

## EXISTENCE OF PERIODIC SOLUTIONS FOR A SECOND ORDER NONLINEAR NEUTRAL DIFFERENTIAL EQUATION WITH FUNCTIONAL DELAY

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ABSTRACT. In this article we study the existence of periodic solutions of the second order nonlinear neutral differential equation with functional delay

$$\frac{d^2}{dt^2}x(t) + p(t) \frac{d}{dt}x(t) + q(t)x^3(t) = \frac{d}{dt}g(t, x(t - \tau(t))) + f(t, x^3(t), x^3(t - \tau(t))).$$

The main tool employed here is the Burton-Krasnoselskii's hybrid fixed point theorem dealing with a sum of two mappings, one is a large contraction and the other is compact.

### 1. INTRODUCTION

Due to their importance in numerous applications, for example, physics, population dynamics, industrial robotics, and other areas, many authors are studying the existence, uniqueness, stability and positivity of solutions for delay differential equations, see [1]-[23] and references therein.

In this paper, we are interested in the analysis of qualitative theory of periodic solutions of delay differential equations. Motivated by the papers [1]-[6], [9]-[11], [16]-[18], [21]-[23] and the references therein, we concentrate on the existence of periodic solutions for the second order nonlinear neutral differential equation

$$\frac{d^2}{dt^2}x(t) + p(t) \frac{d}{dt}x(t) + q(t)x^3(t) = \frac{d}{dt}g(t, x(t - \tau(t))) + f(t, x^3(t), x^3(t - \tau(t))), \quad (1.1)$$

where  $p, q$  are positive continuous real-valued functions. The function  $g : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  is differentiable and  $f : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  is continuous in their respective arguments. To reach our desired end we transform (1.1) into an integral equation and then use Burton-Krasnoselskii's fixed point theorem to show the existence of periodic solutions. The obtained integral equation splits in the sum of two mappings, one is a large contraction and the other is compact.

Note that in our consideration the neutral term  $\frac{d}{dt}g(t, x(t - \tau(t)))$  of (1.1) produces non-linearity in the derivative term  $\frac{d}{dt}x(t - \tau(t))$ . The neutral term  $\frac{d}{dt}x(t - \tau(t))$  in [2] enters linearly. As a consequence, our analysis is different form that in [2].

The organization of this paper is as follows. In Section 2, we introduce some notations and lemmas, and state some preliminary results needed in later sections, then we give the Green's function of (1.1), which plays an important role in this paper. Also, we present the inversion of (1.1) and Burton-Krasnoselskii's fixed point theorem. In Section 3, we present our main result on existence.

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## 2. PRELIMINARIES

For  $T > 0$ , let  $P_T$  be the set of all continuous scalar functions  $x$ , periodic in  $t$  of period  $T$ . Then  $(P_T, \|\cdot\|)$  is a Banach space with the supremum norm

$$\|x\| = \sup_{t \in \mathbb{R}} |x(t)| = \sup_{t \in [0, T]} |x(t)|.$$

Since we are searching for the existence of periodic solutions for equation (1.1), it is natural to assume that

$$p(t+T) = p(t), \quad q(t+T) = q(t), \quad \tau(t+T) = \tau(t), \quad (2.1)$$

where  $\tau$  is a continuous scalar function, and  $\tau(t) \geq \tau^* > 0$ . Also, we assume

$$\int_0^T p(s) ds > 0, \quad \int_0^T q(s) ds > 0. \quad (2.2)$$

Functions  $g(t, x)$  and  $f(t, x, y)$  are periodic in  $t$  with period  $T$ . They are globally Lipschitz continuous in  $x$  and in  $x$  and  $y$ , respectively. That is,

$$g(t+T, x) = g(t, x), \quad f(t+T, x, y) = f(t, x, y), \quad (2.3)$$

and there are positive constants  $k_1, k_2, k_3$  such that

$$|g(t, x) - g(t, y)| \leq k_1 \|x - y\|, \quad (2.4)$$

and

$$|f(t, x, y) - f(t, z, w)| \leq k_2 \|x - z\| + k_3 \|y - w\|. \quad (2.5)$$

**Lemma 2.1.** ([17]) *Suppose that (2.1) and (2.2) hold and*

$$\frac{R_1 \left[ \exp \left( \int_0^T p(u) du \right) - 1 \right]}{Q_1 T} \geq 1, \quad (2.6)$$

