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Existence and global attractivity of periodic solution for impulsive stochastic Volterra-Levin equations*

Dingshi Li[a]†, Daoyi Xu[b]

[a]*School of Mathematics, Southwest Jiaotong University, Chengdu, 610031, P.R. China*

[b]*Yangtze Center of Mathematics, Sichuan University, Chengdu, 610064, P. R. China*

Abstract

In this paper, we consider a class of impulsive stochastic Volterra-Levin equations. By establishing a new integral inequality, some sufficient conditions for the existence and global attractivity of periodic solution for impulsive stochastic Volterra-Levin equations are given. Our results imply that under the appropriate linear periodic impulsive perturbations, the impulsive stochastic Volterra-Levin equations preserve the original periodic property of the nonimpulsive stochastic Volterra-Levin equations. An example is provided to show the effectiveness of the theoretical results.

Keywords: Periodic Solution; Volterra-Levin; Stochastic; Impulsive; Global Attractivity

2000 Mathematics Subject Classifications: 34A37, 34D23.

1 Introduction

Since Itô introduced his stochastic calculus about 50 years ago, the theory of stochastic differential equations has been developed very quickly [1–3]. It is now being recognized to be not only richer than the corresponding theory of differential equations without stochastic perturbation but also represent a more natural framework for mathematical modeling of many real-world phenomena. Now there also exists a well-developed qualitative theory of stochastic differential equations [4–6]. However, not so much has been developed in the direction of the periodically stochastic differential equations. Till now only a few papers have been published on this topic [7–10]. In [10], Xu et al. showed that stochastic differential equations with delay has a periodic solution if its solutions are uniformly bounded and point dissipativity.

Meanwhile, the theory of impulsive differential equations has attracted the interest of many researchers in the past twenty years [11–15] since they provide a natural description of several real processes subject to certain perturbations whose duration is negligible in comparison with the duration of the process. Such processes are often investigated in various fields of science and technology such as physics, population dynamics, ecology, biological systems, optimal control, etc. For details, see [11, 13] and references therein. In [16], the stability of nonlinear stochastic differential delay systems with impulsive are studied by constructing an

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†Corresponding author. Email: lidingshi2006@163.com

impulse control for a nonlinear stochastic differential delay system. Recently, the corresponding theory for the existence of periodic solution for impulsive functional differential equations has been studied by several authors [17–20].

To the best of our knowledge, there are no results on the existence of periodic solution for impulsive stochastic differential equation, which is very important in both theories and applications and also is a very challenging problem. Motivated by the above discussions, in this paper, we will focus on the existence and global attractivity of periodic solution for impulsive stochastic Volterra-Levin equations [21, 22]. First we will establish the equivalence between the solution of impulsive stochastic Volterra-Levin equations and that of a corresponding nonimpulsive stochastic Volterra-Levin equations by the method given in [16]. Then, by establishing a new integral inequality, some sufficient conditions for the existence and global attractivity of periodic solution for impulsive stochastic Volterra-Levin equations are given. Our results imply that under the appropriate linear periodic impulsive perturbations, the impulsive stochastic Volterra-Levin equations preserve the original periodic property of the nonimpulsive stochastic Volterra-Levin equations. An example is provided to show the effectiveness of the theoretical results.

2 Model description and preliminaries

For convenience, we introduce several notations and recall some basic definitions.

$C(X,Y)$ denotes the space of continuous mappings from the topological space X to the topological space Y. Especially, let $C \triangleq C([-\tau,0],R)$ with a norm $\|\varphi\| = \sup\limits_{-\tau\le s\le 0} |\varphi(s)|$ and $|\cdot|$ is the Euclidean norm of a vector $x \in R$, where τ is a positive constant.

$$PC(J,H) = \Big\{\psi(t) : J \to H \mid \psi(t) \text{ is continuous for all but at most countable points } s \in J \\ \text{and at these points } s \in J,\ \psi(s^+) \text{ and } \psi(s^-) \text{ exist, } \psi(s^-) = \psi(s)\Big\},$$

where $J \subset R$ is an interval, H is a complete metric space, $\psi(s^+)$ and $\psi(s^-)$ denote the right-hand and left-hand limit of the function $\psi(s)$, respectively. Especially, let $PC \triangleq PC([-\tau,0],R)$.

Let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\ge 0}, P)$ be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\ge 0}$ satisfying the usual conditions (i.e, it is right continuous and $\mathscr{F}_0$ contains all P-null sets). If $x(t)$ is an R-valued stochastic process on $t \in [t_0-\tau,\infty)$, we let $x_t = x(t+s) : -\tau \le s \le 0$, which is regarded as a PC-valued stochastic process for $t \ge 0$. Denote by $PC^b_{\mathscr{F}_0}([-\tau,0],R)$ ($BC^b_{\mathscr{F}_0}([-\tau,0],R)$) the family of all bounded $\mathscr{F}_0$-measurable, PC-valued (C-valued) random variables ϕ, satisfying $\|\phi\|^p_{L^p} = \sup\limits_{-\tau\le s\le 0} E|\phi(s)|^p < \infty$, where $E[f]$ means the mathematical expectation of f.

For any $\phi \in C$, we define $[\phi(t)]_\tau = \sup_{-\tau\le s\le 0} |\phi(t+s)|$. In the following discussion, we always use the notations

$$\underline{f} = \min_{t\in[0,\omega]} |f(t)|, \qquad \overline{f} = \max_{t\in[0,\omega]} |f(t)|,$$

where $f(t)$ is a continuous ω-periodic function, where $\omega > 0$.

