REPRESENTATION AND STABILITY OF SOLUTIONS OF SYSTEMS OF FUNCTIONAL DIFFERENTIAL EQUATIONS WITH MULTIPLE DELAYS

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ABSTRACT. This paper is devoted to the study of systems of nonlinear functional differential equations with time-dependent coefficients and multiple variable increasing delays represented by functions \( g_i(t) < t \). The solution is found in terms of a piecewise-defined matrix function. Using our representation of the solution and Gronwall’s, Bihari’s and Pinto’s integral inequalities, asymptotic stability results are proved for some classes of nonlinear functional differential equations with multiple variable delays and linear parts given by pairwise permutable constant matrices. The derived theory is illustrated on nontrivial examples.

1. INTRODUCTION

The classical method of steps [8] where the initial value problem

\[
\begin{align*}
\dot{x}(t) &= Bx(t - \tau), \quad t \geq 0 \\
x(t) &= \varphi(t), \quad -\tau \leq t \leq 0
\end{align*}
\]

is solved by subsequent integrating of equation (1.1) on intervals \([0, \tau), [\tau, 2\tau), [2\tau, 3\tau), \ldots\) was renown in 2003 by Khusainov and Shuklin [13]. Applying this method, they constructed so-called delayed matrix exponential

\[
e^{Bt}_{\tau} = \begin{cases} 
\Theta, & t < -\tau, \\
E, & -\tau \leq t < 0, \\
E + Bt + B^2(t-\tau)^2/2 + \cdots + B^k (t-(k-1)\tau)^k/k!, & (k-1)\tau \leq t < k\tau, k \in \mathbb{N}
\end{cases}
\]

where \( \Theta, E \) are the zero and the identity \( N \times N \) matrix, respectively. Let us recall their result.

**Theorem 1.1.** Let \( \varphi \in C^1_{\tau} := C^1([-\tau, 0], \mathbb{R}^N) \) and \( AB = BA \). Then any solution \( x(t) \) of the Cauchy problem consisting of equation

\[
\dot{x}(t) = Ax(t) + Bx(t - \tau), \quad t \geq 0
\]

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and initial condition (1.2) has the form

\[ x(t) = e^{A(t+\tau)}e^{Bt}\varphi(-\tau) + \int_{-\tau}^{0} e^{A(t-\tau-s)}e^{B(t-\tau-s)}e^{Ar}(\varphi'(s) - A\varphi(s))ds, \]

where \( \tilde{B} = e^{-A\tau}B \).

Using the variation of constants formula for retarded differential equations with constant delay [8] and \( e^{Bt} \), they stated the solution of nonhomogeneous equation

\[ \dot{x}(t) = Ax(t) + Bx(t-\tau) + f(t) \]

with continuous function \( f : [0, \infty) \to \mathbb{R}^N \), satisfying initial condition (1.2). Later, their result was used to establish sufficient conditions for the exponential stability of the trivial solution of the nonlinear equation

\[ \dot{x}(t) = Ax(t) + Bx(t-\tau) + f(x(t), x(t-\tau)) \]

with various functions \( f \) in [16]. The results from [13] were generalized to delay differential equations with multiple fixed delays and pairwise permutable matrices in [14] and analogical theory was developed for retarded oscillating systems and difference equations with one or more fixed delays (cf. [7, 11, 12, 15]). Recently, the matrix representation of solutions of systems of differential equations with a single fixed delay was applied to boundary-value problems in [4, 5, 6].

In this paper, we consider the functional differential equation (FDE) with one or multiple time-dependent delays. More precisely, we deal with equations of the form

\[ \dot{x}(t) = B_1(t)x(g_1(t)) + \cdots + B_n(t)x(g_n(t)), \quad t \geq 0 \]

where \( B_i \in C([0, \infty), L(\mathbb{R}^N, \mathbb{R}^N)) \), \( g_i \in G^0 \) for \( i = 1, \ldots, n \) and

\[ G^s := \{ g \in C([s, \infty), \mathbb{R}) \mid g(t) < t \text{ on } [s, \infty), g \text{ is increasing} \}. \]

In Section 2, we derive the solution of a corresponding nonhomogeneous equation. Later, in Section 3 we use the property of commutativity of matrices to transform the nonlinear FDE with multiple delays and linear term \( Ax(t) \) to a nonlinear FDE with multiple delays but without a delay-independent linear term. After this transformation, we can apply the theory of Section 2, and so establish sufficient conditions for the exponential stability of the trivial solution of nonlinear FDE with multiple delays and linear term \( Ax(t) \) added on the right-hand side, supposing that the linear parts are given by pairwise permutable constant matrices. So, in Section 3, we study the exponential stability of the trivial solution of systems of FDEs with linear parts given by pairwise permutable matrices (for stability criteria for scalar equations with variable coefficients see e.g. [1, 2, 10, 19]).

In the whole paper \( \|E\| = 1 \), \( \mathbb{N} \) denotes the set of all positive integers and \( g^k(t), g^{-k}(t) \) for \( k = 2, 3, \ldots \) denote the iterations of functions \( g(t) \), \( g^{-1}(t) \).
respectively, e.g. if \( k = 2 \), then \( g^2(t) = g(g(t)) \) and \( g^{-2}(t) = g^{-1}(g^{-1}(t)) \). Moreover, \( g^0(t) = t \). If \( g : [s, \infty) \to [g(s), \infty) \) is not surjective, we define \( g^{-1}(q) := \infty \) whenever \( q \) is such that \( g(t) < q \) for any \( t \in [s, \infty) \).

**Definition 1.2.** Given continuous function \( F \), under the solution of a general FDE

\[
(1.3) \quad \dot{x}(t) = F(x(g_1(t)), \ldots, x(g_n(t)), t), \quad t \geq 0
\]
satisfying initial condition (1.2) with \( \tau = \min\{g_1(0), \ldots, g_n(0)\} \) we understand function \( x \in C([-\tau, \infty), \mathbb{R}^N) \cap C^1([0, \infty), \mathbb{R}^N) \) (at 0 we take the right-hand derivative) which solves equation (1.3) and satisfies (1.2).

2. Solutions of systems of FDEs

In this section we derive a representation of a solution of FDE with single variable delay using a piecewise-defined matrix function, which is analogous to delayed matrix exponential \( e^{Bt}_\tau \) for equations with constant delay.

Later, we find a solution of FDE with multiple delays as it was done in [14]. Throughout this part, we widely use the method of steps and variation of constants formula for FDEs (cf. [8, 9]). We note that the existence and uniqueness of solutions of problems of this section are obvious. First, we find the fundamental solution of linear FDE with one delay satisfying the below-stated initial condition (2.2).

**Theorem 2.1.** Let \( s \in \mathbb{R}, B \in C([s, \infty), L(\mathbb{R}^N, \mathbb{R}^N)), g \in G^s \). Then the matrix solution of equation

\[
(2.1) \quad \dot{X}(t) = B(t)X(g(t)), \quad t \geq s
\]
satisfying initial condition

\[
(2.2) \quad X(t) = \begin{cases} 
\Theta, & t < s, \\
E, & t = s 
\end{cases}
\]
has the form \( X(t) = X^B_g(t, s) \) where

\[
(2.3) \quad X^B_g(t, s) := \begin{cases} 
\Theta, & t < s, \\
E + \int_{g^{-1}(s)}^{t} B(q_1) dq_1 + \cdots \\
\cdots + \int_{g^{-k}(s)}^{t} B(q_1) \int_{g^{-k-1}(s)}^{g(q_{k-1})} B(q_2) \cdots \int_{g^{-2}(s)}^{g(q_2)} B(q_k) dq_k \cdots dq_1, & s \leq t < g^{-1}(s), \\
g^{-k}(s) \leq t < g^{-(k+1)}(s), & k \in \mathbb{N}.
\end{cases}
\]

**Proof.** From definition of \( X^B_g(t, s) \), the initial condition is immediately verified. If \( s \leq t < g^{-1}(s) \) then \( g(s) \leq g(t) < s \) and

\[
\dot{X}(t) = \dot{E} = \Theta = B(t)\Theta = B(t)X(g(t)).
\]

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Now, let $g^{-k}(s) \leq t < g^{-(k+1)}(s)$ for some $k \in \mathbb{N}$. Then $g^{-(k-1)}(s) \leq g(t) < g^{-k}(s)$ and we get

$$X(t) = B(t) + B(t) \int_{g^{-1}(s)}^{g(t)} B(q_2) dq_2 + \cdots + B(t) \int_{g^{-(k-1)}(s)}^{g(t)} B(q_2) \cdots \int_{g^{-(k+1)}(s)}^{g(t)} B(q_k) dq_k \cdots dq_2 = B(t)X(g(t)).$$

Hence equation (2.1) is verified and the proof is finished. \qed

Now, when we have the fundamental solution, we can derive the solution of corresponding nonhomogeneous equation (see [9]). Without any loss of generality we assume the initial function to be given on $[g(0), 0]$.

**Theorem 2.2.** Let $B \in C([0, \infty), L(\mathbb{R}^N, \mathbb{R}^N))$, $g \in C^0$, $f \in C([0, \infty), \mathbb{R}^N)$, $\varphi \in C([g(0), 0], \mathbb{R}^N)$. Then the solution of the initial value problem

\begin{align}
\dot{x}(t) &= B(t)x(g(t)) + f(t), \quad t \geq 0 \\
x(t) &= \varphi(t), \quad g(0) \leq t \leq 0
\end{align}

has the form

\begin{equation}
x(t) = \begin{cases} 
\varphi(t), & t \in [g(0), 0), \\
X^B_g(t, 0)\varphi(0) + \int_0^t X^B_g(t, s)B(s)\psi(g(s))ds + \int_0^t X^B_g(t, s)f(s)ds, & 0 \leq t
\end{cases}
\end{equation}

where

$$\psi(t) = \begin{cases} 
\varphi(t), & t \in [g(0), 0), \\
0, & t \notin [g(0), 0].
\end{cases}$$

**Proof.** Clearly, $x \in C([g(0), \infty), \mathbb{R}^N) \cap C^1([0, g^{-1}(0)) \cup (g^{-1}(0), \infty), \mathbb{R}^N)$ and it satisfies condition (2.5).

Let $0 \leq t < g^{-1}(0)$. Then for $0 \leq s \leq t$ we get

$$g(0) \leq g(s) \leq g(t) < 0 \leq s \leq t < g^{-1}(0).$$

Consequently, $X^B_g(t, s) = E$ and $\psi(g(s)) = \varphi(g(s))$. Hence from formula (2.6)

\begin{equation}
x(t) = \varphi(0) + \int_0^t B(s)\varphi(g(s))ds + \int_0^t f(s)ds
\end{equation}

what after differentiating with respect to $t$ yields

$$\dot{x}(t) = B(t)\varphi(g(t)) + f(t).$$

Even though $X^B_g(t, s)$ is not $C^1$ at $t = g^{-1}(s)$ (left-hand derivative is $\Theta$, right-hand derivative is $B(g^{-1}(s))$, solution $x(t)$ is $C^1$ at $g^{-1}(0)$. To see
this, we differentiate the solution (2.7) for \( t \in [0, g^{-1}(0)) \) and

\[
x(t) = X^B_g(t, 0) \varphi(0) + \int_0^t X^B_g(t, s) B(s) \psi(g(s)) ds + \int_0^t X^B_g(t, s) f(s) ds
\]

for \( t \in [g^{-1}(0), g^{-2}(0)) \), both at \( t = g^{-1}(0) \). We obtain

\[
\lim_{t \to (g^{-1}(0))^−} \dot{x}(t) = B(g^{-1}(0)) \varphi(0) + f(g^{-1}(0)),
\]

\[
\lim_{t \to (g^{-1}(0))^+} \dot{x}(t) = B(g^{-1}(0)) [\varphi(0) + \psi(0)] + f(g^{-1}(0)).
\]

In fact, we get the equality since \( \psi(0) = 0 \).