where

$$R_1 = \max_{t \in [0, T]} \left| \int_t^{t+T} \frac{\exp \left( \int_t^s p(u) du \right)}{\exp \left( \int_0^T p(u) du \right) - 1} q(s) ds \right|, \quad Q_1 = \left( 1 + \exp \left( \int_0^T p(u) du \right) \right)^2 R_1^2.$$

Then there are continuous  $T$ -periodic functions  $a$  and  $b$  such that  $b(t) > 0$ ,  $\int_0^T a(u) du > 0$  and

$$a(t) + b(t) = p(t), \quad \frac{d}{dt} b(t) + a(t) b(t) = q(t), \quad \text{for } t \in \mathbb{R}.$$

**Lemma 2.2.** ([22]) *Suppose the conditions of Lemma 2.1 hold and  $\phi \in P_T$ . Then the equation*

$$\frac{d^2}{dt^2} x(t) + p(t) \frac{d}{dt} x(t) + q(t) x(t) = \phi(t),$$

has a  $T$ -periodic solution. Moreover, the periodic solution can be expressed by

$$x(t) = \int_t^{t+T} G(t, s) \phi(s) ds,$$

where

$$G(t, s) = \frac{\int_t^s \exp \left[ \int_t^u b(v) dv + \int_u^s a(v) dv \right] du + \int_s^{t+T} \exp \left[ \int_t^u b(v) dv + \int_u^{s+T} a(v) dv \right] du}{\left[ \exp \left( \int_0^T a(u) du \right) - 1 \right] \left[ \exp \left( \int_0^T b(u) du \right) - 1 \right]}.$$

**Corollary 2.3.** ([22]) *Green's function  $G$  satisfies the following properties*

$$\begin{aligned} G(t, t+T) &= G(t, t), \quad G(t+T, s+T) = G(t, s), \\ \frac{\partial}{\partial s} G(t, s) &= a(s) G(t, s) - \frac{\exp\left(\int_t^s b(v) dv\right)}{\exp\left(\int_0^T b(v) dv\right) - 1}, \\ \frac{\partial}{\partial t} G(t, s) &= -b(t) G(t, s) + \frac{\exp\left(\int_t^s a(v) dv\right)}{\exp\left(\int_0^T a(v) dv\right) - 1}. \end{aligned}$$

The following lemma is fundamental to our results.

**Lemma 2.4.** *Suppose (2.1)-(2.3) and (2.6) hold. If  $x \in P_T$ , then  $x$  is a solution of equation (1.1) if and only if*

$$\begin{aligned} x(t) &= \int_t^{t+T} G(t, s) q(s) [x(s) - x^3(s)] ds \\ &\quad + \int_t^{t+T} g(s, x(s - \tau(s))) [E(t, s) - a(s) G(t, s)] + G(t, s) f(s, x^3(s), x^3(s - \tau(s))) ds, \end{aligned} \quad (2.7)$$

where

$$E(t, s) = \frac{\exp\left(\int_t^s b(v) dv\right)}{\exp\left(\int_0^T b(v) dv\right) - 1}. \quad (2.8)$$

*Proof.* Let  $x \in P_T$  be a solution of (1.1). Rewrite (1.1) as

$$\begin{aligned} \frac{d^2}{dt^2} x(t) + p(t) \frac{d}{dt} x(t) + q(t) x(t) &= q(t) [x(t) - x^3(t)] \\ &\quad + \frac{d}{dt} g(t, x(t - \tau(t))) + f(t, x^3(t), x^3(t - \tau(t))). \end{aligned}$$