We consider impulsive stochastic Volterra-Levin equations as follows:

$$\begin{cases} dx(t) = -(\int_{t-\tau}^{t} p(s-t) g(s, x(s))\, ds)dt + \sigma(t)\, dB(t), & t \ge t_0 \ge 0, \quad t \ne t_k, \\ x(t_k^+) = b_k x(t_k), \quad t \ge t_0, \quad t = t_k, \end{cases} \tag{1}$$

with initial condition

$$x_{t_0}(s) = \varphi(s) \in PC^b_{\mathscr{F}_0}([-\tau,0],R), \quad s \in [-\tau,0], \tag{2}$$

where $p \in C([-\tau,0],R)$, $g \in C(R,R)$ and $\sigma \in C([t_0,\infty),R)$.

Remark 2.1. Recently, Appleby [21] and Luo [22] studied the stability of Eq. (1) with $g(t, x(t)) = g(x(t))$ and $b_k = 1$, $k = 1, 2, \cdots$, by using fixed point theory, respectively. In [21,22], for the stability purpose, they assume that $\int_{t_0}^{\infty} e^{4\alpha s}\sigma(s)\, ds < \infty$, or, $\sigma^2(t)\ln t \to 0$ as $t \to \infty$. In this paper, we will assume that $\sigma(t)$ is a periodic function.

Throughout this paper, we make the following assumptions:

(H_1) $t_0 < t_1 < t_2 < \cdots$ are fixed impulsive points with $\lim\limits_{k\to\infty} t_k = \infty$.

(H_2) $\{b_k\}$ is a real sequence and $b_k \neq 0$, $k = 1, 2, \cdots$.

(H_3) $I(t) = \prod\limits_{t_0<t_k<t} b_k$ is a periodic function with period ω, $k = 1, 2, \cdots$, and $m \leq |\prod\limits_{t_0\leq t<t_0+\omega} I(t)| \leq M$.

(H_4) $g(t, x(t))$ and $\sigma(t)$ are periodic continuous functions with periodic ω for $t \geq t_0$.

(H_5) $g(t, x(t))$ is Lipschitz-continuous with Lipschitz constant L. Without loss of generality, we also assume that $g(t, 0) = 0$, $xg(t, x) \geq 0$ and $\lim\limits_{x\to 0} \frac{g(t,x)}{x} = \gamma(t) < \infty$.

Define

$$h(t) := \begin{cases} \frac{g(t,x(t))}{x(t)}, & x(t) \neq 0, \\ \gamma(t), & x(t) = 0 \end{cases}$$

and $\int_{-\tau}^{0} p(s)\, ds = \alpha$, where $\alpha > 0$.

Remark 2.2. Condition (H_5) is similar as the conditions on g and p in [21,22].

Remark 2.3. It follows from (H_4) and (H_5) that function $h(t)$ is nonnegative integral function and satisfies that $\sup\limits_{t\geq t_0} \int_{t-\tau}^{t} h(s)ds = H$ and $\lim\limits_{t\to\infty} \int_{t_0}^{t} h(s)ds = \infty$.

Definition 2.1. A function $x(t)$ defined on $[t_0-\tau, \infty)$ is said to be a solution of Eq. (1) with initial condition (2) if

(a) $x(t)$ is absolutely continuous on each interval $(t_k, t_{k+1}]$, $k = 0, 1, \cdots$;

(b) For any t_k, $k = 1, 2, \cdots$, $x(t_k^+)$ and $x(t_k^-)$ exist and $x(t_k^-) = x(t_k)$;

(c) $x(t)$ satisfies the differential equation of (1) for almost everywhere in $[t_0, \infty)\backslash t_k$ and the impulsive condtions for every $t = t_k$, $k = 1, 2, \cdots$.

(d) $x_{t_0}(s) = \varphi(s)$, $s \in [-\tau, 0]$.

Under Condition (H_5), Eq. (1) can be rewritten as follows:

$$\begin{cases} dx(t) = -\alpha h(t)x(t)\, dt + d\left(\int_{-\tau}^{0} p(s)\int_{t+s}^{t} g(u, x(u))\, duds\right) + \sigma(t)\, dB(t), & t \geq t_0 \geq 0, \quad t \neq t_k, \\ x(t_k^+) = b_k x(t_k), \quad t \geq t_0, & t = t_k. \end{cases} \tag{3}$$

Under the assumptions $(H_1) - (H_5)$, we consider the following system:

$$\begin{aligned} dy(t) &= -\alpha h(t)y(t)\, dt + \prod_{t_0<t_k<t} b_k^{-1} d\left(\int_{-L}^{0} p(s)\int_{t+s}^{t} g\left(u, \prod_{t_0<t_k<u} b_k y(u)\right) duds\right) \\ &\quad + \prod_{t_0<t_k<t} b_k^{-1}\sigma(t)\, dB(t), \quad t \geq t_0, \end{aligned} \tag{4}$$

with initial condition

$$y_{t_0}(s) = \varphi(s), \quad s \in [-\tau, 0]. \tag{5}$$

By a solution $y(t)$ of (4) with initial condition (5), we mean an absolutely continuous function $y(t)$ defined on $[t_0, \infty)$ satisfying (4) a.e. for $t \geq t_0$ and $y(t) = \varphi(t)$ on $[t_0 - \tau, t_0]$.

The following lemma will be useful to prove our results. The proof is similar to that of Lemma 3.1 in [16].

Lemma 2.1. Assume that $(H_1)-(H_5)$ hold. Then

(i) if $y(t)$ is a solution of (4) and (5), then $x(t)=\prod_{t_0<t_k<t} b_k y(t)$ is a solution of (3) and (2) on $[t_0-\tau,\infty)$;

(ii) if $x(t)$ is a solution of (3) and (2), then $y(t)=\prod_{t_0<t_k<t} b_k^{-1} x(t)$ is a solution of (4) and (5) on $[t_0-\tau,\infty)$.

