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Bifurcation diagrams for singularly perturbed system

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Abstract

We consider a singularly perturbed system where the fast dynamic of the unperturbed problem exhibits a trajectory homoclinic to a critical point. We assume that the slow time system is 1-dimensional and it admits a unique critical point, which undergoes a bifurcation as a second parameter varies: transcritical, saddle-node, or pitchfork. In this setting Battelli and Palmer proved the existence of a unique trajectory $(\tilde{x}(t,\varepsilon,\lambda),\tilde{y}(t,\varepsilon,\lambda))$ homoclinic to the slow manifold. The purpose of this paper is to construct curves which divide the 2-dimensional parameters space in different areas where $(\tilde{x}(t,\varepsilon,\lambda),\tilde{y}(t,\varepsilon,\lambda))$ is either homoclinic, heteroclinic, or unbounded. We derive explicit formulas for the tangents of these curves. The results are illustrated by some examples.

Keywords. Singular perturbation, homoclinic trajectory, transcritical bifurcation saddle-node bifurcation.
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1 Introduction

In this paper we consider the following singularly perturbed system:

$$\begin{cases} \dot{x} = \varepsilon f(x,y,\varepsilon,\lambda) \\ \dot{y} = g(x,y,\varepsilon,\lambda) \end{cases} \tag{1.1}$$

where $x \in \mathbb{R}$, $y \in \mathbb{R}^n$ and $(x,y) \in \Omega$, $\Omega \subset \mathbb{R}^{1+n}$ is open, λ and ε are small real parameters and $f(x,y,\varepsilon,\lambda)$, $g(x,y,\varepsilon,\lambda)$ are C^r, bounded with their derivatives, $r \geq 3$. We suppose that the following conditions hold:

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(i) for all $x \in \mathbb{R}$, we have

$$g(x,0,0,0) = 0,$$

(ii) the infimum over $x \in \mathbb{R}$ of the moduli of the real parts of the eigenvalues of the jacobian matrix $\frac{\partial g}{\partial y}(x,0,0,0)$ is greater than a positive number Λ^g.

(iii) the equation

$$\dot{y} = g(0,y,0,0)$$

has a solution $h(t)$ homoclinic to the origin $0 \in \mathbb{R}^n$

(iv) $\dot{h}(t)$ is the unique bounded solution of the linear variational system:

$$\dot{y} = \frac{\partial g}{\partial y}(0,h(t),0,0)y \tag{1.2}$$

up to a scalar multiple.

According to condition (ii), for any $x \in \mathbb{R}$, the linear system $\dot{y} = \frac{\partial g}{\partial y}(x,0,0,0)y$ has an exponential dichotomy on $\mathbb{R}$ with exponent $\Lambda^g > 0$ and projections, say, $P^0(x)$. For simplicity we set $P^0(0) = P^0$. Let $\mathrm{rank}[P^0(x)] = p$, p being the number of eigenvalues of $\frac{\partial g}{\partial y}(x,0,0,0)$ with positive real parts: we stress that p is constant. From assumptions (ii) and (iii) it follows that the linear system (1.2) and its adjoint

$$\dot{y} = -\big[\frac{\partial g}{\partial y}(0,h(t),0,0)\big]^* y \tag{1.3}$$

have exponential dichotomies on both $\mathbb{R}_+$ and $\mathbb{R}_-$; i.e. there are projections $P^\pm$ and $k > 0$ such that

$$\begin{array}{ll} \|Y(t)P^-Y^{-1}(s)\| \le k\mathrm{e}^{-\Lambda^g(t-s)} & \text{if } s \le t \le 0 \\ \|Y(t)(\mathbf{I}-P^-)Y^{-1}(s)\| \le k\mathrm{e}^{-\Lambda^g(s-t)} & \text{if } t \le s \le 0 \\ \|Y(t)P^+Y^{-1}(s)\| \le k\mathrm{e}^{-\Lambda^g(t-s)} & \text{if } 0 \le s \le t \\ \|Y(t)(\mathbf{I}-P^+)Y^{-1}(s)\| \le k\mathrm{e}^{-\Lambda^g(s-t)} & \text{if } 0 \le t \le s \end{array} \tag{1.4}$$

where $Y(t)$ is the fundamental matrix of (1.2), and the analogous estimate hold for (1.3). Here and later we use the shorthand notation $\pm$ to represent both the $+$ and $-$ equations and functions. Observe that $\mathrm{rank}(P^+) = \mathrm{rank}(P^-) = p$ and the projections of the dichotomy of (1.3) on $\mathbb{R}_\pm$ are $\mathbf{I} - [P^\pm]^*$. Moreover from (i)–(iv) it follows that (1.3) has a unique bounded solution on $\mathbb{R}$, up to a multiplicative constant. We denote one of these solutions by $\psi(t)$. Note that $\psi := \psi(0)$ satisfies $\mathcal{N}[P^+]^* \cap \mathcal{R}[P^-]^* = \mathrm{span}(\psi) =$

