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NONEXISTENCE OF POSITIVE SOLUTIONS OF NONLINEAR BOUNDARY VALUE PROBLEMS

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Abstract. We discuss the nonexistence of positive solutions for nonlinear boundary value problems. In particular, we discuss necessary restrictions on parameters in nonlocal problems in order that (strictly) positive solutions exist. We consider cases that can be written in an equivalent integral equation form which covers a wide range of problems. In contrast to previous work, we do not use concavity arguments, instead we use positivity properties of an associated linear operator which uses ideas related to the u_0-positive operators of Krasnosel'skiĭ.

1. Introduction

In recent years there has been much interest in the existence of positive solutions of nonlinear boundary value problems, with a positive nonlinearity f, where the boundary conditions (BCs) can be of local or nonlocal type. A typical second order local problem is

$$-u''(t) = f(t, u(t)),\ t \in (0, 1),\ \ u(0) = 0,\ u(1) = 0, \tag{1.1}$$

but one can consider more general equations such as $-(p(t)u'(t))' + q(t)u(t) = f(t, u(t))$, or more general separated BCs $au(0) - bu'(0) = 0,\ cu(1) + du'(1) = 0$, where a, b, c, d are non-negative and $ac + ad + bc > 0$. A typical fourth order local problem is

$$-u^{(4)}(t) = f(t, u(t)),\ t \in (0, 1),\ \ u(0) = 0,\ u''(0) = 0,\ u(1) = 0,\ u''(1) = 0, \tag{1.2}$$

which can arise from the model of an elastic beam with simply supported ends. The corresponding nonlocal problems are

$$-u''(t) = f(t, u(t)),\ t \in (0, 1),\ \ u(0) = \beta_1[u],\ u(1) = \beta_2[u], \tag{1.3}$$

and

$$\begin{gathered} -u^{(4)}(t) = f(t, u(t)),\ t \in (0, 1), \\ u(0) = \beta_1[u], u''(0) + \beta_2[u] = 0,\ u(1) = \beta_3[u],\ u''(1) + \beta_4[u] = 0, \end{gathered} \tag{1.4}$$

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where a general situation is obtained by taking $\beta_j[u]$ to be positive linear functionals on $C[0,1]$, that is, to be given by Riemann-Stieltjes integrals

$$\beta_j[u] = \int_0^1 u(t)\, dB_j(t), \tag{1.5}$$

where B_j are nondecreasing functions. These nonlocal BCs can be interpreted as feedback controls, see for example [7, 37]. Some of the β_j can be zero, while others are not, so this covers many BCs. A typical example of such a functional is

$$\beta[u] = \sum_{i=1}^{p} \beta_i u(\eta_i) + \int_0^1 b(t)u(t)\, dt, \tag{1.6}$$

where $\eta_i \in (0,1)$, $\beta_i \geq 0$, and $b \in L^1$ with $b \geq 0$; $p = \infty$ is allowed if the series is absolutely convergent. Thus, the very well studied multipoint BCs and integral BCs can be studied in a single framework. Problems with multipoint and with integral BCs have been studied using many types of fixed point theory, particularly Krasnosel'skiĭ's theorem, Leggett-Williams theorem, and fixed point index theory.

Non-resonant cases for Riemann-Stieltjes BCs have been studied in [10] and with a unified theory in [35, 36] using the theory of fixed point index. Some resonant cases are also studied using similar ideas in [39, 40]. It is also possible to discuss existence of positive solutions when $\beta_j[u]$ have some positivity properties but are not necessarily positive for all positive u. This was first observed for some multipoint problems in [6] and then shown for the general case of Riemann-Stieltjes BCs with sign changing Stieltjes measures (that is B_j are functions of bounded variation) in [34, 35, 36].

In this paper we consider only the case of positive functionals and are interested in determining the conditions on the nonlocal terms under which positive solutions do not exist for any $f \geq 0$, corresponding to conditions on the coefficients β_i and the function b in (1.6). This gives the conditions that must be imposed in order to discuss existence of positive solutions. In most previous work these conditions have been determined by the restrictions required in showing, by a direct construction, that the Green's function for the problem exists and that it is non-negative, for example [19, 21, 34]. Our method does not depend on constructing the Green's function for the nonlocal problem but considers the nonlocal problem as a perturbation from the local problem when it is known that the Green's function for the local problem is non-negative. When we have m boundary terms of nonlocal type we can then write the necessary condition succinctly in terms of the spectral radius of an $m \times m$ matrix.

Many papers have given nonexistence results, we mention only a few, for example [2, 41] have used inequalities of the type we use but not with the optimal constants.

Some papers prove another kind of nonexistence result if some parameter multiplying the nonlinearity f is sufficiently large (or sufficiently small), see for example [3, 5].

Some previous works that give necessary conditions on parameters for the existence of positive solutions in some multipoint problems have used arguments involving concavity of solutions. For example, for the so-called "three-point" problem

$$u'' + a(t)f(u(t)) = 0,\ u(0) = 0, u(1) = \alpha u(\eta),\ \eta \in (0, 1),$$

it was shown by Ma [20], by a concavity argument, that if $a \geq 0$ and $f(u) \geq 0$ for $u \geq 0$, then no positive solution can exist if $\alpha\eta > 1$. Similarly for the four-point problem with $a \geq 0$ and $f(u) \geq 0$ for $u \geq 0$,

$$u'' + a(t)f(u(t)) = 0,\ u(0) = \alpha u(\xi), u(1) = \beta u(\eta),\ 0 < \xi, \eta < 1,$$

it was shown by Liu [18], again with concavity arguments, that no positive solution can exist if $\alpha(1 - \xi) > 1$ or if $\beta\eta > 1$. For this problem it was shown in [14] that also there can be no positive solution if $\alpha\xi(1 - \beta) + (1 - \alpha)(1 - \beta\eta) > 0$, using concavity once more.

There are also other kinds of non-existence results, for example [22] discusses some periodic BCs with sign-changing Green's function. A recent paper [9] discusses some nonexistence results for some second order equations with several different three-point BCs. When the form is $u'' + q(t)f(u(t)) = 0$, one of the results of [9] shows that no solution exists satisfying an inequality of the type $f(\|u\|) < c\|u\|$, c is a constant depending on the data of the problem. These are of a different type to our results which either assume only $f(u) \geq 0$ and discuss the allowable data (parameters), or discuss nonexistence of positive solutions for a given nonlinearity f using sharp pointwise inequalities of the type $f(u) \leq cu$ or $f(u) \geq cu$, where c is related to the spectral radius of the associated linear operator.

