Electronic Journal of Qualitative Theory of Differential Equations
2012, No. **59**, 1-16; **http://www.math.u-szeged.hu/ejqtde/**

Existence of solutions in the α-norm for partial functional differential equations with infinite delay

Khalil EZZINBI [1] **and Amor REBEY**

Université Cadi Ayyad, Faculté des Sciences Semlalia, Département de Mathématiques,
BP 2390, Marrakech, Marocco,
Institut Supérieur des Mathématiques Appliquées et de l'Informatique de Kairouan,
Avenue Assad Iben Fourat - 3100 Kairouan, Tunisie.
rebey_amor@yahoo.fr

Abstract: In this work, we prove a result on the local existence of mild solution in the α-norm for some partial functional differential equations with infinite delay. We suppose that the linear part generates a compact analytic semigroup. The nonlinear part is just assumed to be continuous. We use the compactness method, to show the main result of this work. Some application is provided.

Keywords: Analytic semigroup; Fractional power; Infinite delay; Mild solution; Sadovskii's fixed point theorem.
AMS Subject Classifications: 34K30, 45N05.

1 Introduction

The aim of this work is to study the local existence and continuability of solutions for some class of partial functional differential equations with infinite delay and deviating argument in terms involving spatial partial derivatives. As a model for this class we consider the following model

$$\begin{cases} \dfrac{\partial}{\partial t}v(t,x) = \dfrac{\partial^2}{\partial x^2}v(t,x) + a\dfrac{\partial}{\partial x}v(t-r,x) + \displaystyle\int_{-\infty}^{0} g(\theta)v(t+\theta,x)d\theta \\ \qquad\qquad\qquad\qquad + f(\dfrac{\partial}{\partial x}v(t-r,x)) \ \text{ for } t \geq 0 \ \text{ and } x \in [0,\pi], \\ v(t,0) = v(t,\pi) = 0 \quad \text{ for } t \geq 0, \\ v(\theta,x) = v_0(\theta,x) \quad \text{ for } \theta \leq 0 \ \text{ and } x \in [0,\pi], \end{cases} \tag{1}$$

[1] Corresponding author.
E-mail address: ezzinbi@ucam.ac.ma (Khalil Ezzinbi).

where a and r are positive constants, $g :]-\infty, 0] \to \mathbb{R}$ is a positive integrable function, $f : \mathbb{R} \to \mathbb{R}$ is continuous and v_0 is a given initial function. Equation (1) can be written in the following abstract form

$$\begin{cases} \dfrac{d}{dt}u(t) = -Au(t) + F(t,\ u_t) & \text{for } t \geq 0, \\ u_0 \qquad = \varphi \in \mathcal{B}_\alpha, \end{cases} \tag{2}$$

where $-A$ is the infinitesimal generator of an analytic semigroup on a Banach space X. $\mathcal{B}_\alpha$ is a subset of $\mathcal{B}$,where $\mathcal{B}$ is a Banach space of functions mapping from $]-\infty, 0]$ into X and satisfying some axioms that will be introduced later, and for $0 < \alpha < 1$, the operator A^α is the fractional power of A. This operator $(A^\alpha, D(A^\alpha))$ will be described in Section 2. We suppose that F is a continuous function from $\mathbb{R}_+ \times \mathcal{B}_\alpha$ with values in X, where $\mathcal{B}_\alpha$ is defined by

$$\mathcal{B}_\alpha = \{\varphi \in \mathcal{B} : \varphi(\theta) \in D(A^\alpha) \text{ for } \theta \leq 0 \text{ and } A^\alpha\varphi \in \mathcal{B}\},$$

with the norm

$$\|\varphi\|_{\mathcal{B}_\alpha} := \|A^\alpha\varphi\|_{\mathcal{B}} \quad \text{for } \varphi \in \mathcal{B}_\alpha.$$

For every $t \geq 0$, the history function $u_t \in \mathcal{B}_\alpha$ is defined by

$$u_t(\theta) = u(t+\theta) \quad \text{for } \theta \leq 0.$$

In this paper, we will discuss the local and global existence of solutions for Equation (2) where the nonlinear part F is just assumed to be continuous with respect to a fractional power of A in the second variable. Recall that when F is Lipschitz continuous in $\mathcal{B}_\alpha$, Equation (2) has been studied by [6].

The present paper is organized as follows. In Section 2, we study the local existence of mild solutions in the α-norm for Eq. (2). In Section 3, we establish a result about continuation of solutions. Finally, to illustrate our results, we give in Section 4 an application.

2 Local existence of the mild solutions

In this section we study the existence of mild solutions for partial functional differential equations (2). Before that, we collect some useful results. For literature relating to semigroup theory, we suggest Pazy [14], Engel and Nagel [9]. We denote by X a Banach space with norm $\|.\|$ and $-A$ is the infinitesimal generator of a bounded analytic semigroup of linear operator $(T(t))_{t\geq 0}$ on X. We assume without loss of generality that $0 \in \rho(A)$. Note that if the assumption $0 \in \rho(A)$ is not satisfied, one can substitute the operator A by the operator $(A - \sigma I)$ with σ large enough such that $0 \in \rho(A - \sigma)$. This allows us

to define the fractional power A^α for $0 < \alpha < 1$, as a closed linear invertible operator with domain $D(A^\alpha)$ dense in X. The closedness of A^α implies that $D(A^\alpha)$, endowed with the graph norm of A^α, $|x| = \|x\| + \|A^\alpha x\|$, is a Banach space. Since A^α is invertible, its graph norm $|.|$ is equivalent to the norm $\|x\|_\alpha = \|Ax\|$. Thus, $D(A^\alpha)$ equipped with the norm $\|.\|_\alpha$, is a Banach space, which is denoted by X_α. For $0 < \beta \leq \alpha < 1$, the imbedding $X_\alpha \hookrightarrow X_\beta$ is compact if the resolvent operator of A is compact. Also, the following properties are well known.