From Lemma 2.2, we have

$$\begin{aligned} x(t) &= \int_t^{t+T} G(t, s) q(s) [x(s) - x^3(s)] ds \\ &\quad + \int_t^{t+T} G(t, s) \left[ \frac{\partial}{\partial s} g(s, x(s - \tau(s))) + f(s, x^3(s), x^3(s - \tau(s))) \right] ds. \end{aligned} \quad (2.9)$$

Performing an integration by parts, we have

$$\begin{aligned} &\int_t^{t+T} G(t, s) \frac{\partial}{\partial s} g(s, x(s - \tau(s))) ds \\ &= - \int_t^{t+T} \left[ \frac{\partial}{\partial s} G(t, s) \right] g(s, x(s - \tau(s))) ds \\ &= \int_t^{t+T} g(s, x(s - \tau(s))) [E(t, s) - a(s) G(t, s)] ds, \end{aligned} \quad (2.10)$$

where  $E$  is given by (2.8). We obtain (2.7) by substituting (2.10) in (2.9). Since each step is reversible, the converse follows easily. This completes the proof.  $\square$

**Lemma 2.5.** ([22]) Let  $A = \int_0^T p(u) du$ ,  $B = T^2 \exp\left(\frac{1}{T} \int_0^T \ln(q(u)) du\right)$ . If

$$A^2 \geq 4B, \tag{2.11}$$

then we have

$$\begin{aligned} \min \left\{ \int_0^T a(u) du, \int_0^T b(u) du \right\} &\geq \frac{1}{2} \left( A - \sqrt{A^2 - 4B} \right) := l, \\ \max \left\{ \int_0^T a(u) du, \int_0^T b(u) du \right\} &\leq \frac{1}{2} \left( A + \sqrt{A^2 - 4B} \right) := m. \end{aligned}$$

**Corollary 2.6.** ([22]) Functions  $G$  and  $E$  satisfy

$$\frac{T}{(e^m - 1)^2} \leq G(t, s) \leq \frac{T \exp\left(\int_0^T p(u) du\right)}{(e^l - 1)^2}, \quad |E(t, s)| \leq \frac{e^m}{e^l - 1}.$$

In the analysis, we employ a fixed point theorem in which the notion of a large contraction is required as one of the sufficient conditions. The following definition, due to T. A. Burton, can be found in [5], [6].

**Definition 2.7** (Large Contraction). Let  $(\mathbb{M}, d)$  be a metric space and consider  $\mathcal{B} : \mathbb{M} \rightarrow \mathbb{M}$ . Then  $\mathcal{B}$  is said to be a *large contraction* if given  $\phi, \varphi \in \mathbb{M}$  with  $\phi \neq \varphi$  then  $d(\mathcal{B}\phi, \mathcal{B}\varphi) \leq d(\phi, \varphi)$  and if for all  $\varepsilon > 0$ , there exists a  $\delta < 1$  such that

$$[\phi, \varphi \in \mathbb{M}, d(\phi, \varphi) \geq \varepsilon] \Rightarrow d(\mathcal{B}\phi, \mathcal{B}\varphi) \leq \delta d(\phi, \varphi).$$

The next theorem is also a result of T. A. Burton. This captivating theorem, which constitutes a basis for our main result, is a reformulated version of Krasnoselskii's fixed point theorem and has been used successfully in existence and stability in differential equations (see [[5], Theorem 3] and [6]).

**Theorem 2.8** (Burton-Krasnoselskii). Let  $\mathbb{M}$  be a closed bounded convex nonempty subset of a Banach space  $(\mathbb{B}, \|\cdot\|)$ . Suppose that  $\mathcal{A}$  and  $\mathcal{B}$  map  $\mathbb{M}$  into  $\mathbb{M}$  such that

- (i)  $x, y \in \mathbb{M}$ , implies  $\mathcal{A}x + \mathcal{B}y \in \mathbb{M}$ ,
- (ii)  $\mathcal{A}$  is compact and continuous,
- (iii)  $\mathcal{B}$  is a large contraction mapping.

Then there exists  $z \in \mathbb{M}$  with  $z = \mathcal{A}z + \mathcal{B}z$ .