Now if \( g^{-k}(0) \leq t < g^{-(k+1)}(0) \) for some \( k \in \mathbb{N} \), then by differentiating formula (2.6) we obtain

\[
\dot{x}(t) = X^B_g(t, 0) \varphi(0) + B(t) \psi(g(t))
\]

\[
+ \int_0^t \dot{X}_g^B(t, s) B(s) \psi(g(s)) ds + f(t) + \int_0^t \dot{X}_g^B(t, s) f(s) ds
\]

\[
= B(t) X^B_g(t, 0) \varphi(0) + B(t) \psi(g(t)) + B(t) \int_0^t X^B_g(t, s) B(s) \psi(g(s)) ds
\]

\[
+ f(t) + B(t) \int_0^t X^B_g(t, s) f(s) ds
\]

where we used the properties of \( X^B_g(t, s) \) from Theorem 2.1. Next, we apply the identity

\[
\int_0^t X^B_g(g(t), s) F(s) ds = \int_0^{g(t)} X^B_g(g(t), s) F(s) ds
\]

with \( F(s) \) standing for \( B(s) \psi(g(s)) \) or \( f(s) \) to get

\[
\dot{x}(t) = B(t) \left[ X^B_g(g(t), 0) \varphi(0) + \psi(g(t)) + \int_0^{g(t)} X^B_g(g(t), s) B(s) \psi(g(s)) ds
\]

\[
+ \int_0^{g(t)} X^B_g(g(t), s) f(s) ds \right] + f(t).
\]

For \( t \geq g^{-1}(0) \) it holds \( \psi(g(t)) = 0 \) and equation (2.4) is verified. \( \Box \)

**Remark 2.3.** In reality, if \( t \geq g^{-1}(0) \) is fixed, two integrals in solution \( x(t) \) of (2.6) are split into more integrals as \( s \) varies from 0 to \( t \). Note that

\[
\int_0^t X^B_g(t, s) B(s) \psi(g(s)) ds = \int_0^\min(t, g^{-1}(0)) X^B_g(t, s) B(s) \varphi(g(s)) ds.
\]
If we denote
\[ X_l(t, s) := E + \int_{g^{-1}(s)}^{t} B(q_1)dq_1 + \cdots + \int_{g^{-1}(s)}^{t} B(q_1) \int_{g^{-1}(s)}^{g(q_1)} B(q_2) \cdots \int_{g^{-1}(s)}^{g(q_{l-1})} B(q_l)dq_l \cdots dq_1 \]
for \( l = 0, \ldots, k \), i.e. the lower index \( l \) denotes the number of integrals in the sum, then \( x(t) \) can be written as

\[
x(t) = X_k(t, 0)\varphi(0) + \int_{0}^{g^k(t)} X_k(t, s)B(s)\varphi(g(s))ds + \int_{0}^{g^{k-1}(t)} X_{k-1}(t, s)B(s)\varphi(g(s))ds + \int_{0}^{g^{k-1}(t)} X_{k-1}(t, s)f(s)ds + \int_{g(t)}^{t} X_{0}(t, s)f(s)ds
\]

for \( g^{-k}(0) \leq t < g^{-(k+1)}(0), k \in \mathbb{N}, \) i.e.

\[
0 \leq g^k(t) < g^{-1}(0) \leq g^{k-1}(t) < \cdots \leq g^2(t) < g^{-(k-1)}(0) \leq g(t) < g^{-k}(0) \leq t < g^{-(k+1)}(0).
\]

Here we used the form of \( X^B(t, s) \) for fixed \( t \) and variable \( s \) (in (2.3) it was given for fixed \( s \) and variable \( t \)):

(2.8)

\[
X^B(t, s) = \begin{cases} 
\Theta, & t < s, \\
E, & g(t) < s \leq t, \\
E + \int_{g^{-1}(s)}^{t} B(q_1)dq_1 + \cdots + \int_{g^{-k}(s)}^{t} B(q_1) \int_{g^{-1}(s)}^{g(q_1)} B(q_2) \cdots \int_{g^{-1}(s)}^{g(q_{l-1})} B(q_l)dq_l \cdots dq_1, & g^{k+1}(t) < s \leq g^k(t), k \in \mathbb{N}.
\end{cases}
\]

Now we provide an application of Theorem 2.2 on a problem with a bounded delay.

**Example 2.4.** Let us consider the following initial value problem

\[
\dot{x}(t) = B(t)x \left( \frac{1 - 3e^{-t}}{2} \right) + f(t), \quad t \geq 0
\]

\[
x(t) = \varphi(t), \quad -1 \leq t \leq 0.
\]

Here we have \( g : [0, \infty) \rightarrow [-1, 1/2), \ g(t) = (1 - 3e^{-t})/2 \) and \( g^{-1} : [-1, 1/2) \rightarrow [0, \infty), \ g^{-1}(s) = \ln \frac{3}{1-2s}. \) Hence we set \( g^{-1}(q) = \infty \) whenever

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where from (2.3) we get

\[ \dot{\frac{t}{2}} \]

\[ (2.9) \]

\[ \frac{\dot{q}}{2} \]

\[ \frac{\dot{t}}{2} \]

\[ \frac{\dot{s}}{2} \]

\[ \frac{\dot{g}}{2} \]

\[ \frac{\dot{q}}{2} \]

\[ \frac{\dot{t}}{2} \]

\[ \frac{\dot{s}}{2} \]

\[ \frac{\dot{g}}{2} \]

\[ \frac{\dot{q}}{2} \]

\[ \frac{\dot{t}}{2} \]

\[ \frac{\dot{s}}{2} \]

\[ \frac{\dot{g}}{2} \]

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\[ \frac{\dot{s}}{2} \]
Analogically we proceed on other intervals \([g_2^{-k}(s), g_2^{-(k+1)}(s)]\) with \(k = 2, 3, \ldots\). By this process we obtain

\[(2.10)\]

\[
X_{g_1, g_2}(t, s) := \begin{cases} 
\Theta, & t < s, \\
X_{g_1, g_2}^B(t, s), & s \leq t < g_2^{-1}(s), \\
X_{g_1, g_2}^B(t, s) + \int_{g_2^{-1}(s)}^t X_{g_1, g_2}^B(t, q_1) B_2(q_1) X_{g_1}^B(g_2(q_1), s) dq_1 + \cdots \\
\cdots + \int_{g_2^{-k}(s)}^t X_{g_1, g_2}^B(t, q_1) B_2(q_1) \cdots \int_{g_2^{-2}(s)}^t X_{g_1, g_2}^B(g_2(q_{k-1}), q_k) B_2(q_k) X_{g_1}^B(g_2(q_k), s) dq_k \cdots dq_1, \\
g_2^{-k}(s) \leq t < g_2^{-(k+1)}(s), k \in \mathbb{N}.
\end{cases}
\]

**Theorem 2.6.** Let \(s \in \mathbb{R}, B_1, B_2 \in C([s, \infty), L(\mathbb{R}^N, \mathbb{R}^N)), g_1, g_2 \in G^s.

Then \(X(t) = X_{g_1, g_2}^B(t, s)\) is the matrix solution of equation (2.9) satisfying initial condition (2.2).

Proof. At \(t = s\) it holds

\(X(s) = X_{g_1}^B(s, s) = E.\)

Hence initial condition is immediately verified.

Let \(s \leq t < g_2^{-1}(s)\). Then

\(X(t) = X_{g_1}^B(t, s)\)

and \(g_2(t) < s\). So we get

\[
\dot{X}(t) = \dot{X}_{g_1}^B(t, s) = B_1(t) X_{g_1}^B(g_1(t), s) = B_1(t) X(g_1(t)) + B_2(t) X(g_2(t)).
\]

Now, if \(g_2^{-k}(s) \leq t < g_2^{-(k+1)}(s)\) for some \(k \in \mathbb{N}\), then \(g_2^{-(k-1)}(s) \leq g_2(t) < g_2^{-k}(s)\). Accordingly,

\[
\dot{X}(t) = B_1(t) X_{g_1}^B(g_1(t), s) + B_2(t) X_{g_1}^B(g_2(t), s)
\]

\[
+ \int_{g_2^{-1}(s)}^t B_1(t) X_{g_1}^B(g_1(t), q_1) B_2(q_1) X_{g_1}^B(g_2(q_1), s) dq_1 + \cdots
\]

\[
\cdots + B_2(t) \int_{g_2^{-(k-1)}(s)}^{g_2(t)} X_{g_1}^B(g_2(t), q_2) B_2(q_2) \cdots \int_{g_2^{-1}(s)}^{g_2(q_{k-1})} X_{g_1}^B(g_2(q_{k-1}), q_k) dq_k \cdots dq_1
\]

\[
\times B_2(q_k) X_{g_1}^B(g_2(q_k), s) dq_k \cdots dq_2 + \int_{g_2^{-k}(s)}^t B_1(t) X_{g_1}^B(g_1(t), q_1) B_2(q_1)
\]

\[
\times \int_{g_2^{-(k-1)}(s)}^{g_2(t)} X_{g_1}^B(g_2(t), q_2) B_2(q_2) \cdots \int_{g_2^{-1}(s)}^{g_2(q_{k-1})} X_{g_1}^B(g_2(q_{k-1}), q_k) dq_k \cdots dq_1
\]

\[
\times B_2(q_k) X_{g_1}^B(g_2(q_k), s) dq_k \cdots dq_1 = B_1(t) X(g_1(t)) + B_2(t) X(g_2(t))
\]
since \( X_{g_1}^B(g_1(t), q_1) = \Theta \) whenever \( g_1 > g_1(t) \). In conclusion, we have proved that \( X(t) \) solves equation (2.9) for any \( t \geq s \). \( \square \)

**Remark 2.7.** Sometimes, it may be easier to use the “fixed \( t \)” form of \( X_{g_1,B}^B(t, s) \) analogical to (2.8) instead of “fixed \( s \)” given by (2.10).

Matrix function \( X_{g_1,B}^B(t, s) \) has some important properties which are concluded in the next lemma.

