(t, x_1, x_2/r(t)) \end{pmatrix}$$

for all $t \in [t_0, \zeta)$ and $t \ne t_k$, and $X(t_k + 0)$ as well as $X(t_k - 0)$ exist and satisfy $X(t_k + 0) = X(t_k) = I_k X(t_k - 0)$ for all $t_k \in [t_0, \zeta)$.

Let $X(t) = X(t; t_0, X_0)$ be a solution with $X(t_0) = X_0$. It is clear that (3) has null solution when $p(t) \equiv 0$. The null solution of (3) is said to be stable if for any $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon, t_0)$ such that $||X_0|| < \delta$ implies that $X(t)$ exists on $[t_0, \infty)$ and $||X(t)|| < \varepsilon$ for all $t \ge t_0$.

2 Preliminaries

For the convenience, we will view $C(t), D(t), E(t)$ and $F(t, u, v)$ as

$$C(t) = \prod_{t_1 \le t_k \le t} c_k, \ D(t) = \prod_{t_1 \le t_k \le t} d_k, \ E(t) = \prod_{t_1 \le t_k \le t} I_k = \begin{bmatrix} C(t) & 0 \\ 0 & D(t) \end{bmatrix} \tag{4}$$

and

$$F(t, u, v) = f(t, C(t)u, D(t)v) \tag{5}$$

whenever these notations are defined.

Let $U(t) = (u_{ij}(t))$ be any matrix. In this paper the norm of $U(t)$ is defined by the maximum of the row sums of $(|u_{ij}(t)|)$.

First of all, we consider the relation between the solutions of (3) and the solutions of the following equation

$$Y' = E^{-1}(t)A(t)E(t)Y + E^{-1}(t)B(t,Y), \quad t \geq t_0, \tag{6}$$

where

$$Y = (y_1, y_1)^T, \quad B(t,Y) = \begin{pmatrix} 0 \\ p(t) - F\left(t, y_1, y_2/r(t)\right) \end{pmatrix}$$

and F is defined as in (5).

Let $t_0 < \zeta \leq \infty$. By a solution of (6) we mean a continuous function $Y \in C([t_0, \zeta), \mathbb{R}^2)$ which satisfies (6) when $t \neq t_k \in [t_0, \zeta)$.

Now let $Y(t)$ be a solution of (6). Then, by straightforward verifications, we learn that $X(t) = E(t)Y(t)$ satisfies

$$X(t_k) = I_k X(t_k - 0) \quad \text{for all } t_k \in [t_0, \zeta)$$

and it renders (3) into an identity when $t \neq t_k$. Consequently, $X(t) = E(t)Y(t)$ is a solution of (3).

Conversely, suppose that X is a solution of (3). Then, for $Y(t) = E^{-1}(t)X(t)$, we have

$$Y(t_k - 0) = E^{-1}(t_k - 0)X(t_k - 0) = E^{-1}(t_k)x(t_k) = Y(t_k).$$

In addition, it is easy to verify that $Y(t) = E^{-1}(t)X(t)$ satisfies (6) when $t \neq t_k$. So far the following result is obvious.

Lemma 1 *If $X(t)$ is a solution of (3), then $Y(t) = E^{-1}(t)X(t)$ is a solution of (6). Conversely, if $Y(t)$ is a solution of (6), then $X(t) = E(t)Y(t)$ is a solution of (3).*

We next consider the solutions of (6). It is clear that the solutions of (6) exist by the theory of ordinary differential equations [9]. Specially, if $E^{-1}(t)A(t)E(t)$ is divided into

$$E^{-1}(t)A(t)E(t) = A_1(t) + A_2(t),$$

where

$$A_1(t) = \begin{bmatrix} 0 & \frac{C(t)^{-1}D(t)}{r(t)} \\ -\frac{C(t)^{-1}D(t)}{r(t)} & 0 \end{bmatrix}, \; A_2(t) = \begin{bmatrix} 0 & 0 \\ \frac{C(t)^{-1}D(t)}{r(t)} - C(t)D(t)^{-1}a(t) & 0 \end{bmatrix},$$

then, the fundamental matrix of the linear system corresponding to (6):

$$Y' = A_1(t)Y, \quad t \geq t_0$$

is given by

$$\begin{aligned} \Phi(t) &= \exp\left(\int_{t_0}^t A_1(s)\mathrm{d}s\right) \\ &= \begin{bmatrix} \cos\left(\int_{t_0}^t \frac{C(s)^{-1}D(s)}{r(s)}\mathrm{d}s\right) & \sin\left(\int_{t_0}^t \frac{C(s)^{-1}D(s)}{r(s)}\mathrm{d}s\right) \\ -\sin\left(\int_{t_0}^t \frac{C(s)^{-1}D(s)}{r(s)}\mathrm{d}s\right) & \cos\left(\int_{t_0}^t \frac{C(s)^{-1}D(s)}{r(s)}\mathrm{d}s\right) \end{bmatrix}. \end{aligned} \tag{7}$$