Theorem 1 *[14] Let $0 < \alpha < 1$ and $-A$ is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ on X satisfying $0 \in \rho(A)$. Then we have*

i) $T(t) : X \longrightarrow D(A^\alpha)$ *for every* $t > 0$,
ii) $T(t)A^\alpha x = A^\alpha T(t)x$ *for every* $x \in D(A^\alpha)$ *and* $t \geq 0$,
iii) For every $t > 0$, $A^\alpha T(t)$ *is bounded on* X *and there exists* $M_\alpha > 0$ *such that :*

$$\|A^\alpha T(t)\| \leq M_\alpha e^{\omega t} t^{-\alpha}, \tag{3}$$

iv) If $0 < \alpha \leq \beta < 1$, *then* $D(A^\beta) \hookrightarrow D(A^\alpha)$.
v) There exists $N_\alpha > 0$ *such that*

$$\|(T(t) - I)A^{-\alpha}\| \leq N_\alpha t^\alpha \quad \textit{for } t > 0.$$

Recall that $A^{-\alpha}$ is given by the following formulas

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} T(t) dt,$$

where the integral converges in the uniform operator topology for every $\alpha > 0$. Consequently, if $T(t)$ is compact for each $t > 0$, then $A^{-\alpha}$ is compact for every $0 < \alpha < 1$.

In all this paper, we suppose that $(\mathcal{B}, \|.\|_\mathcal{B})$ is a normed linear space of functions mapping $]-\infty, 0]$ into X, and satisfying the following fundamental axioms which have been first introduced by Hale and Kato in [12]:

(A1) There exist a positive constant H and functions $K(.), H(.) : \mathbb{R}^+ \to \mathbb{R}^+$, with K continuous and M locally bounded, such that for any $\sigma \in \mathbb{R}$ and $a > 0$, if $x :]-\infty, \sigma + a] \to X$, $x_\sigma \in \mathcal{B}$, and $x(.)$ is continuous on $[\sigma, \sigma + a]$, then for all t in $[\sigma, \sigma + a]$ the following conditions hold:
(i) $x_t \in \mathcal{B}$,
(ii) $\| x(t) \| \leq H \| x_t \|_\mathcal{B}$,
(iii) $\| x_t \|_\mathcal{B} \leq K(t - \sigma) \sup_{\sigma \leq s \leq t} \| x(s) \| + M(t - \sigma) \| x_\sigma \|_\mathcal{B}$.

(A2) For the function $x(.)$ in **(A1)**, $t \mapsto x_t$ is a $\mathcal{B}$-valuded continuous function for t in $[\sigma, \sigma + a]$.

(B) The space $\mathcal{B}$ is complete.

Now, we make the following hypothesis:

(H1) The operator $-A$ is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ on the Banach space X satisfying $0 \in \rho(A)$.

(H2) The semigroup $(T(t))_{t\geq 0}$ is compact on X. It means that $T(t)$ is compact on X for $t > 0$.

(H3) $A^{-\alpha}\varphi \in \mathcal{B}$ for $\varphi \in \mathcal{B}$, where the function $A^{-\alpha}\varphi$ is defined by

$$(A^{-\alpha}\varphi)(\theta) = A^{-\alpha}(\varphi(\theta)) \text{ for } \theta \leq 0.$$

Lemma 2 *[6] Assume that **(H1)** and **(H3)** hold. Then $\mathcal{B}_\alpha$ is a Banach space.*

Definition 3 *Let $\varphi \in \mathcal{B}_\alpha$. A function $u :]-\infty, a] \to X_\alpha$ is called a mild solution of Eq. (2) if the restriction of $u(.)$ to the interval $[0, a]$ is continuous and*

i) $u(t) = T(t)\varphi(0) + \int_0^t T(t-s)F(s, u_s)ds \quad$ *for* $t \in [0, a]$,

ii) $u_0 = \varphi$.

The main result of this section is the following theorem.

Theorem 4 *Assume that the hypothesis **(H1)**-**(H3)** hold. Let U be an open subset of the Banach space $\mathcal{B}_\alpha$ and $F : [0, a] \times U \to X$ be continuous. Then for each $\varphi \in U$, there exist $b := b_\varphi$ with $0 < b \leq a$ and a mild solution $u \in C([0, b]; X_\alpha)$ of Eq. (2).*

Proof.— The proof of this result is based on the Schauder fixed-point theorem.
Let $\varphi \in U$, there exist constants $r > 0$, $b_1 \in]0, a]$ and $N \geq 0$ such that $\overline{B(\varphi, r)} := \{\phi \in \mathcal{B}_\alpha : \|\phi - \varphi\|_{\mathcal{B}_\alpha} \leq r\} \subseteq U$ and $\|F(s, \phi)\| \leq N$ for all $s \in [0, b_1]$ and $\phi \in \overline{B(\varphi, r)}$.
Consider the function $w :]-\infty, b_1] \to X_\alpha$ defined by

$$w(t) = \begin{cases} T(t)\varphi(0) & \text{for } t \in [0, b_1] \\ \varphi(t) & \text{for } t \leq 0. \end{cases}$$

By Axioms **(A1)(i)** and **(A2)**, there exists $0 < b < b_1$ such that $\|w_t - \varphi\|_{\mathcal{B}_\alpha} < \frac{r}{2}$ for all $t \in [0, b]$. We choose b small enough such that

$$K_b M_\alpha N \int_0^b \frac{e^{\omega s}}{s^\alpha} ds < \frac{r}{2}, \tag{4}$$

where $K_b = \sup_{0 \le t \le b} K(t)$. Let us introduce the space

$$\Omega_b := \{u \in C([0, b]; X_\alpha) : u(0) = \varphi(0)\},$$

endowed with the uniform norm topology. Let $u \in \Omega_b$. We define the extension $\widetilde{u}$ of u on $]-\infty, b]$ by

$$\widetilde{u}(t) = \begin{cases} u(t) & \text{for } t \in [0, b] \\ \varphi(t) & \text{for } t \le 0. \end{cases}$$

