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On positive solutions for a class of nonlocal problems*

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Abstract

In this paper, we study a class of nonlocal semilinear elliptic problems with inhomogeneous strong Allee effect. By means of variational approach, we prove that the problem has at least two positive solutions for large λ under suitable hypotheses about nonlinearity. We also prove some nonexistence results. In particular, we give a positive answer to the conjecture of Liu-Wang-Shi.

Keywords: Positive solutions; Nonlocal problem; Inhomogeneous strong Allee effect

MSC(2000): 35J20; 35J25

1 Introduction

In this paper, we study the following problem

$$\begin{cases} -M\left(\int_\Omega \frac{1}{2}|\nabla u|^2\,dx\right)\Delta u = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$ with $N \geq 1$, the nonlocal coefficient $M(t)$ is a continuous function of $t = \int_\Omega \frac{1}{2}|\nabla u|^2\,dx$. We shall give a positive answer to a conjecture by Liu, Wang and Shi of [1].

The problem (1.1) is related to a model introduced by Kirchhoff [2]. More precisely, Kirchhoff proposed a model given by the equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2\,dx\right)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.2}$$

where ρ, ρ_0, h, E, L are constants, which extends the classical D'Alembert's wave equation, by considering the effect of the changing in the length of the string during the vibration. A

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distinguishing feature of equation (1.2) is that the equation contains a nonlocal coefficient $\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx$, and hence the equation is no longer a pointwise identity. The problem

$$\begin{cases} -\left(a + b\int_\Omega |\nabla u|^2\,dx\right)\Delta u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.3}$$

is related to the stationary analogue of the equation (1.2). Problem (1.3) received much attention only after Lions [3] proposed an abstract framework to the problem. Some important and interesting results can be found, for example, in [4–15].

In the context of population biology, the nonlinear function $f(x,u) \equiv ug(x,u)$ represents a density dependent growth if $g(x,u)$ is a function depending on the population density u. While traditionally $g(x,u)$ is assumed to be declining to reflect the crowding effect of the increasing population, Allee suggested that physiological and demographic precesses often possess an optimal density, with the response decreasing as either higher or lower densities. Such growth pattern is called an Allee effect. If the growth rate per capita is negative when u is small, we call it a strong Allee effect; if the growth rate per capita is small than the maximum but still positive for small u, we call it a weak Allee effect (for detail, see [16] or [17]).

Under the special case of problem (1.3) with $a = 1$, $b = 0$ and $f(x,u)$ satisfies inhomogeneous strong Allee effect growth pattern, Liu, Wang and Shi [1] proved that the problem

$$\begin{cases} -\Delta u = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.4}$$

has at least two positive solutions for large λ if $\int_0^{c(x)} f(x,s)\,ds > 0$ for x in an open subset of Ω, where $c(x) \in C^1(\overline{\Omega})$ such that $f(x,c(x)) = 0$ (see the assumption of $(f2)$). They also prove some nonexistence results. In particular, they conjecture that the nonexistence holds if $\int_0^{c(x)} f(x,s)\,ds \le 0$ for any $x \in \overline{\Omega}$ (see Remark 1.7 of [1]). We also note that the first work for (1.4) to be concerned with the relation between multiplicity of positive solutions and the measure of the bumps of the nonlinearity f is due to Brown and Budin [18].

Motivated by above, we generalize existence and nonexistence results for the semilinear elliptic problem (1.4) to the case of nonlocal semilinear elliptic problem (1.1). More precisely, if $f(x,u)$ satisfies inhomogeneous strong Allee effect growth pattern and the nonlocal coefficient $M(t)$ satisfies some suitable conditions, we establish the existence of at least two positive solutions for the nonlocal problem (1.1) with λ large enough. We also prove some nonexistence results for the nonlocal problem (1.1). In particular, we shall give a positive answer to the conjecture by Liu, Wang and Shi. We note that, in [19], the authors studied the existence of positive solutions for a nonlocal elliptic problem (which different from (1.1)) with homogeneous sign-changing nonlinearity by variational approach.

We point out the nonlocal coefficient $M(t)$ raises some of the essential difficulties. For example, the way of proving the geometry condition of Mountain Pass Theorem in [1] can not be used here because the functional of (1.1) is not C^2 function under our assumptions. In order to overcome this difficulty, we divided Ω into B_1 and B_2 by comparing the value of $c(x)$ with $\underline{b}$, then use Poincaré inequality to prove it (see Lemma 3.3).