In the present paper we will consider a general case which covers equations of an arbitrary order with local and nonlocal BCs. We make use of the set-up developed in [36]. In particular we will deduce the above mentioned results of [9, 18, 20] without using concavity arguments. We utilise positivity properties of an associated linear operator, which properties are closely related to the u_0-positivity property studied in detail by Krasnosel'skiĭ [12], with a modification introduced and studied in some recent papers by the author [31, 32]. Hence our results can be applied to more general equations as well as more general BCs.

Since our discussion uses an integral equation set-up, our results apply not only to standard types of differential equations of an arbitrary integer order but also to many fractional differential equations which have a similar integral equation version. As we have not searched the literature on fractional problems we have not given references to the vast amount of work on that topic.

This methodology can also be used together with the theory of fixed point index in the discussion of existence results, and when combined with non-existence results shows that some hypotheses are sharp, see for example [31, 32], but we do not discuss existence results in this paper.

This work is partly a review of known results which can be found in several different papers of the author. We give here some more precise versions using a single method, in particular we give explicit conditions needed for a nonlocal problem of arbitrary order with two nonlocal BCs. We illustrate the general results with some new examples for second order equations with two nonlocal BCs and for a fourth order problem with four nonlocal BCs.

2. Preliminaries

We review the set-up that occurs frequently in the study of positive solutions of boundary value problems (BVPs) for ordinary differential equations, for example,

$$u''(t) + f(t, u(t)) = 0, \text{ or } u^{(4)}(t) = f(t, u(t)), \; t \in (0, 1),$$

or more complicated ones, with various kinds of boundary conditions (BCs) of local or nonlocal type, see for example, [36, 37]. It is supposed that the local BVP is not at resonance and the local problem has a non-negative Green's function.

A subset K of a Banach space X is called a cone if K is closed and $x, y \in K$ and $\alpha \geq 0$ imply that $x + y \in K$ and $\alpha x \in K$, and $K \cap (-K) = \{0\}$. We always suppose that $K \neq \{0\}$. A cone defines a partial order by $x \preceq_K y \iff y - x \in K$. A cone is said to be *reproducing* if $X = K - K$ and to be *total* if $X = \overline{K - K}$.

In the space $C[0, 1]$ of real-valued continuous functions on $[0, 1]$, endowed with the usual supremum norm, $\|u\| := \sup\{|u(t)| : t \in [0, 1]\}$, the standard cone of non-negative functions $P := \{u \in C[0, 1] : u(t) \geq 0, \; t \in [0, 1]\}$ is well known (write $u = u^+ - u^-$) to be reproducing.

Studying positive solutions of a non-resonant BVP can often be done by finding fixed points, in some sub-cone K of the cone P, of the nonlinear integral operator

$$Nu(t) = \int_0^1 G(t, s) f(s, u(s))\, ds. \tag{2.1}$$

If the nonlinearity is of the more complicated form $g(t)f(t, u)$ with a possibly singular term g (usually integrable), then we may replace the kernel (Green's function) $G(t, s)$ by $\widetilde{G}(t, s) = G(t, s)g(s)$, so in the theory we only need to consider the form (2.1) with sufficiently general hypotheses on G.

Under mild conditions this defines a compact map N in the space $C[0,1]$ and, when $G \geq 0$ and $f \geq 0$, the theory of fixed point index can often be applied to prove existence of multiple fixed points of N in a sub-cone of P, that is positive solutions of the BVP.

The rather weak conditions that we now impose on G, f are similar to ones in the papers [35, 36, 38].

(C_1) The kernel $G \geq 0$ is measurable, and for every $\tau \in [0,1]$ we have

$$\lim_{t\to\tau} |G(t,s) - G(\tau,s)| = 0 \text{ for almost every (a.e.) } s \in [0,1].$$

(C_2) There exist a non-negative function $\Phi \in L^1$ with $\Phi(s) > 0$ for a.e. $s \in (0,1)$, and $c \in P \setminus \{0\}$ such that

$$c(t)\Phi(s) \leq G(t,s) \leq \Phi(s), \text{ for } 0 \leq t, s \leq 1. \tag{2.2}$$

For a subinterval $J = [t_0, t_1]$ of $[0,1]$ let $c_J := \min\{c(t) : t \in J\}$; since $c \in P \setminus \{0\}$, there exist intervals J with $c_J > 0$.

(C_3) The nonlinearity $f : [0,1] \times [0,\infty) \to [0,\infty)$ satisfies Carathéodory conditions, that is, $f(\cdot, u)$ is measurable for each fixed $u \geq 0$ and $f(t,\cdot)$ is continuous for a. e. $t \in [0,1]$, and for each $r > 0$, there exists ϕ^r such that

$$f(t,u) \leq \phi^r(t) \text{ for all } u \in [0,r] \text{ and a. e. } t \in [0,1], \text{ where } \Phi\phi^r \in L^1.$$

Clearly, $(C_1), (C_2)$ are satisfied if $G(t,s) = \hat{G}(t,s)g(s)$ where $\hat{G}$ is continuous and $g \in L^1$ with suitable positivity properties. A precursor of condition (C_2) was used in [17]. The condition (C_2) is frequently satisfied by ordinary differential equations with both local and nonlocal boundary conditions, see, for example, [36] for a quite general situation.

For a subinterval $J = [t_0, t_1] \subseteq [0,1]$ such that $c_J := \min\{c(t) : t \in J\} > 0$, we define cones K_c, K_J by

$$K_c := \{u \in P : u(t) \geq c(t)\|u\|,\ t \in [0,1]\}, \tag{2.3}$$

$$K_J := \{u \in P : u(t) \geq c_J\|u\|,\ t \in J\}. \tag{2.4}$$

It is clear that $K_c \subset K_J$. When we consider the cone K_J we will always suppose that $c_J > 0$. These cones, especially the second, have been studied by many authors in the study of existence of multiple positive solutions of boundary value problems. We mention only a few such contributions, for the first cone see, for example, [15, 16], for the second see [4, 35, 36, 38].

These cones fit the hypotheses $(C_1), (C_2)$, in fact, under those conditions both N and the associated linear operator L defined by $Lu(t) = \int_0^1 G(t,s)u(s)\,ds$ map P into K_c, the routine arguments have been given many times, see, for example, [17, 36, 32].

Consider the example the BVP $u^{(4)} = g(t)f(t, u(t))$ with BCs

$$u(0) = \beta_1[u],\ u''(0) + \beta_2[u] = 0,\ u(1) = \beta_3[u],\ u''(1) + \beta_4[u] = 0. \tag{2.5}$$

Let γ_j be the solution of $\gamma_j^{(4)} = 0$ with modified BCs (2.5) where $\beta_j[u]$ is replaced by 1 and $\beta_i[u]$ for $i \neq j$ is replaced by 0; thus $\gamma_1(0) = 1,\ \gamma_1''(0) = 0,\ \gamma_1(1) = 0,\ \gamma_1''(1) = 0$ and $\gamma_2, \gamma_3, \gamma_4$ are defined analogously. Then γ_i can be found explicitly and are positive on $(0, 1)$; for a similar problem see Example 5.5 below.