Proof. (i) Suppose that $y(t)$ is a solution of (4) on $[t_0,\infty)$, then we have for any $t\neq t_k,\ k=1,2,\cdots,$

$$\begin{aligned}
dx(t) &= \prod_{t_0<t_k<t} b_k dy(t) \\
&= \prod_{t_0<t_k<t} b_k \left[-\alpha h(t)y(t)dt+\prod_{t_0<t_k<t} b_k^{-1} d\left(\int_{-L}^{0} p(s)\int_{t+s}^{t} g\left(u,\prod_{t_0<t_k<u} b_k y(u)\right)duds\right)\right] \\
&\quad +\sigma(t)dB(t) \\
&= -\alpha h(t)x(t)dt+d\left(\int_{-\tau}^{0} p(s)\int_{t+s}^{t} g(u,x(u))duds\right)+\sigma(t)dB(t), \quad t\geq t_0,
\end{aligned}$$

which implies that $x(t)$ satisfies the first equation of (3) for almost everywhere in $[t_0,\infty)\backslash t_k$.

On the other hand, for every $t=t_k,\ k=1,2,\cdots,$

$$x\left(t_k^+\right)=\lim_{t\to t_k^+}\prod_{t_0<t_j<t} b_j y(t)=\prod_{t_0<t_j\leq t_k} b_j y(t)$$

and

$$x(t_k)=\prod_{t_0<t_j<t_k} b_j y(t).$$

this means that, for every $t=t_k,\ k=1,2,\cdots,$

$$x\left(t_k^+\right)=b_k x(t_k).$$

Therefore, we arrive at a conclusion that $x(t)$ is the solution of (3) corresponding to initial condition (2). In fact, if $y(t)$ is the solution of (4) with initial condition (5), then $x(t)=\prod_{t_0<t_k<t} b_k y(t)=y(t)=\varphi(t)$ on $[t_0-\tau,t_0]$.

(ii) Since $x(t)$ is a solution of (3) and (2), so $x(t)$ is absolutely continuous on each interval $(t_k,t_{k+1}),\ k=1,2,\cdots$. Therefore, $y(t)=\prod_{t_0<t_k<t} b_k^{-1}x(t)$ is absolutely continuous on $(t_k,t_{k+1}),\ k=1,2,\cdots$. What's more, it follows that, for any $t=t_k,\ k=1,2,\cdots,$

$$\begin{aligned}
y\left(t_k^+\right) &= \lim_{t\to t_k^+}\prod_{t_0<t_j<t} b_j^{-1}x(t) \\
&= \prod_{t_0<t_j\leq t_k} b_j^{-1}x\left(t_k^+\right) \\
&= \prod_{t_0<t_j<t_k} b_j^{-1}x(t_k) \\
&= y(t_k)
\end{aligned}$$

and

$$\begin{aligned}
y\left(t_k^-\right) &= \lim_{t\to t_k^-}\prod_{t_0<t_j<t} b_j^{-1}x(t) \\
&= \prod_{t_0<t_j<t_k} b_j^{-1}x\left(t_k^-\right) \\
&= y(t_k),
\end{aligned}$$

which implies that $y(t)$ is continuous and easy to prove absolutely continuous on $[t_0, \infty)$. Now, similar proof in the case (i), we can easily check that $y(t) = \prod_{t_0 < t_k < t} b_k^{-1} x(t)$ is the solution of (4) on $[t_0 - \tau, \infty)$ corresponding to the initial condition (5).

From the above analysis, we know that the conclusion of Lemma 2.1 is true. The proof is complete.

□

We assume that for any $\varphi \in PC^b_{\mathscr{F}_0}([-\tau, 0], R)$, there exists a unique solution of (3). Later on we shall often denote the solution of (3) by $x(t) = x(t, t_0, \varphi)$, or $x_t(t_0, \varphi)$ for all t_0 and $\varphi \in PC^b_{\mathscr{F}_0}([-\tau, 0], R)$. By Lemma 2.1, for any $\varphi \in PC^b_{\mathscr{F}_0}([-\tau, 0], R)$, there exists a unique solution of (4). We also shall often denote the solution of (4) by $y(t) = x(t, t_0, \varphi)$, or $y_t(t_0, \varphi)$ for all t_0 and $\varphi \in PC^b_{\mathscr{F}_0}([-\tau, 0], R)$.

Definition 2.2. A stochastic process $x_t(s)$ is said to be periodic with period ω if its finite dimensional distributions are periodic with periodic ω, i.e., for any positive integer m and any moments of time $t_1, \ldots, t_m$, the joint distributions of the random variables $x_{t_{1+k\omega}}(s), \ldots, x_{t_{m+k\omega}}(s)$ are independent of k, $(k = \pm 1, \pm 2, \cdots)$.

Remark 2.4. By the definition of periodicity, if $x_t(s)$ is an ω-periodic stochastic process, then its mathematic expectation and variance are ω-periodic [8, p49].

Definition 2.3. The set $S \subset PC^b_{\mathscr{F}_0}([-\tau, 0], R)$ is called a global attracting set of (3), if for any initial value $\varphi \in PC^b_{\mathscr{F}_0}([-\tau, 0], R)$, we have

$$\operatorname{dist}(x_t(t_0, \varphi), S) \to 0 \quad \text{as} \quad t \to \infty,$$

where

$$\operatorname{dist}(\eta, S) = \inf_{\gamma \in S} \rho(\eta, \gamma) \quad \text{for} \quad \eta \in PC^b_{\mathscr{F}_0}([-\tau, 0], R),$$

where $\rho(\cdot, \cdot)$ is any distance in $PC^b_{\mathscr{F}_0}([-\tau, 0], R)$.

Definition 2.4. The periodic solution $x(t, t_0, \varphi)$ with the initial condition $\varphi \in PC^b_{\mathscr{F}_0}([-\tau, 0], R^n)$ of Eq. (3) is called globally attractive if for any solution $x(t, t_0, \varphi_1)$ with the initial condition $\varphi_1 \in PC^b_{\mathscr{F}_0}([-\tau, 0], R^n)$ of Eq. (3),

$$E|x(t, t_0, \varphi) - x(t, t_0, \varphi_1)| \to 0 \quad \text{as} \quad t \to \infty.$$

Remark 2.5. Similarly as Definition 2.2-2.4, the periodicity, attracting set and global attractivity of the solution of (4) can be defined.