**Lemma 2.8.** Let \( s \in \mathbb{R}, B_1, B_2 \in C([s, \infty), L(\mathbb{R}^N, \mathbb{R}^N)), g_1, g_2 \in G^n \). Then the following statements hold true for any \( t \in \mathbb{R} \):

1. if \( B_1 = \Theta \) then \( X_{g_1,B}^B(t, s) = X_{g_2}^B(t, s) \),
2. if \( B_2 = \Theta \) then \( X_{g_1,B}^B(t, s) = X_{g_1}^B(t, s) \),
3. if \( g_1(t) = g_2(t) \) for all \( t \in [s, \infty) \), then \( X_{g_1,B}^B(t, s) = X_{g_1}^B(t, s) \),
4. \( X_{g_1,B}^B(t, s) = X_{g_2,B}^B(t, s) \).

**Proof.** All statements of the lemma follow from the uniqueness of a solution of a corresponding initial value problem. For instance in 1., both sides of the identity solve equation

\[
\dot{X}(t) = B_1(t)X(g_1(t)) + B_2(t)X(g_2(t)) = B_2(t)X(g_2(t))
\]

together with initial condition (2.2). \( \square \)

As before, we obtain a result on the solution of nonhomogeneous equation, this time with two delays.

**Theorem 2.9.** Let \( B_1, B_2 \in C([0, \infty), L(\mathbb{R}^N, \mathbb{R}^N)), g_1, g_2 \in G^n, f \in C([0, \infty), \mathbb{R}^N) \),

\( \gamma := \min\{g_1(0), g_2(0)\}, \varphi \in C([\gamma, 0], \mathbb{R}^N) \). Then the solution of the initial value problem

\[
\begin{align*}
\dot{x}(t) &= B_1(t)x(g_1(t)) + B_2(t)x(g_2(t)) + f(t), & t \geq 0 \\
x(t) &= \varphi(t), & \gamma \leq t \leq 0
\end{align*}
\]

has the form

\[
\begin{align*}
x(t) &= \begin{cases} 
\varphi(t), & \gamma \leq t < 0, \\
X_{g_1,B}^B(t, 0)\varphi(0) + \int_0^t X_{g_1,B}^B(t, s) [B_1(s)\varphi(g_1(s)) + B_2(s)\varphi(g_2(s))]ds + \int_0^t X_{g_1,B}^B(t, s)f(s)ds, & 0 \leq t
\end{cases}
\end{align*}
\]

where

\[
\psi(t) = \begin{cases} 
\varphi(t), & t \in [\gamma, 0), \\
0, & t \notin [\gamma, 0).
\end{cases}
\]

**Proof.** Clearly, the initial condition is satisfied. In verification of equation (2.11) we consider four cases with respect to \( t \).
Let \(0 \leq t < \min\{g_1^{-1}(0), g_2^{-1}(0)\}\). Then \(X_{g_1,g_2}^{B_1,B_2}(t,s) = E\) for \(s \in [0,t]\) since
\[g_2(t) < 0 \leq s < g_2^{-1}(0), \quad g_1(t) < 0 \leq s < g_1^{-1}(0).\]

Thus from (2.13) we get
\[x(t) = \varphi(0) + \int_0^t B_1(s)\varphi(g_1(s)) + B_2(s)\varphi(g_2(s))ds + \int_0^t f(s)ds\]
which is a solution of equation (2.11) since \(x(g_i(t)) = \varphi(g_i(t))\) for \(i = 1, 2\).

If \(g_1^{-1}(0) \leq t < g_2^{-1}(0)\) then \(g_2(t) < 0 \leq s < g_2^{-1}(0)\) for \(s \in [0,t]\).
Therefore \(x(g_2(t)) = \varphi(g_2(t)), \ X_{g_1,g_2}^{B_1,B_2}(t,s) = X_{g_1}^{B_1}(t,s)\) and we obtain
\[x(t) = X_{g_1}^{B_1}(t,0)\varphi(0) + \int_0^t X_{g_1}^{B_1}(t,s)[B_1(s)\psi(g_1(s)) + B_2(s)\varphi(g_2(s))]ds + \int_0^t X_{g_1}^{B_1}(t,s)f(s)ds.\]
After differentiating
\[\dot{x}(t) = B_1(t)x(q_1(t)) + B_2(t)\varphi(g_2(t)) + f(t),\]
so one can see that \(x(t)\) really solves (2.11).

The case \(g_2^{-1}(0) \leq t < g_1^{-1}(0)\) can be proved analogically to the previous one using the change \(X_{g_1,g_2}^{B_1,B_2}(t,s) = X_{g_2,g_1}^{B_2,B_1}(t,s)\) from Lemma 2.8.

Finally, if \(\max\{g_1^{-1}(0), g_2^{-1}(0)\} \leq t\) then \(\psi(g_1(t)) = \psi(g_2(t)) = 0\) and direct differentiation of (2.13) gives the desired result. \(\square\)

Now, we apply formula (2.13) on a problem with concrete unbounded delays.

**Example 2.10.** Let us consider the following initial value problem
\[\dot{x}(t) = B_1(t)x(t-1) + B_2(t)x(\sqrt{t} - 1), \quad t \geq 0\]
\[x(t) = \varphi(t), \quad -1 \leq t \leq 0.\]

In this case \(g_1(t) = t - 1, g_1^{-1}(s) = s + 1, g_2(t) = \sqrt{t} - 1, g_2^{-1}(s) = (s + 1)^2\) and we can assume \(s \geq 0\). Hence by (2.3),
\[
X_{g_1}^{B_1}(t,s) = \begin{cases} 
\Theta, & t < s, \\
E, & s \leq t < s + 1, \\
E + \int_{s+1}^t B_1(q_1)dq_1, & s + 1 \leq t < s + 2, \\
E + \int_{s+1}^{s+2} B_1(q_1)dq_1 + \int_s^{s+2} B_1(q_1)\int_{s+1}^{q_1-1} B_1(q_2)dq_2dq_1, & s + 2 \leq t < s + 3, \\
\ldots, & s + 3 \leq t.
\end{cases}
\]

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Next, from (2.10)

\( (2.15) \)

\[
X_{B_1,B_2}^{B_1,B_2}(t,s) = \begin{cases}
\Theta, & t < s, \\
E, & \text{on } M_1, \\
F_2(t,s) := E + \int_{s+1}^t B_1(q_1)dq_1, & \text{on } M_2, \\
F_3(t,s) := E + \int_{s+1}^t B_1(q_1)dq_1 + \int_{s+2}^{q_1-1} B_1(q_2)dq_2dq_1, & \text{on } M_3, \\
F_4(t,s) := E + \int_{s+1}^t B_1(q_1)dq_1 + \int_{s+1}^{t+1} B_1(q_2)dq_2dq_1, & \text{on } M_4, \\
F_5(t,s) := E + \int_{s+1}^t B_1(q_1)dq_1 + \int_{s+2}^{q_1-1} B_1(q_2)dq_2dq_1 + \int_{s+1}^{t+1} B_1(q_2)dq_2dq_1, & \text{on } M_5, \\
+ \int_{s+1}^{t+1} B_2(q_1)dq_1, & \text{on } M_6, \\
\cdots,
\end{cases}
\]

where

\[
M_1 = \{(t,s) \in \mathbb{R}^2_+ | s \leq t < (s+1)^2, s \leq t < s + 1 \},
\]

\[
M_2 = \{(t,s) \in \mathbb{R}^2_+ | s \leq t < (s+1)^2, s + 1 \leq t < s + 2 \},
\]

\[
M_3 = \{(t,s) \in \mathbb{R}^2_+ | s \leq t < (s+1)^2, s + 2 \leq t < s + 3 \},
\]

\[
M_4 = \{(t,s) \in \mathbb{R}^2_+ | (s+1)^2 \leq t < ((s+1)^2 + 1)^2, s + 1 \leq t < s + 2 \},
\]

\[
M_5 = \{(t,s) \in \mathbb{R}^2_+ | (s+1)^2 + 1 \leq t < ((s+1)^2 + 1)^2, s + 2 \leq t < s + 3 \},
\]

\[
M_6 = \{(t,s) \in \mathbb{R}^2_+ | (s+1)^2 \leq t < (s+1)^2 + 1, s + 2 \leq t < s + 3 \}
\]

with \( \mathbb{R}^2_+ = [0,\infty) \times [0,\infty) \). For the convenience, these sets are sketched in Figure 1.

**Corollary 2.11.** The solution of Example 2.10 has the form

\( (2.16) \)

\[
x(t) = \begin{cases}
\varphi(t), & -1 \leq t < 0, \\
\varphi(0) + \int_0^t B(s)ds, & 0 \leq t < 1, \\
F_4(t,0)\varphi(0) + \int_0^{t-1} F_4(t,s)B(s)ds + \int_{t-1}^{t-1} F_2(t,s)B(s)ds + \int_{t-1}^{t-1} B(s)ds, & 1 \leq t < 2, \\
F_5(t,0)\varphi(0) + \int_0^{t-1} F_5(t,s)B(s)ds + \int_{t-1}^{t-1} F_6(t,s)B(s)ds + \int_{t-1}^{t-1} F_2(t,s)B(s)ds + \int_{t-1}^{t-1} B(s)ds, & 2 \leq t < (3 + \sqrt{5})/2, \\
F_5(t,0)\varphi(0) + \int_0^{t-1} F_5(t,s)B(s)ds + \int_{t-1}^{t-1} F_6(t,s)B(s)ds + \int_{t-1}^{t-1} F_2(t,s)B(s)ds, & (3 + \sqrt{5})/2 \leq t < 3, \\
\cdots,
\end{cases}
\]

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where \( B(s) = B_1(s)\varphi(s-1) + B_2(s)\varphi(\sqrt{s}-1) \) and \( F_2(t,s), \ldots, F_6(t,s) \) are given in (2.15).

For the graph of the solution with concrete functions \( B_1, B_2 \in C([0, \infty), \mathbb{R}) \), \( \varphi \in C([-1,0], \mathbb{R}) \) see Figure 1.

One can proceed inductively from \( X_{g_1, g_2}^{B_1, B_2}(t,s) \) and, with the aid of the latter theorem, construct the fundamental solution of FDE with any finite number \( n \geq 3 \) of variable delays. So one obtains

(2.17)

\[
X_{g_1, \ldots, g_n}^{B_1, \ldots, B_n}(t,s) := \begin{cases} 
\Theta, & t < s, \\
Y(t,s), & s \leq t < g_n^{-1}(s), \\
Y(t,s) + \int_{g_n^{-1}(s)}^{t} Y(t,q_1)B_n(q_1)Y(g_n(q_1),s)dq_1 + \cdots \\
\cdots + \int_{g_n^{-1}(s)}^{t} Y(t,q_1)B_n(q_1)\int_{g_n(q_1)^{k-1}(s)}^{g_n(q_1)} Y(g_n(q_1),q_2)B_n(q_2)\cdots \\
\cdots \times \int_{g_n^{-1}(s)}^{g_n(q_{k-1})} Y(g_n(q_{k-1}),q_k)B_n(q_k)Y(g_n(q_k),s)dq_1 \cdots dq_k, & g_n^{-k}(s) \leq t < g_n^{-(k+1)}(s), k \in \mathbb{N}.
\end{cases}
\]

where \( Y(t,s) = X_{g_1, \ldots, g_n}^{B_1, \ldots, B_n^{-1}}(t,s) \).