$[\mathcal{R}P^+ \cap \mathcal{N}P^-]^\perp$; we assume w.l.o.g. that $|\psi(0)| = 1$.
As a second remark we observe that condition (i) implies the existence of a so called local "slow manifold" $\mathcal{M}^c$. I.e. there is a function $v(x, \varepsilon, \lambda)$ defined for x, ε, λ small enough, such that $v(x, 0, 0) \equiv 0$ and the manifold $y = v(x, \varepsilon, \lambda)$ is an invariant centre manifold for the flow of (1.1) (see for example [2, 11]). Moreover $v(x, \varepsilon, \lambda)$ is C^{r-1}, bounded with its derivatives. Using the flow of (1.1) we can pass from the local manifold $y = v(x, \varepsilon, \lambda)$ to a global slow manifold for system (1.1) which will be denoted by $\mathcal{M}^c = \mathcal{M}^c(\varepsilon, \lambda)$. Note that if we assume (as in the examples in section 4) that $g(x, 0, \varepsilon, \lambda) = 0$, then $v(x, \varepsilon, \lambda) \equiv 0$.

Let $x_c(t, \xi, \varepsilon, \lambda)$ be the solution of the scalar ODE:

$$\dot{x} = f(x, v(x, \varepsilon, \lambda), \varepsilon, \lambda) \qquad x(0) = \xi \tag{1.5}$$

So $(x_c(t, \xi, \varepsilon, \lambda), v(x_c(t, \xi, \varepsilon, \lambda), \varepsilon, \lambda))$ describes the flow on the slow manifold $\mathcal{M}^c$, and (1.5) is the so called "slow time" system.

The behavior of homoclinic and heteroclinic trajectories subject to singular perturbation has been studied in several papers, see e.g. [1, 2, 4, 5, 6, 11, 13]. In particular in [6] the authors built up a theory to prove the existence of solutions homoclinic to $\mathcal{M}^c$, for the perturbed problem (1.1) assuming conditions (i)–(iv) and giving transversality conditions of several different types. They refine previous results obtained in [4].

This paper is thought as a sequel of [6]. Here we assume that the "slow time" system (1.5) is one-dimensional so that there is a unique solution $(\tilde{x}(t, \varepsilon, \lambda), \tilde{y}(t, \varepsilon, \lambda))$ homoclinic to $\mathcal{M}^c$. Moreover we assume that (1.5) undergoes a bifurcation for $\varepsilon = 0$ as λ changes sign. We mainly focus on the transcritical and saddle-node case, i.e. we assume f has one of the following form:

$$\begin{aligned} f(x, v(x, \varepsilon, \lambda), \varepsilon, \lambda) &= x^2 - b(\varepsilon)\lambda^2 + O(x^3) && (1.6) \\ f(x, v(x, \varepsilon, \lambda), \varepsilon, \lambda) &= x^2 - a(\varepsilon)\lambda + O(x^3) && (1.7) \end{aligned}$$

where $a(\varepsilon)$ and $b(\varepsilon)$ are positive C^{r-1} functions and the terms contained in $O(x^3)$ are C^{r-1} in x, ε and C^{r-2} in λ. The aim of this paper is to derive further Melnikov conditions which enable us to divide the ε, λ space into different sets in which $(\tilde{x}(t, \varepsilon, \lambda), \tilde{y}(t, \varepsilon, \lambda))$ has different behavior: it is homoclinic, heteroclinic or it does not converge to critical points either in the past or in the future. We stress that we have explicit formulas for the derivatives of the curves defining the border of these sets. This is the content of Theorems 3.2, 3.5 which regard respectively the case where (1.5) undergoes a transcritical or a saddle-node bifurcation.

We emphasize that the assumptions (1.6) and (1.7) on f are generic. In fact assume $f = 0$, $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial^2 f}{\partial x^2} \neq 0$ for $(x, y, \varepsilon, \lambda) = (0, 0, 0, 0)$. Following subsection 11.2 of [12] and recalling that $v(x, \varepsilon, \lambda) = O(|\varepsilon| + |\lambda|)$, we see that, if $\frac{\partial f}{\partial \lambda} + \frac{\partial f}{\partial y}\frac{\partial v}{\partial \lambda} \neq 0$ at $(x, y, \varepsilon, \lambda) = (0, 0, 0, 0)$, we can find a new parameter $\bar{\lambda} = \bar{\lambda}(\varepsilon, \lambda)$ with C^{r-2} dependence on ε and λ and a C^{r-2} change of variables $\bar{x} = \bar{x}(x, \varepsilon, \lambda)$, so that (1.5) takes the form