Remark 2.6. From Lemma 2.1, we can easily obtain that if the periodic solution of (4) is globally attractive, then the periodic solution of (3) is also globally attractive.

Definition 2.5. The solutions $y_t(t_0, \varphi)$ of (4) are said to be

(i) p-uniformly bounded, if for each $\alpha > 0$, $t_0 \in R$, there exists a positive constant $\theta = \theta(\alpha)$ which is independent of t_0 such that $\|\varphi\|^p_{L^P} \leq \alpha$ implies $E[\|y_t(t_0, \varphi)\|^p] \leq \theta, \ \ t \geq t_0$;

(ii) p-point dissipative, if there is a constant $N > 0$, for any point $\varphi \in BC^b_{\mathscr{F}_0}([-\tau, 0], R^n)$, there exists $T(t_0, \varphi)$ such that

$$E[\|y_t(t_0, \varphi)\|^p] \leq N, \quad t \geq t_0 + T(t_0, \varphi).$$

We recall the following result [10, Theorem 3.5] which lays the foundation for the existence of periodic solution to Eq. (4).

Lemma 2.2. Under Conditions $(H_1) - (H_5)$, assume that the solutions of Eq. (4) are p-uniformly bounded and p-point dissipative for $p > 2$, then there is an ω-periodic solution.

Lemma 2.3. Let $u(t) \in C(R, R_+)$ be a solution of the delay integral inequality

$$\begin{cases} u(t) \le \eta_1 e^{-\delta \int_{t_0}^t h(v)dv} + \eta_2 [u(t)]_\tau + \eta_3 \int_{t_0}^t e^{-\delta \int_s^t h(v)dv} h(s) [u(s)]_\tau \, ds + \eta_4, \quad t \ge t_0, \\ u(t) \le \phi(t), \quad t \in [t_0 - \tau, t_0], \end{cases} \tag{6}$$

where η_1, η_2, η_3 and η_4 are nonnegative constants, $\delta > 0$, $h(t)$ is a nonnegative integral function, $\sup\limits_{t \ge t_0} \int_{t-\tau}^t h(s)ds = H$. $\phi(s) \in C([-\tau, 0], R_+)$, $s \in [-\tau, 0]$. If $\Upsilon = \eta_2 + \eta_3/\delta < 1$, then there are positive constants $\lambda < \delta$ and N such that

$$u(t) \le N e^{-\lambda \int_{t_0}^t h(v)dv} + (1-\Upsilon)^{-1}\eta_4, \quad t \ge t_0, \tag{7}$$

where λ and N are determined by

$$[\phi(t_0)]_\tau < N \quad \text{and} \quad \frac{\eta_1}{N} + e^{\lambda H}\eta_2 + e^{\lambda H}\frac{\eta_3}{\delta - \lambda} < 1. \tag{8}$$

Proof. From the conditions $\eta_2 + \eta_3/\delta < 1$ and $\phi(s) \in C([-\tau, 0], R_+)$, $s \in [-\tau, 0]$, by using continuity, we obtain there exist positive constants λ and N such that (8) holds. In order to prove (7), we first prove for any $d > 1$,

$$u(t) < dN e^{-\lambda \int_{t_0}^t h(v)dv} + (1-\Upsilon)^{-1}\eta_4, \quad t \ge t_0. \tag{9}$$

If (9) is not true, from the fact that $[\phi(t_0)]_\tau \le N$ and $u(t)$ is continuous, then there must be a $t_1 > t_0$ such that

$$u(t_1) = dN e^{-\lambda \int_{t_0}^{t_1} h(v)dv} + (1-\Upsilon)^{-1}\eta_4, \tag{10}$$

$$u(t) \le dN e^{-\lambda \int_{t_0}^t h(v)dv} + (1-\Upsilon)^{-1}\eta_4, \quad t_0 - \tau \le t \le t_1. \tag{11}$$

Hence, it follows from (6), (8) and (11) that

$$\begin{aligned}
u(t_1) &\le \eta_1 e^{-\delta \int_{t_0}^{t_1} h(v)dv} + \eta_2 [u(t_1)]_\tau + \eta_3 \int_{t_0}^{t_1} e^{-\delta \int_s^{t_1} h(v)dv} h(s) [u(s)]_\tau ds + \eta_4 \\
&\le \eta_1 e^{-\delta \int_{t_0}^{t_1} h(v)dv} + \eta_2 \left[dN e^{\lambda \int_{t_1-\tau}^{t_1} h(v)dv} e^{-\lambda \int_{t_0}^{t_1} h(v)dv} + (1-\Upsilon)^{-1}\eta_4 \right] \\
&\quad + \eta_3 \int_{t_0}^{t_1} e^{-\delta \int_s^{t_1} h(v)dv} h(s) \left[dN e^{\lambda \int_{s-\tau}^{s} h(v)dv} e^{-\lambda \int_{t_0}^{s} h(v)dv} + (1-\Upsilon)^{-1}\eta_4 \right] ds + \eta_4 \\
&\le \eta_1 e^{-\delta \int_{t_0}^{t_1} h(v)dv} + \eta_2 \left[dN e^{\lambda H} e^{-\lambda \int_{t_0}^{t_1} h(v)dv} + (1-\Upsilon)^{-1}\eta_4 \right] \\
&\quad + \eta_3 \int_{t_0}^{t_1} e^{-\delta \int_s^{t_1} h(v)dv} h(s) \left[dN e^{\lambda H} e^{-\lambda \int_{t_0}^{s} h(v)dv} + (1-\Upsilon)^{-1}\eta_4 \right] ds + \eta_4 \\
&\le \eta_1 e^{-\lambda \int_{t_0}^{t_1} h(v)dv} + \eta_2 dN e^{\lambda H} e^{-\lambda \int_{t_0}^{t_1} h(v)dv} + \eta_3 \int_{t_0}^{t_1} e^{-\delta \int_s^{t_1} h(v)dv} h(s) \, dN e^{\lambda H} e^{-\lambda \int_{t_0}^{s} h(v)dv} ds \\
&\quad + \left(\eta_2 + \frac{\eta_3}{\delta} \right) (1-\Upsilon)^{-1}\eta_4 + \eta_4 \\
&< \left(\frac{\eta_1}{N} + \eta_2 e^{\lambda H} + \eta_3 e^{\lambda H} \int_{t_0}^{t_1} e^{-(\delta-\lambda)\int_s^{t_1} h(v)dv} h(s) \, ds \right) dN e^{-\lambda \int_{t_0}^{t_1} h(v)dv} + (1-\Upsilon)^{-1}\eta_4 \\
&\le \left(\frac{\eta_1}{N} + \eta_2 e^{\lambda H} + \frac{\eta_3}{\delta - \lambda} e^{\lambda H} \right) dN e^{-\lambda \int_{t_0}^{t_1} h(v)dv} + (1-\Upsilon)^{-1}\eta_4 \\
&< dN e^{-\lambda \int_{t_0}^{t_1} h(v)dv} + (1-\Upsilon)^{-1}\eta_4,
\end{aligned}$$