**Theorem 2.12.** Let \( s \in \mathbb{R}, 3 \leq n \in \mathbb{N}, B_i \in C([s, \infty), L(\mathbb{R}^N, \mathbb{R}^N)), g_i \in G^s \) for \( i = 1, \ldots, n \). Then \( X(t) = X_{g_1, \ldots, g_n}^{B_1, \ldots, B_n}(t,s) \) is the matrix solution of equation

(2.18)\[ \dot{X}(t) = B_1(t)X(g_1(t)) + \cdots + B_n(t)X(g_n(t)), \quad t \geq s \]

satisfying initial condition (2.2).

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Proof. The case \( n = 2 \) was proved in Theorem 2.6. So here we suppose that the statement is true for \( n - 1 \) and we show that it holds also for \( n \).

Clearly, at \( t = s \)

\[
X_{g_1,\ldots,g_n}^{B_1,\ldots,B_n}(s, s) = X_{g_1,\ldots,g_n}^{B_1,\ldots,B_{n-1}}(s, s) = \cdots = X_{g_1}^{B_1}(s, s) = E
\]

and initial condition is verified. For the simplicity we denote \( Y(t, s) = X_{g_1,\ldots,g_n}^{B_1,\ldots,B_{n-1}}(t, s) \) in the rest of the proof.

If \( s \leq t < g_n^{-1}(s) \) then \( g_n(t) < s \) and

\[
\begin{align*}
\dot{X}(t) &= \dot{Y}(t, s) = B_1(t)Y(g_1(t), s) + \cdots + B_{n-1}(t)Y(g_{n-1}(t), s) \\
&= B_1(t)X(g_1(t)) + \cdots + B_{n-1}(t)X(g_{n-1}(t)) + B_n(t)X(g_n(t))
\end{align*}
\]

since \( X(g_n(t)) = \Theta \).

Now, let \( g_n^{-k}(s) \leq t < g_n^{-k+1}(s) \) for some \( k \in \mathbb{N} \). Then from (2.17) using the inductive hypothesis, we get for the derivative

\[
\begin{align*}
\dot{X}(t) &= B_1(t)Y(g_1(t), s) + \cdots + B_{n-1}(t)Y(g_{n-1}(t), s) + B_n(t)Y(g_n(t), s) \\
&+ \int_{g_n^{-1}(s)}^{g_n(t)} [B_1(t)Y(g_1(t), q_1) + \cdots + B_{n-1}(t)Y(g_{n-1}(t), q_1)]B_n(q_1)Y(g_n(q_1), s)dq_1 \\
&+ \cdots + B_n(t)\int_{g_n^{-k}(s)}^{g_n(t)} Y(q_n(t), q_2)B_n(q_2)\cdots\int_{g_n^{-1}(s)}^{g_n(q_{k-1})} Y(q_n(q_{k-1}), q_k)B_n(q_k) \\
&\times Y(q_n(q_n), s)dq_k \cdots dq_2 + \int_{g_n^{-k}(s)}^{g_n(t)} [B_1(t)Y(g_1(t), q_1) + \cdots + B_{n-1}(t)Y(g_{n-1}(t), q_1)] \\
&\times B_n(q_1)\cdots\int_{g_n^{-1}(s)}^{g_n(q_{k-1})} Y(q_n(q_{k-1}), q_k)B_n(q_k)Y(q_n(q_n), s)dq_k \cdots dq_1.
\end{align*}
\]

By collecting terms beginning with \( B_i(t) \) we obtain for each \( i = 1, \ldots, n-1 \) exactly \( B_i(t)X(g_i(t)) \) since \( Y(g_i(t), q_i) = \Theta \) for \( g_i(t) < q_i \) (hence the upper boundary of integrals is changed from \( t \) to \( g_i(t) \)). Next, \( g_n^{-k}(s) \leq g_n(t) < g_n^{-k+1}(s) \), thus collecting terms beginning with \( B_n(t) \) yields \( B_n(t)X(g_n(t)) \) (in comparison to \( X(t) \), the number of integrals in \( X(g_n(t)) \) is decreased by one). In conclusion, the last identity is precisely the equation which \( X(t) \) has to satisfy. \( \square \)

Matrix function \( X_{g_1,\ldots,g_n}^{B_1,\ldots,B_n}(t, s) \) has properties that are analogous to those of \( X_{g_1,g_2}^{B_1,B_2}(t, s) \) provided in Lemma 2.8. We conclude them into a lemma without a proof.

**Lemma 2.13.** Let \( s \in \mathbb{R} \), \( 3 \leq n \in \mathbb{N} \), \( B_i \in C([s, \infty), L(\mathbb{R}^N, \mathbb{R}^N)) \), \( g_i \in G^s \) for \( i = 1, \ldots, n \). Then the following statements hold true for any \( t \in \mathbb{R} \):

1. If \( B_i = \Theta \) for some \( i \in \{1, \ldots, n\} \), then

\[
X_{g_1,\ldots,g_n}^{B_1,\ldots,B_{i-1},B_{i+1},\ldots,B_n}(t, s) = X_{g_1,\ldots,g_i\cdot g_{i+1},\ldots,g_n}^{B_1,\ldots,B_{i-1},B_{i+1},\ldots,B_n}(t, s),
\]

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(2) if \( g_i(t) = g_j(t) \) for any \( t \in [s, \infty) \) and some \( i < j, i, j \in \{1, \ldots, n\} \),

\[
X_{g_1, \ldots, g_i} \cdots B_{i-1, B_i} \cdots B_i^{-1} \cdots B_{n-1, B_n} (t, s) = X_{g_1, \ldots, g_i} \cdots B_{i-1, B_i} \cdots B_{i+1} \cdots B_n (t, s),
\]

(3) for any bijective mapping \( \sigma : \{1, \ldots, n\} \to \{1, \ldots, n\} \)

\[
X_{g_1, \ldots, g_n} (t, s) = X_{g_{\sigma(1)}, \ldots, g_{\sigma(n)}} (t, s).
\]

The statement on the solution of the nonhomogeneous initial value problem with \( n \) delays follows (cf. [9]).

**Theorem 2.14.** Let \( 3 \leq n \in \mathbb{N}, B_i \in C([0, \infty), L(\mathbb{R}^N, \mathbb{R}^N)), g_i \in C^0 \) for \( i = 1, \ldots, n, f \in C([0, \infty), \mathbb{R}^N), \gamma := \min\{g_1(0), \ldots, g_n(0)\}, \varphi \in C([\gamma, 0], \mathbb{R}^N) \). Then the solution of the equation

\[
\dot{x}(t) = B_1(t)x(g_1(t)) + \cdots + B_n(t)x(g_n(t)) + f(t), \quad t \geq 0
\]

satisfying initial condition (2.12) has the form

\[
x(t) = \begin{cases} 
\varphi(t), & \gamma \leq t < 0, \\
\varphi(g_1(t)) + \int_0^t X_{g_1, \ldots, g_n} (s, t) [B_1(s)\varphi(g_1(s)) + \cdots + B_n(s)\varphi(g_n(s))] ds + \int_0^t X_{g_1, \ldots, g_n} (s, t) f(s) ds, & 0 \leq t
\end{cases}
\]

where \( \psi(t) \) is given by (2.14).

**Proof.** The proof is similar to the proof of Theorem 2.9, so we omit some details. Note that

\[
\psi(g_i(t)) = \begin{cases} 
\varphi(g_i(t)), & t < g_i^{-1}(0), \\
0, & g_i^{-1}(0) \leq t
\end{cases}
\]

for each \( i = 1, \ldots, n \).

If \( 0 \leq t < \min\{g_1^{-1}(0), \ldots, g_n^{-1}(0)\} \) then \( X_{g_1, \ldots, g_n} (t, s) = E \) for \( s \in [0, t] \) and \( x(t) \) has the form

\[
x(t) = \varphi(0) + \int_0^t [B_1(s)\varphi(g_1(s)) + \cdots + B_n(s)\varphi(g_n(s))] ds + \int_0^t f(s) ds
\]

what solves equation (2.19).

If there are the nonempty sets

\[
M_1 = \{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}, \quad M_2 = \{1, \ldots, n\} \setminus M_1
\]

such that \( g_{i_k}^{-1}(0) \leq t < g_{j_k}^{-1}(0) \) for each \( i \in M_1, j \in M_2 \), then applying Lemma 2.13 we obtain \( X_{g_1, \ldots, g_n} (t, s) = X_{g_{\sigma(1)}, \ldots, g_{\sigma(n)}} (t, s) \) for \( s \in [0, t] \) and

\[
\psi(g_i(t)) = \begin{cases} 
0, & i \in M_1, \\
\varphi(g_i(t)), & i \in M_2
\end{cases}
\]

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Consequently, we rewrite $x(t)$ as

$$x(t) = X(t,0)\varphi(0) + \int_0^t X(t,s) \left[ \sum_{i \in M_1} B_i(s)\psi(g_i(s)) + \sum_{j \in M_2} B_j(s)\varphi(g_j(s)) \right] ds$$

$$+ \int_0^t X(t,s)f(s)ds$$

where $X(t,s) = X_{g_1^{-1} \cdots g_k^{-1}}^B(t,s)$. Then for the derivative it holds

$$\dot{x}(t) = \sum_{i \in M_1} B_i(t)X(g_i(t),0)\varphi(0) + \sum_{j \in M_2} B_j(t)\varphi(g_j(t))$$

$$+ \int_0^t \sum_{i \in M_1} B_i(t)X(g_i(t),s) \left[ \sum_{j \in M_1} B_j(s)\psi(g_j(s)) + \sum_{j \in M_2} B_j(s)\varphi(g_j(s)) \right] ds$$

$$+ f(t) + \int_0^t \sum_{i \in M_1} B_i(t)X(g_i(t),s)f(s)ds.$$
Proof. Denote \( y(t) = e^{-At}x(t) \). Then from (2.21), (2.12)

\[
\begin{align*}
\dot{y}(t) &= \tilde{B}_1(t)y(g_1(t)) + \cdots + \tilde{B}_n(t)y(g_n(t)) + \tilde{f}(t), \quad t \geq 0 \\
y(t) &= \tilde{\varphi}(t), \quad \gamma \leq t \leq 0
\end{align*}
\]

where \( \tilde{f}(t) = e^{-At}f(t) \), \( \tilde{\varphi}(t) = e^{-At}\varphi(t) \). Applying Theorem 2.14 to this problem yields

\[
y(t) = \begin{cases}
\tilde{\varphi}(t), & \gamma \leq t < 0, \\
X_{g_1, \ldots, g_n}^\tilde{B}_1, \ldots, \tilde{B}_n(t, 0)\varphi(0) + \int_0^t X_{g_1, \ldots, g_n}^\tilde{B}_1, \ldots, \tilde{B}_n(t, s) \left[ \tilde{B}_1(s)\tilde{\psi}(g_1(s)) \\
\quad \cdots + \tilde{B}_n(s)\tilde{\psi}(g_n(s)) \right] ds + \int_0^t X_{g_1, \ldots, g_n}^\tilde{B}_1, \ldots, \tilde{B}_n(t, s)\tilde{f}(s)ds, & 0 \leq t
\end{cases}
\]

where

\[
\tilde{\psi}(t) = \begin{cases}
\tilde{\varphi}(t), & t \in [\gamma, 0), \\
0, & t \notin [\gamma, 0).
\end{cases}
\]

Note that \( \tilde{\psi}(t) = e^{-At}\psi(t) \) for any \( t \in \mathbb{R} \) and \( \tilde{B}_1(s)\tilde{\psi}(g_i(s)) = e^{-As}B_i\psi(g_i(s)) \). When one returns to \( x(t) \), the formula (2.22) is obtained. \( \square \)

3. Exponential stability of nonlinear FDEs

In this section, we apply the theory derived in the preceding section to establish criteria for the exponential stability of the trivial solution of nonlinear FDE with multiple variable delays where the linear parts are given by pairwise permutable constant matrices. First, we estimate the fundamental solutions \( X_\beta^\beta(t, s) \) and \( X_{g_1, \ldots, g_n}^\tilde{B}_1, \ldots, \tilde{B}_n(t, s) \) with the aid of the next lemma.