The rest of this paper is organized as follows. In Section 2, we present our main results and some necessary preliminary lemmata. In Sections 3, we use variational method and sub-supersolution method to prove the main results. In Section 4, we prove the conjecture of Liu, Wang and Shi's and give some examples which satisfy our hypotheses.

2 Main results and preliminaries

In this section, we give our main results and some necessary preliminary lemmata which will be used later. For simplicity we write $X = H_0^1(\Omega)$ with the norm $\|u\| = \left(\int_\Omega |\nabla u|^2\, dx\right)^{1/2}$.

Hereafter, $f(x,t)$ and $M(t)$ are always supposed to verify the following assumptions:

$(f1)$ $f(x,u) \in C\left(\overline{\Omega} \times \mathbb{R}^+\right)$ and $f(x,\cdot) \in C^1\left(\mathbb{R}^+\right)$ for any $x \in \overline{\Omega}$;

$(f2)$ There exist $b(x) \in C(\overline{\Omega})$, $c(x) \in C^1(\overline{\Omega})$ such that $0 < b(x) < c(x)$ and $f(x,0) = f(x,b(x)) = f(x,c(x)) = 0$ for any $x \in \overline{\Omega}$;

$(f3)$ For a.e. $x \in \overline{\Omega}$, $f(x,s) < 0$ for any $s \in (0,b(x)) \cup (c(x),+\infty)$ and $f(x,s) > 0$ for any $s \in (b(x),c(x))$;

(M) $\exists m_0 > 0$ such that

$$M(t) \geq m_0 \text{ for all } t \geq 0.$$

Remark 2.1. Note that the weak maximum principle (Theorem 8.1 of [20]) and strong maximum principle (Theorem 8.1 of [20]) also hold for the nonlocal problem (1.1) because $M(t)$ satisfies the assumption (M).

Definition 2.1. We say that $u \in X$ is a weak solution of (1.1), if

$$M\left(\int_\Omega \frac{1}{2}|\nabla u|^2\, dx\right) \int_\Omega \nabla u \nabla \varphi\, dx = \lambda \int_\Omega f(x,u)\varphi\, dx$$

for any $\varphi \in X$.

Define

$$\Phi(u) = \widehat{M}\left(\int_\Omega \frac{1}{2}|\nabla u|^2\, dx\right),\ \Psi(u) = \int_\Omega F(x,u)\, dx,$$

where $\widehat{M}(t) = \int_0^t M(s)\, ds$, $F(x,u) = \int_0^u f(x,t)\, dt$. We redefine $f(x,s)$, such that $f(x,s) \equiv 0$ when $s \in (-\infty,0) \cup (c(x),\infty)$, but it does not change the positive solution set of (1.1) since any positive solution of (1.1) satisfies $0 \leq u(x) \leq c(x)$ for all $x \in \Omega$. Indeed, suppose on the contrary that there exists a positive solution $v(x)$ of (1.1) and a point $x_0 \in \Omega$ such that $v(x_0) > c(x_0)$. From the regularity assumptions on $f(x,u)$, any weak u of (1.1) is a classical solution of (1.1) (see [21, 22]), i.e., $u \in C^2(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$ with some $\alpha \in (0,1)$. So $v \in C^2(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$. Hence, there exists a measurable subset S of Ω with positive measure such that $v(x) > c(x)$ on S. Let $v_0(x) = v(x)$ if $x \in S$ and $v_0(x) = 0$ if $x \in \Omega \setminus \overline{S}$. Clearly, v_0 is also a solution of (1.1) and $f(x,v_0) \leq 0$ for a.e. $x \in \overline{\Omega}$. The weak maximum principle (Theorem 8.1 of [20]) implies $v_0(x) \leq 0$ in Ω. So $v(x) \leq 0$ in Ω. This is a contradiction. Then the energy functional $I_\lambda(u) = \Phi(u) - \lambda\Psi(u) : X \to \mathbb{R}$ associated with problem (1.1) is well-defined. Then it is easy to see that $I_\lambda \in C^1(X,\mathbb{R})$ is weakly lower semi-continuous and $u \in X$ is a weak solution of (1.1) if and only if u is a critical point of I_λ. By the definition of modified $f(x,u)$ and an argument similar to above (note that the measure of S may be zero in this case), any solution u of (1.1) is either zero or satisfies $0 < u(x) < c(x)$ for all