$$\dot{\bar{x}} = -\bar{\lambda}(\varepsilon, \lambda) + c(\varepsilon)\bar{x}^2 + O(\bar{x}^3),$$

where $c(\varepsilon) > 0$ is C^{r-1} (possibly reversing time, i.e. passing from t to $-t$), and $O(\bar{x}^3)$ is C^{r-1} in x and ε and C^{r-2} in λ. Hence we reduce to the case where f has the form (1.7), and (1.5) undergoes a saddle-node bifurcation. When $\frac{\partial f}{\partial \lambda} + \frac{\partial f}{\partial y}\frac{\partial v}{\partial \lambda} = 0$ at $(x, y, \varepsilon, \lambda) = (0, 0, 0, 0)$ (e.g. if for some physical reasons the origin of the system (1.5) is forced to be a critical point for any λ), generically we have a transcritical bifurcation. In such a case, up to a C^{r-2} change of parameters and variables we can pass from (1.5) to

$$\dot{\bar{x}} = -\bar{x}\bar{\lambda}(\varepsilon, \lambda) + c(\varepsilon)\bar{x}^2 + O(\bar{x}^3),$$

see again subsection 11.2 of [12]. Then passing from $\bar{x}$ to $\tilde{x} = \bar{x} - \frac{\bar{\lambda}}{2c(\varepsilon)}$, we reduce to the case where f has the form (1.6), and (1.5) undergoes a transcritical bifurcation. We emphasize that in all the change of parameters we leave unchanged the singular parameter ε.

Let $u(\varepsilon, \lambda)$, $s(\varepsilon, \lambda)$ be the zeroes of $f(x, v(x, \varepsilon, \lambda), \varepsilon, \lambda) = 0$, and denote by $U(\varepsilon, \lambda) = (u(\varepsilon, \lambda), v(u(\varepsilon, \lambda), \varepsilon, \lambda))$, $S(\varepsilon, \lambda) = (s(\varepsilon, \lambda), v(s(\varepsilon, \lambda), \varepsilon, \lambda))$ the critical points of (1.1). When f is either of the form (1.6) or (1.7), (1.5) admits two critical points for $\lambda > 0 = \varepsilon$, i.e. $u(0, \lambda), s(0, \lambda) \in \mathbb{R}$: u is unstable while s is stable. If f is as in (1.6), the critical points $u(\varepsilon, \lambda)$ and $s(\varepsilon, \lambda)$ of (1.5) reverse their stability properties as we pass from $\lambda > 0$ to $\lambda < 0$, while if it is as in (1.7) then $u(\varepsilon, \lambda)$ and $s(\varepsilon, \lambda)$ are distinct for $\lambda > 0$, they coincide for $\lambda = 0$ and they do not exist for $\lambda < 0$.

Our purpose is to find trajectories of (1.1) which are close for any $t \in \mathbb{R}$ to the homoclinic trajectory $(0, h(t))$ of the unperturbed system. We use the implicit function theorem to construct Melnikov conditions which ensure the existence of such trajectories, and which allow to say if they are homoclinic, heteroclinic or unbounded. The techniques can be applied also to bifurcations of higher order, i.e. when the first nonzero term of the expansion of f in x has degree 3 or more (obviously in this case we need to assume f at least C^4 or more in the x variable). However in such a case to obtain a complete unfolding of the singularity more parameters are needed. In fact we just sketch the case of pitchfork bifurcation (which however appears frequently when f, for some physical reasons, is odd in x for any ε and λ). Again, following subsection 11.2 of [12], we see that, up to changes in variables and parameters, we may reduce to f of the form

$$f(x, v(x,\varepsilon,\lambda), \varepsilon, \lambda) = [x^2 - a(\varepsilon)\lambda][x - b(\varepsilon)\lambda] + o(x^3) \tag{1.8}$$

where $a(\varepsilon)$ and $b(\varepsilon)$ are C^{r-1} positive functions and the $o(x^3)$ is C^{r-1} in ε and C^{r-2} in λ.