Lemma 3.1. If \( s \in \mathbb{R} \) and \( f \in C([s, \infty), \mathbb{R}) \), then

\[
\int_s^t f(q_1)\int_s^{q_1} f(q_2)\cdots\int_s^{q_{k-1}} f(q_k) dq_k \cdots dq_1 = \frac{1}{k!} \left( \int_s^t f(q) dq \right)^k
\]

for each \( k \in \mathbb{N}, t \in \mathbb{R} \).

Proof. We prove the lemma via induction with respect to \( k \). Denote

\[
F_k(t) = \int_s^t f(q_1)\int_s^{q_1} f(q_2)\cdots\int_s^{q_{k-1}} f(q_k) dq_k \cdots dq_1
\]

\[
G_k(t) = \frac{1}{k!} \left( \int_s^t f(q) dq \right)^k.
\]

Clearly \( F_1(t) = G_1(t) \). Let \( F_{k-1}(t) = G_{k-1}(t) \). Then \( F_k(s) = 0 = G_k(s) \) and

\[
F'_k(t) = f(t)F_{k-1}(t) = f(t)G_{k-1}(t) = \frac{f(t)}{(k-1)!} \left( \int_s^t f(q) dq \right)^{k-1} = G'_k(t).
\]

\( \square \)

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Lemma 3.2. Let \( s \in \mathbb{R} \), \( B \in C([s, \infty), L(\mathbb{R}^N, \mathbb{R}^N)) \), \( g \in G^s \). Then
\[
\|X_g^B(t, s)\| \leq \exp\left\{ \int_s^t \|B(q)\| dq \right\}
\]
for any \( t \geq s \).

Proof. It is sufficient to prove the statement for \( g : [s, \infty) \to [g(s), \infty) \) surjective. By this, the other case is also covered.

Let \( t \geq g^{-1}(s) \) be arbitrary and fixed, \( k \in \mathbb{N} \) be such that \( g^{-k}(s) \leq t < g^{-(k+1)}(s) \). Then from (2.3) we know that
\[
X_g^B(t, s) = E + X_1(t, s) + \cdots + X_k(t, s)
\]
where
\[
X_j(t, s) = \int_{g^{-j}(s)}^t B(q_1) \int_{g^{-(j-1)}(s)}^{g(q_1)} B(q_2) \cdots \int_{g^{-1}(s)}^{g(q_{j-1})} B(q_j) dq_j \cdots dq_1
\]
for \( j = 1, \ldots, k \). Since \( g \) is increasing and according to Lemma 3.1 we derive
\[
\|X_j(t, s)\| \leq \int_{g^{-j}(s)}^t \|B(q_1)\| \int_{g^{-(j-1)}(s)}^{g(q_1)} \|B(q_2)\| \cdots \int_{g^{-1}(s)}^{g(q_{j-1})} \|B(q_j)\| dq_j \cdots dq_1
\]
\[
\leq \int_s^t \|B(q_1)\| \int_s^{q_1} \|B(q_2)\| \cdots \int_s^{q_{j-1}} \|B(q_j)\| dq_j \cdots dq_1 = \frac{1}{j!} \left( \int_s^t \|B(q)\| dq \right)^j.
\]
As a consequence,
\[
\|X_g^B(t, s)\| \leq 1 + \|X_1(t, s)\| + \cdots + \|X_k(t, s)\|
\]
\[
\leq \sum_{j=0}^k \frac{1}{j!} \left( \int_s^t \|B(q)\| dq \right)^j \leq \sum_{j=0}^\infty \frac{1}{j!} \left( \int_s^t \|B(q)\| dq \right)^j = \exp \left\{ \int_s^t \|B(q)\| dq \right\}.
\]
Obviously, the last estimate holds for each \( k \in \mathbb{N} \) and hence for any \( t \geq g^{-1}(s) \).

If \( s \leq t < g^{-1}(s) \) then
\[
\|X_g^B(t, s)\| = \|E\| = 1 \leq \exp \left\{ \int_s^t \|B(q)\| dq \right\}
\]
so it remains true for such \( t \). \( \Box \)

Lemma 3.3. Let \( s \in \mathbb{R} \), \( 2 \leq n \in \mathbb{N} \), \( B_i \in C([s, \infty), L(\mathbb{R}^N, \mathbb{R}^N)) \), \( g_i \in G^s \) for \( i = 1, \ldots, n \). Then
\[
\|X_{g_1, \ldots, g_n}^{B_1, \ldots, B_n}(t, s)\| \leq \exp \left\{ \int_s^t \sum_{i=1}^n \|B_i(q)\| dq \right\}
\]
for any \( t \geq s \).
Proof. As before, it is enough to prove the lemma for \( g_i : [s, \infty) \rightarrow [g(s), \infty) \) surjective for each \( i = 1, \ldots, n \).

We show that if the statement holds for \( n - 1 \) delays, then it is true for \( n \). Let \( k \in \mathbb{N} \) be such that \( g_n^{-k}(s) \leq t < g_n^{-(k+1)}(s) \) for arbitrary and fixed \( t \geq g_n^{-1}(s) \). Then from (2.17) we know that

\[
X_{g_1, \ldots, g_n}^n(t, s) = Y(t, s) + X_1(t, s) + \cdots + X_k(t, s)
\]

where

\[
X_j(t, s) = \int_{g_n^{-j}(s)}^t Y(t, q_1)B_n(q_1) \int_{g_n^{-(j-1)}(s)}^{g_n(q_2)} Y(g_n(q_1), q_2)B_n(q_2) \cdots \cdot \cdot \cdot \int_{g_n^{-1}(s)}^{g_n(q_j-1)} Y(g_n(q_j-1), q_j)B_n(q_j)Y(g_n(q_j), s)dq_j \ldots dq_1
\]

for \( j = 1, \ldots, k \) and \( Y(t, s) = X_{g_1, \ldots, g_n}^n(t, s) \). For \( X_j(t, s) \) we get

(3.1)

\[
\|X_j(t, s)\| \leq \int_{g_n^{-j}(s)}^t \|Y(t, q_1)\|\|B_n(q_1)\| \int_{g_n^{-(j-1)}(s)}^{g_n(q_1)} \|Y(g_n(q_1), q_2)\|\|B_n(q_2)\| \cdots \cdot \cdot \cdot \int_{g_n^{-1}(s)}^{g_n(q_j-1)} \|Y(g_n(q_j-1), q_j)\|\|B_n(q_j)\|\|Y(g_n(q_j), s)\|dq_j \ldots dq_1.
\]

Applying the inductive hypothesis, we know that

\[
\|Y(t, s)\| \leq \exp \left\{ \int_s^{t} \sum_{i=1}^{n-1} \|B_i(q)\|dq \right\}.
\]

Thus we estimate the right-hand side of inequality (3.1) by

\[
\int_{g_n^{-j}(s)}^t \|B_n(q_1)\| \int_{g_n^{-(j-1)}(s)}^{g_n(q_1)} \|B_n(q_2)\| \cdots \int_{g_n^{-1}(s)}^{g_n(q_j-1)} \|B_n(q_j)\| \times Z(t, q_1, \ldots, q_j, s)dq_j \ldots dq_1
\]

for each \( j = 1, \ldots, k \), where for \( \beta(q) = \sum_{i=1}^{n-1} \|B_i(q)\| \)

\[
Z(t, q_1, \ldots, q_j, s) = \exp \left\{ \int_{q_1}^t \beta(q)dq + \int_{q_2}^{q_1} \beta(q)dq + \cdots + \int_{q_j}^{q_{j-1}} \beta(q)dq + \int_{s}^{q_j} \beta(q)dq \right\} \leq \exp \left\{ \int_s^t \beta(q)dq \right\}.
\]

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Therefore we get
\[
\|X_j(t, s)\| \leq \exp \left\{ \int_s^t \beta(q) dq \right\} \int_s^t \|B_n(q_1)\| \int_{g_{n-1}(q_1)}^{g_{n}(q_{j-1})} \|B_n(q_j)\| \cdots \times \int_{g_{n-1}(s)}^{g_{n}(q_j)} \|B_n(q_j)\| dq_j \cdots dq_1
\]
for each \( j = 1, \ldots, k \). In conclusion,
\[
\|X_{g_1,\ldots,g_n}^B(t, s)\| \leq \exp \left\{ \int_s^t \beta(q) dq \right\} \left( 1 + \int_{g_{n-1}^{-1}(s)}^t \|B_n(q_1)\| dq_1 + \cdots + \int_{g_{n-1}^{-1}(s)}^t \|B_n(q_1)\| \int_{g_{n-1}^{-1}(s)}^{g_{n-1}^{-1}(s)} \|B_n(q_2)\| \cdots \int_{g_{n-1}^{-1}(s)}^{g_{n-1}^{-1}(s)} \|B_n(q_k)\| dq_k \cdots dq_1 \right).
\]
Finally, applying Lemma 3.1 as in the proof of Lemma 3.2 we obtain
\[
\|X_{g_1,\ldots,g_n}^B(t, s)\| \leq \exp \left\{ \int_s^t \beta(q) + \|B_n(q)\| dq \right\}.
\]
Thus the statement is proved for \( t \geq g_{n}^{-1}(s) \). Analogically, one can prove it for \( t \geq g_i^{-1}(s) \) for any \( i = 1, \ldots, n - 1 \) by the change of order described in Lemma 2.13. If \( s \leq t < \min\{g_1^{-1}(s), \ldots, g_n^{-1}(s)\} \), \( X_{g_1,\ldots,g_n}^B(t, s) = E \), hence the statement holds. \( \square \)

Now, we define what exactly we shall understand under the notion of exponential stability. Then we use the estimations of fundamental solutions to derive the sufficient conditions for the exponential stability of FDEs with different types of nonlinearities (see [14, 15, 16, 17] for analogical criteria for delay differential and difference equations with constant delays).