We define the set $\Omega_b(\varphi)$ by

$$\Omega_b(\varphi) := \{u \in \Omega_b : \ \|\widetilde{u}_t - \varphi\|_{\mathcal{B}_\alpha} \le r \ for \ t \in [0, b]\}.$$

Let $v(t) = T(t)\varphi(0)$ for $t \in [0, b]$. Its extension $\widetilde{v} :]-\infty, b] \to X_\alpha$ is defined by

$$\widetilde{v}(t) = \begin{cases} v(t) & \text{for } t \in [0, b] \\ \varphi(t) & \text{for } t \le 0. \end{cases}$$

It is easy to see that $\widetilde{v}$ is the restriction of w on $]-\infty, b]$ and v is an element of $\Omega_b(\varphi)$. Then $\Omega_b(\varphi)$ is a nonempty.
$\Omega_b(\varphi)$ is closed convex in $C([0, b], X_\alpha)$. To prove that. Let $(u^n)_{n \ge 0}$ be in $\Omega_b(\varphi)$ with $\lim\limits_{n \to +\infty} u^n = u$ in $C([0, b]; X_\alpha)$. The Axioms **(A1)(iii)** implies that for any $t \in [0, b], \ n \in \mathbb{N}$, we have

$$\begin{aligned} \|\widetilde{u}_t^n - \widetilde{u}_t\|_{\mathcal{B}_\alpha} &\le K(t) \sup_{0 \le s \le t} \|\widetilde{u}^n(s) - \widetilde{u}(s)\|_\alpha \\ &\le K_b \sup_{0 \le s \le b} \|u^n(s) - u(s)\|_\alpha \to 0 \text{ as } n \to +\infty. \end{aligned}$$

From this together with the inequality

$$\|\widetilde{u}_t - \varphi\|_{\mathcal{B}_\alpha} \le \|\widetilde{u}_t - \widetilde{u}_t^n\|_{\mathcal{B}_\alpha} + \|\widetilde{u}_t^n - \varphi\|_{\mathcal{B}_\alpha} \text{ for any } n \in \mathbb{N},$$

we deduce that $\|\widetilde{u}_t - \varphi\|_{\mathcal{B}_\alpha} \le r$. Consequently, $u \in \Omega_b(\varphi)$.
By using the triangular inequality, it is clear that $\lambda u_1 + (1 - \lambda) u_2 \in \Omega_b(\varphi)$, for

any u_1, $u_2 \in \Omega_b(\varphi)$ and $\lambda \in [0,1]$. Then $\Omega_b(\varphi)$ is closed bounded convex set. Consider now the mapping defined on $\Omega_b(\varphi)$ by

$$H(u)(t) = T(t)\varphi(0) + \int_0^t T(t-s)F(s,\widetilde{u}_s)ds \text{ for } t \in [0,b].$$

We claim that H maps $\Omega_b(\varphi)$ into $\Omega_b(\varphi)$. In fact, let $u \in \Omega_b(\varphi)$ and $0 \leq t_0 < t \leq b$. Then

$$\begin{aligned} Hu(t) - Hu(t_0) &= (T(t) - T(t_0))\varphi(0) + \int_{t_0}^t T(t-s)F(s,\widetilde{u}_s)ds \\ &\quad + \int_0^{t_0} (T(t-s) - T(t_0-s))F(s,\widetilde{u}_s)ds \\ &= (T(t) - T(t_0))\varphi(0) + \int_{t_0}^t T(t-s)F(s,\widetilde{u}_s)ds \\ &\quad + (T(t-t_0) - I)\int_0^{t_0} T(t_0-s)F(s,\widetilde{u}_s)ds. \end{aligned}$$

We obtain that

$$\begin{aligned} \|Hu(t) - Hu(t_0)\|_\alpha &\leq \|(T(t) - T(t_0))A^\alpha\varphi(0)\| + M_\alpha N \int_{t_0}^t \frac{e^{\omega(t-s)}}{(t-s)^\alpha}ds \\ &\quad + \|(T(t-t_0) - I)\int_0^{t_0} A^\alpha T(t_0-s)F(s,\widetilde{u}_s)ds\| \to 0 \text{ as } t \to {t_0}^+. \end{aligned}$$

Using similar argument for $0 \leq t < t_0 \leq b$, we conclude that

$$\|Hu(t) - Hu(t_0)\|_\alpha \to 0 \text{ as } t \to {t_0}^-.$$

This implies that $Hu \in C([0,b];X_\alpha)$. Now, we claim that $\widetilde{H(u)}_t \in \overline{B(\varphi,r)}$ for $t \in [0,b]$. Let $u \in \Omega_b(\varphi)$ and $t \in [0,b]$. Then

$$\widetilde{Hu}(t) = \begin{cases} v(t) + y(t) & \text{for } t \in [0,b] \\ \varphi(t) & \text{for } t \leq 0, \end{cases}$$

where

$$y(t) = \begin{cases} \displaystyle\int_0^t T(t-s)F(s,\widetilde{u}_s)ds & \text{for } t \in [0,b] \\ 0 & \text{for } t \leq 0. \end{cases}$$

Simple computations yield $\widetilde{(Hu)}_t = \widetilde{v}_t - y_t$ for $t \in [0,b]$. Then, we get for any $t \in [0,b]$

$$\begin{aligned}\|\widetilde{(Hu)}_t - \varphi\|_{\mathcal{B}_\alpha} &\leq \|\widetilde{v}_t - \varphi\|_{\mathcal{B}_\alpha} + \|y_t\|_{\mathcal{B}_\alpha} \\ &\leq \frac{r}{2} + K(t) \sup_{0\leq s\leq t} \|y(s)\|_\alpha \\ &\leq \frac{r}{2} + K_b M_\alpha N \int_0^b \frac{e^{\omega\tau}}{\tau^\alpha} d\tau.\end{aligned}$$