If u satisfies $u(t) = \sum_{i=1}^4 \beta_i[u]\gamma_i(t) + N_0u(t)$ then u is a solution of the BVP, where $N_0u(t) = \int_0^1 G_0(t, s)g(s)f(s, u(s))\,ds$ corresponds to the local problem (when all $\beta_i[u]$ are identically 0).

In general we study positive fixed points of the integral operator

$$Nu(t) = Bu(t) + N_0u(t) := \sum_{i=1}^{m} \beta_i[u]\gamma_i(t) + \int_0^1 G_0(t, s)f(s, u(s))\,ds \tag{2.6}$$

where we shall suppose that G_0, f satisfy the hypotheses (C_1)-(C_3) above with functions c_0, Φ_0 in (C_2). The terms $\beta_i[u]$ are positive bounded linear functionals on $C[0, 1]$, thus given by Riemann-Stieltjes integrals as in (1.5). Here m may be any number between 0 and the order of the underlying differential equation, that is, if some term $\beta_i[u]$ is identically zero it can, and should, be excluded from the calculations.

It is well known, using the Arzelà-Ascoli theorem, that N_0 is a compact (completely continuous) operator in $C[0, 1]$, see for example Proposition 3.1 of Chapter 5 of [24]; B has finite rank and so is compact, hence N is compact.

In this paper we only consider positive linear functionals β_i and impose the following assumptions on the 'boundary terms'.

- (C_4) For each i, B_i is a non-decreasing function and $\mathcal{G}_i(s) \geq 0$ for a. e. $s \in [0, 1]$, where $\mathcal{G}_i(s) := \int_0^1 G_0(t, s)\,dB_i(t)$. Note that $\mathcal{G}_i(s)$ exists for a. e. s by (C_1).
- (C_5) The functions γ_i are continuous non-negative functions, positive on $(0, 1)$ and are linearly independent, that is, $\sum_{i=1}^m a_i\gamma_i(t) \equiv 0$ implies that $a_i = 0$ for every i; hence there exist positive functions c_i, $i = 1, \ldots, m$, such that $\gamma_i(t) \geq c_i(t)\|\gamma_i\|$ namely $c_i(t) = \gamma_i(t)/\|\gamma_i\|$.

Let $[B]$ denote the $m \times m$ matrix whose (i, j)-th entry is $\beta_i[\gamma_j]$; then $[B]$ is non-negative, that is, it has non-negative entries. It is shown in [36] that the operator B and the matrix $[B]$ are closely related, for example B and $[B]$ have equal spectral radii, $r(B) = r([B])$, in particular $r(B)$ can be calculated.

Starting with the form (2.6), it is shown in [36] that if $r(B) < 1$ ($r(B) = 1$ is the resonant case), then the Green's function exists, that is $Nu(t) = \int_0^1 G(t, s)f(s, u(s)\,ds$. Using some vector notation, writing $\langle\beta, \gamma\rangle := \sum_{i=1}^m \beta_i\gamma_i$ for the inner product in $\mathbb{R}^m$,

G can be written

$$G(t,s) := \langle (I-[B])^{-1}\mathcal{G}(s), \gamma(t)\rangle + G_0(t,s), \tag{2.7}$$

where $\mathcal{G}(s)$, $\gamma(t)$ denote vector functions with components $\mathcal{G}_i(s)$ and $\gamma_i(t)$, respectively. Moreover, the conditions $(C_1)-(C_2)$ are valid for the new Green's function with explicit modified functions c and Φ, where $c(t) = \min\{c_i(t), i = 0, \cdots, m\}$ and N maps P into K_c.

It is possible to discuss existence using either (2.6) or (2.7): see [8] for an example of the first approach and [36] for the second approach.

It was shown in [36] that if $f \geq 0$ then positive solutions do not exist if B satisfies a positivity assumption, called u_0-positive (see below), and also $r(B) > 1$. Hence $r(B) < 1$ is required in order to find positive solutions in the non-resonant case. We will extend this result slightly in the present paper using the notion of a linear operator being u_0-positive relative to two cones as introduced by this author in [31] and further studied in [32]. We also give illustrative examples. Using the same ideas we also give nonexistence results when the nonlinearity satisfies conditions of the type $f(t,u) \geq au$ or $f(t,u) \leq bu$ for all $u \geq 0$, in one case the u_0-positivity condition is not needed.

3. The u_0-positivity property

A useful concept due to Krasnosel'skiĭ, [11, 12, 13] is that of a u_0-positive linear operator on a cone.

In a recent paper [31], we gave a modification of this definition. We suppose that we have two cones in a Banach space X, $K_0 \subset K_1$ and we let $\preceq$ denote the partial order defined by the larger cone K_1, that is, $x \preceq y \iff y - x \in K_1$. We say that L is *positive* if $L(K_1) \subset K_1$,

Our modified definition reads as follows.

Definition 3.1. Let $K_0 \subset K_1$ be cones as above. A positive bounded linear operator $L : X \to X$ is said to be *u_0-positive relative to the cones* (K_0, K_1), if there exists $u_0 \in K_1 \setminus \{0\}$, such that for every $u \in K_0 \setminus \{0\}$ there are constants $k_2(u) \geq k_1(u) > 0$ such that

$$k_1(u)u_0 \preceq Lu \preceq k_2(u)u_0.$$

When $K_0 = K_1$ we recover the original definition in [11, 13]. This is stronger than requiring that L is positive and is satisfied if L is u_0-positive on K_1 according to the original definition.

The idea behind our modified definition is that we wish to exploit the extra properties satisfied by elements of the smaller cone K_0 but only use the weaker K_1-ordering.

In the recent paper [31], we proved a comparison theorem which is similar to one given by Keener and Travis [11], which was itself a sharpening of some results of Krasnosel'skiĭ [12], § 2.5.5. Some applications of the Keener-Travis theorem to some nonlinear problems were given in [29, 30].