**Definition 3.4.** Let \( n \in \mathbb{N} \), \( A, B_1, \ldots, B_n \) be pairwise permutable \( N \times N \) constant matrices, i.e. \( AB_i = B_iA \), \( B(IB_j = B_jB_i \) for each \( i, j \in \{1, \ldots, n\} \), \( g_i \in G^0 \) for \( i = 1, \ldots, n \), \( \gamma := \min\{g_1(0), \ldots, g_n(0)\} \), \( \varphi \in C([\gamma, 0]; \mathbb{R}^N) \) and
\[
f: \mathbb{R}^N \times \cdots \times \mathbb{R}^N \rightarrow \mathbb{R}^{n+1}
\]
be a given continuous function. A solution \( x_\varphi : [\gamma, \infty) \rightarrow \mathbb{R}^N \) of the equation
\[
\dot{x}(t) = Ax(t) + B_1x(g_1(t)) + \cdots + B_nx(g_n(t)) + f(x(t), x(g_1(t)), \ldots, x(g_n(t))), \quad t \geq 0
\]
with initial condition (2.12) is called exponentially stable, if there exist positive constants \( c_1, c_2, \delta \), depending on \( A, B_1, \ldots, B_n, f \) and \( \|\varphi\| = \max_{t \in [\gamma, 0]} \|\varphi(t)\| \), such that
\[
\|x_\eta(t) - x_\varphi(t)\| \leq c_1e^{-\delta t}, \quad t \geq 0
\]
for any solution \( x_\eta(t) \) of the equation (3.2) satisfying the initial condition

\[
x_\eta(t) = \eta(t), \quad \gamma \leq t \leq 0
\]

with \( \eta \in C([\gamma, 0], \mathbb{R}^N) \), \( \| \eta - \varphi \| < \delta \).

**Theorem 3.5.** Let \( n \in \mathbb{N} \), \( A, B_1, \ldots, B_n \) be pairwise permutable \( N \times N \) constant matrices, i.e. \( AB_i = B_iA \), \( B_iB_j = B_jB_i \) for each \( i, j \in \{1, \ldots, n\} \), \( A \) have eigenvalues \( \lambda_1, \ldots, \lambda_N \) such that

\[
\text{Re} \lambda_1 \leq \cdots \leq \text{Re} \lambda_N \leq -k < 0,
\]

\( g_i \in G^0 \) for \( i = 1, \ldots, n \) and there be \( k_1 < k \) such that

\[
\int_0^t \sum_{i=1}^n \| \tilde{B}_i(q) \| dq \leq k_1 t
\]

for all \( t \geq 0 \), where \( \tilde{B}_i(t) = e^{-A(t-n(t))}B_i \) for \( i = 1, \ldots, n \). Then if \( f(x) = o(||x||) \), the trivial solution of equation

\[
(3.3) \quad \dot{x}(t) = Ax(t) + B_1x(g_1(t)) + \cdots + B_nx(g_n(t)) + f(x(t))
\]

is exponentially stable.

**Proof.** Let \( \gamma := \min\{g_1(0), \ldots, g_n(0)\} \), \( \varphi \in C([\gamma, 0], \mathbb{R}^N) \). According to Theorem 2.15 the solution of equation (3.3) satisfying condition (2.12) has the form

\[
x(t) = X_{g_1(0), \ldots, g_n(0)} (t, 0)e^{At}\varphi(0) + \int_0^t X_{g_1(t), \ldots, g_n(t)} (t, s) e^{A(t-s)} \left[ B_1\psi(g_1(s)) + \cdots + B_n\psi(g_n(s)) \right] ds + \int_0^t X_{g_1(t), \ldots, g_n(t)} (t, s) e^{A(t-s)} f(x(s)) ds
\]

for \( t \geq 0 \), where \( \psi(t) \) is given by (2.14). From the property of eigenvalues of \( A \) it follows that there are positive constants \( k, K \) such that \( ||e^{At}|| \leq Ke^{-kt} \) for all \( t \geq 0 \). Next, since \( f(x) = o(||x||) \), for any \( P > 0 \) there is \( \delta > 0 \) such that if \( ||x|| < \delta \), then \( ||f(x)|| < P ||x|| \). Applying these two estimations, Lemma 3.3 and assuming that \( ||x(s)|| \) is sufficiently small for \( s \in [0, t], t \geq 0 \) we derive

\[
||x(t)|| \leq Ke_0 \int_0^t e^{\beta(s) - k(t-s)} ||\varphi(0)|| + K \sum_{i=1}^n ||B_i|| \int_0^t e^{\int_0^s \beta(s) - k(t-s)} ||\psi(g_i(s))|| ds
\]

\[
+ KP \int_0^t e^{\int_0^t \beta(s) - k(t-s)} ||x(s)|| ds
\]

with \( \beta(t) = \sum_{i=1}^n ||\tilde{B}_i(t)||. \) Denoting \( u(t) = \exp\{kt - \int_0^t \beta(s) ds\} ||x(t)|| \) we get the estimate for \( u(t) \)

\[
(3.4) \quad u(t) \leq K ||\varphi(0)|| + K \sum_{i=1}^n ||B_i|| \int_0^t e^{\int_0^s \beta(s) - k(t-s)} ||\psi(g_i(s))|| ds + KP \int_0^t u(s) ds.
\]

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From (3.4) we get

\textbf{Theorem 3.6.} Let the assumptions of Theorem 3.5 be fulfilled. Moreover, (3.3) is exponentially stable. □

Let 

\[ S \]

\[ M \]

\[ i.e. \] for any 

\[ P \]

\[ t \]

\[ f(t) \]

\[ \eta \]

\[ \psi \]

\[ k_{t} \]

\[ t \]

\[ u(t) \]

\[ \|x(t)\| \]

\[ \exp \left \{ KPt - kt + \int_{0}^{t} \beta(q) dq \right \} \]

\[ M e^{(KP - k_{1} - K) t} \]

Therefore, if \( P < \frac{k_{1}-k_{t}}{K} \) then for max\{\|\varphi(0)\|, M\} < \delta it holds \( \|x(t)\| \leq Me^{-n t} \) for all \( t \geq 0 \) with \( \eta = k_{t} - k_{1} - KP > 0 \), i.e. the trivial solution of (3.3) is exponentially stable. □

\textbf{Theorem 3.6.} Let the assumptions of Theorem 3.5 be fulfilled. Moreover, let \( S_{i} := \sup_{t \geq 0} t - g_{i}(t) < \infty \) for each \( i = 1, \ldots, n \). If

\[ f(x, y_{1}, \ldots, y_{n}) = o(\|x\| + \|y_{1}\| + \cdots + \|y_{n}\|), \]

\[ i.e. \] for any \( P > 0 \) there exists \( \delta > 0 \) such that

\[ \|x\|, \|y_{1}\|, \ldots, \|y_{n}\| < \delta \quad \Rightarrow \quad \|f(x, y_{1}, \ldots, y_{n})\| < P(\|x\| + \|y_{1}\| + \cdots + \|y_{n}\|), \]

then the trivial solution of equation (3.2) is exponentially stable.

\textbf{Proof.} Let \( \gamma := \min\{g_{1}(0), \ldots, g_{n}(0)\} \), \( \varphi \in C([\gamma, 0], \mathbb{R}^{N}) \). Denote

\[ E(t) = \begin{cases} 
0, & t < 0, \\
kt - \int_{0}^{t} \beta(q) dq, & t \geq 0,
\end{cases} \quad \beta(t) = \sum_{i=1}^{n} \|\tilde{B}_{i}(t)\|, \quad u(t) = e^{E(t)}\|x(t)\|, \]

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with $x(t)$ being the solution of equation (3.2) together with condition (2.12), which is known to have a form (see Theorem 2.15)

$$
x(t) = X_{g_1 \ldots g_n}(t, 0) e^{At} \varphi(0) + \int_0^t X_{g_1 \ldots g_n}(t, s) e^{A(t-s)} [B_1 \psi(g_1(s)) + \ldots + B_n \psi(g_n(s))] ds + \int_0^t X_{g_1 \ldots g_n}(t, s) e^{A(t-s)} f(x(s), x(g_1(s)), \ldots, x(g_n(s))) ds
$$

for $t \geq 0$, where $\psi(t)$ is given by (2.14). Note that $E(t)$ is a continuous function defined on the whole $\mathbb{R}$. Analogically to the proof of Theorem 3.5, we derive estimation

$$
u(t) \leq M + KP \int_0^t \nu(s) ds + KP \sum_{i=1}^n \int_0^t e^{E(s)-E(g_i(s))} \nu(g_i(s)) ds
$$

with $M$ given by (3.5), assuming $\|\varphi\|$ and $\|x(s)\|$ to be sufficiently small for all $s \in [0, t]$, $t \geq 0$. Denoting $h(t)$ the nondecreasing continuous function defined on $[0, \infty)$ and given by

$$
h(t) := c + KP \int_0^t \nu(s) ds + KP \sum_{i=1}^n \int_0^t e^{E(s)-E(g_i(s))} \nu(g_i(s)) ds
$$

where $c = \max\{M, \|\varphi\|\}$, we get the inequality $\nu(t) \leq h(t)$ on $[0, \infty)$. Let us estimate for $s \in [0, t]$:

$$
u(g_i(s)) \leq \sup_{0 \leq \sigma \leq s} \nu(g_i(\sigma)) \leq \sup_{0 \leq \sigma \leq g_i^{-1}(0)} \nu(g_i(\sigma)) + \sup_{g_i^{-1}(0) \leq \sigma \leq s} \nu(g_i(\sigma)) \leq \|\varphi\| + \sup_{0 \leq \sigma \leq s} \nu(\sigma) \leq 2h(s),
$$

(3.7) by the property of a nondecreasing function $h$. Thus we obtain

$$
h(t) \leq c + KP \int_0^t h(s) ds + 2KP \sum_{i=1}^n \int_0^t e^{E(s)-E(g_i(s))} h(s) ds
$$

for all $t \geq 0$. Furthermore, for each $i \in \{1, \ldots, n\}$, if $0 \leq s < g_i^{-1}(0)$, then

(3.8) $E(s) - E(g_i(s)) = ks - \int_0^s \beta(q) dq \leq kg_i^{-1}(0) = k(t - g_i(t))|_{t=g_i^{-1}(0)} \leq kS_i$

and if $s \geq g_i^{-1}(0)$,

(3.9) $E(s) - E(g_i(s)) = k(s - g_i(s)) - \int_{g_i(s)}^s \beta(q) dq \leq k(s - g_i(s)) \leq kS_i.$

Therefore

$$
h(t) \leq c + KP \left(1 + 2 \sum_{i=1}^n e^{kS_i}\right) \int_0^t h(s) ds
$$

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whenever $t \geq 0$. Finally, from Gronwall’s inequality

$$u(t) \leq h(t) \leq c \exp \left\{ KP \left( 1 + 2 \sum_{i=1}^{n} e^{kS_i} \right) t \right\}$$

for all $t \geq 0$. Hence

$$\|x(t)\| \leq c \exp \left\{ KP \left( 1 + 2 \sum_{i=1}^{n} e^{kS_i} \right) t - kt + \int_{0}^{t} \beta(q) dq \right\}$$

$$\leq c \exp \left\{ \left( KP \left( 1 + 2 \sum_{i=1}^{n} e^{kS_i} \right) - k + k_1 \right) t \right\}.$$ 

So, if $P < \frac{k-k_1}{K(1+2\sum_{i=1}^{n} e^{kS_i})}$, then for $c < \delta$ (that is for $\|\varphi\|$ sufficiently small) the solution $x(t)$ satisfies $\|x(t)\| \leq ce^{-\eta t}$ with $\eta = k - k_1 - KP(1+2\sum_{i=1}^{n} e^{kS_i}) > 0$, i.e. the trivial solution of (3.2) is exponentially stable. □

In further work we shall write $\omega_1 \propto \omega_2$ for functions $\omega_1, \omega_2 : A \rightarrow \mathbb{R} \setminus \{0\}$, $A \subset \mathbb{R}$, if the function $\frac{\omega_1}{\omega_2}$ is nondecreasing on $A$.