Let $Y(t)$ be a solution of (6) with $Y(t_0) = Y_0$, then it satisfies that

$$Y(t) = \Phi(t)Y_0 + \int_{t_0}^t \Phi(t)\Phi^{-1}(s)\left\{A_2(s)Y(s) + E^{-1}(s)B(s, Y(s))\right\}\mathrm{d}s. \tag{8}$$

As a special case, we consider

$$\begin{cases} f(t,u,v) = \widetilde{f}(t,u), \\ r \in C^2([t_0,\infty),(0,\infty)) \text{ and} \\ r'(t_k) = 0 \text{ or } c_k = d_k \text{ for all } k \in \mathbb{N}. \end{cases} \tag{9}$$

For example, we consider

$$r(t) = 2 + t + \sin t, \quad t \geq 0 \text{ and } t_k = (2k-1)\pi.$$

Then it holds that $r'(t_k) = 0$ for all $k \in \mathbb{N}$.

At this stage we set

$$x_1 = x \text{ and } x_2 = x_1' + \frac{r'x_1}{2r}.$$

Then, similarly to [1, 8], from (2) it follows that when $t \geq t_0$ and $t \neq t_k$,

$$\begin{cases} x_1' = -\frac{r'(t)}{2r(t)}x_1 + \frac{a(t)}{r(t)}x_2 + \left(1 - \frac{a(t)}{r(t)}\right)x_2, \\ x_2' = -\frac{a(t)}{r(t)}x_1 - \frac{r'(t)}{2r(t)}x_2 + \left[\left(\frac{r'(t)}{2r(t)}\right)' + \left(\frac{r'(t)}{2r(t)}\right)^2\right]x_1 + \frac{p(t)}{r(t)} - \frac{\widetilde{f}(t,x_1)}{r(t)}. \end{cases}$$

Hence (2) is equivalent to

$$\begin{cases} X' = \widetilde{A}_1(t)X + \widetilde{A}_2(t)X + \frac{1}{r(t)}\begin{pmatrix} 0 \\ p(t) - \widetilde{f}(t,x_1) \end{pmatrix}, \ t \geq t_0, \ t \neq t_k, \\ X(t_k) = I_k X(t_k - 0), \ k \in \mathbb{N}, \end{cases} \tag{10}$$

where X and I_k are defined as in (3), and

$$\widetilde{A}_1(t) = \begin{bmatrix} -\frac{r'(t)}{2r(t)} & \frac{a(t)}{r(t)} \\ -\frac{a(t)}{r(t)} & -\frac{r'(t)}{2r(t)} \end{bmatrix}, \widetilde{A}_2(t) = \begin{bmatrix} 0 & 1-\frac{a(t)}{r(t)} \\ \left(\frac{r'(t)}{2r(t)}\right)' + \left(\frac{r'(t)}{2r(t)}\right)^2 & 0 \end{bmatrix}.$$

Analogously to (6), we consider the following equation

$$Y' = E^{-1}(t)\widetilde{A}_1(t)E(t)Y + E^{-1}(t)\widetilde{A}_2(t)E(t)Y + E^{-1}(t)\widetilde{B}(t,Y),\ t \geq t_0, \quad (11)$$

where $Y = (y_1, y_2)^T$, and

$$\widetilde{B}(t,Y) = \frac{1}{r(t)} \begin{pmatrix} 0 \\ p(t) - \widetilde{f}\,(t, C(t)y_1) \end{pmatrix}.$$

Since the relation between the solutions of (10) and the solutions of (11) is similar to Lemma 1, we wish to refrain from the repeating statements.

Let us set

$$\widetilde{A}_{11}(t) = \begin{bmatrix} -\frac{r'(t)}{2r(t)} & \frac{C(t)^{-1}D(t)a(t)}{r(t)} \\ -\frac{C(t)^{-1}D(t)a(t)}{r(t)} & -\frac{r'(t)}{2r(t)} \end{bmatrix}$$

as well as

$$\widetilde{A}_{12}(t) = \begin{bmatrix} 0 & 0 \\ \frac{\left(C(t)^{-1}D(t) - C(t)D(t)^{-1}\right)a(t)}{r(t)} & 0 \end{bmatrix}.$$

Then it follows that

$$E^{-1}(t)\widetilde{A}_1(t)E(t) = \widetilde{A}_{11}(t) + \widetilde{A}_{12}(t).$$

Now we take into account the following system corresponding to (11):

$$Y' = \widetilde{A}_{11}(t)Y,\ t \geq t_0. \quad (12)$$

It is easy to verify that the fundamental matrix of (12) is given by

$$\widetilde{\Phi}(t) = e^{-\int_{t_0}^t \frac{r'(t)}{2r(t)}\mathrm{d}s} \begin{bmatrix} \cos\left(\int_{t_0}^t \frac{C(s)^{-1}D(s)a(s)}{r(s)}\mathrm{d}s\right) & \sin\left(\int_{t_0}^t \frac{C(s)^{-1}D(s)a(s)}{r(s)}\mathrm{d}s\right) \\ -\sin\left(\int_{t_0}^t \frac{C(s)^{-1}D(s)a(s)}{r(s)}\mathrm{d}s\right) & \cos\left(\int_{t_0}^t \frac{C(s)^{-1}D(s)a(s)}{r(s)}\mathrm{d}s\right) \end{bmatrix}.$$