which contradicts to the equality (10). So (9) holds for all $t \ge t_0$. Letting $d \to 1$ in (9), we have (7). The proof is complete. □

If $\eta_4 = 0$, we can easily get the following corollary:

Corollary 2.1. Assume that all conditions of Lemma 2.3 hold and $\lim\limits_{t \to \infty} \int_{t_0}^t h(s)ds = \infty$. Then all solutions of the inequality (6) convergence to zero.

3 Main results

To obtain the existence and global attractivity of periodic solution of Eq. (1), we introduce the following assumption.

(H_6) There exist positive constants $p > 2$ and I such that

$$\Upsilon_1 = 4^{p-1}\left(L\frac{M}{m}\int_{-\tau}^{0}|p(s)s|\,ds\right)^p + 4^{p-1}\left(L\frac{M}{\alpha m}\int_{-\tau}^{0}|p(s)s|\,ds\right)^p < 1,$$

and

$$\int_{t_0}^{\infty} e^{-2\alpha\int_s^t h(u)du}\sigma^2(s)\,ds \le I.$$

Theorem 3.1. Suppose that $(H_1)-(H_6)$ hold, then the system (1) must have a periodic solution, which is globally attractive and in the attracting set $S = \{\varphi \in PC^b_{\mathscr{F}_0}([-\tau,0],R) \,|\, \|\varphi\|^p_{L^p} \le m(1-\Upsilon_1)^{-1}J_1\}$, where $J_1 = 4^{p-1}\left(\frac{1}{m}\right)^p (p(p-1)/2)^{p/2} I^{\frac{p}{2}}$.

Proof. By the method of variation parameter, we have from (4) that for $t \ge t_0$,

$$\begin{aligned} y(t) &= e^{-\alpha\int_{t_0}^t h(u)du}\left(\varphi(0) - \int_{-\tau}^{0} p(s)\int_{t_0+s}^{t_0} g(u,y(u))\,duds\right) \\ &\quad + \prod_{t_0<t_k<t} b_k^{-1}\int_{-\tau}^{0} p(s)\int_{t+s}^{t} g\left(u, \prod_{t_0<t_k<u} b_k y(u)\right)duds \\ &\quad - \int_{t_0}^{t} e^{-\alpha\int_v^t h(u)du}h(v)\prod_{t_0<t_k<v} b_k^{-1}\int_{-\tau}^{0} p(s)\int_{v+s}^{v} g\left(u,\prod_{t_0<t_k<u} b_k y(u)\right)dudsdv \\ &\quad + \int_{t_0}^{t} e^{-\alpha\int_s^t h(u)du}\prod_{t_0<t_k<s} b_k^{-1}\sigma(s)\,dB(s) \\ &=: I_1(t) + I_2(t) + I_3(t) + I_4(t). \end{aligned} \tag{12}$$

By using the inequality $(a+b+c+d)^p \le 4^{p-1}(a^p+b^p+c^p+d^p)$ for any positive real numbers a, b, c and d, taking expectations, we find for all $t \ge t_0$,

$$E|y(t)|^p \le 4^{p-1}E\left(|I_1(t)|^p + |I_2(t)|^p + |I_3(t)|^p + |I_4(t)|^p\right). \tag{13}$$

We first evaluate the first term of the right-hand side as follows:

$$\begin{aligned} E|I_1(t)|^p &= E\left|e^{-\alpha\int_{t_0}^t h(u)du}\left(\varphi(0) - \int_{-\tau}^{0} p(s)\int_{t_0+s}^{t_0} g(u,\varphi(u))\,duds\right)\right|^p \\ &\le 2^{p-1}e^{-\alpha p\int_{t_0}^t h(u)du}\left(E|\varphi(0)|^p + E\left|\int_{-\tau}^{0} p(s)\int_{t_0+s}^{t_0} g(u,\varphi(u))\,duds\right|^p\right) \\ &\le 2^{p-1}e^{-\alpha p\int_{t_0}^t h(u)du}\left(E|\varphi(0)|^p + \left(L\|\varphi\|\int_{-\tau}^{0}|p(s)s|\,ds\right)^p\right). \end{aligned} \tag{14}$$