By the estimate (4), we deduce that

$$\|\widetilde{H(u)}_t - \varphi\|_{\mathcal{B}_\alpha} \leq r \text{ for all } t \in [0, b].$$

Finally, we have proved that $H(\Omega_b(\varphi)) \subseteq \Omega_b(\varphi)$.
We will prove now the continuity of H. Let $(u^n)_{n\geq 1}$ be a convergent sequence in $\Omega_b(\varphi)$ with $\lim_{n\to\infty} u^n = u$, we obtain $\lim_{n\to\infty} \widetilde{u}^n = \widetilde{u}$. Then, for $t \in [0, b]$ we have

$$\|Hu^n(t) - Hu(t)\|_\alpha \leq M_\alpha \int_0^t e^{\omega(t-s)}(t-s)^{-\alpha} \|F(s, \widetilde{u}_s^n) - F(s, \widetilde{u}_s)\| ds. \quad (5)$$

By Axioms **(A1)(iii)** and **(A2)** we have the mapping $(s, u) \mapsto \widetilde{u}_s$ is continuous in $[0, b] \times \Omega_b(\varphi)$. On the other hand the set $\{\widetilde{u}\} \cup \{\widetilde{u}^n : n \geq 1\}$ is compact. Hence, the set $\Lambda = \{(s, \widetilde{u}_s^n), (s, \widetilde{u}_s) : s \in [0, b], n \geq 1\}$ is compact in $[0, b] \times \mathcal{B}_\alpha$. By Heine's theorem implies that F is uniformly continuous in Λ. Accordingly, since $(u^n)_{n\geq 1}$ converge to u, we have

$$\|Hu^n - Hu\|_\infty \leq M_\alpha \int_0^b \frac{e^{\omega s}}{s^\alpha} ds \sup_{s\in[0,b]} \|F(s, \widetilde{u}_s^n) - F(s, \widetilde{u}_s)\| \to 0 \text{ as } n \to +\infty. \quad (6)$$

Then, we obtain that $(Hu^n)_{n\geq 1}$ converge to Hu. And this yields the continuity of H.
We will prove now that, for each $0 < t \leq b$, the set $\left\{ \int_0^t T(t-s)F(s, \widetilde{u}_s)ds,\ u \in \Omega_b(\varphi) \right\}$ is relatively compact in X_α.
Let $t \in]0, b]$ fixed, and $\beta > 0$ such that $\alpha < \beta < 1$, we have

$$\| A^\beta \int_0^t T(t-s)F(s, \widetilde{u}_s)ds \| \leq M_\beta N \int_0^b \frac{e^{\omega s}}{s^\beta} ds.$$

Then for fixed $t \in]0, b]$

$$\left\{ A^\beta \int_0^t T(t-s)F(s, \widetilde{u}_s)ds,\ u \in \Omega_b(\varphi) \right\}$$

is bounded in X. By **(H2)** and Theorem 1, we deduce that $A^{-\beta} : X \to X_\alpha$ is compact. Consequently

$$\left\{ \int_0^t T(t-s)F(s, \widetilde{u}_s)ds,\ u \in \Omega_b(\varphi) \right\}$$

is relatively compact set in X_α.
We will show that $\{Hu(t),\ u \in \Omega_b(\varphi)\}$ is an equicontinuous family of functions. Let $u \in \Omega_b(\varphi)$ and $0 \leq t_1 < t_2 \leq b$. Then

$$\begin{aligned} Hu(t_2) - Hu(t_1) &= (T(t_2) - T(t_1))\varphi(0) + \int_{t_1}^{t_2} T(t_2 - s)F(s, \widetilde{u}_s)ds \\ &\quad + \int_0^{t_1} (T(t_2 - s) - T(t_1 - s))F(s, \widetilde{u}_s)ds \\ &= (T(t_2) - T(t_1))\varphi(0) + \int_{t_1}^{t_2} T(t_2 - s)F(s, \widetilde{u}_s)ds \\ &\quad + (T(t_2 - t_1) - I)\int_0^{t_1} T(t_1 - s)F(s, \widetilde{u}_s)ds. \end{aligned}$$

We obtain that

$$\begin{aligned} \|Hu(t_2) - Hu(t_1)\|_\alpha \leq & \|(T(t_2) - T(t_1))\varphi(0)\|_\alpha + M_\alpha N \int_{t_1}^{t_2} \frac{e^{\omega s}}{s^\alpha} ds \\ & + \|(T(t_2 - t_1) - I)\int_0^{t_1} A^\alpha T(t_1 - s)F(s, \widetilde{u}_s)ds\|. \end{aligned}$$

We claim that the first part tend to zero as $|t_2 - t_1| \to 0$, since for $t_1 > 0$ the set

$$\left\{ \int_0^{t_1} A^\alpha T(t_1 - s)F(s, \widetilde{u}_s)ds :\ u \in \Omega_b(\varphi) \right\}$$

is relatively compact in X, there is a compact set $\widetilde{K}$ in X such that

$$\int_0^{t_1} A^\alpha T(t_1 - s)F(s, \widetilde{u}_s)ds \in \widetilde{K} \text{ for any } u \in \Omega_b(\varphi).$$

By Banach-Steinhaus's theorem, we have

$$\left\| (T(t_2 - t_1) - I)\int_0^{t_1} A^\alpha T(t_1 - s)F(s, \widetilde{u}_s)ds \right\| \to 0 \text{ as } t_2 \to t_1,$$

uniformly in $u \in \Omega_b(\varphi)$. Using similar argument for $0 \leq t_2 < t_1 \leq b$, we can conclude that $\{Hu(t),\ u \in \Omega_b(\varphi)\}$ is equicontinuous. By Ascoli-Arzela's theorem, we deduce $\{Hu(.),\ u \in \Omega_b(\varphi)\}$ is relatively compact in $C([0, b], X_\alpha)$. Now by Schauder's fixed point theorem, we get that H has a fixed point u in $\Omega_b(\varphi)$, which implies that u is a mild solutions of Equation (2) on $[0, b]$. This ends the proof of Theorem. ■

3 Global continuation of the solutions

In order to define the mild solution in its maximal interval of existence, we add the following condition

(H4) $F : [0, +\infty[\times\mathcal{B}_\alpha \to X$ is continuous and takes bounded sets of $[0, +\infty[\times\mathcal{B}_\alpha$ into bounded sets in X.