**Theorem 3.7.** Let the assumptions of Theorem 3.5 be fulfilled and $S_i := \sup_{t \geq 0} t - g_i(t) < \infty$ for each $i = 1, \ldots, n$. If

$$f(x, y_1, \ldots, y_n) = o\left(\|x\|^{\gamma_0} + \|y_1\|^{\gamma_1} + \cdots + \|y_n\|^{\gamma_n}\right)$$

for given constants $1 < \gamma_0, \gamma_1, \ldots, \gamma_n$, i.e. for any $P > 0$ there exists $\delta > 0$ such that if $\|x\|, \|y_1\|, \ldots, \|y_n\| < \delta$ then

$$\|f(x, y_1, \ldots, y_n)\| < P \left(\|x\|^{\gamma_0} + \|y_1\|^{\gamma_1} + \cdots + \|y_n\|^{\gamma_n}\right),$$

then the trivial solution of equation (3.2) is exponentially stable.

**Proof.** Let $\gamma := \min\{g_1(0), \ldots, g_n(0)\}, \varphi \in C([\gamma, 0], \mathbb{R}^N)$ and $x(t)$ be a solution of equation (3.2) satisfying initial condition (2.12). As before, assuming $\|\varphi\|, \|x(s)\|$ to be sufficiently small for all $s \geq 0$ and using notations (3.5), (3.6) we obtain inequality

$$u(t) \leq M + KP \int_{0}^{t} e^{E(s)(1-\gamma_0)} u(s)^{\gamma_0} ds + KP \sum_{i=1}^{n} \int_{0}^{t} e^{E(s)-\gamma_iE(g_i(s))} u(g_i(s))^{\gamma_i} ds.$$

Denote

$$h(t) := c + KP \int_{0}^{t} e^{E(s)(1-\gamma_0)} u(s)^{\gamma_0} ds + KP \sum_{i=1}^{n} \int_{0}^{t} e^{E(s)-\gamma_iE(g_i(s))} u(g_i(s))^{\gamma_i} ds$$

with $c = \max\{\|\varphi\|, M\}$. Clearly, $u(t) \leq h(t)$ for all $t \geq 0$ and, arguing like in (3.7), also $u(g_i(t)) \leq 2h(t)$ for all $t \geq 0$ and each $i = 1, \ldots, n$. Next, EJQTDE, 2012 No. 54, p. 23
\( E(t)(1-\gamma_0) \leq (k-k_1)(1-\gamma_0)t \) for all \( t \geq 0 \) and by (3.8), (3.9)

\[
E(t) - \gamma_i E(g_i(t)) = \begin{cases} 
  E(t) \leq kS_i, & 0 \leq t < g_i^{-1}(0), \\
  E(t) - E(g_i(t)) + E(g_i(t))(1-\gamma_i) \leq kS_i + (k-k_1)(1-\gamma_i)g_i(t), & t \geq g_i^{-1}(0).
\end{cases}
\]

Therefore

\[
h(t) \leq c + \sum_{i=0}^{n} \int_{0}^{t} \lambda_i(s)\omega_i(u(s))ds
\]

where \( \lambda_0(t) = KPe^{(k-k_1)(1-\gamma_0)t} \),

\[
\lambda_i(t) = \begin{cases} 
  2^n KPe^{kS_i}, & 0 \leq t < g_i^{-1}(0), \\
  2^n KPe^{kS_i+(k-k_1)(1-\gamma_i)g_i(t)}, & t \geq g_i^{-1}(0), \quad i = 1, \ldots, n
\end{cases}
\]

are continuous and positive functions on \([0, \infty)\) and \( \omega_i(z) = z^{\gamma_i} \) for \( i = 0, \ldots, n \). Accordingly,

\[
h(t) \leq c + \sum_{i=0}^{n} \int_{0}^{t} \lambda_i(s)\omega_i(h(s))ds
\]

for all \( t \geq 0 \). Without any loss of generality we can assume that \( \gamma_0, \ldots, \gamma_n \) form a nondecreasing sequence, i.e. \( 1 < \gamma_0 \leq \gamma_1 \leq \cdots \leq \gamma_n \) (in the other case we change the notation for them and also for corresponding coefficients \( \lambda_i(s) \)).

If \( \gamma_0 = \gamma_n \), the statement follows from Bihari's lemma [3, 18]. Indeed, in this case

\[
h(t) \leq c + \int_{0}^{t} \lambda(s)\omega(h(s))ds =: H(t)
\]

with \( \lambda(t) = \sum_{i=0}^{n} \lambda_i(t) \), \( \omega(z) = \omega_0(z) = \cdots = \omega_n(z) \). Then

\[
W(H(t)) \leq \int_{0}^{t} \lambda(s)ds, \quad W(u) = \int_{c}^{u} \frac{dz}{\omega(z)}
\]

since \( W(H(0)) = W(c) = 0 \). Let \( P \) be such small that

\[
\|\lambda\| := \int_{0}^{\infty} \lambda(s)ds < \int_{c}^{\infty} \frac{dz}{\omega(z)} \quad (< \infty).
\]

Consequently, \( W(H(t)) < \|\lambda\| \) for all \( t \geq 0 \) and

\[
u(t) \leq h(t) \leq H(t) < W^{-1}(\|\lambda\|) =: C < \infty.
\]

Hence we have

\[
\|x(t)\| \leq C \exp \left\{ \int_{0}^{t} \beta(q)dq - kt \right\} \leq Ce^{-(k-k_1)t}
\]
whenever $t \geq 0$. Now, if $\|\varphi\|$ is sufficiently small, then $c$ is small and

$$W(C) = \|\lambda\| < \int_c^\delta \frac{dz}{\omega(z)} = W(\delta).$$

Thus $C < \delta$ and the exponential stability of the trivial solution follows.

In the other case, when $\gamma_0 < \gamma_n$, Pinto’s inequality [18] is applied. Note that $\omega_0 \propto \cdots \propto \omega_n$. Let $P$ be such small that

$$\|\lambda_i\| := \int_0^\infty \lambda_i(s)ds < \int_c^\delta \frac{dz}{\omega_i(z)}; \quad i = 0, \ldots, n$$

where

$$c_{i-1} = c, \quad c_i = W_i^{-1}(W_i(c_{i-1}) + \|\lambda_i\|), \quad i = 0, \ldots, n-1,$$

$$W_i(u) = \int_{u_i}^u \frac{dz}{\omega_i(z)}; \quad u, u_i > 0, \quad i = 0, \ldots, n.$$

Then Pinto’s inequality yields

$$u(t) \leq h(t) \leq W_n^{-1} \left( W_n(c_{n-1}) + \int_0^t \lambda_n(s)ds \right) \leq W_n^{-1} \left( W_n(c_{n-1}) + \|\lambda_n\| \right).$$

Here the right-hand side is constant for all $t \geq 0$ and we denote it by $C$. The trivial solution of equation (3.2) is exponentially stable if $C < \delta$. So it remains to verify, if this inequality can be assured by making $\|\varphi\|$ sufficiently small. From definition of $C$ we know that $C < \delta$ if

$$\|\lambda_n\| < \int_{c_{n-1}}^\delta \frac{dz}{\omega_n(z)},$$

i.e. if $c_{n-1} < \delta_{n-1} \leq \delta$ for $\delta_{n-1} > 0$ sufficiently small. Analogically, this is satisfied if $c_{n-2} < \delta_{n-2} \leq \delta_{n-1}$ with $\delta_{n-2} > 0$ small. Finally, we obtain that $C < \delta$ if $c = c_{i-1} < \delta_{i-1} \leq \delta_0 \leq \cdots \leq \delta_{n-1} \leq \delta$ with $\delta_{i-1} > 0$ sufficiently small. So if $\|\varphi\|$ is sufficiently small, the trivial solution is really exponentially stable. This completes the proof. \hfill \Box

We have also a result for nonautonomous nonlinear FDEs:

**Theorem 3.8.** Let the assumptions of Theorem 3.5 be fulfilled and $m_i \in \mathbb{N}$, $\gamma_{ij} > 1$ for $i = 0, \ldots, n$, $j = 1, \ldots, m_i$ be given constants. Assume that there
exists a nonnegative function \( r \in C([0, \infty), \mathbb{R}) \) satisfying
\[
\int_0^\infty r(s) \exp \left\{ \left( k s - \sum_{j=1}^n \int_0^s \| B_j(q) \| dq \right) \left( 1 - \gamma_0 \right) \right\} ds < \infty,
\]
\[
\int_{\delta^{-1}(0)}^\infty r(s) \exp \left\{ k(s - \gamma_i g_i(s)) - \sum_{j=1}^n \left( \int_0^s \| B_j(q) \| dq \right) - \gamma_i \int_0^s \| B_j(q) \| dq \right\} ds < \infty
\]
for \( i = 1, \ldots, n \) where \( \gamma_i = \min\{\gamma_{i1}, \ldots, \gamma_{im_i}\} \), \( i = 0, \ldots, n \), such that for any positive constants \( a_{ij} \), \( i = 0, \ldots, n \), \( j = 1, \ldots, m_i \) there is \( \delta > 0 \) such that if \( \| x \|, \| y_1 \|, \ldots, \| y_n \| < \delta \) then
\[
\| f(t, x, y_1, \ldots, y_n) \| \leq r(t) \left[ \sum_{j=1}^{m_0} a_{0j} \| x \|^{\varphi_{0j}} + \sum_{i=1}^{n} \sum_{j=1}^{m_i} a_{ij} \| y_i \|^{\gamma_{ij}} \right].
\]
Then the trivial solution of equation
\[
(3.10) \quad \dot{x}(t) = Ax(t) + B_1 x(g_1(t)) + \cdots + B_n x(g_n(t)) + f(t, x(t), x(g_1(t)), \ldots, x(g_n(t))), \quad t \geq 0
\]
is exponentially stable.