As to the second term, by (H_5) and (H_3), we have

$$\begin{aligned} E|I_2(t)|^p &= E\left|\prod_{t_0<t_k<t} b_k^{-1}\int_{-\tau}^{0} p(s)\int_{t+s}^{t} g\left(u,\prod_{t_0<t_k<u} b_k y(u)\right)duds\right|^p \\ &\le E\left(\frac{1}{m}\int_{-\tau}^{0}|p(s)|\int_{t+s}^{t} L\left|\prod_{t_0<t_k<u} b_k y(u)\right|duds\right)^p \\ &\le E\left(L\frac{M}{m}\int_{-\tau}^{0}|p(s)|\int_{t+s}^{t}|y(u)|\,duds\right)^p \\ &\le \left(L\frac{M}{m}\int_{-\tau}^{0}|p(s)s|\,ds\right)^p [E|y(t)|^p]_\tau. \end{aligned} \tag{15}$$

As to the third term, by Hölder inequality, (H_5) and (H_3), we have

$$\begin{aligned}
E|I_3(t)|^p &= E\left|\int_{t_0}^{t} e^{-\alpha\int_v^t h(u)du} h(v) \prod_{t_0<t_k<v} b_k^{-1} \int_{-\tau}^{0} p(s)\int_{v+s}^{v} g\left(u, \prod_{t_0<t_k<u} b_k y(u)\right) dudsdv\right|^p \\
&\le \left(L\frac{M}{m}\int_{-\tau}^{0}|p(s)s|\,ds\right)^p E\left[\int_{t_0}^{t} e^{-\alpha\int_v^t h(u)du} h(v)\,[y(v)]_\tau dv\right]^p \\
&= \left(L\frac{M}{m}\int_{-\tau}^{0}|p(s)s|\,ds\right)^p E\left[\int_{t_0}^{t}\left(e^{-\alpha\int_v^t h(u)du}h(v)\right)^{\frac{p-1}{p}}\left(e^{-\alpha\int_v^t h(u)du}h(v)\right)^{\frac{1}{p}}[y(v)]_\tau dv\right]^p \\
&\le \left(L\frac{M}{m}\int_{-\tau}^{0}|p(s)s|\,ds\right)^p \left(\int_{t_0}^{t} e^{-\alpha\int_v^t h(u)du}h(v)\,dv\right)^{p-1}\int_{t_0}^{t} e^{-\alpha\int_v^t h(u)du}h(v)\,[E|y(v)|^p]_\tau dv \\
&\le \alpha\left(L\frac{M}{\alpha m}\int_{-\tau}^{0}|p(s)s|\,ds\right)^p \int_{t_0}^{t} e^{-\alpha\int_v^t h(u)du}h(v)\,[E|y(v)|^p]_\tau dv. \qquad (16)
\end{aligned}$$

As far as the last term is concerned, using an estimate on the Itô integral established in [24, Proposition 1.9], Hölder inequality, (H_5) and (H_3), we obtain:

$$\begin{aligned}
E|I_4(t)|^p &= E\left|\int_{t_0}^{t} e^{-\alpha\int_s^t h(u)du}\prod_{t_0<t_k<s} b_k^{-1}\sigma(s)\,dB(s)\right|^p \\
&\le \left(\frac{1}{m}\right)^p E\left|\int_{t_0}^{t} e^{-\alpha\int_s^t h(u)du}\sigma(s)dB(s)\right|^p \\
&\le \left(\frac{1}{m}\right)^p c_p\left(\int_{t_0}^{t}\left(e^{-\alpha p\int_s^t h(u)du}|\sigma(s)|^p\right)^{\frac{2}{p}} ds\right)^{\frac{p}{2}} \\
&= \left(\frac{1}{m}\right)^p c_p\left(\int_{t_0}^{t} e^{-2\alpha\int_s^t h(u)du}\sigma^2(s)ds\right)^{\frac{p}{2}} \\
&\le \left(\frac{1}{m}\right)^p c_p I^{\frac{p}{2}}, \qquad (17)
\end{aligned}$$

where $c_p = (p(p-1)/2)^{p/2}$.

It follows from (13)-(17) that

$$\begin{aligned}
E|y(t)|^p \le\ & 8^{p-1}e^{-\alpha\int_{t_0}^t h(u)du}\left(E|\varphi(0)|^p + \left(L\,\|\varphi\|\int_{-\tau}^{0}|p(s)s|\,ds\right)^p\right) \\
&+ 4^{p-1}\left(L\frac{M}{m}\int_{-\tau}^{0}|p(s)s|\,ds\right)^p [E|y(t)|^p]_\tau \\
&+ 4^{p-1}\alpha\left(L\frac{M}{\alpha m}\int_{-\tau}^{0}|p(s)s|\,ds\right)^p \int_{t_0}^{t} e^{-\alpha\int_s^t h(u)du}h(s)\,E[|y(s)|^p]_\tau ds \\
&+ 4^{p-1}\left(\frac{1}{m}\right)^p c_p I^{\frac{p}{2}}. \qquad (18)
\end{aligned}$$

From Lemma 2.3 and Condition (H_6), the solutions of (4) are p-uniformly bounded and $S_1 = \{\varphi \in PC^b_{\mathscr{F}_0}([-\tau,0],R)\,|\,\|\varphi\|^p_{L^p} \le (1-\Upsilon_1)^{-1}J_1\}$ is an attracting set of (4) (i.e., the family of all solutions of (4) is p-point dissipative). From Lemma 2.2, then system (4) must exist an ω-periodic solution. It follows from Lemma 2.1, (H_3) and the equivalence between (1) and (3) that the system (1) must have an ω-periodic solution.

In view of (ii) of Lemma 2.1 and (H_3), it's easy to see that

$$S = \{\varphi \in PC^b_{\mathscr{F}_0}([-\tau,0],R)\,|\,\frac{1}{m}\|\varphi\|^p_{L^p} \le (1-\Upsilon_1)^{-1}J_1\}$$

i,e,

$$S=\{\varphi\in PC^{b}_{\mathscr{F}_0}\left([-\tau,0],R\right)|\|\varphi\|^{p}_{L^p}\le m(1-\Upsilon_1)^{-1}J_1\}$$

is an attracting set of (1)

Denote $y^*(t)$ be the ω-periodic solution and $y(t)$ be an arbitrary solution of Eq. (4).