Theorem 5 *Assume that **(H1)-(H4)** hold. Then there exists a maximal interval $[0, t_{max}[$ and a mild solution of Eq. (2) defined on $[0, t_{max}[$. However, if $t_{max} < +\infty$, then $\limsup_{t\to t_{max}} \|u(t, \varphi)\|_\alpha = +\infty$.*

Proof.— On can see that the mild solution of Eq. (2) is defined on $[0, t_{max}[$. Assume that $t_{max} < +\infty$ and $\limsup_{t\to t_{max}} \|u(t, \varphi)\|_\alpha < +\infty$. Then there exists $L > 0$ such that $\|u(t, \varphi)\|_\alpha < L$ for $t \in [0, t_{max}[$. Then, from Axiom **(A1)(iii)** there exists constant $r > 0$ such that $\|u_s(., \varphi)\|_{\mathcal{B}_\alpha} \leq r$, for all $s \in [0, t_{max}[$ and consequently by hypothesis **(H4)** there exists $R > 0$ such that $\|F(s, u_s)\| \leq R$, for all $s \in [0, t_{max}[$. Let $u : [t_0, t_{max}[\to X_\alpha$ ($t_0 \in]0, t_{max}[$) be the restriction of $u(., \varphi)$ to $[t_0, t_{max}[$. Consider $t \in [t_0, t_{max}[$ and β such that $\alpha < \beta < 1$. Then

$$\begin{aligned}\|u(t)\|_\beta &\leq \|A^{\beta-\alpha}T(t)A^\alpha\varphi(0)\| + \|\int_0^t A^\beta T(t-s)F(s, u_s)ds\| \\ &\leq M_{\beta-\alpha}\frac{e^{\omega t}}{t^{\beta-\alpha}}\|\varphi(0)\|_\alpha + M_\beta R\int_0^t \frac{e^{\omega s}}{s^\beta}ds.\end{aligned}$$

Thus, $\|u(t)\|_\beta$ is bounded on $[t_0, t_{max}[$. Now, for $t_0 \leq t < t+h < t_{max}$, we have

$$\begin{aligned}u(t+h) - u(t) &= T(t+h)\varphi(0) - T(t)\varphi(0) + \int_0^{t+h} T(t+h-s)F(s, u_s)ds \\ &\quad - \int_0^t T(t-s)F(s, u_s)ds \\ &= T(t)[(T(h) - I)\varphi(0)] + (T(h) - I)\int_0^t T(t-s)F(s, u_s)ds \\ &\quad + \int_t^{t+h} T(t+h-s)F(s, u_s)ds \\ &= (T(h) - I)u(t) + \int_t^{t+h} T(t+h-s)F(s, u_s)ds.\end{aligned}$$

Taking the α-norm, we obtain

$$\begin{aligned}\|u(t+h)-u(t)\|_\alpha &\leq \|(T(h)-I)A^{-(\beta-\alpha)}A^\beta u(t)\| + RM_\alpha \int_t^{t+h} \frac{e^{\omega(t+h-s)}}{(t+h-s)^\alpha} ds \\ &\leq N_{\beta-\alpha} h^{\beta-\alpha} \|u(t)\|_\beta + RM_\alpha \int_t^{t+h} \frac{e^{\omega(t+h-s)}}{(t+h-s)^\alpha} ds \\ &\leq N_{\beta-\alpha} h^{\beta-\alpha} \|u(t)\|_\beta + RM_\alpha \int_0^h \frac{e^{\omega s}}{s^\alpha} ds \\ &\leq N_{\beta-\alpha} h^{\beta-\alpha} \|u(t)\|_\beta + RM_\alpha \max\{1, e^{\omega t_{max}}\} \frac{h^{1-\alpha}}{1-\alpha}.\end{aligned}$$

This implies that

$$\|u(t+h)-u(t)\|_\alpha \to 0 \quad \text{as} \quad h \to 0$$

uniformly with respect to $t \in [t_0, t_{max}[$. Which implies that u is uniformly continuous on $[t_0, t_{max}[$. Consequently, $u(., \varphi)$ can be extended to the right to t_{max}, which contradicts the maximality of $[0, t_{max}[$. This completes the proof of the theorem. ■

The following result provides sufficient conditions for global solutions to Eq. (2).

Corollary 6 *Under the same assumptions as in Theorem 4, if there exist k_1, $k_2 \in C(\mathbb{R}_+, \mathbb{R}_+)$ such that $\|F(t,\varphi)\| \leq k_1(t)\|\varphi\|_{\mathcal{B}_\alpha} + k_2(t)$ for $\varphi \in \mathcal{B}_\alpha$ and $t \geq 0$, then Eq. (2) admits global solutions.*

The proof of this corollary is based on the following lemma, whose proof can be found in Lemma 6.7 of [14].