Subsequently, the solution Y of (11) with $Y(t_0) = Y_0$ satisfies that

$$\begin{aligned} Y(t) &= \widetilde{\Phi}(t)Y_0 + \int_{t_0}^t \widetilde{\Phi}(t)\widetilde{\Phi}^{-1}(s)\widetilde{A}_{12}(s)Y(s)\mathrm{d}s + \\ &\int_{t_0}^t \widetilde{\Phi}(t)\widetilde{\Phi}^{-1}(s)\left\{E^{-1}(s)\widetilde{A}_2(s)E(s)Y(s) + E^{-1}(s)\widetilde{B}(s,Y(s))\right\}\mathrm{d}s,\ t \geq t_0. \end{aligned} \quad (13)$$

3 Main results

In the sequel we give the stability criteria for (2). Recall that the definitions of $C(t)$, $D(t)$ and $E(t)$ have been defined in (4). For simplicity, we introduce another notations as follows. Let $\lambda_1(c)$ and $\lambda_2(c)$ be denoted, respectively, by

$$\lambda_1(c) \le \inf_{t\ge t_0} C(t) = \inf_{t\ge t_0} \prod_{t_1\le t_k\le t} c_k,\ \ \lambda_2(c) \ge \sup_{t\ge t_0} C(t) = \sup_{t\ge t_0} \prod_{t_1\le t_k\le t} c_k.$$

The notations $\lambda_1(d)$ and $\lambda_2(d)$ can be defined similarly.

Theorem 1 *Suppose that the following conditions hold:*
(i) $p(t) \equiv 0$, $\int_{t_0}^{\infty} b_1(s)\mathrm{d}s < \infty$, $\int_{t_0}^{\infty} \frac{b_2(s)}{|r(s)|^{\alpha}}\mathrm{d}s < \infty$ *and*

$$\int_{t_0}^{\infty} \left| \frac{C(s)^{-1}D(s)}{r(s)} - C(s)D(s)^{-1}a(s) \right| \mathrm{d}s < \infty,$$

(ii) $\lambda_1(c)\cdot\lambda_1(d) > 0$ *and* $\lambda_2(c)+\lambda_2(d) < \infty$.
Then the null solution of (3) is stable.

Proof. Let $Y(t)$ be a solution of (6) satisfying $Y(t_0) = Y_0$. Then, Lemma 1 implies that $X(t) = E(t)Y(t)$ is a solution of (3) with the initial condition $X(t_0) = Y_0$. Set

$$\begin{aligned} M &= \max\left\{ \lambda_1(c)^{-1},\ \lambda_1(d)^{-1},\ \lambda_2(c),\ \lambda_2(d),\ \int_{t_0}^{\infty} b_1(s)\mathrm{d}s,\ \int_{t_0}^{\infty} \frac{b_2(s)}{|r(s)|^{\alpha}}\mathrm{d}s, \right. \\ &\left. \int_{t_0}^{\infty} \left| \frac{C(s)^{-1}D(s)}{r(s)} - C(s)D(s)^{-1}a(s) \right| \mathrm{d}s \right\}. \end{aligned} \tag{14}$$

Note that $p(t) \equiv 0$ implies that the function B in (8) satisfies

$$\begin{aligned} ||B(s,Y(s))|| &\le |f(s, C(s)y_1(s), D(s)y_2(s)/r(s))| \\ &\le b_1(s)|x_1(s)|^{\alpha} + \frac{b_2(s)}{|r(s)|^{\alpha}}|x_2(s)|^{\alpha} \\ &\le b_1(s)||X(s)||^{\alpha} + \frac{b_2(s)}{|r(s)|^{\alpha}}||X(s)||^{\alpha}, \end{aligned}$$

where we have imposed the relation $X(s) = E(s)Y(s)$ and the assumption (H3) on the function f for the second step. Note further that for all $t \ge t_0$,

$\Phi(t)$ as in (7) satisfies $||\Phi(t)||, ||\Phi^{-1}(t)|| \leq \sqrt{2}$, and $||E(t)||, ||E^{-1}(t)|| \leq M$, we now multiply (8) by $E(t)$ and then obtain

$$\begin{aligned} ||X(t)|| \ \leq \ & \sqrt{2}M||Y_0|| + \\ & 2M^2 \int_{t_0}^{t} \left| \frac{C(s)^{-1}D(s)}{r(s)} - C(s)D(s)^{-1}a(s) \right| \cdot ||X(s)||\mathrm{d}s + \\ & 2M^2 \int_{t_0}^{t} b_1(s)||X(s)||^{\alpha}\mathrm{d}s + 2M^2 \int_{t_0}^{t} \frac{b_2(s)}{|r(s)|^{\alpha}}||X(s)||^{\alpha}\mathrm{d}s. \end{aligned} \tag{15}$$

Let $w(t)$ be defined by

$$w(t) = \max\left\{ b_1(t) + \frac{b_2(t)}{|r(t)|^{\alpha}}, \ \left| \frac{C(t)^{-1}D(t)}{r(t)} - C(t)D(t)^{-1}a(t) \right| \right\}. \tag{16}$$

Then, since $w(t) \leq b_1(t) + \frac{b_2(t)}{|r(t)|^{\alpha}} + \left| \frac{C(t)^{-1}D(t)}{r(t)} - C(t)D(t)^{-1}a(t) \right|$, it holds that $\int_{t_0}^{\infty} w(s)\mathrm{d}s \leq 3M$.