We rewrite the Eq. (4) by

$$\begin{aligned} d\left(y\left(t\right)-y^{*}\left(t\right)\right) &= -\alpha h\left(t\right)\left(y\left(t\right)-y^{*}\left(t\right)\right)dt \\ &\quad+\prod_{t_0<t_k<t} b_k^{-1} d\left(\int_{-L}^{0} p\left(s\right)\int_{t+s}^{t}\left(g\left(u,\prod_{t_0<t_k<u}\left(b_k y\left(u\right)\right)\right.\right.\right. \\ &\quad\left.\left.\left.-g\left(t,\prod_{t_0<t_k<u} b_k y^{*}\left(u\right)\right)\right)duds\right),\quad t\ge t_0. \end{aligned} \tag{19}$$

Proceeding as the proof of the existence of periodic solution, we have

$$\begin{aligned} E|y\left(t\right)|^{p} &\le 6^{p-1}e^{-\alpha\int_{t_0}^{t}h(u)du}\left(E|\varphi\left(0\right)|^{p}+\left(L\left\|\varphi\right\|\int_{-\tau}^{0}|p\left(s\right)s|\,ds\right)^{p}\right) \\ &\quad+3^{p-1}\left(L\frac{M}{m}\int_{-\tau}^{0}|p\left(s\right)s|\,ds\right)^{p}[E|y\left(t\right)|^{p}]_{\tau} \\ &\quad+3^{p-1}\left(L\frac{M}{m}\int_{-\tau}^{0}|p\left(s\right)s|\,ds\right)^{p}\int_{t_0}^{t}e^{-\alpha\int_{s}^{t}h(u)du}h\left(s\right)E[|y\left(s\right)|^{p}]_{\tau}ds. \end{aligned} \tag{20}$$

From Corollary 2.1 and Condition (H_6), we get that the periodic solution is globally attractive . And the proof is completed.

□

If $b_k=1,\ k=1,2,\cdots$, the system (1) becomes the system without impulses

$$dx\left(t\right)=-\int_{t-\tau}^{t}p\left(s-t\right)g\left(s,x\left(s\right)\right)ds+\sigma\left(t\right)dB\left(t\right). \tag{21}$$

Corollary 3.1. Suppose that $(H_4),(H_5)$ and (H_6) with $m=M=1$ hold, then the system (21) must have a periodic solution, which is globally attractive and in the attracting set $S_2=\{\varphi\in BC^{b}_{\mathscr{F}_0}([-\tau,0],R)|\|\varphi\|^{p}_{L^p}\le (I-\Upsilon_2)^{-1}J_2\}$, where $J_2=4^{p-1}(p\left(p-1\right)/2)^{p/2}I^{\frac{p}{2}}$.

Proof. The proof is similar to that of Theorem 3.1, so we omit it here. □

4 Example

Example 4.1. Consider the impulsive stochastic Volterra-Levin equations

$$dx\left(t\right)=-\int_{t-1}^{t}e^{-(t-s)}\left|4^{-\frac{2}{3}}\cos\frac{\pi}{2}s\right|x\left(s\right)ds+\cos\frac{\pi}{2}tdB\left(t\right),\quad t\ge 0,\quad t\ne t_k, \tag{22}$$

with

$$x\left(t_k^{+}\right)=b_k x\left(t_k\right),$$

where $b_k\ne 0,\ t_k=k,\ k=1,2,\ldots$.

It is obvious that

$$h\left(t\right)=\left|\frac{1}{4}\cos\frac{\pi}{2}t\right|,\quad \sigma\left(t\right)=\cos\frac{\pi}{2}t,\quad \alpha=\int_{-\tau}^{0}p\left(s\right)ds=\int_{-\tau}^{0}e^{s}ds=1-\frac{1}{e},$$

$L=\frac{1}{4}$ and $\tau=1$.

Case 5.1. Let $b_k=1$, $k=1,2,\cdots$, then Eq. (22) becomes nonimpulsive stochastic Volterra-Levin equations. Taking $p=3$, we have

$$\begin{aligned}\Upsilon_2 &= 4^2\left(L\int_{-\tau}^{0}|p(s)\,s|\,ds\right)^3+4^2\left(\frac{L}{\alpha}\int_{-\tau}^{0}|p(s)\,s|\,ds\right)^3\\ &= 4^2\left(\frac{1}{4}\int_{-\tau}^{0}|e^s s|\,ds\right)^3+4^2\left(\frac{1}{4}\frac{e}{e-1}\int_{-\tau}^{0}|e^s s|\,ds\right)^3\\ &= \left(\frac{e-2}{4e}\right)^3+\left(\frac{e-2}{4(e-1)}\right)^3<1\end{aligned}$$

and

$$\begin{aligned}\int_0^\infty e^{-2\alpha\int_s^t h(u)du}\sigma^2(s)\,ds &= \int_0^\infty e^{-\frac{2(e-1)}{e}\int_s^t\left|\frac{1}{4}\cos\frac{\pi}{2}u\right|du}\cos^2\frac{\pi}{2}sds\\ &\le \int_0^\infty e^{-\frac{1}{4}\frac{2(e-1)}{e}\int_s^t\left|\cos\frac{\pi}{2}u\right|du}\left|\cos\frac{\pi}{2}s\right|ds=\frac{2e}{e-1}\end{aligned}$$

It follows from Corollary 3.1 that Eq. (22) has a 4-periodic solution, which is globally attractive .