Theorem 3.2 ([31])**.** *Let $K_0 \subset K_1$ be cones in a Banach space X, and let $\preceq$ denote the partial order of K_1. Suppose that L_1, L_2 are bounded linear operators and that at least one is u_0-positive relative to (K_0, K_1). If there exist*

$$\begin{aligned} &u_1 \in K_0 \setminus \{0\},\ \lambda_1 > 0,\ \text{ such that }\ \lambda_1 u_1 \preceq L_1 u_1, \quad \text{ and} \\ &u_2 \in K_0 \setminus \{0\},\ \lambda_2 > 0,\ \text{ such that }\ \lambda_2 u_2 \succeq L_2 u_2, \end{aligned} \tag{3.1}$$

and $L_1 u_j \preceq L_2 u_j$ for $j = 1, 2$, then $\lambda_1 \le \lambda_2$. If, in addition, $L_j(K_1 \setminus \{0\}) \subset K_0 \setminus \{0\}$ and if $\lambda_1 = \lambda_2$ in (3.1)*, then it follows that u_1 is a (positive) scalar multiple of u_2.*

This is most often applied when there is only one linear operator L and one of u_j is an eigenfunction of L corresponding to a positive eigenvalue λ_j.

There is a simple known result, which has been rediscovered many times, but we do not know the original source. It gives a comparison result in one direction and requires no u_0-positivity hypotheses on L and no restriction on K. For completeness we include the simple proof. The spectral radius of a linear operator L is denoted $r(L)$.

Theorem 3.3. *Let L be a bounded linear operator in a Banach space X and let K be a cone in X. Suppose that $L(K) \subset K$ and there exist $\lambda_0 > 0$ and $v \in K \setminus \{0\}$ such that $Lv \succeq_K \lambda_0 v$. Then it follows that $r(L) \ge \lambda_0$.*

Proof. If not, we have $0 \le r(L) < \lambda_0$. Hence L/λ_0 maps K into K and $r(L/\lambda_0) < 1$. As is well known, from the Neumann series, $(I - L/\lambda_0)^{-1}$ then maps K into K. We have $L(v/\lambda_0) \succeq_K v$ that is $(I - L/\lambda_0)(-v) \in K$, hence $-v \in K$ so that $v = 0$. This contradiction shows that $r(L) \ge \lambda_0$. □

Remark 3.4. Theorem 3.3 does not prove that L has an eigenvalue $\lambda \ge \lambda_0$ with eigenfunction in K; in fact simple examples show that there need be no such eigenvalue (see, for example, [1, 32]). If L is compact (also termed completely continuous) then L does have such an eigenvalue as shown long ago by Krasnosel'skiĭ [12]. If K is a total cone, it then follows by the Kreĭn-Rutman theorem that the spectral radius $r(L)$ is an eigenvalue of L with eigenfunction in K. When, in addition, L is u_0-positive relative to (K_0, K_1) and $r(L)$ is an eigenvalue of L with eigenvector in K_0, the result of Theorem 3.3 is a consequence of Theorem 3.2, and then also $r(L)$ is the unique positive eigenvalue with eigenfunction in K, see [12, 31]. Nussbaum [28] has given an extension of the Kreĭn-Rutman theorem where compactness is replaced by $r_{\text{ess}}(L) < r(L)$, where $r_{\text{ess}}(L)$ denotes the essential spectral radius of L. Extensions of Krasnosel'skiĭ's result have been given for condensing operators in [1] and for some nonlinear 1-homogeneous operators in [28]; a new short proof for linear condensing operators using fixed point index theory is given in [32].

There is no similar result in the other direction, that is, if L is a positive linear operator and

$$\text{there exist } \lambda_0 > 0 \text{ and } v \in K \setminus \{0\} \text{ such that } Lv \preceq_K \lambda_0 v, \tag{3.2}$$

then it cannot be inferred that $r(L) \leq \lambda_0$, without some extra condition. A simple example in $\mathbb{R}^2$ with cone $K = \{(x, y) : x \geq 0, y \geq 0\}$ is

$$L(x, y) := (2x, x + y). \tag{3.3}$$

Then $L(0, 1) = (0, 1)$ so (3.2) holds with $\lambda_0 = 1$ but $r(L) = 2$ and is an eigenvalue. The example also shows that compactness is not a sufficient extra condition.

We now give a new result that gives a positive inference under some compactness and u_0-positivity assumptions.

Definition 3.5. Let X be a Banach space and let K_0, K_1 be cones in X with $K_0 \subset K_1$. We say that a linear operator L_1 is a *minorant* of L if $L_1 u \preceq Lu$ (the ordering of K_1) for all $u \in K_1$.

Theorem 3.6. *Let L be a compact linear operator with $L(K_1) \subset K_1$ and suppose there exist bounded linear minorants L_n with $L_n \to L$ in the operator norm where each L_n is u_n-positive relative to (K_0, K_1). Assume that $r(L_n)$ is an eigenvalue of L_n with eigenfunction $\varphi_n \in K_0$. If there exist $\lambda_0 > 0$ and $v \in K_0 \setminus \{0\}$ such that $Lv \preceq \lambda_0 v$. then it follows that $r(L) \leq \lambda_0$.*

Proof. We may suppose that $r(L) > 0$. We have $L_n \varphi_n = r(L_n)\varphi_n$ and $Lv \preceq \lambda_0 v$. As L_n is u_n-positive relative to (K_0, K_1), the comparison theorem, Theorem 3.2, gives $r(L_n) \leq \lambda_0$ for each n. By Lemma 2 of Nussbaum [26], $r(L_n) \to r(L)$ and therefore $r(L) \leq \lambda_0$. □

Remark 3.7. (1) The hypotheses hold taking $L_n = L$ if L is u_0-positive relative to (K_0, K_1) and $r(L)$ is an eigenvalue of L with eigenfunction in K_0, for example if the cone K_1 is total, and $L(K_1) \subset K_0$.
(2) The same proof shows that the result holds if instead of compactness of L it is assumed that $r_{\text{ess}}(L) < r(L)$, where $r_{\text{ess}}(L)$ denotes the essential spectral radius of L, since in a personal communication to this author in 2006, Professor R.D. Nussbaum remarked that the proof in [26] actually shows that if L_n is a sequence of bounded linear operators on a Banach space and $L_n \to L$ in the operator norm and $r_{\text{ess}}(L) < r(L)$, then $r(L_n) \to r(L)$ as $n \to \infty$. Although there are several inequivalent definitions of

'essential spectrum', see [23], it was shown in [25] that the radius is the same whatever definition is employed.

The reason behind these assumptions is that they fit naturally into our set-up. In fact, for $X = C[0,1]$, when $Lu(t) = \int_0^1 G(t,s)u(s)\,ds$ and the conditions $(C_1), (C_2)$ hold then defining L_n by

$$L_n u(t) = \int_{t_n}^{1-t_n} G(t,s)u(s)\,ds, \quad \text{where } \ 0 < t_n < 1/2, \tag{3.4}$$

it follows that L_n are minorants of L, and, if $t_n \to 0$, then $L_n \to L$ in the operator norm. Moreover, each L_n is u_n-positive relative to (K_c, P) provided $c(t) > 0$ for $t \in (0,1)$. This last fact was essentially first proved in [31] with a small refinement in [32]. For completeness we include the short proof here.