The assumptions used in the main Theorems are the following:

(v)

$$\int_{-\infty}^{\infty} \psi^*(t) \frac{\partial g}{\partial x}(0, h(t), 0, 0) dt \neq 0$$

(vi)

$$B_0 := \frac{\int_{-\infty}^{\infty} \psi^*(t) \frac{\partial g}{\partial \lambda}(0, h(t), 0, 0) dt}{\int_{-\infty}^{\infty} \psi^*(t) \frac{\partial g}{\partial x}(0, h(t), 0, 0) dt} \neq \pm \frac{\partial u}{\partial \lambda}(0,0) = \pm\sqrt{b(0)}\,,$$

The paper is divided as follows. In section 2 we briefly review some facts, proved in [6]: we construct the solutions asymptotic to the slow manifold $\mathcal{M}^c$, then we match them via implicit function theorem, to construct a solution $(\tilde{x}(t,\varepsilon,\lambda), \tilde{y}(t,\varepsilon,\lambda))$ homoclinic to $\mathcal{M}^c$. In section 3 we show which is the behavior of $(\tilde{x}(t,\varepsilon,\lambda), \tilde{y}(t,\varepsilon,\lambda))$ as ε and λ varies, in the transcritical and in the saddle-node case (subsections 3.1 and 3.2 respectively).
So we give sufficient conditions in order to have homoclinic, heteroclinic or no bounded solutions close to $(0, h(t))$, as the parameters vary: this is the content of Theorems 3.2 and 3.5. We emphasize that condition (v) is needed to construct $(\tilde{x}(t,\varepsilon,\lambda), \tilde{y}(t,\varepsilon,\lambda))$, condition (vi) is needed just in the (1.6) case, to understand the behavior of such a trajectory: no further condition is needed for (1.7). Finally we explain how the same methods can be extended to describe pitchfork and higher degree bifurcations in subsection 3.3. We illustrate our results drawing some bifurcation diagrams. Finally in section 4 we construct examples for which we can explicitly compute the derivatives of the bifurcation curves appearing in the diagrams. We conclude the introduction with a remark concerning the regularity of the functions used and constructed.

Remark 1.1. We stress that the loss of two orders of regularity just depends on the following facts: one order is due to the construction of the slow manifold, the other to the change of parameters that drives (1.5) either in the form (1.6) or (1.7). If we assume $g(x, 0, \varepsilon, \lambda) \equiv 0$ so that the slow manifold reduces to $y = v(x, \varepsilon, \lambda) \equiv 0$, and we assume that f satisfies directly either (1.6) or (1.7), there is no loss of regularity and we may always assume f and g just C^r with $r \geq 1$, and all the functions introduced would be C^r as well.

2 The centre-stable and centre-unstable manifolds

In this section we define the local centre-stable and centre-unstable manifolds and we recall their smoothness properties. These manifolds are (locally) invariant manifolds of solutions that approach the slow manifolds $y = v(x, \varepsilon, \lambda)$ at an exponential rate. In [5, 6] the following result has been proved.

Theorem 2.1. *[6] Let f and g be bounded C^r functions, $r \geq 2$, with bounded derivatives, satisfying conditions (i)-(iv) of the Introduction and let the numbers β and σ satisfy $0 < r\sigma < \beta < \Lambda^g$. Then, given suitably small positive numbers μ_1 and μ_2, there exist positive numbers ρ_0, λ_0, $\varepsilon_0(< 2\sigma/N$, where N is a bound for the derivatives of $f(x, 0, 0, 0))$, such that for $|\varepsilon| \leq \varepsilon_0$, $|\lambda| \leq \lambda_0$, $|\xi^{\pm}| \leq \rho_0$, $\zeta^+ \in \mathcal{R}(P^+)$, $|\zeta^+| \leq \mu_1$, $\zeta^- \in \mathcal{N}(P^-)$, $|\zeta^-| \leq \mu_2$, there exists a unique solution*

$$(x^{\pm}(t), y^{\pm}(t)) = (x^{\pm}(t, \xi^{\pm}, \zeta^{\pm}, \lambda), y^{\pm}(t, \xi^{\pm}, \zeta^{\pm}, \varepsilon, \lambda))$$

of (1.1) defined respectively for $t \geq 0$ and for $t \leq 0$ such that

$$e^{|\beta t|}|x^+(t) - x_c(\varepsilon t, \xi^+, \varepsilon, \lambda)| \leq \mu_1, \quad e^{|\beta t|}|y^+(t) - v(x^+(t), \varepsilon, \lambda)| \leq \mu_1 \tag{2.1}$$

for $t \geq 0$, and

$$e^{|\beta t|}|x^-(t) - x_c(\varepsilon t, \xi^-, \varepsilon, \lambda)| \leq \mu_2, \quad e^{|\beta t|}|y^-(t) - v(x^-(t), \varepsilon, \lambda)| \leq \mu_2 \tag{2.2}$$

for $t \leq 0$, and

$$P^+[y^+(0) - v(x^+(0), \varepsilon, \lambda)] = \zeta^+, \quad (\mathbf{I} - P^-)[y^-(0) - v(x^-(0), \varepsilon, \lambda)] = \zeta^- \tag{2.3}$$