$x \in \Omega$. Moreover, we have

$$\begin{aligned} I'_\lambda(u)v &= M\left(\int_\Omega \frac{1}{2}|\nabla u|^2\,dx\right)\int_\Omega \nabla u \nabla v\,dx - \lambda\int_\Omega f(x,u)v\,dx \\ &= \Phi'(u)v - \lambda\Psi'(u) \text{ for any } v \in X. \end{aligned}$$

From (M) and Lemma 4.1 of [23] we can easily see that Φ' is of (S_+) type, i.e., if $u_n \rightharpoonup u$ in X and $\overline{\lim\limits_{n\to+\infty}}\,(\Phi'(u_n) - \Phi'(u), u_n - u) \le 0$, then $u_n \to u$ in X. Lemma 1.2 of [1] implies that Ψ' is weak-strong continuous, i.e., $u_n \rightharpoonup u$ implies $\Psi'(u_n) \to \Psi'(u)$. So I'_λ is of (S_+) type.

Our main existence result is as follows:

Theorem 2.1. *If $M(t)$ satisfies (M) and $f(x,u)$ satisfies $(f1)$–$(f3)$, and Ω_1 is an open subset of Ω such that*

$$\int_0^{c(x)} f(x,s)\,ds > 0 \tag{2.1}$$

for $x \in \Omega_1$, then for λ large enough, (1.1) has at least two positive solutions, and (1.1) has no solution for small λ.

In order to prove our main existence result we need the following lemma:

Lemma 2.1 (see [1]). *Suppose that f satisfies $(f1)$–$(f3)$. If $u(x)$ is an integrable function in Ω, and there is a measurable subset Ω_0 of Ω with positive measure, such that*

$$\int_0^{c(x)} f(x,s)\,ds > 0 \text{ in } \Omega_0 \text{ and } \int_0^{c(x)} f(x,s)\,ds \le 0 \text{ in } \Omega\setminus\Omega_0,$$

then

$$\int_0^{u(x)} f(x,s)\,ds \le \int_0^{c(x)} f(x,s)\,ds \text{ in } \Omega_0 \text{ and } \int_0^{u(x)} f(x,s)\,ds \le 0 \text{ in } \Omega\setminus\Omega_0,$$

Now we turn to the nonexistence of the positive solutions of (1.1) when (2.1) does not hold for any $x \in \overline{\Omega}$. We define $\overline{c} = \max_{x\in\overline{\Omega}} c(x)$, $\overline{f}(u) = \max_{x\in\overline{\Omega}} f(x,u)$. Our main nonexistence result is

Theorem 2.2. *If $\int_0^{\overline{c}} \overline{f}(u)\,du \le 0$, then (1.1) has no positive solution for any $\lambda > 0$.*

In order to prove our main nonexistence result, we recall a theorem in [24] for (1.1) with the special case of $M(t) \equiv 1$ and $f(x,u) \equiv f(u)$. In fact, the theorem also holds for the nonlocal problem (1.1) with $f(x,u) \equiv f(u)$. Because the proof is similar to that of [24], we omit it here (for detail, see the proof of Theorem 1 in [24]). Let us assume that $f : \mathbb{R} \to \mathbb{R}$ is a C^1 function and let the following conditions hold: there exist $0 \le s_0 < s_1 < s_2$, such that

$$\begin{cases} f(s_i) = 0, & i = 1,2, \\ f(s_0) \le 0, & \\ f(s) < 0, & s_0 < s < s_1, \\ f(s) > 0, & s_1 < s < s_2 \end{cases} \tag{2.2}$$

and let

$$\int_{s_0}^{s_2} f(s)\,ds \leq 0. \tag{2.3}$$

We have the following lemma.

Lemma 2.2. *Assume that f satisfies (2.2) and (2.3). Let Ω be a bounded domain with smooth boundary. If (1.1) with $f(x,u) \equiv f(u)$ has a positive solution u, then u cannot satisfy*

$$\begin{cases} u_{\max} = \max_{x\in\Omega} u(x) \in (s_1, s_2)\,, \\ u(x) > 0, \qquad x \in \Omega. \end{cases} \tag{2.4}$$

Remark 2.2. Note that our assumptions $(f1)$–$(f3)$ are weaker than $(f1)$–$(f4)$ of [1] even in the case of $M(t) \equiv 1$. In fact, from $(f1)$–$(f3)$, we can easily see that there exists a positive constant β such that $f(x,s) \leq \beta s$ for any $s \geq 0$ and a.e. $x \in \Omega$, i.e., the condition $(f4)$ of [1]. We do not need the conditions of $b(x) \in C^{1,\alpha}(\Omega)(0 < \alpha < 1)$ and $f(\cdot,u) \in C^{1,\alpha}(\overline{\Omega})$ for any $u \geq 0$ because we do not need energy functional of (1.1) is a C^2 function in our proof.