Proof. Let \( \gamma := \min\{g_1(0), \ldots, g_n(0)\} \), \( \varphi \in C([\gamma, 0], \mathbb{R}^N) \). For the solution \( x(t) \) of equation (3.10) satisfying initial condition (2.12) we assume that \( \| \varphi \| \) and \( \| x(s) \| \) are sufficiently small for all \( s \geq 0 \). In the notation of (3.5), (3.6) we can write
\[
u(t) \leq M + K \sum_{j=1}^{m_0} a_{0j} \int_0^t r(s) e^{E(s)(1-\gamma_{0j})} u(s)^{\gamma_{0j}} ds
\]
\[
+ K \sum_{i=1}^{n} \sum_{j=1}^{m_i} a_{ij} \int_0^t r(s) e^{E(s)-\gamma_{ij} E(g_i(s))} u(g_i(s))^{\gamma_{ij}} ds.
\]
Next, we denote
\[
\lambda_{0j}(t) = K a_{0j} r(t) e^{E(t)(1-\gamma_{0j})}, \quad j = 1, \ldots, m_0,
\]
\[
\lambda_{ij}(t) = K a_{ij} r(t) e^{E(t)-\gamma_{ij} E(g_i(t))}, \quad i = 1, \ldots, n, \ j = 1, \ldots, m_i
\]
to arrive at
\[
u(t) \leq M + \sum_{j=1}^{m_0} \int_0^t \lambda_{0j}(s) u(s)^{\gamma_{0j}} ds + \sum_{i=1}^{n} \sum_{j=1}^{m_i} \int_0^t \lambda_{ij}(s) u(g_i(s))^{\gamma_{ij}} ds.
\]

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for all $t \geq 0$. If

$$h(t) := c + \sum_{j=1}^{m_0} \int_0^t \lambda_{0j}(s) u(s)^{\gamma_{0j}} ds + \sum_{i=1}^{n} \sum_{j=1}^{m_i} \int_0^t \lambda_{ij}(s) u(g_i(s))^{\gamma_{ij}} ds$$

with $c = \max\{M, \|\varphi\|\}$, then $u(t) \leq h(t)$ for all $t \geq 0$. Analogically to (3.7), $u(g_i(t)) \leq 2h(t)$ whenever $t \geq 0$ and $i = 1, \ldots, n$. Therefore

$$h(t) \leq c + \sum_{i=0}^{n} \sum_{j=1}^{m_i} \int_0^t \mu_{ij}(s) h(s)^{\gamma_{ij}} ds$$

with $\mu_{0j}(t) = \lambda_{0j}(t)$ for $j = 1, \ldots, m_0$ and $\mu_{ij}(t) = 2^{\gamma_{ij}} \lambda_{ij}(t)$ for $i = 1, \ldots, n$, $j = 1, \ldots, m_i$.

Now we collect the coefficients $\mu_{ij}(s)$ by the same exponents and create an increasing sequence of exponents. So we get exponents $1 < \delta_1 < \cdots < \delta_p$ such that $1 \leq p \leq m_0 + \cdots + m_n$ and $\{\delta_i\}_{i=1}^{p} = \{\gamma_{i1}, \ldots, \gamma_{im_i}\}_{i=0}^{n}$. Moreover, denoting $\Omega := \{(i,1), \ldots, (i,m_i)\}_{i=0}^{n}$ the set of all indices, for each $i \in \{1, \ldots, p\}$: if $\delta_k = \gamma_{i_1j_1} = \cdots = \gamma_{i_kj_k}$, $\{(i_1, j_1)\}_{i=1}^{k} \subset \Omega$ and $\delta_k \neq \gamma_{ij}$ for all $(i,j) \in \Omega \setminus \{(i_i, j_1)\}_{i=1}^{k}$ (i.e. the set $\{(i_i, j_1)\}_{i=1}^{k}$ is the maximal subset of index set $\Omega$ such that $\delta_k = \gamma_{ij}$ for each $(i,j)$ from this subset), then we define $\nu_k(t) := \sum_{i=1}^{k} \mu_{i_1j_1}(t)$. Now, for $\omega_i(z) := z^{\delta_i}, i = 1, \ldots, p$ we get the sequence $\omega_1 \propto \cdots \propto \omega_p$ and $h(t)$ fulfills

$$h(t) \leq c + \sum_{i=1}^{p} \int_0^t \nu_i(s) \omega_i(h(s)) ds.$$ 

The proof can be finished exactly as the previous one using Bihari’s inequality if $p = 1$, or Pinto’s inequality if $p > 1$. In addition, the assumptions of the theorem establish the convergence of $\int_0^\infty \nu_i(s) ds$, $i = 1, \ldots, p$, which is important for the mentioned inequalities (see proof of Theorem 3.7 or [18]).

Finally, we apply one of the derived stability criteria on a simple biological model with delayed birthrates, concerning two species whose predator-prey-position is periodically changed in time.

**Example 3.9.** Let us consider the following system

$$\begin{align*}
\dot{x}_1(t) &= -\alpha_1 x_1(t) + \beta_1 x_1(t - e^{-t}) - \gamma x_1(t) x_2(t) \sin t \\
\dot{x}_2(t) &= -\alpha_2 x_2(t) + \beta_2 x_2(t - e^{-2t}) + \gamma x_1(t) x_2(t) \sin t
\end{align*}$$

(3.11)

with parameters $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma > 0$ such that $\alpha_1 \leq \alpha_2$.

This time, we have delay functions $g_1(t) = t - e^{-t}$, $g_2(t) = t - e^{-2t}$, matrices

$$A = \begin{pmatrix} -\alpha_1 & 0 \\ 0 & -\alpha_2 \end{pmatrix}, \quad B_1 = \begin{pmatrix} \beta_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 \\ 0 & \beta_2 \end{pmatrix}$$
and nonlinear function $f : [0, \infty) \times \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f(t, x) = (-\gamma x_1 x_2 \sin t, \gamma x_1 x_2 \sin t)$$

for vector $x = (x_1, x_2)$. It is easy to see that $A, B_1, B_2$ are pairwise permutable and

$$\|f(t, x)\| \leq \gamma |\sin t| \sqrt{2} \|x\|^2$$

for all $(t, x) \in [0, \infty) \times \mathbb{R}^2$ and vector norm $\|x\| = \sqrt{x_1^2 + x_2^2}$. We have the next lemma.

**Lemma 3.10.** Let $1 < \gamma_0 < 2$ be arbitrary and fixed. Then for any $a > 0$ there exists $\delta > 0$ such that if $\|x\| < \delta$, then

$$\|f(t, x)\| \leq a \gamma |\sin t| \sqrt{2} \|x\|^{\gamma_0}.$$

**Proof.** Clearly, if $\|x\| = 0$, then $\|f(t, x)\| = 0$. Now, let $a > 0$ be fixed and $0 < \|x\| < \delta$. Accordingly,

$$\frac{\|f(t, x)\|}{\|x\|^\gamma_0} \leq \frac{\gamma |\sin t|}{\sqrt{2}} \|x\|^{2-\gamma_0} < \frac{\gamma |\sin t|}{\sqrt{2}} \delta^{2-\gamma_0}.$$

Hence, it is sufficient to set $\delta = a^{-1/\gamma_0}$ to obtain the statement of the lemma. \(\square\)

Using the matrix norm $\|C\| = \sqrt{\sum |c_{ij}|^2}$ (in order to satisfy the basic assumption $\|E\| = 1$), we obtain $\|\tilde{B}_1(t)\| = \beta_1 e^{\alpha_1 e^{-t}} / \sqrt{2}$ and $\|\tilde{B}_2(t)\| = \beta_2 e^{\alpha_2 e^{-2t}} / \sqrt{2}$. Consequently,

$$\beta(t) = \|\tilde{B}_1(t)\| + \|\tilde{B}_2(t)\| \leq \frac{(\beta_1 + \beta_2) e^{\alpha_2}}{\sqrt{2}}$$

for all $t \geq 0$. So we can take $k_1 = (\beta_1 + \beta_2) e^{\alpha_2} / \sqrt{2}$ and condition $\int_0^t \beta(q) dq \leq k_1 t$ is satisfied.

**Corollary 3.11.** If $(\beta_1 + \beta_2) e^{\alpha_2} < \sqrt{2} \alpha_1$, then system (3.11) has exponentially stable trivial solution.

**Proof.** The corollary follows from Theorem 3.8. We show that all assumptions of this theorem are fulfilled. Let $1 < \gamma_0 < 2$ be arbitrary and fixed, and $r(t) := \frac{\gamma |\sin t|}{\sqrt{2}}$. Then, by the assumption of the corollary, $k_1 < \alpha_1$ and

$$\int_0^\infty r(s) \exp \left\{ \left( \alpha_1 s - \int_0^s \beta(q) dq \right) (1 - \gamma_0) \right\} ds$$

$$\leq \int_0^\infty r(s) \exp \left\{ (k_1 - \alpha_1)(\gamma_0 - 1) s \right\} ds$$

$$\leq \frac{\gamma}{\sqrt{2}} \int_0^\infty e^{(k_1 - \alpha_1)(\gamma_0 - 1)s} ds = \frac{\gamma}{\sqrt{2}(\gamma_0 - 1)(\alpha_1 - k_1)} < \infty.$$
Next, since \( s - 1 \leq g_i(s) < s \) and \( \beta(s) \geq \frac{\beta_1 + \beta_2}{\sqrt{2}} \) for \( i = 1, 2, s \geq 0 \), we have
\[
\int_{g_i(0)}^{\infty} r(s) \exp \left\{ \alpha_1 (s - \gamma_i g_i(s)) - \int_0^s \beta(q) dq + \gamma_i \int_0^{g_i(s)} \beta(q) dq \right\} ds \\
\leq \frac{\gamma}{\sqrt{2}} \int_{g_i(0)}^{\infty} \exp \left\{ (k_1 - \alpha_1) \gamma_i g_i(s) + \alpha_1 s - \int_0^s \beta(q) dq \right\} ds \\
\leq \frac{\gamma}{\sqrt{2}} \int_{g_i(0)}^{\infty} \exp \left\{ (k_1 - \alpha_1) \gamma_i g_i(s) + \left( \alpha_1 - \frac{\beta_1 + \beta_2}{\sqrt{2}} \right) s \right\} ds \\
\leq \frac{\gamma e^{(\alpha_1 - k_1)\gamma_i}}{\sqrt{2}} \int_{g_i(0)}^{\infty} \exp \left\{ \left( k_1 - \alpha_1 \right) \gamma_i + \left( \alpha_1 - \frac{\beta_1 + \beta_2}{\sqrt{2}} \right) s \right\} ds
\]
for \( i = 1, 2 \). The right-hand side of the latter inequality has the form
\[
\frac{\gamma e^{(\alpha_1 - k_1)\gamma_i}}{\sqrt{2}} \int_{g_i(0)}^{\infty} e^{-\eta s} ds < \infty
\]
with \( \eta = (\alpha_1 - k_1)\gamma_i + \frac{\beta_1 + \beta_2}{\sqrt{2}} - \alpha_1 > 0 \) whenever
\[
\gamma_i > \frac{\sqrt{2} \alpha_1 - (\beta_1 + \beta_2)}{\sqrt{2} \alpha_1 - (\beta_1 + \beta_2)e^{\alpha_2}} \quad (> 1).\]
Clearly, when we set \( f(t, x_1, y_1, y_2) := f(t, x) \), then by Lemma 3.10
\[
\| f(t, x_1, y_1, y_2) \| \leq \alpha r(t) (\| x_1 \|^{\gamma_0} + \| y_1 \|^{\gamma_1} + \| y_2 \|^{\gamma_2}).
\]
By that, the proof is finished. \( \square \)

REFERENCES


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