Moreover $y^{\pm}(t, \xi^{\pm}, \zeta^{\pm}, \varepsilon, \lambda) - v(x^{\pm}(t, \xi^{\pm}, \zeta^{\pm}, \varepsilon, \lambda), \varepsilon, \lambda)$ and $x^{\pm}(t, \xi^{\pm}, \zeta^{\pm}, \varepsilon, \lambda) - x_c(\varepsilon t, \xi^{\pm}, \varepsilon, \lambda)$ are C^{r-1} in the parameters $(\xi^{\pm}, \zeta^{\pm}, \varepsilon, \lambda)$ and for $k \leq r-1$ their k^{th} derivatives also satisfy the estimate (2.2) with β replaced by $\beta - k\sigma$ and μ_1 and μ_2 replaced by possibly larger constants. Also there is $N_1 > 0$ such that for $t \leq 0$

$$\begin{aligned} &e^{|\beta t|}|x^-(t, \xi^-, \zeta^-, \varepsilon, \lambda) - x_c(\varepsilon t, \xi^-, \varepsilon, \lambda)| \leq N_1|\varepsilon||\zeta^-|, \\ &e^{|\beta t|}|y^-(t, \xi^-, \zeta^-, \varepsilon, \lambda) - v(x^-(t, \xi^-, \zeta^-, \varepsilon, \lambda), \varepsilon, \lambda))| \leq N_1|\zeta^-|. \end{aligned} \tag{2.4}$$

and for $t \geq 0$

$$\begin{aligned} &e^{|\beta t|}|x^+(t, \xi^+, \zeta^+, \varepsilon, \lambda) - x_c(\varepsilon t, \xi^+, \varepsilon, \lambda)| \leq N_1|\varepsilon||\zeta^+|, \\ &e^{|\beta t|}|y^+(t, \xi^+, \zeta^+, \varepsilon, \lambda) - v(x^+(t, \xi^+, \zeta^+, \varepsilon, \lambda), \varepsilon, \lambda))| \leq N_1|\zeta^+|. \end{aligned} \tag{2.5}$$

Following section 2.1 in [6], using Theorem 2.1 we define the local centre-unstable and centre-stable manifolds near the origin in $\mathbb{R}^{n+1}$ as follows

$$\begin{aligned}\mathcal{M}^{cu}_{loc} &:= \{(x^-(0,\xi^-,\zeta^-,\varepsilon,\lambda), y^-(0,\xi^-,\zeta^-,\varepsilon,\lambda)) \;:\; |\zeta^-|<\mu_0,\, |\xi^-|<\rho_0\},\\ \mathcal{M}^{cs}_{loc} &:= \{(x^+(0,\xi^+,\zeta^+,\varepsilon,\lambda), y^+(0,\xi^+,\zeta^+,\varepsilon,\lambda)) \;:\; |\zeta^+|<\mu_0,\, |\xi^+|<\rho_0\}.\end{aligned}$$

In [6] it has been proved that $\mathcal{M}^{cu}_{loc}$ and $\mathcal{M}^{cs}_{loc}$ are respectively negatively and positively invariant for (1.1). Thus, going respectively forward and backward in t, we can construct from $\mathcal{M}^{cu}_{loc}$ and $\mathcal{M}^{cs}_{loc}$ the global manifold $\mathcal{M}^{cu}$ and $\mathcal{M}^{cs}$, see Lemma 2.3 in section 2.2 in [6]. Therefore $\mathcal{M}^{cu}$ and $\mathcal{M}^{cs}$ are respectively $p+1$ and $n-p+1$ dimensional immersed manifolds of $\mathbb{R}^{n+1}$, made up by the trajectories asymptotic to $\mathcal{M}^c$ resp. in the past and in the future.

Following the discussion after Theorems 2.1 and 2.2 in [6], we see that the k^{th} derivatives of $x^+(t,\xi,\zeta^+,\varepsilon,\lambda)$ and of $x^-(t,\xi,\zeta^-,\varepsilon,\lambda)$ with respect to $(\xi,\zeta^\pm,\varepsilon,\lambda)$ are bounded above in absolute value by $C_k e^{(k+1)\sigma|t|}$ for $t\in\mathbb{R}$ respectively, where C_k is a constant and $\sigma > N\varepsilon_0$ is a positive number that satisfies $0<r\sigma<\beta<\Lambda^g$. Finally, because of uniqueness of $(x^\pm(t,\xi^\pm,\zeta^\pm,\varepsilon,\lambda), y^\pm(t,\xi,\zeta^\pm,\varepsilon,\lambda))$, we see that the following properties hold:

$$\begin{aligned}x^\pm(t,\xi^\pm,v(\xi^\pm,\varepsilon,\lambda),\varepsilon,\lambda) &= x_c(\varepsilon t,\xi^\pm,\varepsilon,\lambda),\\ y^\pm(t,\xi^\pm,v(\xi^\pm,\varepsilon,\lambda),\varepsilon,\lambda) &= v(x_c(\varepsilon t,\xi^\pm,\varepsilon,\lambda),\varepsilon,\lambda)\end{aligned} \tag{2.6}$$

and

$$x^\pm(t,\xi^\pm,\zeta^\pm,0,\lambda) = \xi^\pm \tag{2.7}$$

see [6]. Since $x_c(0,\xi,\varepsilon,\lambda)=\xi$, we see that the slow manifold $\mathcal{M}^c$ defined by $y=v(\xi,\varepsilon,\lambda)$ is contained in the intersection between $\mathcal{M}^{cu}$ and $\mathcal{M}^{cs}$.

Exploiting section 2.3 in [6] we can define a foliation of $\mathcal{M}^{cu}_{loc}$ and $\mathcal{M}^{cs}_{loc}$ as follows. Let $|\xi|$ be sufficiently small, we set

$$\begin{aligned}\mathcal{M}^{cu}(\xi) &:= \{(x^-(t,\xi,\zeta^-,\varepsilon,\lambda), y^-(t,\xi,\zeta^-,\varepsilon,\lambda)) \mid |\zeta^-|<\mu_0,\, \zeta^-\in\mathcal{N}P^-, t\in\mathbb{R}\}\\ \mathcal{M}^{cs}(\xi) &:= \{(x^+(t,\xi,\zeta^+,\varepsilon,\lambda), y^+(t,\xi,\zeta^+,\varepsilon,\lambda)) \mid |\zeta^+|<\mu_0,\, \zeta^+\in\mathcal{R}P^+, t\in\mathbb{R}\}.\end{aligned}$$

Using the flow of (1.1) we can remove the smallness assumption on ξ (but we get $\mu_1=\mu_1(|\xi|)$, $\mu_2=\mu_2(|\xi|)$, $N_1=N_1(|\xi|)$ in the estimates (2.2), (2.1), (2.4), (2.5)).

From section 2.3 in [6] we see that that $\mathcal{M}^{cu}(\xi)$ and $\mathcal{M}^{cs}(\xi)$ are p and $n-p$ manifolds for any $\xi\in\mathbb{R}$, and that $\mathcal{M}^{cu}=\cup_{\xi\in\mathbb{R}}\mathcal{M}^{cu}(\xi)$, $\mathcal{M}^{cs}=\cup_{\xi\in\mathbb{R}}\mathcal{M}^{cs}(\xi)$, are the global centre-unstable and centre-stable manifolds defined above. Moreover given $\bar\xi,\tilde\xi\in\mathbb{R}$ then either $\mathcal{M}^{cu}(\bar\xi)$ and $\mathcal{M}^{cu}(\tilde\xi)$ coincide or they do not intersect; similarly either $\mathcal{M}^{cs}(\bar\xi)$ and $\mathcal{M}^{cs}(\tilde\xi)$ coincide or they do not intersect: thus $\mathcal{M}^{cu}(\xi)$ and $\mathcal{M}^{cs}(\xi)$ define indeed foliations for $\mathcal{M}^{cu}$ and $\mathcal{M}^{cs}$.

We denote by $B(\zeta,\rho)$ the ball with centre $\zeta \in \mathbb{R}^{n+1}$ and radius $\rho > 0$. Let $A \subset \mathbb{R}^{n+1}$ be a set, we define $\mathrm{dist}(\zeta, A) = \inf\{|\zeta - \eta| \mid \eta \in A\}$. We borrow from [6] a theorem which ensures the existence of a solution of (1.1) homoclinic to $\mathcal{M}^c$.

Theorem 2.2. *[6] Let f and g be bounded C^r functions, $r \geq 2$, with bounded derivatives, satisfying conditions (i)–(v) of the Introduction. Then there exist positive numbers λ_0, ε_0 such that for any $|\varepsilon| < \varepsilon_0$, $|\lambda| < \lambda_0$ there is a solution $(\tilde{x}(t,\varepsilon,\lambda), \tilde{y}(t,\varepsilon,\lambda)) \in (\mathcal{M}^{cs} \cap \mathcal{M}^{cu}) \setminus \mathcal{M}^c$ satisfying*

$$\lim_{|t|\to\infty} dist\big((\tilde{x}(\varepsilon,\lambda,t), \tilde{y}(\varepsilon,\lambda,t)), \mathcal{M}^c\big) = 0$$

Moreover there is a neighborhood Ω^0 of $(0, h(0))$ such that, if $(x(t), y(t)) \in (\mathcal{M}^{cs} \cap \mathcal{M}^{cu})$ and $(x(0), y(0)) \in \Omega^0$, then $(x(t), y(t)) \equiv (\tilde{x}(t,\varepsilon,\lambda), \tilde{y}(t,\varepsilon,\lambda))$, so local uniqueness is ensured.