Remark 2.3. The condition of $f(x,\cdot) \in C^1(\mathbb{R}^+)$ for any $x \in \overline{\Omega}$ can be relaxed to $f(x,\cdot)$ is locally lipschitz in $\mathbb{R}^+$ for any $x \in \overline{\Omega}$. In fact, Lemma 2.2 also holds when $f : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function because the symmetry results of [25] hold under this weaker condition.

3 Proofs of main results

In this section we shall prove Theorem 2.1 and 2.2.

Lemma 3.1. *If $M(t)$ satisfies (M), and $f(x,u)$ satisfies $(f1)$–$(f3)$ and (2.1), then for λ large enough, $I_\lambda(\cdot)$ has a global minimum point u_1 such that $I_\lambda(u_1) < 0$.*

Proof. Since $\int_0^{c(x)} f(x,s)\,ds > 0$ in Ω_1, then there exists a measurable set $\Omega_0 \subset \Omega$ with positive measure, such that $\int_0^{c(x)} f(x,s)\,ds > 0$ in Ω_0 and $\int_0^{c(x)} f(x,s)\,ds \leq 0$ in $\Omega \backslash \Omega_0$. From

(M) and the definition of $\widehat{M}(t)$, we have $\widehat{M}(t) \geq m_0 t$. In view of Lemma 2.1, we have that

$$\begin{aligned}
I_\lambda(u) &= \widehat{M}\left(\int_\Omega \frac{1}{2}|\nabla u|^2\,dx\right) - \lambda\int_\Omega F(x,u)\,dx \\
&\geq m_0\int_\Omega \frac{1}{2}|\nabla u|^2\,dx - \lambda\int_\Omega\left(\int_0^{u(x)} f(x,s)\,ds\right)dx \\
&\geq m_0\int_\Omega \frac{1}{2}|\nabla u|^2\,dx - \lambda\int_{\Omega_0}\left(\int_0^{u(x)} f(x,s)\,ds\right)dx - \lambda\int_{\Omega\setminus\Omega_0}\left(\int_0^{u(x)} f(x,s)\,ds\right)dx \\
&\geq m_0\int_\Omega \frac{1}{2}|\nabla u|^2\,dx - \lambda\int_{\Omega_0}\left(\int_0^{c(x)} f(x,s)\,ds\right)dx \\
&\geq m_0\int_\Omega \frac{1}{2}|\nabla u|^2\,dx - \lambda\int_{\Omega_0} A_1\,dx \\
&= \frac{m_0}{2}\|u\|^2 - \lambda\,|\Omega_0|\,A_1 \to +\infty \quad \text{as} \quad \|u\| \to +\infty, \qquad (3.1)
\end{aligned}$$

where $A_1 = \max_{\overline{\Omega}_0\times[0,\bar{c}]}|F(x,s)|$. It follows that I_λ is coercive and bounded from below. Since I_λ is weakly lower semi-continuous, I_λ has a global minimum point u_1 in X.

Next we shall prove $I_\lambda(u_1) < 0$, thus u_1 is a positive solution of (1.1). In fact, we only need to verify that when λ is large there exists a $u_0 \in X$, such that $I_\lambda(u_0) < 0 = I_\lambda(0)$. We define $u_0(x) = 0$ in $\Omega\setminus\Omega_{1\varepsilon}$, and $u_0(x) = c(x)$ in Ω_1 and properly in $\Omega_{1\varepsilon}\setminus\Omega_1$ such that $u_0 \in X$ and $0 \leq u_0(x) \leq c(x)$, where $\Omega_{1\varepsilon} = \{x \in \Omega : \mathrm{dist}(x,\Omega_1) \leq \varepsilon\}$. Then we have