We will use this theorem to prove the existence of periodic solutions for equation (1.1). We begin with the following proposition (see [5], [6]) and for convenience we present its proof.

**Proposition 2.9.** If  $\|\cdot\|$  is the maximum norm,

$$\mathbb{M} = \left\{ \varphi \in P_T : \|\varphi\| \leq \sqrt{3}/3 \right\},$$

and  $(\mathfrak{B}\varphi)(t) = \varphi(t) - \varphi^3(t)$ , then  $\mathfrak{B}$  is a large contraction of the set  $\mathbb{M}$ .

*Proof.* For each  $t \in \mathbb{R}$  we have for the real functions  $\varphi, \psi$

$$\begin{aligned} |(\mathfrak{B}\varphi)(t) - (\mathfrak{B}\psi)(t)| \\ = |\varphi(t) - \psi(t)| \left| 1 - (\varphi^2(t) + \varphi(t)\psi(t) + \psi^2(t)) \right|. \end{aligned}$$

On the other hand,

$$\begin{aligned} |\varphi(t) - \psi(t)|^2 &= \varphi^2(t) - 2\varphi(t)\psi(t) + \psi^2(t) \\ &\leq 2(\varphi^2(t) + \psi^2(t)). \end{aligned}$$

Using  $\varphi^2(t) + \psi^2(t) < 1$  we have

$$\begin{aligned} &|(\mathfrak{B}\varphi)(t) - (\mathfrak{B}\psi)(t)| \\ &\leq |\varphi(t) - \psi(t)| [1 - (\varphi^2(t) + \psi^2(t)) + |\varphi(t)\psi(t)|] \\ &\leq |\varphi(t) - \psi(t)| \left[ 1 - (\varphi^2(t) + \psi^2(t)) + \frac{\varphi^2(t) + \psi^2(t)}{2} \right] \\ &\leq |\varphi(t) - \psi(t)| \left[ 1 - \frac{\varphi^2(t) + \psi^2(t)}{2} \right] \\ &\leq \|\varphi - \psi\|. \end{aligned}$$

Then

$$\|\mathfrak{B}\varphi - \mathfrak{B}\psi\| \leq \|\varphi - \psi\|.$$

Now, let  $\epsilon \in (0, 1)$  be given and let  $\varphi, \psi \in \mathbb{M}$  with  $\|\varphi - \psi\| \geq \epsilon$ .

a) Suppose that for some  $t$  we have

$$\epsilon/2 \leq |\varphi(t) - \psi(t)|.$$

Then

$$(\epsilon/2)^2 \leq |\varphi(t) - \psi(t)|^2 \leq 2(\varphi^2(t) + \psi^2(t)),$$

that is

$$\varphi^2(t) + \psi^2(t) \geq \epsilon^2/8.$$

For all such  $t$  we have

$$\begin{aligned} |(\mathfrak{B}\varphi)(t) - (\mathfrak{B}\psi)(t)| &\leq |\varphi(t) - \psi(t)| \left[ 1 - \frac{\epsilon^2}{16} \right] \\ &\leq \left[ 1 - \frac{\epsilon^2}{16} \right] \|\varphi - \psi\|. \end{aligned}$$

b) Suppose that for some  $t$  we have

$$|\varphi(t) - \psi(t)| \leq \epsilon/2,$$

then

$$|(\mathfrak{B}\varphi)(t) - (\mathfrak{B}\psi)(t)| \leq |\varphi(t) - \psi(t)| \leq (1/2)\|\varphi - \psi\|.$$

So, for all  $t$  we have

$$|(\mathfrak{B}\varphi)(t) - (\mathfrak{B}\psi)(t)| \leq \max \left\{ 1/2, 1 - \frac{\epsilon^2}{16} \right\} \|\varphi - \psi\|.$$