Case 1. Suppose that $\alpha > 1$. Then we have from (15)–(16) that

$$||X(t)|| \ \leq \ \sqrt{2}M||Y_0|| + 2M^2 \int_{t_0}^{t} w(s)||X(s)||\mathrm{d}s + 2M^2 \int_{t_0}^{t} w(s)||X(s)||^{\alpha}\mathrm{d}s. \tag{17}$$

Let $R(t)$ be defined by

$$R(t) = \sqrt{2}M||Y_0|| + 2M^2 \int_{t_0}^{t} w(s)||X(s)||\mathrm{d}s + 2M^2 \int_{t_0}^{t} w(s)||X(s)||^{\alpha}\mathrm{d}s.$$

Then, it is readily obtained that

$$R'(t) \leq 2M^2 w(t)R(t) + 2M^2 w(t)R(t)^{\alpha}. \tag{18}$$

Multiplying (18) by $(1-\alpha)R(t)^{-\alpha}\exp\left(2M^2(\alpha-1)\int_{t_0}^{t} w(s)\mathrm{d}s\right)$, we arrive at

$$\left[(1 + R(t)^{1-\alpha})e^{2M^2(\alpha-1)\int_{t_0}^{t} w(s)\mathrm{d}s}\right]' \geq 0,$$

which means that

$$(1 + R(t)^{1-\alpha})e^{2M^2(\alpha-1)\int_{t_0}^{t} w(s)\mathrm{d}s} \geq 1 + M_0^{1-\alpha}, \tag{19}$$

here $M_0 = \sqrt{2}M||Y_0||$. In view of (19), we have when

$$M_0 < \left(e^{6M^3(\alpha-1)} - 1\right)^{1/(1-\alpha)},$$

$$R(t) \leq \left\{[1 + M_0^{1-\alpha}]e^{6M^3(1-\alpha)} - 1\right\}^{\frac{1}{1-\alpha}}. \tag{20}$$

From (20) and (17) we learn that $X(t)$ is defined on $[t_0, \infty)$ (see [9, Chapter 2]). Furthermore, when M_0 is small enough, i.e., $||Y_0||$ is small enough, (17) and (20) imply that $||X(t)||$ is also small enough on $[t_0, \infty)$, which indicates the null solution of (3) is stable.

Case 2. Suppose that $\alpha = 1$. We now have, from (15)–(16), that

$$||X(t)|| \leq \sqrt{2}M||Y_0|| + 4M^2 \int_{t_0}^{t} w(s)||X(s)||\mathrm{d}s,$$

which, together with the Bellman's Inequality, induces

$$\begin{aligned} ||X(t)|| &\leq \sqrt{2}M||Y_0||e^{4M^2 \int_{t_0}^{t} w(s)\mathrm{d}s} \\ &\leq \sqrt{2}M||Y_0||e^{12M^3}, \end{aligned}$$

and this, likewise, means that the null solution of (3) is stable. The proof is complete.

We observe that when $c_k \equiv d_k$ on $\mathbb{N}$, $C(t)^{-1}D(t) = C(t)D(t)^{-1} = 1$ for all $t \geq t_0$. Hence the following result is clear.

Corollary 1 *Suppose that the following conditions hold:*
(i) $p(t) \equiv 0$, $\int_{t_0}^{\infty} b_1(s)\mathrm{d}s < \infty$, $\int_{t_0}^{\infty} \frac{b_2(s)}{|r(s)|^\alpha}\mathrm{d}s < \infty$ *and*

$$\int_{t_0}^{\infty} \left|\frac{1}{r(s)} - a(s)\right| \mathrm{d}s < \infty, \text{ and}$$

(ii) $c_k = d_k$ *for all* $k \in \mathbb{N}$, $\lambda_1(c) > 0$ *and* $\lambda_2(c) < \infty$.
Then the null solution of (3) is stable.

We notice that, by similar arguments, we may show that the solution $X(t; t_0, X_0)$ of (3) exists on $[t_{,0}, \infty)$ for any $X_0 \in \mathbb{R}^2$ under the provisions in Theorem 1. Now we consider the case that $p(t)$ is of constant sign and is not identically zero. In this case we impose the assumption (in (H2)) that $f(t,u,v)$ is monotone decreasing in u and learn that the vector function

$$\begin{pmatrix} 0 \\ -f\left(t, x_1, x_2/r(t)\right) \end{pmatrix}$$

is quasi-monotone increasing on $\mathbb{R}^2$.