$$\begin{aligned}
I_\lambda(u_0) &= \widehat{M}\left(\int_\Omega \frac{1}{2}|\nabla u_0|^2\,dx\right) - \lambda\int_\Omega F(x,u_0)\,dx \\
&= \widehat{M}\left(\int_\Omega \frac{1}{2}|\nabla u_0|^2\,dx\right) - \lambda\int_{\Omega_1} F(x,c(x))\,dx - \lambda\int_{\Omega\setminus\Omega_1} F(x,u_0)\,dx \\
&= \widehat{M}\left(\int_\Omega \frac{1}{2}|\nabla u_0|^2\,dx\right) - \lambda\int_{\Omega_1} F(x,c(x))\,dx - \lambda\int_{\Omega_{1\varepsilon}\setminus\Omega_1} F(x,u_0)\,dx \\
&\leq \widehat{M}\left(\int_\Omega \frac{1}{2}|\nabla u_0|^2\,dx\right) - \lambda\int_{\Omega_1} F(x,c(x))\,dx - \lambda\int_{\Omega_{1\varepsilon}\setminus\Omega_1} (-A_2)\,dx \\
&\leq \widehat{M}\left(\int_\Omega \frac{1}{2}|\nabla u_0|^2\,dx\right) - \lambda\int_{\Omega_1} F(x,c(x))\,dx - \lambda\left[-A_2\left(|\Omega_{1\varepsilon}| - |\Omega_1|\right)\right], \qquad (3.2)
\end{aligned}$$

where $A_2 = \max_{\overline{\Omega}_{1\varepsilon}\times[0,\bar{c}]}|F(x,s)|$. Since $\int_0^{c(x)} f(x,s)\,ds > 0$ when $x \in \Omega_1$ and $\int_0^{c(x)} f(x,s)\,ds$ is continuous, then there must exist an open subset Ω_2 with $\overline{\Omega}_2 \subset \Omega_1$ and $\delta > 0$, such that $|\Omega_2| > 0$ and $\int_0^{c(x)} f(x,s)\,ds \geq \delta$ for $x \in \Omega_2$. Choose ε small enough, such that $\delta\,|\Omega_2| + A_2\left(|\Omega_1| - |\Omega_{1\varepsilon}|\right) > 0$. These facts with (3.2) implies that

$$I_\lambda(u_0) \leq \widehat{M}\left(\int_\Omega \frac{1}{2}|\nabla u_0|^2\,dx\right) - \lambda\left[\delta\,|\Omega_2| + A_2\left(|\Omega_1| - |\Omega_{1\varepsilon}|\right)\right].$$

Therefore when λ large enough, $I_\lambda(u_0) < 0$, and consequently when λ is large enough, (1.1) has a positive solution $u_1(x)$ satisfying $I_\lambda(u_1) = \inf_{u\in X} I_\lambda(u) < 0$. ■

Next, we use Mountain Pass Theorem to prove that (1.1) has another positive solution u_2. Firstly, we prove $I_\lambda(u)$ satisfies Palais-Smale condition.

Definition 3.1. We say that I_λ satisfies (P.S.) condition in X, if any sequence $\{u_n\} \subset X$ such that $\{I_\lambda(u_n)\}$ is bounded and $I'_\lambda(u_n) \to 0$ as $n \to +\infty$, has a convergent subsequence, where (P.S.) means Palais-Smale.

Lemma 3.2. *If $M(t)$ satisfies (M), f satisfies (f1)–(f3) and (2.1), then I_λ satisfies (P.S.) condition.*

Proof. Suppose that $\{u_n\} \subset X$, $|I_\lambda(u_n)| \le c_0$ and $I'_\lambda(u_n) \to 0$ as $n \to +\infty$. In view of (3.1), we have

$$c_0 \ge I_\lambda(u_n) \ge \frac{m_0}{2}\|u_n\|^2 - \lambda|\Omega_0| A_1.$$

Hence, $\{\|u_n\|\}$ is bounded. Without loss of generality, we assume that $u_n \rightharpoonup u$, then

$$I'(u_n)(u_n - u) \to 0.$$

Therefore, we have $u_n \to u$ by the (S_+) property of I'_λ. ■

Lemma 3.3. *If $M(t)$ satisfies (M), f satisfies (f1)–(f3), then there exist $\rho > 0$ and $\gamma > 0$ such that $I_\lambda(u) \ge \gamma$ for every $u \in X$ with $\|u\| = \rho$.*