Hence, for each  $\epsilon > 0$ , if  $\delta = \max \left\{ 1/2, 1 - \frac{\epsilon^2}{16} \right\} < 1$ , then

$$\|\mathfrak{B}\varphi - \mathfrak{B}\psi\| \leq \delta \|\varphi - \psi\|.$$

Consequently,  $\mathfrak{B}$  is a large contraction. □

### 3. EXISTENCE OF PERIODIC SOLUTIONS

To apply Theorem 2.8, we need to define a Banach space  $\mathbb{B}$ , a bounded convex subset  $\mathbb{M}$  of  $\mathbb{B}$  and construct two mappings, one is a large contraction and the other is compact. So, we let  $(\mathbb{B}, \|\cdot\|) = (P_T, \|\cdot\|)$  and  $\mathbb{M} = \{\varphi \in \mathbb{B} : \|\varphi\| \leq L\}$ , where  $L = \sqrt{3}/3$ . We express equation (2.7) as

$$\varphi(t) = (\mathcal{B}\varphi)(t) + (\mathcal{A}\varphi)(t) := (H\varphi)(t),$$

where  $\mathcal{A}, \mathcal{B} : \mathbb{M} \rightarrow \mathbb{B}$  are defined by

$$(\mathcal{A}\varphi)(t) = \int_t^{t+T} g(s, \varphi(s - \tau(s))) [E(t, s) - a(s)G(t, s)] + G(t, s) f(s, \varphi^3(s), \varphi^3(s - \tau(s))) ds, \quad (3.1)$$

and

$$(\mathcal{B}\varphi)(t) = \int_t^{t+T} G(t, s) q(s) [\varphi(s) - \varphi^3(s)] ds. \quad (3.2)$$

To simplify notations, we introduce the following constants

$$\begin{aligned} \alpha &= \frac{T \exp\left(\int_0^T p(u) du\right)}{(e^T - 1)^2}, \quad \beta = \frac{e^m}{e^T - 1}, \quad \sigma = \max_{t \in [0, T]} |q(t)|, \quad \lambda = \max_{t \in [0, T]} |b(t)|, \\ \mu &= \max_{t \in [0, T]} |a(t)|, \quad \rho_1 = \max_{t \in [0, T]} |g(t, 0)|, \quad \rho_2 = \max_{t \in [0, T]} |f(t, 0, 0)|. \end{aligned} \quad (3.3)$$

We need the following assumptions

$$\alpha\sigma T \leq 1, \quad (3.4)$$

$$JT [(k_1L + \rho_1)(\beta + \mu\alpha) + \alpha((k_2 + k_3)L^3 + \rho_2)] \leq L, \quad (3.5)$$

where  $J$  is constants with  $J \geq 3$ .

We shall prove that the mapping  $H$  has a fixed point which solves (1.1), whenever its derivative exists.

**Lemma 3.1.** *Suppose that conditions (2.1)-(2.6), (2.11) and (3.5) hold. Then  $\mathcal{A} : \mathbb{M} \rightarrow \mathbb{M}$  is compact.*

*Proof.* Let  $\mathcal{A}$  be defined by (3.1). Obviously,  $\mathcal{A}\varphi$  is continuous and it is easy to show that  $(\mathcal{A}\varphi)(t + T) = (\mathcal{A}\varphi)(t)$ . Observe that in view of (2.4) and (2.5), we have

$$\begin{aligned} |g(t, x)| &\leq |g(t, x) - g(t, 0) + g(t, 0)| \\ &\leq |g(t, x) - g(t, 0)| + |g(t, 0)| \\ &\leq k_1 \|x\| + \rho_1. \end{aligned}$$

Similarly,

$$\begin{aligned} |f(t, x, y)| &\leq |f(t, x, y) - f(t, 0, 0) + f(t, 0, 0)| \\ &\leq |f(t, x, y) - f(t, 0, 0)| + |f(t, 0, 0)| \\ &\leq k_2 \|x\| + k_3 \|y\| + \rho_2. \end{aligned}$$

So, for any  $\varphi \in \mathbb{M}$ , we have

$$\begin{aligned} & |(\mathcal{A}\varphi)(t)| \\ & \leq \int_t^{t+T} |g(s, \varphi(s - \tau(s)))| [|E(t, s)| + |a(s)| |G(t, s)|] + |G(t, s)| |f(s, \varphi^3(s), \varphi^3(s - \tau(s)))| ds \\ & \leq \int_t^{t+T} (k_1 L + \rho_1)(\beta + \mu\alpha) + \alpha((k_2 + k_3)L^3 + \rho_2) ds \\ & \leq T[(k_1 L + \rho_1)(\beta + \mu\alpha) + \alpha((k_2 + k_3)L^3 + \rho_2)] \leq \frac{L}{J} < L. \end{aligned}$$

That is  $\mathcal{A}\varphi \in \mathbb{M}$ .