Lemma 7 *[14] Let $v : [0,a] \to [0,\infty[$ be continuous. If there are positive constants A, $B, 0 < \alpha < 1$ such that*

$$v(t) \leq A + B \int_0^t \frac{v(s)}{(t-s)^\alpha} ds \quad \text{for } t \in [0,a]$$

then, there is a constant C such that

$$v(t) \leq C \quad \text{for } t \in [0,a].$$

Proof.— Assume that $t_{max} < +\infty$. Let $M := \sup_{0 \leq t \leq t_{max}} \|T(t)\|$. Then for every $t \in [0, t_{max}[$, we have

$$\begin{aligned}\|u(t)\|_\alpha &\leq \|T(t)A^\alpha \varphi(0)\| + \|\int_0^t A^\alpha T(t-s)F(s,u_s)ds\| \\ &\leq MH\|\varphi\|_{\mathcal{B}_\alpha} + M_\alpha \int_0^t \frac{e^{\omega(t-s)}}{(t-s)^\alpha} \Big(k_1(s)\|u_s\|_{\mathcal{B}_\alpha} + k_2(s)\Big) ds.\end{aligned}$$

Therefore, by **(A1)(iii)**,

$$\|u(t)\|_\alpha \leq A + B\int_0^t \frac{1}{(t-s)^\alpha} \sup_{0\leq\sigma\leq s} \|u(\sigma)\|_\alpha ds, \tag{7}$$

where

$$A = MH\|\varphi\|_{\mathcal{B}_\alpha} + M_\alpha\Big[\sup_{0\leq s\leq t_{max}} k_2(s) + \|\varphi\|_{\mathcal{B}_\alpha} \sup_{0\leq s\leq t_{max}} \Big(k_1(s)M(s)\Big)\Big]\int_0^{t_{max}} \frac{e^{\omega s}}{s^\alpha} ds,$$

and

$$B = M_\alpha \max\{1, e^{\omega t_{max}}\} \sup_{0\leq s\leq t_{max}} \Big(k_1(s)K(s)\Big).$$

We claim that the function $g : s \mapsto \int_0^s \frac{1}{(s-\sigma)^\alpha} \sup_{0\leq\tau\leq\sigma} \|u(\tau)\|_\alpha d\sigma$ is nondecreasing on $[0,t]$. Let $s, s' \in [0,t]$ be such that $s < s'$. Then

$$\begin{aligned}
g(s) &= \int_0^s \frac{1}{(s-\sigma)^\alpha} \sup_{0\leq\tau\leq\sigma} \|u(\tau)\|_\alpha d\sigma \\
&= \int_0^s \frac{1}{\sigma^\alpha} \sup_{0\leq\tau\leq s-\sigma} \|u(\tau)\|_\alpha d\sigma \\
&\leq \int_0^s \frac{1}{\sigma^\alpha} \sup_{0\leq\tau\leq s'-\sigma} \|u(\tau)\|_\alpha d\sigma \\
&\leq \int_0^{s'} \frac{1}{\sigma^\alpha} \sup_{0\leq\tau\leq s'-\sigma} \|u(\tau)\|_\alpha d\sigma = g(s').
\end{aligned}$$

Therefore g is nondecreasing on $[0,t]$ and $\sup_{0\leq s\leq t} g(s) = g(t)$. Then by the inequality (7), we obtain

$$\sup_{0\leq s\leq t} \|u(s)\|_\alpha \leq A + B\int_0^t \frac{1}{(t-s)^\alpha} \sup_{0\leq\sigma\leq s} \|u(\sigma)\|_\alpha ds,$$

By Lemma 7, there is a constant C such that

$$\sup_{0\leq s\leq t} \|u(s)\|_\alpha \leq C,$$

which implies that $\sup_{0\leq s<t_{max}} \|u(s)\|_\alpha < \infty$, and the proof is complete. ■

4 Application

Consider the following functional differential equation

$$\begin{cases} \dfrac{\partial}{\partial t}v(t,x) = \dfrac{\partial^2}{\partial x^2}v(t,x) + a\dfrac{\partial}{\partial x}v(t-r,x) + \displaystyle\int_{-\infty}^{0} g(\theta)v(t+\theta,x)d\theta \\ \qquad\qquad\qquad\qquad\qquad\qquad +f(\dfrac{\partial}{\partial x}v(t-r,x)) \text{ for } t \geq 0 \text{ and} x \in [0,\pi], \\ v(t,0) = v(t,\pi) = 0 \quad \text{for } t \geq 0, \\ v(\theta,x) = v_0(\theta,x) \quad \text{for } \theta \leq 0 \text{ and } x \in [0,\pi], \end{cases} \tag{8}$$

where a and r are positive constants, $g :]-\infty, 0] \to \mathbb{R}$ is a positive integrable function, $f : \mathbb{R} \to \mathbb{R}$ and $v_0 :]-\infty, 0] \times [0,\pi] \to \mathbb{R}$ are continuous. Let $X = \mathrm{L}^2([0,\pi];\mathbb{R})$ and $A : D(A) \subset X \to X$ defined by $Ay = -y''$ with domain $D(A) = H^2[0,\pi] \cap H_0^1[0,\pi]$. Then $Ay = \sum\limits_{n=1}^{\infty} n^2(y,e_n)e_n$ for $y \in D(A)$, where $\{e_n(s) = \sqrt{\frac{2}{\pi}} sin\ ns,\ n \geq 1\}$, is the orthonormal set of eigenvectors of A. For each $y \in D(A^{\frac{1}{2}}) := \{y \in X : \sum\limits_{n=1}^{\infty} n(y,e_n)e_n \in X\}$ the operator $A^{\frac{1}{2}}$ is given by $A^{\frac{1}{2}}y = \sum_{n=1}^{\infty} n(y,e_n)e_n$.

Lemma 8 *[17] If $y \in D(A^{\frac{1}{2}})$, then y is absolutly continuous, $y' \in X$ and*

$$\|y'\|_X = \|A^{\frac{1}{2}}y\|_X.$$

It is well known that $-A$ is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ on X given by $T(t)x = \sum\limits_{n=1}^{\infty} e^{-n^2 t}(x,e_n)e_n,\ x \in X$. It follows from this last expression that $(T(t))_{t\geq 0}$ is a compact semigroup on X. This implies that Assumption **(H1)** and **(H2)** are satisfied. Let, for $\gamma > 0$,

$$\mathcal{B} = C_\gamma = \{\varphi \in C(]-\infty,0];X) : \lim_{\theta\to-\infty} e^{\gamma\theta}\varphi(\theta) \text{ exists in } X\},$$

with the norm

$$\|\varphi\|_\gamma = \sup_{\theta\leq 0} e^{\gamma\theta}\|\varphi(\theta)\| \text{ for } \varphi \in C_\gamma.$$

This space satisfies axioms **(A1)**, **(A2)** and **(B)**. The norm in $\mathcal{B}_{\frac{1}{2}}$ is given by

$$\|\varphi\|_{\mathcal{B}_{\frac{1}{2}}} = \sup_{\theta\leq 0} e^{\gamma\theta}\|A^{\frac{1}{2}}\varphi(\theta)\| = \sup_{\theta\leq 0} e^{\gamma\theta}\sqrt{\int_0^\pi \left(\frac{\partial}{\partial x}(\varphi)(\theta)(x)\right)^2 dx}$$

Assume that,

(H5) $: e^{-2\gamma}g \in L^2(\mathbb{R}^-)$.