Recall the Comparison Theorem [10]. Briefly speaking, if $F(t,x) : \mathbb{R}^{2+1} \to \mathbb{R}^2$ and F is quasi-monotone increasing in x, and if ψ is the maximal solution of $x' = F(t,x)$ with $\psi(t_0) = \psi_0$ for $t \geq t_0$, then $\varphi' \leq F(t,\varphi)$ with $\varphi(t_0) \leq \psi_0$ for $t \geq t_0$ implies that $\varphi(t) \leq \psi(t)$ for $t \geq t_0$. Hence, with the aid of comparison theorem we may also show that the solution $X(t;t_0,X_0)$ of (3) exists on $[t_0,\infty)$ for any $X_0 \in \mathbb{R}^2$. For simplicity we ignore the details of proof.

Next we consider the boundedness for (3).

Theorem 2 *Suppose that the following conditions hold:*
(i) $\int_{t_0}^{\infty} b_1(s)\mathrm{d}s < \infty$*,* $\int_{t_0}^{\infty} \frac{b_2(s)}{|r(s)|^{\alpha}}\mathrm{d}s < \infty$*,* $\int_{t_0}^{\infty} |p(s)|\mathrm{d}s < \infty$ *and*

$$\int_{t_0}^{\infty} \left| \frac{C(s)^{-1}D(s)}{r(s)} - C(s)D(s)^{-1}a(s) \right| \mathrm{d}s < \infty,$$

(ii) $\lambda_1(c) \cdot \lambda_1(d) > 0$ *and* $\lambda_2(c) + \lambda_2(d) < \infty$*.*
Then every solution of (3) is bounded.

Proof. We first assume that $Y(t)$ is the solution of (6) with $Y(t_0) = Y_0$. Then $X(t) = E(t)Y(t)$ is a solution of (3). Let M be defined as in (14) and

$$\int_{t_0}^{\infty} |p(s)|\mathrm{d}s \leq M.$$

Furthermore, let $w(t)$ be defined as in (16). Note that the function B in (8) satisfies that

$$||B(s,Y(s))|| \leq |p(s)| + b_1(s)||X(s)||^{\alpha} + \frac{b_2(s)}{|r(s)|^{\alpha}}||X(s)||^{\alpha}$$

for the time being.

Case 1. Suppose that $\alpha = 1$. Then, similarly to (15) we have

$$||X(t)|| \;\leq\; \sqrt{2}M||Y_0|| + 2M^3 +$$

$$
\begin{aligned}
& 2M^2 \int_{t_0}^{t} \left| \frac{C(s)^{-1}D(s)}{r(s)} - C(s)D(s)^{-1}a(s) \right| \cdot ||X(s)||\mathrm{d}s + \\
& 2M^2 \int_{t_0}^{t} b_1(s)||X(s)||\mathrm{d}s + 2M^2 \int_{t_0}^{t} \frac{b_2(s)}{|r(s)|^\alpha}||X(s)||\mathrm{d}s \\
\leq \;& \sqrt{2}M||Y_0|| + 2M^3 + 4M^2 \int_{t_0}^{t} w(s)||X(s)||\mathrm{d}s. \qquad (21)
\end{aligned}
$$

By means of Bellman's Inequality and (21) one arrives at

$$
\begin{aligned}
||X(t)|| &\leq \left(\sqrt{2}M||Y_0|| + 2M^3\right) e^{4M^2 \int_{t_0}^{t} w(s)\mathrm{d}s} \\
&\leq \left(\sqrt{2}M||Y_0|| + 2M^3\right) e^{12M^3}
\end{aligned}
$$

which shows that every solution of (3) is bounded when $\alpha = 1$.

Case 2. Suppose that $\alpha > 1$. For any given $\varepsilon > 0$, we take $T > t_0$ so that

$$
\int_T^\infty w(s)\mathrm{d}s \leq \varepsilon \quad \text{and} \quad \int_T^\infty |p(s)|\mathrm{d}s \leq \varepsilon.
$$

Analogously to (8) we have

$$
\begin{aligned}
Y(t) \;=\; & \Phi(t)\Phi^{-1}(T)Y(T) + \\
& \int_T^t \Phi(t)\Phi^{-1}(s)\left[A_2(s)Y(s) + E^{-1}(s)B(s, Y(s))\right]\mathrm{d}s, \quad t \geq T.
\end{aligned}
$$

which leads to

$$
\begin{aligned}
||X(t)|| \;\leq\; & 2M^2||Y(T)|| + 2M^2\varepsilon + 2M^2 \int_T^t w(t)||X(s)||\mathrm{d}s \\
& + 2M^2 \int_T^t w(t)||X(s)||^\alpha \mathrm{d}s =: R(t), \quad t \geq T. \qquad (22)
\end{aligned}
$$