To see that  $\mathcal{A}$  is continuous, we let  $\varphi, \psi \in \mathbb{M}$ . Given  $\varepsilon > 0$ , take  $\eta = \varepsilon/N$  with  $N = T[k_1(\beta\theta + \mu\alpha) + 3\alpha(k_2 + k_3)L^2]$  where  $k_1, k_2$  and  $k_3$  are given by (2.4) and (2.5). Now, for  $\|\varphi - \psi\| < \eta$ , we obtain

$$\begin{aligned} \|\mathcal{A}\varphi - \mathcal{A}\psi\| & \leq \int_t^{t+T} [k_1(\beta + \mu\alpha)\|\varphi - \psi\| + \alpha(k_2 + k_3)\|\varphi^3 - \psi^3\|] ds \\ & \leq N\|\varphi - \psi\| < \varepsilon. \end{aligned}$$

This proves that  $\mathcal{A}$  is continuous.

To show that the image of  $\mathcal{A}$  is contained in a compact set. Let  $\varphi_n \in \mathbb{M}$ , where  $n$  is a positive integer. Then, as above, we see that

$$\|\mathcal{A}\varphi_n\| \leq L.$$

Next we calculate  $\frac{d}{dt}(\mathcal{A}\varphi_n)(t)$  and show that it is uniformly bounded. By making use of (2.1), (2.2) and (2.3) we obtain by taking the derivative in (3.1) that

$$\begin{aligned} & \frac{d}{dt}(\mathcal{A}\varphi_n)(t) \\ & = \frac{\exp\left(\int_t^{t+T} b(v) dv\right) - 1}{\exp\left(\int_0^T b(v) dv\right) - 1} g(t, \varphi_n(t - \tau(t))) \\ & + \int_t^{t+T} g(s, \varphi_n(s - \tau(s))) \left[ -b(t)E(t, s) - a(s) \left( -b(t)G(t, s) + \frac{\exp\left(\int_t^s a(v) dv\right)}{\exp\left(\int_0^T a(v) dv\right) - 1} \right) \right] ds \\ & + \int_t^{t+T} \left( -b(t)G(t, s) + \frac{\exp\left(\int_t^s a(v) dv\right)}{\exp\left(\int_0^T a(v) dv\right) - 1} \right) f(s, \varphi_n^3(s), \varphi_n^3(s - \tau(s))) ds. \end{aligned}$$

Consequently, by invoking (2.4), (2.5) and (3.3), we obtain

$$\begin{aligned} & \left| \frac{d}{dt}(\mathcal{A}\varphi_n)(t) \right| \\ & \leq \beta(k_1 L + \rho_1) + T[(k_1 L + \rho_1)(\lambda\beta + \mu(\lambda\alpha + \beta)) + (\lambda\alpha + \beta)((k_2 + k_3)L^3 + \rho_2)] \\ & \leq D, \end{aligned}$$

for some positive constant  $D$ . Hence the sequence  $(\mathcal{A}\varphi_n)$  is uniformly bounded and equicontinuous. The Ascoli-Arzelà theorem implies that a subsequence  $(\mathcal{A}\varphi_{n_k})$  of  $(\mathcal{A}\varphi_n)$  converges

uniformly to a continuous  $T$ -periodic function. Thus  $\mathcal{A}$  is continuous and  $\mathcal{A}(\mathbb{M})$  is contained in a compact subset of  $\mathbb{M}$ .  $\square$