Theorem 3.8. *Let G satisfy $(C_1)-(C_2)$ and let $J = [t_0, t_1]$ and $c_J = \min\{c(t) : t \in J\}$ and suppose $c_J > 0$. Let L_J be defined on $C[0,1]$ by $L_J u(t) = \int_{t_0}^{t_1} G(t,s)u(s)\,ds$. Then L_J is u_0-positive relative to (K_c, P) for $u_0(t) := \int_{t_0}^{t_1} G(t,s)\,ds$. Furthermore $r(L_J) > 0$ and so $r(L_J)$ is an eigenvalue of L_J with eigenfunction in K_c by the Kreĭn-Rutman theorem.*

Proof. Let $u \in K_c \setminus \{0\}$. Then we have

$$L_J u(t) = \int_{t_0}^{t_1} G(t,s)u(s)\,ds \le \Big(\int_{t_0}^{t_1} G(t,s)\,ds\Big)\|u\| = \|u\|u_0(t),$$

and

$$L_J u(t) = \int_{t_0}^{t_1} G(t,s)u(s)\,ds \ge \Big(\int_{t_0}^{t_1} G(t,s)\,ds\Big)c_J\|u\| = c_J\|u\|u_0(t).$$

We note that, for $t \in J$, $u_0(t) \ge \int_{t_0}^{t_1} c_J\Phi(s)\,ds > 0$, so $u_0 \ne 0$. Also, $(C_1)-(C_2)$ imply that u_0 is continuous. Using (C_2) we have

$$L_J c(t) = \int_{t_0}^{t_1} G(t,s)c(s)\,ds \ge c(t)\int_{t_0}^{t_1} \Phi(s)c(s)\,ds,$$

that is $L_J c \succeq \lambda_0 c$ for $\lambda_0 = \int_{t_0}^{t_1} \Phi(s)c(s)\,ds > 0$. By Theorem 3.3, $r(L_J) \ge \lambda_0 > 0$. □

The result that L_J is u_0-positive relative to two cones was an important motivation for our introducing the concept in [31], since it has not been possible to prove that L itself is u_0-positive without some assumptions in addition to $(C_2)-(C_2)$. A simple additional assumption is either of the 'symmetry' assumptions $G(t,s) = G(s,t)$ or $G(t,s) = G(1-s, 1-t)$, for all t, $s \in [0,1]$, as shown in Corollary 7.5 of [36].

4. Non-existence results

We now give nonexistence results using the above ideas.

Theorem 4.1. *(i) Suppose that $0 \le f(t,u) \le au$ for almost all $t \in [0,1]$ and all $u > 0$ where $a < \mu(L) = 1/r(L)$. Then the equation $u = Nu$ has no solution in $P \setminus \{0\}$.*

(ii) Suppose that $f(t,u) \ge bu$ for almost all $t \in [0,1]$ and all $u > 0$ with $b > \mu(L)$. Then the equation $u = Nu$ has no solution in $P \setminus \{0\}$.

Proof. (i) If $u \in P \setminus \{0\}$ is a solution of $u = Nu$ then

$$u(t) = Nu(t) = \int_0^1 G(t,s) f(s,u(s))\, ds \le \int_0^1 G(t,s) a u(s)\, ds = aLu(t),$$

that is, $u \preceq aLu$, By Theorem 3.3 this implies $ar(L) \ge 1$, a contradiction. The proof of (ii) is almost identical using Theorems 3.6 and 3.8. □

A short proof of part (i) is essentially given by Nussbaum in Proposition 2 of [27] with a simple argument. A similar result is proved in [36] assuming for part (ii) that L is u_0-positive (as in [13], that is relative to (P,P)).

If L is u_0-positive relative to (K_c, P) then the hypotheses can be sharpened. The following result is essentially shown in [32], a version using the original definition of u_0-positive is in [30]. We give the proof here for completeness.

Theorem 4.2. *Let L be u_0-positive relative to (K_c, P), and suppose $r(L) > 0$.*
(i) Suppose that $0 \le f(t,u) < \mu(L)u$ for almost all $t \in [0,1]$ and all $u > 0$, where $\mu(L) = 1/r(L)$. Then the equation $u = Nu$ has no solution in $P \setminus \{0\}$.
(ii) If $f(t,u) > \mu(L)u$ for almost all $t \in [0,1]$ and all $u > 0$, then the equation $u = Nu$ has no solution in $P \setminus \{0\}$.

Proof. (i) By the Kreĭn-Rutman theorem, since P is a total cone, $r(L)$ is an eigenvalue of L with eigenfunction $\varphi \in P$, and since $L(P) \subset K_c$, it follows that $\varphi \in K_c$. If $u = Nu$ for some $u \in P \setminus \{0\}$ we then have

$$u = Nu \preceq \mu(L)Lu, \quad \text{thus} \quad r(L)u \preceq Lu, \quad \text{and} \quad r(L)\varphi = L\varphi.$$

Since N maps P into K_c, we have $u \in K_c$. By the comparison theorem, Theorem 3.2, u is a positive scalar multiple of φ and thus $Lu = r(L)u$. We therefore have $u = Nu = \mu(L)Lu$. However, this is impossible since $u \in K_c \setminus \{0\}$ implies $u(s) > 0$ for s on some sub-interval of $(0,1)$ and, for those $t \in (0,1)$ for which $c(t) > 0$, we have $G(t,s) \ge c(t)\Phi(s) > 0$ for a.e. $s \in (0,1)$ and hence

$$Nu(t) = \int_0^1 G(t,s) f(s,u(s))\, ds < \mu(L) \int_0^1 G(t,s) u(s)\, ds.$$

The proof of (ii) is almost identical and so is omitted. □

We now discuss positive solutions of nonlocal BVPs which we consider as positive fixed points of N where

$$Nu(t) = Bu(t) + N_0u(t) := \sum_{i=1}^{m} \beta_i[u]\gamma_i(t) + \int_0^1 G_0(t,s)f(s,u(s))\,ds.$$

Our aim is to find necessary conditions on B in order that positive solutions can exist.

Theorem 4.3. *Let B be u_0-positive relative to (K_c, P).*
(a) If $r(B) > 1$ and $f(t,u) \geq 0$ for all $u \geq 0$ and a.e. $t \in [0,1]$, or if
(b) (the resonance case) $r(B) = 1$ and $f(t,u) > 0$ for $u > 0$ and a.e. $t \in [0,1]$,
then the nonlocal BVP

$$u(t) = Bu(t) + N_0u(t) = \langle \beta[u], \gamma(t)\rangle + \int_0^1 G(t,s)f(s,u(s))\,ds \tag{4.1}$$

has no nonzero solution in K_c.