By the same manner as (19), we have from (22) that

$$
\left(1 + R(t)^{1-\alpha}\right) e^{2M^2(\alpha-1)\int_T^t w(s)\mathrm{d}s} \geq 1 + M_0^{1-\alpha}, \quad t \geq T, \qquad (23)
$$

where $M_0 = R(T) = 2M^2(||Y(T)|| + \varepsilon)$. Thus it holds from (23) that

$$
R(t)^{1-\alpha} \geq \left(1 + M_0^{1-\alpha}\right) e^{2M^2(1-\alpha)\int_T^t w(s)\mathrm{d}s} - 1, \quad t \geq T,
$$

which results in

$$R(t) \leq \left\{ \left[M_0^{1-\alpha} + 1 \right] e^{2M^2(1-\alpha)\varepsilon} - 1 \right\}^{\frac{1}{1-\alpha}}, \quad t \geq T, \tag{24}$$

where

$$M_0 < \left(e^{2M^2(\alpha-1)\varepsilon} - 1 \right)^{-\frac{1}{\alpha-1}}.$$

Since ε is arbitrary, we can ensure that (24) is valid for $Y(T)$. Further, from (22) and (24) we learn that $R(t)$ is bounded on $[T, \infty)$, which implies that the solution $X(t)$ of (3) is bounded on $[t_0, \infty)$ when $\alpha > 1$. The proof is complete.

The following result is concerned with the asymptotic behavior of (10) (or (2)) under the assumptions (9). It is based on the fact that the solution $X(t; t_0, x_0)$ of (10) exists on $[t_0, \infty)$ for any $X_0 \in \mathbb{R}^2$. The reasons are similar to the proof of Theorem 1 and the statements before entering Theorem 2, and therefore we skip them.

Theorem 3 *Suppose that the following conditions hold:*
(i) the hypothesis (9) is fulfilled, $\frac{r'(t)}{r(t)} \geq 0$ *on* $[t_0, \infty)$, $\int_{t_0}^{\infty} \frac{r'(s)}{r(s)} \mathrm{d}s = \infty$ *and* $\int_{t_0}^{\infty} \left| \frac{p(s)}{r(s)} \right| \mathrm{d}s < \infty$*;*
(ii) there exists a constant $M > 0$ *such that*

$$\max \left\{ \lambda_1(c)^{-1},\ \lambda_1(d)^{-1},\ \lambda_2(c),\ \lambda_2(d) \right\} \leq M;$$

(iii) there exist positive constants γ_1 *and* γ_2 *such that*

$$\max \left\{ \frac{b_1(t)}{\gamma_1 r(t)},\ \frac{||\widetilde{A}_{12}(t)|| + ||\widetilde{A}_2(t)||}{\gamma_2} \right\} \leq \frac{r'(t)}{r(t)}.$$

Then, every solution of (10) tends to zero as $t \to \infty$ *if one of the following conditions holds:*
(c1) $\alpha > 1$ *and* $4M^2\gamma_2 < 1$*; or*
(c2) $\alpha = 1$ *and* $4M^2(\gamma_2 + \gamma_1) < 1$.

Proof. Note that $b_2(t)$ in assumption(H3) vanishes when $f(t, u, v) = \widetilde{f}(t, u)$. Let $\beta(t)$ be designated by

$$\beta(t) = \frac{r'(t)}{2r(t)}.$$

For any given $\varepsilon > 0$, we may take $T > t_0$ so that $\int_T^\infty \left|\frac{p(s)}{r(s)}\right| \mathrm{d}s \le \varepsilon$. Let $Y(t)$ be a solution of (11) with $Y(T) = Y_0$. For the simplicity, we assume that $\widetilde{\Phi}^{-1}(T) \le M$. Analogously to (13) we now have

$$\begin{aligned} Y(t) &= \widetilde{\Phi}(t)\widetilde{\Phi}^{-1}(T)Y_0 + \\ &\int_T^t \widetilde{\Phi}(t)\widetilde{\Phi}^{-1}(s)\left\{E^{-1}(s)\widetilde{A}_2(s)E(s)Y(s) + E^{-1}(s)\widetilde{B}(s,Y(s))\right\}\mathrm{d}s,\ t \ge T, \end{aligned}$$

which, with the help of $b_2(t) \equiv 0$, infers that

$$\begin{aligned} ||X(t)|| &\le \sqrt{2}M^2||Y_0||e^{-\int_{t_0}^t \beta(s)\mathrm{d}s} + 2M^2\varepsilon + \\ &\quad 2M^2\int_T^t e^{-\int_s^t \beta(u)\mathrm{d}u}\left(||\widetilde{A}_{12}(t)|| + ||\widetilde{A}_2(t)||\right)||X(s)||\mathrm{d}s + \\ &\quad 2M^2\int_T^t e^{-\int_s^t \beta(u)\mathrm{d}u}\frac{b_1(s)}{|r(s)|}||X(s)||^\alpha \mathrm{d}s \\ &=: \widetilde{R}(t), \quad t \ge T, \end{aligned} \tag{25}$$

where we have imposed the inequalities

$$||\widetilde{\Phi}(t)|| \le \sqrt{2}\exp\left(-\int_{t_0}^t \beta(s)\mathrm{d}s\right) \quad \text{and}$$

$$||\widetilde{\Phi}(t)\widetilde{\Phi}^{-1}(s)|| \le 2\exp\left(-\int_s^t \beta(u)\mathrm{d}u\right).$$