We sketch the proof since some details will be useful later on. To prove theorem 2.2 Battelli and Palmer in [6] look for a bifurcation function whose zeroes correspond to solutions of the system

$$\begin{cases} x^+(-T,\xi^+,\zeta^+,\varepsilon,\lambda) = x^-(T,\xi^-,\zeta^-,\varepsilon,\lambda) = \xi \\ y^+(-T,\xi^+,\zeta^+,\varepsilon,\lambda) = y^-(T,\xi^-,\zeta^-,\varepsilon,\lambda) \end{cases} \tag{2.8}$$

where $T > 0$, and $|\xi^\pm| < \rho_0$. Set

$$K(\xi^+,\xi^-,\zeta^+,\zeta^-,\varepsilon,\lambda) := y^+(-T,\xi^+,\zeta^+,\varepsilon,\lambda) - y^-(T,\xi^-,\zeta^-,\varepsilon,\lambda)$$

They apply Lyapunov-Schmidt reduction to system (2.8) and rewrite it as follows

$$\begin{cases} x^+(-T,\xi^+,\zeta^+,\varepsilon,\lambda) = x^-(T,\xi^-,\zeta^-,\varepsilon,\lambda) = \xi \\ K(\xi^+,\xi^-,\zeta^+,\zeta^-,\varepsilon,\lambda) - [\psi^* K(\xi^+,\xi^-,\zeta^+,\zeta^-,\varepsilon,\lambda)]\psi = 0 \\ \psi^* K(\xi^+,\xi^-,\zeta^+,\zeta^-,\varepsilon,\lambda) = 0 \end{cases} \tag{2.9}$$

Using several times the implicit function theorem and exponential dichotomy estimates, they express $\xi^\pm$ as functions of the variables $(\xi,\zeta^\pm,\varepsilon,\lambda)$, then $\zeta^\pm$ as functions of the remaining variables, and they end up with unique C^{r-1} functions $\bar{\zeta}^\pm(\xi,\varepsilon,\lambda)$, and $\bar{\xi}^\pm(\xi,\varepsilon,\lambda)$ with the following properties, see pages 448-453 in [6] for more details. Set

$$\begin{array}{l} \bar{x}^\pm(t,\xi,\varepsilon,\lambda) := x^\pm(t,\bar{\xi}^\pm(\xi,\varepsilon,\lambda),\bar{\zeta}^\pm(\xi,\varepsilon,\lambda),\varepsilon,\lambda) \\ \bar{y}^\pm(t,\xi,\varepsilon,\lambda) := y^\pm(t,\bar{\xi}^\pm(\xi,\varepsilon,\lambda),\bar{\zeta}^\pm(\xi,\varepsilon,\lambda),\varepsilon,\lambda) \end{array},$$

Then $(\bar{x}^{\pm}(t,\xi,\varepsilon,\lambda), \bar{y}^{\pm}(t,\xi,\varepsilon,\lambda))$ are the unique solutions of the first two equations in (2.9). Hence we are left with solving the bifurcation equation:

$$G(\xi,\varepsilon,\lambda) = \psi^*[\bar{y}^+(-T,\xi,\varepsilon,\lambda) - \bar{y}^-(T,\xi,\varepsilon,\lambda)] = 0 \tag{2.10}$$

Following [6] and using (v) we see that

$$\frac{\partial}{\partial\xi}G(0,0,0) = -\textstyle\int_{-\infty}^{+\infty}\psi^*(t)\frac{\partial g}{\partial x}(0,h(t),0,0)dt \neq 0 \tag{2.11}$$

$$\frac{\partial}{\partial\lambda}G(0,0,0) = -\int_{-\infty}^{+\infty}\psi^*(t)\frac{\partial g}{\partial\lambda}(0,h(t),0,0)\mathrm{d}t \tag{2.12}$$

$$\frac{\partial G}{\partial\varepsilon}(0,0,0) = -\textstyle\int_{-\infty}^{\infty}\psi^*(s)\left[\frac{\partial g}{\partial\varepsilon}(s) + \frac{\partial g}{\partial x}(s)\left(\int_0^s f(t)dt\right)\right]ds \tag{2.13}$$

where $g(s)$ stands for $g(0,h(s),0,0)$, $f(s)$ for $f(0,h(s),0,0)$. Therefore, they apply again the Implicit Function Theorem and obtain a C^{r-1} function $\tilde{\xi}(\varepsilon,\lambda)$ such that $G(\tilde{\xi}(\varepsilon,\lambda),\varepsilon,\lambda) \equiv 0$ and