Proof. If $u \in K_c$ is a solution then $u = Bu + N_0u \succeq Bu$ so, by Theorem 3.6, $r(B) \leq 1$. When $r(B) = 1$ the comparison theorem Theorem 3.2 gives u must be a multiple of the normalised eigenfunction φ of B corresponding to the eigenvalue $r(B) = 1$. Thus we have $u = Bu$, hence, from $u = Bu + N_0u$, we must have $N_0u = 0$, therefore $u = 0$. □

Thus, if we want to consider an existence result for positive solutions when $f(t,u) > 0$ for $u > 0$, it is necessary to assume that $r(B) < 1$. If $r(B) = 1$ it is known that it is usually necessary to have f changing sign for positive solutions to exist. Positive solutions can exist in some special cases when $f \geq 0$. For some simple necessary and sufficient conditions in some such cases see [33].

A natural question is to determine when B is u_0-positive. One simple answer is the following easily checked criterion, which is an important reason why we only consider positive functionals β_i in this paper.

Theorem 4.4. *Let $\beta_i[c] > 0$ for each $i = 1, \ldots, m$. Then B is u_0-positive relative to (K_c, P) for $u_0 = \sum_{i=1}^m \gamma_i$.*

Proof. For $u \in K_c \setminus \{0\}$, $c(t)\|u\| \leq u(t) \leq \|u\|$ so $\beta_i[c]\|u\| \leq \beta_i[u] \leq \beta_i[\hat{1}]\|u\|$, where $\hat{1}$ denotes the constant function with value 1. Thus we have

$$\min_{i=1,\cdots,m} \beta_i[c]\|u\| \sum_{i=1}^{m} \gamma_i \leq Bu \leq \max_{i=1,\cdots,m} \beta_i[\hat{1}]\|u\| \sum_{i=1}^{m} \gamma_i.$$

□

When in the theory we choose $c = \min\{c_0, c_1, \ldots, c_m\}$, as is usual, since $c_i(t) = \gamma_i(t)/\|\gamma\|$, $i = 1, \ldots, m$, this criterion means that the matrix $[B]$, whose (i,j)-th entry is $\beta_i[\gamma_j]$, has positive entries.

Firstly we see what the non-existence criterion of Theorem 4.3 means for problems with only one nonlocal term; we obtain an easily checked explicit condition. The nonlinear map N can be written

$$Nu(t) = \beta[u]\gamma(t) + N_0 u(t)$$

and the condition is simply $0 \le \beta[\gamma] < 1$. For example, for the fourth order problem

$$u^{(4)}t) = f(t, u(t)),\ u(0) = 0, u''(0) = 0, u(1) = \beta[u], u''(1) = 0,$$

where $\beta[u] = \int_0^1 u(t)dB(t)$, it is easily checked that $\gamma(t) = t$ so the condition is $\int_0^1 t dB(t) < 1$. Similarly for the fourth order problem

$$u^{(4)}t) = f(t, u(t)),\ u(0) = 0, u''(0) = 0, u(1) = 0, u''(1) + \beta[u] = 0,$$

it is easily checked that $\gamma(t) = (t - t^3)/6$ so the condition is $\int_0^1 (t - t^3)dB(t) < 6$.

For the case of two nonlocal BCs we will see that, using some elementary results concerning non-negative matrices, it is possible to determine explicit criteria for the non-existence of positive solutions without calculating eigenvalues to find $r([B])$ (though, of course, that can be done).

The following simple result is known; for completeness we include a short proof. We write det to denote the determinant of a matrix.

Lemma 4.5. *For an $m \times m$ non-negative matrix $[B]$*

$$r([B]) < 1 \implies \det(I - [B]) > 0.$$

The converse is false.

Proof. For each $t \in [0, 1]$, $r([B]) < 1$ implies that $r(t[B]) < 1$. Thus $I - t[B]$ is invertible so $\det(I - t[B]) \neq 0$ for all $t \in [0, 1]$. Since $\det(I - t[B])$ is a polynomial in t and $\det(I) = 1$, we have $\det(I - t[B]) > 0$ for each $t \in [0, 1]$, in particular, $\det(I - [B]) > 0$. There are many non-negative matrices $[B]$ where $\det(I - [B]) > 0$ but $r([B]) > 1$, one simple example is $\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$. □

When $[B]$ is a non-negative 2×2 matrix we give a necessary and sufficient condition.

Theorem 4.6. *Let $[B] = (b_{ij})$ be a non-negative 2×2 matrix. Then we have*

$$r([B]) < 1 \iff b_{11} < 1,\ b_{22} < 1,\ \det(I - [B]) > 0.$$

Proof. Suppose that $r([B]) < 1$, then $\det(I - [B]) > 0$, that is

$$(1 - b_{11})(1 - b_{22}) - b_{12}b_{21} > 0. \tag{4.2}$$

The following inequalities are well-known for the non-negative matrix $[B]$:

$$\min\{b_{11}+b_{12}, b_{21}+b_{22}\} \le r([B]) \le \max\{b_{11}+b_{12}, b_{21}+b_{22}\}. \tag{4.3}$$

Hence we cannot have both $b_{11} \ge 1$ and $b_{22} \ge 1$ since this would imply $r([B]) \ge 1$. Therefore, from (4.2), it follows that $1 - b_{11} > 0$ and $1 - b_{22} > 0$.

For the converse, now suppose that $b_{11} < 1$, $b_{22} < 1$ and $\det(I-[B]) > 0$. We assume that $r := r([B]) > 0$ else the result is trivial. By the Perron-Frobenius theorem, r is an eigenvalue of $[B]$ and the second eigenvalue, λ (say), is real and satisfies $|\lambda| \le r$. Since $\lambda + r = \mathrm{tr}([B])$, the trace of $[B]$, and $\lambda r = \det([B])$, the inequality $\det(I - [B]) > 0$ can be written

$$1 - \mathrm{tr}([B] + \det([B]) > 0, \quad \text{equivalently,} \quad (1-\lambda)(1-r) > 0.$$

Therefore, either both $\lambda > 1$ and $r > 1$, or else both $\lambda < 1$ and $r < 1$. Since $\mathrm{tr}([B]) = b_{11} + b_{22} < 2$, the second alternative must hold, thus $r([B]) < 1$. □

Very similar arguments show the following for the resonance case.