Proof. We define $\underline{b} = \min_{x\in\overline{\Omega}} b(x)$. For any $u(x) \in X$, we also define $B_1 = \{x \in \Omega : u(x) < \underline{b}\}$, $B_2 = \{x \in \Omega : u(x) \ge \underline{b}\}$. It is well known that the embedding of $X \hookrightarrow L^p(\Omega)$ is continuous when $2 < p \le 2^*$, where 2^* is the critical exponent. By Poincaré's inequality, we have that

$$\underline{b}\,|B_2|^{\frac{1}{p}} \le \left(\int_{B_2} u^p\,dx\right)^{\frac{1}{p}} \le \left(\int_\Omega |u|^p\,dx\right)^{\frac{1}{p}} \le c_1\left(\int_\Omega |\nabla u|^2\,dx\right)^{\frac{1}{2}} = c_1\|u\|,$$

where c_1 is the embedding constant of $X \hookrightarrow L^p(\Omega)$. Thus, we have

$$\begin{aligned}
I_\lambda(u) &= \widehat{M}\left(\int_\Omega \frac{1}{2}|\nabla u|^2\,dx\right) - \lambda\int_\Omega F(x,u)\,dx \\
&\ge \frac{m_0}{2}\|u\|^2 - \lambda\int_{B_1} F(x,u)\,dx - \lambda\int_{B_2} F(x,u)\,dx \\
&\ge \frac{m_0}{2}\|u\|^2 - \lambda\int_{B_2} F(x,u)\,dx \\
&\ge \frac{m_0}{2}\|u\|^2 - \lambda A_3|B_2| \ge \frac{m_0}{2}\|u\|^2 - \lambda A_3\left(\frac{c_1}{\underline{b}}\right)^p\|u\|^p \\
&= \|u\|^2\left(\frac{m_0}{2} - \lambda A_3\left(\frac{c_1}{\underline{b}}\right)^p\|u\|^{p-2}\right),
\end{aligned}$$

where $A_3 = \max_{(x,s)\in\overline{B}_2\times[\underline{b},\overline{c}]}|F(x,s)|$. Therefore, there exists $\frac{m_0\underline{b}^p}{2\lambda A_3 c_1^p} > \rho > 0$ such that $I_\lambda(u) \ge \rho^2\left(\frac{m_0}{2} - \lambda A_3\left(\frac{c_1}{\underline{b}}\right)^p\rho^{p-2}\right) := \gamma > 0$ for every $\|u\| = \rho$ and fixed λ. ■

Proof of Theorem 2.1 concluded. Firstly, let us show that I_λ satisfies the conditions of Mountain Pass Theorem (see Theorem 2.10 of [26]). By Lemma 3.2, I_λ satisfies (P.S.)

condition in X. By Lemma 3.3, for fixed $\lambda > 0$, there exist $\min\left\{\|u_0\|, \frac{m_0 b^p}{2\lambda A_2 c_1^p}\right\} > \rho > 0$, $\gamma > 0$ such that $I_\lambda(u) \geq \gamma > 0$ for every $\|u\| = \rho$, where u_0 comes from (3.2). On the other hand, since $I_\lambda(0) = 0$ and from the proof of Lemma 3.1, there exists $u_0 \in X$ such that $I_\lambda(u_0) < 0$ and $\|u_0\| > \rho$. So from Mountain Pass Theorem, I_λ has another critical point u_2 such that

$$I_\lambda(u_2) \geq \gamma > 0 > I_\lambda(u_1).$$

Therefore, u_2 is another positive solution of (1.1).

Finally, we show that (1.1) has no positive solution when λ is small. We assume that (1.1) has a positive solution u, let $(\Lambda_1, \varphi_1(x))$ be the principal eigen-pair of the problem

$$\begin{cases} -\Delta\phi = \Lambda\phi & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.3}$$

such that $\varphi_1(x) > 0$ in Ω. We rewrite (1.1) as the following form

$$\begin{cases} -\Delta u = \lambda \frac{f(x,u)}{M\left(\int_\Omega \frac{1}{2}|\nabla u|^2\, dx\right)} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.4}$$

Multiplying (3.3) by u, multiplying (3.4) by φ_1, subtracting and integrating in Ω, we obtain

$$0 = \int_\Omega \left[\Lambda_1 u\varphi_1 - \lambda\varphi_1 \frac{f(x,u)}{M(t)}\right] dx = \int_\Omega \frac{u\varphi_1}{M(t)} \left[M(t)\Lambda_1 - \lambda\frac{f(x,u)}{u}\right] dx, \tag{3.5}$$

where $t = \int_\Omega \frac{1}{2}|\nabla u|^2\, dx$. If $\lambda < m_0\Lambda_1/\beta$, then by Remark 2.2, we have

$$M(t)\Lambda_1 - \lambda\frac{f(x,u)}{u} \geq m_0\Lambda_1 - \lambda\frac{f(x,u)}{u} > m_0\Lambda_1 - \lambda\beta > 0.$$