Let

$$\begin{cases} u(t)(x) = v(t,x) \text{ for } t \geq 0 \text{ and } x \in [0,\pi], \\ \varphi(\theta)(x) = v_0(\theta,x) \text{ for } \theta \leq 0 \text{ and } x \in [0,\pi], \\ (F(\varphi))(x) = a\varphi(-r)'(x) + \displaystyle\int_{-\infty}^{0} g(\theta)\varphi(\theta)(x)d\theta + f(\varphi(-r)'(x)) \text{ for } \varphi \in \mathcal{B}_{\frac{1}{2}} \text{ and } x \in [0,\pi]. \end{cases}$$

Then, Eq. (8) takes the following abstract form

$$\begin{cases} \dfrac{d}{dt}u(t) = -Au(t) + F(t,\ u_t) \quad \text{for } t \geq 0, \\ u_0 \qquad = \varphi \in \mathcal{B}_{\frac{1}{2}}. \end{cases} \tag{9}$$

F can be decomposed as follows: $F = F_1 + F_2 + F_3$, where

$$\begin{cases} (F_1(\varphi))(x) = a\varphi(-r)'(x), \\ (F_2(\varphi))(x) = \displaystyle\int_{-\infty}^{0} g(\theta)\varphi(\theta)(x)d\theta, \\ (F_3(\varphi))(x) = f(\varphi(-r)'(x)). \end{cases}$$

Let $\varphi \in \mathcal{B}_{\frac{1}{2}}$, we consider a sequence $(\varphi_n)_n$ convergent to φ in $\mathcal{B}_{\frac{1}{2}}$, then we have

$$\begin{aligned} \|F_1(\varphi_n) - F_1(\varphi)\|_X^2 &= a^2 \int_0^{\pi} \Big|\varphi_n(-r)(x) - \varphi(-r)(x)\Big)\Big|^2 dx \\ &= a^2 \|A^{\frac{1}{2}}(\varphi_n(-r) - \varphi(-r))\|^2 \\ &\leq a^2 e^{2\gamma r} \sup_{\theta \leq 0} e^{2\gamma\theta} \|A^{\frac{1}{2}}(\varphi_n(-r) - \varphi(-r))\|^2 \\ &\leq a^2 e^{2\gamma r} \|\varphi_n - \varphi\|_{\mathcal{B}_{\frac{1}{2}}}^2, \end{aligned} \tag{10}$$

and

$$\begin{aligned} \|F_2(\varphi_n) - F_2(\varphi)\|_X^2 &= \int_0^{\pi} \Big| \int_{-\infty}^{0} g(\theta)\Big(\varphi_n(\theta)(x) - \varphi(\theta)(x)\Big)d\theta\Big|^2 dx \\ &\leq \int_0^{\pi} \Big(\int_{-\infty}^{0} g(\theta)^2 e^{-4\gamma\theta} d\theta\Big) \int_{-\infty}^{0} e^{4\gamma\theta}\Big(\varphi_n(\theta)(x) - \varphi(\theta)(x)\Big)^2 d\theta dx \\ &\leq \Big(\int_{-\infty}^{0} g(\theta)^2 e^{-4\gamma\theta} d\theta\Big) \int_0^{\pi} \int_{-\infty}^{0} e^{4\gamma\theta}\Big(\varphi_n(\theta)(x) - \varphi(\theta)(x)\Big)^2 d\theta dx \\ &\leq \frac{1}{2\gamma}\Big(\int_{-\infty}^{0} g(\theta)^2 e^{-4\gamma\theta} d\theta\Big) \sup_{\theta \leq 0}\{e^{2\gamma\theta} \int_0^{\pi} \Big(\varphi_n(\theta)(x) - \varphi(\theta)(x)\Big)^2 dx\} \\ &\leq \frac{1}{2\gamma}\Big(\int_{-\infty}^{0} g(\theta)^2 e^{-4\gamma\theta} d\theta\Big) \|\varphi_n - \varphi\|_{\mathcal{B}_{\frac{1}{2}}}^2. \end{aligned} \tag{11}$$

Then (10) and (11) imply that $F_1 + F_2$ is continuous on $\mathcal{B}_{\frac{1}{2}}$.
Since

$$\|F_3(\varphi_n) - F_3(\varphi)\|_X^2 = \int_0^\pi |f(\varphi_n(-r)'(x)) - f(\varphi(-r)'(x))|^2 dx.$$

And

$$\begin{aligned}\|A^{\frac{1}{2}}(\varphi_n(-r) - \varphi(-r))\| &\leq e^{\gamma r} \sup_{\theta \leq 0} e^{\gamma\theta} \|A^{\frac{1}{2}}(\varphi_n(\theta) - \varphi(\theta))\| \\ &= e^{\gamma r} \|\varphi_n - \varphi\|_{\mathcal{B}_{\frac{1}{2}}}^2 \to 0 \text{ as } n \to \infty.\end{aligned}$$

Then

$$\frac{\partial}{\partial x}\varphi_n(-r) \to \frac{\partial}{\partial x}\varphi(-r) \text{ as } n \to \infty$$

in $L^2[0, \pi]$. We conclude by the following well known result in L^p spaces convergence. More details can be found in Theorem IV.9 of [7].