Theorem 4.7. *Let $[B] = (b_{ij})$ be a 2×2 non-negative matrix. Then we have*

$$r([B]) = 1 \quad \Longleftrightarrow \quad b_{11} \le 1,\ b_{22} \le 1,\ \det(I - [B]) = 0.$$

Proof. Suppose that $r([B]) = 1$, then 1 is an eigenvalue of $[B]$ and $\det(I - [B]) = 0$, that is

$$(1 - b_{11})(1 - b_{22}) - b_{12}b_{21} = 0. \tag{4.4}$$

As previously, using the inequality (4.3), we must have $b_{11} \le 1$ and $b_{22} \le 1$. For the converse, $\det(I - [B]) = 0$ implies that 1 is an eigenvalue of $[B]$ and, writing λ for the second eigenvalue, we have $0 \le \mathrm{tr}([B]) = 1 + \lambda \le 2$. Hence $-1 \le \lambda \le 1$ so $r([B]) = 1$. □

Theorem 4.8. *For a boundary value problem with two nonlocal BCs involving positive linear functionals $\beta_i[u]$, with $\beta_i(c) > 0$, let $Bu(t) = \sum_{i=1}^{2} \beta_i[u]\gamma_i(t)$. For a positive solution of $u = Bu + N_0u(t)$ to exist when $f(t,u) > 0$ for $u > 0$, it is necessary that*

$$\begin{aligned} &0 \le \beta_1[\gamma_1] < 1, \quad 0 \le \beta_2[\gamma_2] < 1, \quad \textit{and} \\ &(1 - \beta_1[\gamma_1])(1 - \beta_2[\gamma_2]) - \beta_1[\gamma_2]\beta_2[\gamma_1] > 0. \end{aligned}$$

The third condition is $\det(I - [B]) > 0$ *where*

$$[B] = \begin{bmatrix} \beta_1[\gamma_1] & \beta_1[\gamma_2] \\ \beta_2[\gamma_1] & \beta_2[\gamma_2] \end{bmatrix}.$$

Proof. By Theorem 4.4, B is u_0-positive. Since we assume $f(t,u) > 0$ for $u > 0$, by Theorem 4.3 it is necessary that $r(B) < 1$. As shown in [36], $r(B) = r([B])$ where $[B]$ is the non-negative 2×2 matrix written above. The result now follows from the criteria in Theorem 4.6. □

5. Examples

We first see how our result Theorem 4.8 recovers known results. For problems with $f(t,u) > 0$ for $u > 0$, we will determine the allowable parameter region for which positive solutions may exist, equivalently, the excluded region where there can be no positive solution.

Example 5.1. Consider the four-point problem

$$u''(t) + g(t)f(t,u(t)) = 0,\ t \in (0,1),\ \ u(0) = \alpha u(\xi),\ u(1) = \beta u(\eta),$$

where $\eta, \xi \in (0,1)$, α, β are positive constants and we suppose that $f(t,u) > 0$ for $u > 0$. Then $\gamma_1(t) = 1 - t$, $\gamma_2(t) = t$, $\beta_1[u] = \alpha u(\xi)$, $\beta_2[u] = \beta u(\eta)$, and $c(t) = \min\{t, 1-t\}$. Therefore, by Theorem 4.8, the conditions are

$$\alpha(1-\xi) < 1,\ \ \beta\eta < 1,\ \ \big(1 - \alpha(1-\xi)\big)(1-\beta\eta) - \alpha\xi\beta(1-\eta) > 0,$$

which can be written

$$\alpha(1-\xi) < 1,\ \ \beta\eta < 1,\ \ \alpha\xi(1-\beta) + (1-\alpha)(1-\beta\eta) > 0.$$

It was shown in [14], by a geometrical argument using concavity ideas, that for $f \geq 0$, $\alpha\xi(1-\beta)+(1-\alpha)(1-\beta\eta) \geq 0$ is a necessary condition. It had been shown earlier in [18], again using concavity arguments, that no positive solutions exist if either $\alpha(1-\xi) > 1$ or $\beta\eta > 1$. Since we assume $f(t,u) > 0$ for $u > 0$ our result is a little more precise.

We now give a simple example with integral boundary conditions where our result can be applied but concavity arguments are not applicable.

Example 5.2. Suppose that $f(t,u) > 0$ for $u > 0$. Consider the BVP

$$-u''(t) + \omega^2 u(t) = f(t,u(t)),\ \ u(0) = \beta_1[u]\ \ u(1) = \beta_2[u],$$

where $\omega > 0$ and

$$\beta_1[u] = \beta_1 \int_0^1 u(s)\,ds,\ \ \beta_2[u] := \beta_2 \int_0^1 u(s)\,ds,\ \beta_i \text{ are positive constants.}$$

Here $\gamma_1(t) = \dfrac{\sinh(\omega(1-t))}{\sinh(\omega)}$, $\gamma_2(t) = \dfrac{\sinh(\omega t)}{\sinh(\omega)}$. Hence, the matrix $[B] = (\beta_i[\gamma_j])$ is given by

$$[B] = \begin{bmatrix} \beta_1(\cosh(\omega)-1)/(\omega\sinh(\omega)) & \beta_1(\cosh(\omega)-1)/(\omega\sinh(\omega)) \\ \beta_2(\cosh(\omega)-1)/(\omega\sinh(\omega)) & \beta_2(\cosh(\omega)-1)/(\omega\sinh(\omega)) \end{bmatrix}.$$

The conditions on the parameters for which positive solutions may exist can now be read off from Theorem 4.8 (or by finding the eigenvalues), and simplify to

$$\beta_1 + \beta_2 < \frac{\omega \sinh(\omega)}{\cosh(\omega)-1}.$$

The following example is a little more complicated and we use it to show that our method allows us to find the appropriate conditions in these cases, and also to illustrate what happens to the conditions when the problem is considered in different ways.

Example 5.3. Suppose that $f(t,u) > 0$ for $u > 0$. Consider the BVP

$$-u''(t) - \omega^2 u(t) = f(t,u(t)), \;\; u(0) = \beta_1[u] \;\; u(1) = \beta_2[u],$$

where $0 < \omega < \pi$ and

$$\beta_1[u] = \beta_1 \int_0^1 u(s)\,ds, \;\; \beta_2[u] := \beta_2 \int_0^1 s\,u(s)\,ds.$$

Using the theory we have $\gamma_1(t) = \dfrac{\sin(\omega(1-t))}{\sin(\omega)}$, $\gamma_2(t) = \dfrac{\sin(\omega t)}{\sin(\omega)}$, and γ_i are positive on $(0,1)$ since we take $\omega < \pi$. Hence, the matrix $[B] = (\beta_i[\gamma_j])$ is given by

$$[B] = \begin{bmatrix} \beta_1(1-\cos(\omega))/(\omega\sin(\omega)) & \beta_1(1-\cos(\omega))/(\omega\sin(\omega)) \\ \beta_2(\omega-\sin(\omega))/(\sin(\omega)\omega^2) & \beta_2(\sin(\omega)-\cos(\omega)\omega)/(\sin(\omega)\omega^2) \end{bmatrix}.$$

The conditions can now be read off from Theorem 4.8. For definiteness we make the simple choice $\omega = \pi/2$. The conditions are then

$$\beta_1 < \pi/2, \; \beta_2 < \pi^2/4, \text{ and } (1-2\beta_1/\pi)(1-4\beta_2/\pi^2) - 8\beta_1\beta_2(\pi/2-1)/\pi^3 > 0. \quad (5.1)$$

This determines a region in the first quadrant of the (β_1,β_2)-plane bounded by the curve determined by $(1-2\beta_1/\pi)(1-4\beta_2/\pi^2) - 8\beta_1\beta_2(\pi/2-1)/\pi^3 = 0$.