Case 1. Suppose that $\alpha > 1$ and $4M^2\gamma_2 < 1$. Then (25) conduces to

$$\begin{aligned} \widetilde{R}(t) &= \sqrt{2}M^2||Y_0||e^{-\int_{t_0}^t \beta(s)\mathrm{d}s} + 2M^2\varepsilon + \\ &\quad 4M^2\int_T^t e^{-\int_s^t \beta(u)\mathrm{d}u}\beta(s)\left(\gamma_2||X(s)|| + \gamma_1||X(s)||^\alpha\right)\mathrm{d}s, \quad t \ge T, \end{aligned}$$

which implies that

$$\widetilde{R}'(t) \le \left(4M^2\gamma_2 - 1\right)\beta(t)\widetilde{R}(t) + 4M^2\gamma_1\beta(t)\widetilde{R}(t)^\alpha, \quad t \ge T. \tag{26}$$

Now multiplying (26) by $(1-\alpha)\widetilde{R}(t)^{-\alpha}e^{(1-\alpha)(1-4M^2\gamma_2)\int_{t_0}^t \beta(s)\mathrm{d}s}$ we obtain

$$\left\{\left[\widetilde{R}(t)^{1-\alpha} - \frac{4M^2\gamma_1}{1-4M^2\gamma_2}\right]e^{(1-\alpha)(1-4M^2\gamma_2)\int_{t_0}^t \beta(s)\mathrm{d}s}\right\}' \ge 0,$$

which infers that

$$\left[\widetilde{R}(t)^{1-\alpha} - \frac{4M^2\gamma_1}{1-4M^2\gamma_2}\right] e^{(1-\alpha)(1-2M^2\gamma_2)\int_{t_0}^t \beta(s)\mathrm{d}s} \geq M_0^{1-\alpha} - \frac{4M^2\gamma_1}{1-4M^2\gamma_2}, \quad (27)$$

where $M_0 = \widetilde{R}(T) = M^2(\sqrt{2}||Y_0||e^{-\int_{t_0}^T \beta(s)\mathrm{d}s} + 2\varepsilon)$. In view of (27) we learn that when

$$||Y_0|| < \frac{1}{\sqrt{2}M^2}\left[\left(\frac{1-4M^2\gamma_2}{4M^2\gamma_1}\right)^{\alpha-1} - 2M^2\varepsilon\right] e^{\int_{t_0}^T \beta(s)\mathrm{d}s},$$

it follows that

$$\widetilde{R}(t) \leq \frac{1}{\left\{\left[M_0^{1-\alpha} - \frac{4M^2\gamma_1}{1-4M^2\gamma_2}\right] e^{(\alpha-1)(1-4M^2\gamma_2)\int_{t_0}^t \beta(s)\mathrm{d}s} + \frac{4M^2\gamma_1}{1-4M^2\gamma_2}\right\}^{\alpha-1}}. \quad (28)$$

Associated with the condition $\int_{t_0}^\infty \frac{r'(s)}{r(s)}\mathrm{d}s = \infty$, (28) induces $\lim_{t\to\infty} ||\widetilde{R}(t)|| = 0$ and hence $\lim_{t\to\infty} ||X(t)|| = 0$.

Case 2. Suppose that $\alpha = 1$ and $4M^2(\gamma_2 + \gamma_1) < 1$. In this case it follows from (25) that

$$\begin{aligned}\widetilde{R}(t) &= \sqrt{2}||Y_0||e^{-\int_{t_0}^t \beta(s)\mathrm{d}s} + 2M^2\varepsilon + \\ &\quad 4M^2\int_T^t e^{-\int_s^t \beta(u)\mathrm{d}u}\beta(s)\left(\gamma_2+\gamma_1\right)||X(s)||\mathrm{d}s, \quad t \geq T,\end{aligned}$$

and hence

$$\widetilde{R}'(t) \leq \left[4M^2(\gamma_2+\gamma_1) - 1\right]\beta(t)\widetilde{R}(t),$$

which yields that

$$\widetilde{R}(t) \leq \widetilde{R}(T)e^{[4M^2(\gamma_2+\gamma_1)-1]\int_{t_0}^t \beta(s)\mathrm{d}s}.$$

This, together with the hypothesis $\int_{t_0}^\infty \frac{r'(s)}{r(s)}\mathrm{d}s = \infty$, leads to $||X(t)|| \to 0$ as $t \to \infty$. The proof is complete.

4 Examples

In this section we give three examples to illustrate our main results.