Theorem 9 *[7] Let $1 < p < \infty$, Ω an open set in $\mathbb{R}^n$ and $(f_n)_{n\geq 0}$ a sequence in $L^p(\Omega)$. Suppose that $f_n \to f$ as $n \to \infty$ in $L^p(\Omega)$. Then, there exist a subsequence $(f_{n_k})_{k\geq 0}$ of $(f_n)_{n\geq 0}$ and $h \in L^p(\Omega)$ such that*

i) $f_{n_k} \to f$ a.e. in Ω,
ii) $|f_{n_k}(x)| \leq |h(x)|$ $\forall k$ a.e. in Ω.

Then using Theorem 9, we deduce that there exists a subsequence $(\varphi_{n_k})_k$ and $g_1 \in L^2(0, \pi)$ such that

$$\frac{\partial}{\partial x}\varphi_{n_k}(-r)(x) \to \frac{\partial}{\partial x}\varphi(-r)(x) \text{ as } k \to \infty \text{ a.e,}$$

and

$$|\frac{\partial}{\partial x}\varphi_{n_k}(-r)(x)| \leq |g_1(x)| \text{ a.e.}$$

By the continuity of f, we obtain

$$f\Big(\frac{\partial}{\partial x}\varphi_{n_k}(-r)(x)\Big) \to f\Big(\frac{\partial}{\partial x}\varphi(-r)(x)\Big) \text{ as } k \to \infty.$$

If we suppose that $|f(t)| \leq a|t| + b$. By the Lebesgue dominated convergence theorem, we deduce

$$f\Big(\frac{\partial}{\partial x}\varphi_{n_k}(-r)\Big) \to f\Big(\frac{\partial}{\partial x}\varphi(-r)\Big) \text{ as } k \to \infty$$

in $L^2[0, \pi]$. Since the limit does not depend on the subsequence $(\varphi_{n_k})_k$, then we obtain

$$F_3(\varphi_n) \to F_3(\varphi) \text{ as } n \to \infty$$

in $L^2[0,\pi]$. We deduce that F_3 is continuous on $\mathcal{B}_{\frac{1}{2}}$, which implies that F is continuous on $\mathcal{B}_{\frac{1}{2}}$. Consequently, Theorem 4 ensures the existence of a maximal interval of existence $[0, t_{max}[$ and a mild solution $v(t,x)$ on $[0, t_{max}[\times[0,\pi]$. Also, under the assumption that $|f(t)| \leq a|t| + b$, we establish that $t_{max} = \infty$ by applying Corollary 6.

Acknowledgements
The authors would like to thank the referee for his careful reading of the paper. His valuable suggestions and critical remarks made numerous improvements throughout.

References

[1] M. Adimy and K. Ezzinbi, Existence and stability in the α-norm for partial functional differential equations of neutral type, Annali di Matematica Pura ed Applicata, 185, (2006), 437-460.

[2] M. Adimy, K. Ezzinbi, M. Laklach, Existence of solutions for a class of partial neutral differential equations, Comptes Rendus de l'Académie des Sciences, Série I. Mathématique, 330, (2000), 957-962.

[3] M. Adimy, H. Bouzahir, K. Ezzinbi, Existence for a class of partial functional differential equations with infinite delay, Nonlinear Analysis, Theory, Methods and Applications, 46, (2001), 91-112.

[4] M. Adimy, H. Bouzahir, K. Ezzinbi, Local existence and stability for some partial functional differential equations with infinite delay, Nonlinear Analysis, Theory, Methods and Applications, 48, (2002), 323-348.

[5] M. Adimy, H. Bouzahir, K. Ezzinbi, Existence and stability for some partial neutral functional differential equations with infinite delay, Journal of Mathematical Analysis and Applications, 294, (2004), 438-461.

[6] R. Benkhalti, K. Ezzinbi, Existence and stability in the α-norm for some partial functional differential equations with infinite delay, Differential and Integral Equations, Vol. 19, No. 5, (2006), 545-572.

[7] H. Brezis, Analyse Fonctionnelle Théorie et Application, Masson, 1987.

[8] J-C. Chang, H. Liu, Existence of solutions for a class of neutral partial differential equations with nonlocal conditions in the α-norm, Nonliner Analysis, Theory, Methods and Applications, 70, (2009), 3076-3083.

[9] K. Engel and D. Nagel, One-parameter Semigroups for Linear Evolution Equations, Graduate Texts in Mathematics, Springer-Verlag, 194, 2000.

[10] D. Henry, Geometric Theory of Semilinear Parabolic Partial Differential Equations, Springer-Verlag, Berlin/New York, 1989.

[11] K. Ezzinbi, X. Fu, K. Hilal, Existence and regularity in the α-norm for some neutral partial differential equations with nonlocal conditions, Nonlinear Analysis, Theory, Methods and Applications, 67, (2006), 1613-1622.

[12] J. K. Hale, J. Kato, Phase space for retarded equations with infinite delay, Funkcial. Ekvac .21, (1978), 11-41.

[13] Y. Hino, S. Murakami, and T. Naito, Functional Differential Equations with Infinite Delay, Lectures Notes, Springer-Verlag, Vol. 1473, 1991.

[14] A. Pazy, Semigroups of Linear Operators and Application to Partial Differental Equation, Applied Mathematical Sciences, Springer-Verlag, New York, Vol. 44, 1983.

[15] C. C. Travis and G. F. Webb, Existence and Stability for Partial Functional Differential Equations, Journal of Mathmatical Analysis and Application, 200, (1974), 395-419.

[16] C. C. Travis and G. F. Webb, Existence, stability, and compactness in the α-norm for partial functional differential equations, Transation of the American Mathematical Society, 240, (1978), 129-143.

[17] C. C. Travis and G. F. Webb, Partial differential equations with deviating arguments in the time variable, Journal of Mathematical Analysis and Applications, 56, (1976), 397-409.

[18] J. Wu, Theory and Applications for Partial Functional Differential Function, Applied Mathematical Sciences, Springer-Verlag, Vol. 119, 1996.

(Received December 2, 2011)