Now we look at the example in another quite natural way. It can be written

$$-u''(t) = \tilde{f}(t,u(t)) := f(t,u(t)) + \omega^2 u(t), \;\; u(0) = \beta_1[u] \;\; u(1) = \beta_2[u],$$

with

$$\beta_1[u] = \beta_1 \int_0^1 u(s)\,ds, \;\; \beta_2[u] := \beta_2 \int_0^1 s\,u(s)\,ds.$$

Considering the problem in the form

$$-u''(t) = \tilde{f}(t,u(t)), \;\; u(0) = \beta_1[u] \;\; u(1) = \beta_2[u]$$

we have $\tilde{\gamma}_1(t) = 1 - t$, $\tilde{\gamma}_2(t) = t$ and hence the matrix $[\tilde{B}] = (\beta_i[\tilde{\gamma}_j])$ is given by

$$[\tilde{B}] = \begin{bmatrix} \beta_1/2 & \beta_1/2 \\ \beta_2/6 & \beta_2/3 \end{bmatrix}$$

The conditions are now $\beta_1 < 2, \beta_2 < 3$, $(1 - \beta_1/2)(1 - \beta_2/3) - \beta_1\beta_2/12 > 0$. This determines a larger region than found in (5.1) corresponding to a smaller excluded region.

Remark 5.4. The explanation of this apparently paradoxical result is that the non-existence result, which determines the size of the excluded region in the (β_1, β_2)-plane, applies for all $f \geq 0$. When we consider $\tilde{f}$ we have the extra property that $\tilde{f}(t, u) \geq \omega^2 u$ and, by Theorem 4.1, there is a corresponding modification to Theorem 4.3 with condition of the form $r(B + \omega^2 L) > 1$, where $Lu(t) = \int_0^1 G(t, s)u(s)\, ds$, which would increase the size of the excluded region. In other words, changing the form of the equation by adding $\omega^2 u$ to both sides apparently gives a smaller excluded region, that is, a larger allowable parameter region, but, in fact, this is a false impression. Of course this shift can be useful for obtaining simpler expressions, and can also be applied when, instead of assuming $f(t, u) \geq 0$, it is assumed that $f(t, u) + \omega^2 u \geq 0$, especially in the case when the original problem is at resonance (see [39, 40]).

Example 5.5. We now give an example for a fourth order equation with four nonlocal terms, a similar example with "three-point" BCs is given in [36] to illustrate existence results. Consider the problem

$$u^{(4)}(t) = g(t)f(t, u(t)), \quad t \in (0, 1), \tag{5.2}$$

with the nonlocal BCs

$$u(0) = \beta_1[u], \; u'(0) = \beta_2[u], \; u(1) = \beta_3[u], \; u''(1) + \beta_4[u] = 0. \tag{5.3}$$

Other sets of BCs can be treated similarly. This local problem models an elastic beam with clamped end at 0 and hinged (simply supported) end at 1; the nonlocal problem can be thought of as having controllers at the endpoints responding to feedback from measurements of the displacements along parts of the beam.

In this case we have

$$\gamma_1(t) = 1 - \frac{3}{2}t^2 + \frac{1}{2}t^3,\ \gamma_2(t) = t - \frac{3}{2}t^2 + \frac{1}{2}t^3,$$
$$\gamma_3(t) = \frac{3}{2}t^2 - \frac{1}{2}t^3,\ \gamma_4(t) = \frac{1}{4}t^2(1-t).$$
$$c_1(t) = 1 - \frac{3}{2}t^2 + \frac{1}{2}t^3,\ c_2(t) = 3\sqrt{3}(t - \frac{3}{2}t^2 + \frac{1}{2}t^3),$$
$$c_3(t) = \frac{3}{2}t^2 - \frac{1}{2}t^3,\ c_4(t) = \frac{27}{4}t^2(1-t).$$

For the local problem, when all the β_i are replaced by zero, it was shown in [36] that

$$c_0(t) = \min\Big\{\frac{27}{4}t^2(1-t), \frac{3\sqrt{3}}{2}t(1-t)(2-t)\Big\}.$$

Noting that $c_0(t) = \min\{c_2(t), c_4(t)\}$, and comparing the functions $c_1, \dots, c_4$, the final answer is

$$c(t) = \min\{c_1(t), c_3(t)\} = \min\Big\{1 - \frac{3}{2}t^2 + \frac{1}{2}t^3, \frac{3}{2}t^2 - \frac{1}{2}t^3\Big\}.$$

We now assume $\beta_i[c_1]$ and $\beta_i[c_3]$ are both positive. Then the necessary condition is $r([B]) < 1$ where $[B]$ is the 4×4 matrix with (i,j) entry $\beta_i[\gamma_j]$. In general, this condition may be tricky to interpret for individual functionals, but, in any explicit example, it can easily be checked whether or not the necessary condition is satisfied.

For an explicit, but particularly simple, example, we now take

$$\beta_j[u] = b_j \int_0^1 u(s)\,ds, \quad \text{where } b_j > 0, j = 1, \cdots, 4. \tag{5.4}$$

Then by some integrations we obtain

$$[B] = \begin{bmatrix} 5b_1/8 & b_1/8 & 3b_1/8 & b_1/48 \\ 5b_2/8 & b_2/8 & 3b_2/8 & b_2/48 \\ 5b_3/8 & b_3/8 & 3b_3/8 & b_3/48 \\ 5b_4/8 & b_4/8 & 3b_4/8 & b_4/48 \end{bmatrix},$$

and $r([B]) = 5b_1/8 + b_2/8 + 3b_3/8 + b_4/48$ (note that $[B]$ has rank one). The necessary condition for existence of positive solutions is thus

$$30b_1 + 6b_2 + 18b_3 + b_4 < 48. \tag{5.5}$$

For example, no positive solution exists for $(b_1, b_2, b_3, b_4) = (1, 1/100, 1, 1/100)$. If in (5.4) some of the b_i are zero then the corresponding β_i is to be excluded from the computation and a smaller matrix should be considered.

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