That contradicts (3.5). So for small λ, (1.1) has no positive solution. ■

Proof of Theorem 2.2. The proof is similar to [1]. For the sake of completeness, we include it here. If there exists a positive solution (λ, u_*) for (1.1), then u_* is a subsolution of

$$\begin{cases} M\left(\int_\Omega \frac{1}{2}|\nabla u|^2\, dx\right)\Delta u + \lambda\overline{f}(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.6}$$

since $M\left(\int_\Omega \frac{1}{2}|\nabla u_*|^2\, dx\right)\Delta u_* + \lambda\overline{f}(u_*) \geq M\left(\int_\Omega \frac{1}{2}|\nabla u_*|^2\, dx\right)\Delta u_* + \lambda f(x, u_*)$. And $\overline{c}$ is supersolution of (3.6). So by the standard comparison arguments, (3.6) has a positive solution $\overline{u}$ such that $u_* \leq \overline{u} \leq \overline{c}$. But if we let $s_0 = 0$, $s_1 = \underline{b}$ and $s_2 = \overline{c}$, $\overline{f}$ satisfies (2.2) and (2.3), then by Lemma 2.2, (3.6) has no positive solution. This is a contradiction. So (1.1) has no positive solution if $\int_0^{\overline{c}} \overline{f}(u)\, du \leq 0$. ■

4 Proof of a conjecture and some examples

In this section we shall prove the conjecture of Liu, Wang and Shi and give some typical consequences of Theorem 2.1 and 2.2.

In [1], Liu, Wang and Shi conjecture that the nonexistence holds with a weaker condition:

$$\int_0^{c(x)} f(x,s)\,ds \le 0 \text{ for any } x \in \overline{\Omega}. \tag{4.1}$$

In fact, as we will see in the following proposition, the condition (4.1) is more strong than $\int_0^{\overline{c}} \overline{f}(s)\,ds \le 0$. Therefore, by Theorem 2.2, the conjecture is right.

Proposition 4.1. *If $f(x,u)$ satisfies $(f1)$–$(f3)$ and $\int_0^{c(x)} f(x,s)\,ds \le 0$ for any $x \in \overline{\Omega}$, we have $\int_0^{\overline{c}} \overline{f}(s)\,ds \le 0$.*

Proof. From $(f1)$–$(f3)$, we can easily see that $f(x,s) \le 0$ when $s \in [c(x), \overline{c}]$. Thus, we have $\int_{c(x)}^{\overline{c}} f(x,s)\,ds \le 0$. Then, for any $x \in \overline{\Omega}$, we have

$$0 \ge \int_0^{c(x)} f(x,s)\,ds = \int_0^{\overline{c}} f(x,s)\,ds - \int_{c(x)}^{\overline{c}} f(x,s)\,ds \ge \int_0^{\overline{c}} f(x,s)\,ds.$$

In particular, $\int_0^{\overline{c}} \overline{f}(s)\,ds \le 0$. ■

Now, we give some examples which satisfy our hypotheses.

Example 4.1. Let $M(t) = a + bt$ with $t = \int_\Omega \frac{1}{2}|\nabla u|^2\,dx$, here a, b are two positive constants and $f(x,u) = u(u - b(x))(c(x) - u)$ with $b(x) \in C(\overline{\Omega})$, $c(x) \in C^1(\overline{\Omega})$ such that $0 < b(x) < c(x)$ for any $x \in \overline{\Omega}$. It is clear that $M(t)$ and $f(x,u)$ verify our assumptions (M) and $(f1)$–$(f3)$.

Example 4.2. We consider a special case of Example 4.1:

$$\begin{cases} \Delta u + \lambda u(u - b(x))(c(x) - u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.2}$$

where $b(x) \in C(\overline{\Omega})$, $c(x) \in C^1(\overline{\Omega})$ such that $0 < b(x) < c(x)$ for any $x \in \overline{\Omega}$. We have known that $f(x,u)$ satisfies $(f1)$–$(f3)$ from Example 4.1. Moreover, we have

$$\begin{aligned} \int_0^{c(x)} f(x,s)\,ds &= \int_0^{c(x)} s(s - b(x))(c(x) - s)\,ds \\ &= \frac{1}{12}[c(x)]^3(c(x) - 2b(x)). \end{aligned}$$

Then by Theorem 2.1, if there exists an open subset $\Omega_1 \subset \Omega$, such that $c(x) > 2b(x)$ in Ω_1, then (4.2) has at least two positive solutions for large λ.