Case 5.2. Let $b_k=2^{sin\frac{\pi}{2}k}$. Then $I(t)=\prod\limits_{0<t_k<t}b_k=\prod\limits_{0<t_k<t}2^{\sin\frac{\pi}{2}k}$. Now we claim that (H_3) holds. In fact

$$\begin{aligned}I(t+4) &= \prod_{0<t_k<4}2^{\sin\frac{\pi}{2}k}\cdot\prod_{4<t_k<4+t}2^{\sin\frac{\pi}{2}k}\\ &= 2^{\sum\limits_{k=1}^{4}\sin\frac{\pi}{2}k}\cdot\prod_{0<t_k<t}2^{\sin\frac{\pi}{2}(k-4)}\\ &= 2^{\sum\limits_{k=1}^{4}\sin\frac{\pi}{2}k}\cdot\prod_{0<t_k<t}2^{\sin\frac{\pi}{2}k}=2^0\cdot I(t)=I(t),\end{aligned}$$

which implies that $I(t)$ is a periodic function with period 4. By simple computation, we know that $1\le\prod\limits_{0<t_k<t}b_k\le 2$. That is, $m=1$ and $M=2$. Taking $p=3$, we have

$$\begin{aligned}\Upsilon_1 &= 4^2\left(L\frac{M}{m}\int_{-\tau}^{0}|p(s)\,s|\,ds\right)^3+4^2\left(L\frac{M}{\alpha m}\int_{-\tau}^{0}|p(s)\,s|\,ds\right)^3\\ &= 4^2\left(2\cdot\frac{1}{4}\int_{-\tau}^{0}|e^s s|\,ds\right)^3+4^2\left(2\cdot\frac{1}{4}\frac{e}{e-1}\int_{-\tau}^{0}|e^s s|\,ds\right)^3\\ &= 2\left(\left(\frac{e-2}{e}\right)^3+\left(\frac{e-2}{e-1}\right)^3\right)<1.\end{aligned}$$

It follows from Theorem 3.1 that Eq. (22) has a 4-periodic solution, which is globally attractive.

References

[1] K. Itô, On stochastic differential equations, Mem. Amer. Math. Soc., NO.4, 1951.

[2] L.E. Bertram, P.E. Sarachik, Stability of Circuits with randomly time-varying parameters, IRE. Trans. Circuit Theory, CT-6, Special supplement, 1959, 260-270.

[3] K.N. Lu, B. Schmalfuss, Invariant manifolds for stochastic wave equations, J. Differential Equations, 2007, 236, 460-492.

[4] X.R. Mao, Exponential Stability of Stochastic Differential Equations, Marcel Dekker, Inc, New York, 1994.

[5] X.X. Liao, X.R. Mao, Exponential stability and instability of stochastic neural networks, Stochast. Anal. Appl. 1996,14(2), 165-185.

[6] K. Itô, M. Nisio, On stationary solutions of a stochastic differential equation, J. Math. Kyoto University. 1964, 4,1, 1-75.

[7] R.Z. Has'minskii, On the dissipativity of random processes defined by differential equations, Problems of Information Transmission, 1965, 1(1), 88-104.

[8] R.Z. Has'minskii, Stochastic Stability of Differential Equations, Sijthoff and Noordhoff, Maryland, 1980.

[9] B.G. Zhang, On the periodic solution of $n-$dimensional stochastic population models, Stochast. Anal. Appl. 2000, 18(2), 323-331.

[10] D.Y. Xu, Y.M. Huang, Z.G. Yang, Existence theorems for periodic Markov process and stochastic functional differential equations, Discrete Contin. Dyn. Syst., 24 (3) (2009) 1005-1023.

[11] D.D. Bainov, P.S. Simeonov, Systems with Impulse Effect, Theory and Applications, Ellis Horwood Series in Mathematics and its Applications, Ellis Horwood, Chichister, 1989.

[12] G. Ballinger, X. Liu, Existence and uniqueness results for impulsive delay differential equations, Dynamice Continuous Discrete Impulse Systems 5 (1999) 579-591.

[13] V. Lakshmikantham, D.D. Bainov, P.S. Simenov, Theory of Impulsive Differential Equations, World Scientific, Singapore, 1989.

[14] X.Z. Liu, G. Ballinger, Uniform asymptotic stability of impulsive delay differential equations, Computers Math. Appl. 41 (2001) 903-915.

[15] X.Z. Liu, G. Ballinger, Existence and continuability of solutions for differential equations with delays and state-dependent impulses, Nonlinear Anal. TMA 51 (2002) 633-647.

[16] C. Li, J.T. Sun, Stability analysis of nonlinear stochastic differential delay systems under impulsive control, Phys. Lett. A 374 (2010) 1154-1158.

[17] H.F. Huo, W.T. Li, X. Liu, Existence and global attractivity of positive periodic solution of an impulsive delay differential equation, Appl. Anal. 83 (2004) 1279-1290.

[18] W.T. Li, Y.H. Fan, Existence and global attractivity of positive periodic solutions for the impulsive delay Nicholson's blowflies model, J. Comput. Appl. Math. 201 (2007) 55-68.

[19] J.R. Yan, Existence and global attractivity of positive periodic solution for an impulsive Lasota-Wazewska model, J. Math. Anal. Appl. 279 (2003) 111-120.

[20] Z.C. Yang, D.Y. Xu, Existence and exponential stability of periodic solution for impulsive delay differential equations and applications, Nonlinear Anal. 64 (2006) 130-145.

[21] J.A.D. Appleby, Fixed points, stability and harmless stochastic perturbations, preprint.

[22] J.W. Luo, Fixed points and exponential stability for stochastic Volterra-Levin equations, J. Comput. Appl. Math. 234 (2010) 934-940.

[23] X.R. Mao, Stochastic Differential Equations and Applications, Horwood, Chichester, 1997.

[24] A. Ichikawa, Stability of semilinear stochastic evolution equations. J. Math. Anal. Appl. 90 (1) 1982, 12-44.

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