Example 1 *Consider the following equation*

$$\begin{cases} (e^{2t}x')' + \frac{1}{t^2+1}x - \frac{|x|^{\alpha-1}x}{t^2+1} + e^t|x'|^{\alpha-1}x' = \frac{q}{t^2+1}, t \geq 0,\ t \neq k, \\ x(k) = \left(1 + \frac{1}{k^2}\right)x(k-0),\ x'(k) = \left(1 - \frac{1}{(k+1)^2}\right)x'(k-0),\ k = 1, 2, 3, \cdots, \end{cases} \tag{29}$$

where $\alpha \geq 1$ *and* q *is a constant.*

Let $c_k = 1 + \frac{1}{k^2}$ *and* $d_k = 1 - \frac{1}{(k+1)^2}$. *Note that*

$$C(t) = (1+1)\left(1 + \frac{1}{2^2}\right)\ldots\left(1 + \frac{1}{[t]^2}\right),$$

then we have

$$\ln C(t) = \sum_{k=1}^{[t]} \ln\left(1 + \frac{1}{k^2}\right) \leq \sum_{k=1}^{\infty} \ln\left(1 + \frac{1}{k^2}\right)$$

and hence we we can take

$$\lambda_1(c) = 2 \quad and \quad \lambda_2(c) = e^{\sum_{k=1}^{\infty} \ln\left(1+\frac{1}{k^2}\right)}.$$

Similarly, by calculating we have

$$\lambda_1(d) = e^{\sum_{k=1}^{\infty} \ln\left(1-\frac{1}{(k+1)^2}\right)} \quad and \quad \lambda_2(d) = \frac{1}{2}.$$

Note further that when $f(t, u, v) = -\frac{|u|^{\alpha-1}u}{t^2+1} + e^t|v|^{\alpha-1}v$, *and*

$$r(t) = e^{2t},\ a(t) = b_1(t) = \frac{1}{t^2+1},\ b_2(t) = e^t \quad and \quad p(t) = \frac{q}{t^2+1},$$

all the conditions in Theorem 1-2 are verified. Hence, by Theorem 1 we learn the null solution of (29) is stable when $q = 0$ *and, by Theorem 2, every solution of (29) is bounded.*

Example 2 *Consider the equation*

$$\begin{cases} \left(e^{\frac{t}{10}}x'\right)' + e^{\frac{t}{10}}x - \frac{e^{\frac{t}{10}}}{10^3}x^\alpha = 1, t \geq 0,\ t \neq k, \\ x(k) = e^{\frac{(-1)^k}{k}}x(k-0),\ x'(k) = e^{\frac{(-1)^k}{k}}x'(k-0),\ k = 1, 2, 3, \cdots, \end{cases} \tag{30}$$

where α *is a positive odd integer and, in this case* $f(t, u, v) = -\frac{e^{\frac{t}{10}}}{10^3}u^\alpha$.

For $c_k = d_k = e^{\frac{(-1)^k}{k}}$, *by the definitions of* $C(t)$, $D(t)$ *and* $\widetilde{A}_{12}(t)$ *in (4)–(5) and (13) it is easy to see that* $||\widetilde{A}_{12}(t)|| = 0$ *and*

$$C(t) = D(t) = e^{-1+\frac{1}{2}-\frac{1}{3}+\ldots+\frac{(-1)^{[t]}}{[t]}} \to e^{\ln\frac{1}{2}} \quad as \quad t \to \infty.$$

Hence we can take $M = 3$ *so that* $M \geq \max\{\lambda_1(c)^{-1},\ \lambda_1(d)^{-1},\ \lambda_2(c),\ \lambda_2(d)\}$.

In addition, if we take $r(t) = a(t) = e^{\frac{t}{10}}$, $p(t) = 1$, $\gamma_1 = \frac{1}{400}$ *and* $\gamma_2 = \frac{1}{40}$, *then* $4M^2\gamma_2 < 1$ *and* $4M^2(\gamma_1 + \gamma_2) < 1$. *Furthermore, it follows that*

$$\frac{r'(t)}{r(r)} = \frac{1}{10}, \ \frac{b(t)}{\gamma_1 r(r)} = \frac{1}{10}$$

as well as

$$\widetilde{A}_2(t) = \begin{bmatrix} 0 & 1 - \frac{a(t)}{r(t)} \\ \left(\frac{r'(t)}{2r(t)}\right)' + \left(\frac{r'(t)}{2r(t)}\right)^2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \frac{1}{20^2} & 0 \end{bmatrix}$$

and therefore

$$\frac{||\widetilde{A}_{12}(t)|| + ||\widetilde{A}_2(t)||}{\gamma_2} = \frac{1}{10}.$$

Now by Theorem 3 we see that every solution of (30) tends to zero as $t \to \infty$.

Example 3 *Let us consider the special form of (2) as follows*

$$x'' + x = 0, \quad t \geq t_0. \tag{31}$$

It is clear that we fail to use ***Theorem A*** *to consider the stability of (31) since the function* $h(t) \equiv 0$ *in (1). Now we turn to use our results. At present* $r(t) \equiv 1$, $a(t) \equiv 1$, $c_k = d_k = 1$ *and* $f(t, u, v) \equiv 0$ *in (2), so all the conditions in Corollary 1 are fulfilled and hence the null solution of (31) is stable.*

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