If $c(x) \equiv 1$ for all $x \in \overline{\Omega}$, we obtain

$$\begin{aligned} \int_0^1 \overline{f}(s)\,ds &= \int_0^1 \max_{x \in \overline{\Omega}} s(s - b(x))(1 - s)\,ds \\ &= \int_0^1 \max_{x \in \overline{\Omega}} \left[s^2 - s^3 + b(x)\left(s^2 - s\right)\right]\,ds \\ &= \frac{1}{12} - \frac{\underline{b}}{6}, \end{aligned}$$

since $s^2 - s \le 0$ for $s \in [0,1]$. Then by Theorem 2.2, if $\underline{b} = \min_{x\in\overline{\Omega}} b(x) \ge 1/2$, then (4.2) has no positive solution for any $\lambda > 0$.

Example 4.3. Let $M(t) \equiv 1$ and $f(x,s) = s(s-1)(c(x)-s)$ with $3/2 \le c(x)$ for any $x \in \overline{\Omega}$. We can easily obtain that

$$\begin{aligned}
\int_0^{\overline{c}} \overline{f}(s)\,ds &= \int_0^{\overline{c}} \max_{x\in\overline{\Omega}} s(s-1)(c(x)-s)\,ds \\
&= \int_0^{\overline{c}} \left(c(x)s^2 - s^3 + s^2 - c(x)s\right)\,ds \\
&= \frac{\overline{c}^3}{3} - \frac{\overline{c}^4}{4} + \int_0^{\overline{c}} \max_{x\in\overline{\Omega}} c(x)\left(s^2 - s\right)\,ds \\
&= \frac{\overline{c}^3}{3} - \frac{\overline{c}^4}{4} + \max_{x\in\overline{\Omega}} c(x)\left(\frac{\overline{c}^3}{3} - \frac{\overline{c}^2}{2}\right)\,ds \\
&= \frac{\overline{c}^3}{3} - \frac{\overline{c}^4}{4} + \overline{c}\left(\frac{\overline{c}^3}{3} - \frac{\overline{c}^2}{2}\right)\,ds \\
&= \frac{\overline{c}^3}{12}[\overline{c} - 2].
\end{aligned}$$

So $\int_0^{\overline{c}} \overline{f}(s)\,ds \le 0$ if and only if $\overline{c} \le 2$.

On the other hand, we have

$$\begin{aligned}
\int_0^{c(x)} f(x,s)\,ds &= \int_0^{\overline{c}} s(s-1)(c(x)-s)\,ds - \int_{c(x)}^{\overline{c}} s(s-1)(c(x)-s)\,ds \\
&\ge \int_0^{\overline{c}} s(s-1)(c(x)-s)\,ds \\
&= -\frac{\overline{c}^4}{4} + \frac{1+c(x)}{3}\overline{c}^3 - \frac{c(x)}{2}\overline{c}^2.
\end{aligned}$$

If $\int_0^{c(x)} f(x,s)\,ds \le 0$ for any $x \in \overline{\Omega}$, we have

$$\begin{aligned}
0 &\ge -\frac{\overline{c}^4}{4} + \frac{1+c(x)}{3}\overline{c}^3 - \frac{c(x)}{2}\overline{c}^2 \\
&\Rightarrow 4(1+c(x))\overline{c} - 6c(x) \le 3\overline{c}^2.
\end{aligned}$$

In particular, we have

$$4(1+\overline{c})\overline{c} - 6\overline{c} \le 3\overline{c}^2 \Rightarrow \overline{c} \le 2.$$

However, it is clear that

$$\int_0^{\overline{c}} \overline{f}(s)\,ds \le 0 \not\Rightarrow \int_0^{c(x)} f(x,s)\,ds \le 0 \text{ for any } x \in \overline{\Omega}.$$

Therefore, the condition "$\int_0^{c(x)} f(x,s)\,ds \le 0$ for any $x \in \overline{\Omega}$" is more strong than the condition "$\int_0^{\overline{c}} \overline{f}(s)\,ds \le 0$" in this example, which verifies Proposition 4.1 by a concrete example.

Remark 4.1. In [27], Dancer and Yan proved when $c(x) \equiv 1$ and $\{x \in \Omega : b(x) < 1/2\}$ is of positive measure, then (4.2) may have many positive solutions of local minimum type. The results of Example 4.2 shows that the condition $\int_0^1 \overline{f}(s)\,ds \leq 0$ is optimal for the nonexistence of positive solution of (4.2). However, we do not know whether $\int_0^{\overline{c}} \overline{f}(s)\,ds \leq 0$ is optimal for the nonexistence of positive solution of (1.1).

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