**Lemma 3.2.** *For  $\mathcal{B}$  defined in (3.2), suppose that (3.4) holds. Then  $\mathcal{B} : \mathbb{M} \rightarrow \mathbb{M}$  is a large contraction.*

*Proof.* Let  $\mathcal{B}$  be defined by (3.2). Obviously,  $\mathcal{B}\varphi$  is continuous and it is easy to show that  $(\mathcal{B}\varphi)(t+T) = (\mathcal{B}\varphi)(t)$ . So, for any  $\varphi \in \mathbb{M}$ , we have

$$\begin{aligned} |(\mathcal{B}\varphi)(t)| &\leq \int_t^{t+T} |G(t,s)| |q(s)| |\varphi(s) - \varphi^3(s)| ds \\ &\leq \alpha\sigma T \|\varphi - \varphi^3\|. \end{aligned}$$

Since  $\|\varphi\| \leq L$ , we have  $\|\varphi - \varphi^3\| \leq (2\sqrt{3})/9 < L$ . So, for any  $\varphi \in M$ , we have

$$\|\mathcal{B}\varphi\| \leq L.$$

Thus  $\mathcal{B}\varphi \in \mathbb{M}$ . Consequently, we have  $\mathcal{B} : \mathbb{M} \rightarrow \mathbb{M}$ .

It remains to show that  $\mathcal{B}$  is large contraction. From the proof of Proposition 2.9 we have for  $\varphi, \psi \in \mathbb{M}$ , with  $\varphi \neq \psi$

$$|(\mathcal{B}\varphi)(t) - (\mathcal{B}\psi)(t)| \leq \alpha\sigma T \|\varphi - \psi\| \leq \|\varphi - \psi\|.$$

Then  $\|\mathcal{B}\varphi - \mathcal{B}\psi\| \leq \|\varphi - \psi\|$ . Now, let  $\epsilon \in (0, 1)$  be given and let  $\varphi, \psi \in \mathbb{M}$  with  $\|\varphi - \psi\| \geq \epsilon$ . From the proof of the Proposition 2.9 we have found  $\delta < 1$  such that

$$|(\mathcal{B}\varphi)(t) - (\mathcal{B}\psi)(t)| \leq \alpha\sigma T \delta \|\varphi - \psi\| \leq \delta \|\varphi - \psi\|.$$

Then  $\|\mathcal{B}\varphi - \mathcal{B}\psi\| \leq \delta \|\varphi - \psi\|$ . Consequently,  $\mathcal{B}$  is a large contraction.  $\square$

**Theorem 3.3.** *Let  $(P_T, \|\cdot\|)$  be the Banach space of continuous  $T$ -periodic real valued functions and  $\mathbb{M} = \{\varphi \in P_T : \|\varphi\| \leq L\}$ , where  $L = \sqrt{3}/3$ . Suppose (2.1)-(2.6), (2.11), (3.4) and (3.5) hold. Then equation (1.1) has a  $T$ -periodic solution  $\varphi$  in the subset  $\mathbb{M}$ .*

*Proof.* By Lemma 3.1, the operator  $\mathcal{A} : \mathbb{M} \rightarrow \mathbb{M}$  is compact and continuous. Also, from Lemma 3.2, the operator  $\mathcal{B} : \mathbb{M} \rightarrow \mathbb{M}$  is a large contraction. Moreover, if  $\varphi, \psi \in M$ , we see that

$$\|\mathcal{A}\varphi + \mathcal{B}\psi\| \leq \|\mathcal{A}\varphi\| + \|\mathcal{B}\psi\| \leq L/J + (2\sqrt{3})/9 \leq L.$$

Thus  $\mathcal{A}\varphi + \mathcal{B}\psi \in \mathbb{M}$ .

Clearly, all the hypotheses of the Burton-Krasnoselskii theorem are satisfied. Thus there exists a fixed point  $\varphi \in \mathbb{M}$  such that  $\varphi = \mathcal{A}\varphi + \mathcal{B}\varphi$ . By Lemma 2.4 this fixed point is a solution of (1.1) and the proof is complete.  $\square$

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