$$\frac{\partial}{\partial\lambda}\tilde{\xi}(0,0) = B_0 := -\frac{\int_{-\infty}^{\infty}\psi^*(t)\frac{\partial g}{\partial\lambda}(0,h(t),0,0)dt}{\int_{-\infty}^{\infty}\psi^*(t)\frac{\partial g}{\partial x}(0,h(t),0,0)dt} \tag{2.14}$$

$$\frac{\partial}{\partial\varepsilon}\tilde{\xi}(0,0) = -\frac{\int_{-\infty}^{\infty}\psi^*(s)\frac{\partial g}{\partial\varepsilon}(s)ds + \int_{-\infty}^{\infty}(\psi^*(s)\frac{\partial g}{\partial x}(s)\int_0^s f(t)dt)ds}{\int_{-\infty}^{\infty}\psi^*(s)\frac{\partial g}{\partial x}(s)ds}. \tag{2.15}$$

So, if (v) holds, for any (ε,λ) small enough there is a unique solution of (1.1) which is homoclinic to the slow manifold $\mathcal{M}^c$, i.e.:

$$\begin{aligned}
\tilde{x}(t,\varepsilon,\lambda) &= \begin{cases} \bar{x}^+(t-T,\tilde{\xi}(\varepsilon,\lambda),\varepsilon,\lambda) & t \geq 0, \\ \bar{x}^-(t+T,\tilde{\xi}(\varepsilon,\lambda),\varepsilon,\lambda) & t \leq 0. \end{cases} \\
\tilde{y}(t,\varepsilon,\lambda) &= \begin{cases} \bar{y}^+(t-T,\tilde{\xi}(\varepsilon,\lambda),\varepsilon,\lambda) & t \geq 0, \\ \bar{y}^-(t+T,\tilde{\xi}(\varepsilon,\lambda),\varepsilon,\lambda) & t \leq 0. \end{cases}
\end{aligned} \tag{2.16}$$

This concludes the proof of Theorem 2.2. Let us denote by

$$\tilde{\xi}^{\pm}(\varepsilon,\lambda) := \tilde{\xi}^{\pm}(\tilde{\xi}(\varepsilon,\lambda),\varepsilon,\lambda);$$

observe that $\tilde{\xi}^{\pm}(0,\lambda) = (\tilde{\xi}(0,\lambda),0,\lambda)$ but for $\varepsilon \neq 0$ the three functions $\tilde{\xi}(\varepsilon,\lambda), \tilde{\xi}^+(\varepsilon,\lambda), \tilde{\xi}^-(\varepsilon,\lambda)$ are distinct. We recall that $\tilde{\xi}(\varepsilon,\lambda)$ is obtained solving (2.8), so it is a value assumed by x^+ and x^-, more precisely

$$\begin{array}{rcccl}
\tilde{x}^+(0,\varepsilon,\lambda) &=& x^+(-T,\tilde{\xi}^+(\varepsilon,\lambda),\bar{\zeta}^+(\tilde{\xi}(\varepsilon,\lambda),\varepsilon,\lambda),\varepsilon,\lambda) &=& \tilde{\xi}(\varepsilon,\lambda) \\
\tilde{x}^-(0,\varepsilon,\lambda) &=& x^-(T,\tilde{\xi}^-(\varepsilon,\lambda),\bar{\zeta}^-(\tilde{\xi}(\varepsilon,\lambda),\varepsilon,\lambda),\varepsilon,\lambda) &=& \tilde{\xi}(\varepsilon,\lambda)
\end{array};$$

while $\tilde{\xi}^+(\varepsilon,\lambda)$ (respectively $\tilde{\xi}^-(\varepsilon,\lambda)$) is the initial condition of the solution $x_c(\varepsilon t, \tilde{\xi}^+(\varepsilon,\lambda),\varepsilon,\lambda)$ (resp. $x_c(\varepsilon t, \tilde{\xi}^-(\varepsilon,\lambda),\varepsilon,\lambda)$) of the slow time system (1.5) followed by $\tilde{x}^+(t,\varepsilon,\lambda)$ for $t>0$ (resp. followed by $\tilde{x}^-(t,\varepsilon,\lambda)$ for $t<0$), see Theorem 2.1.

In the next section we need the derivatives of $\tilde{\xi}^\pm(\varepsilon,\lambda)$ with respect to ε and λ. For this purpose we evaluate first the derivatives of $\bar{\xi}^\pm(\xi,\varepsilon,\lambda)$; in fact it is easy to check that $\bar{\xi}^\pm(\xi,0,\lambda)=\xi$ for any λ so the derivative with respect to λ is null. Following again section 3.1 in [6] we see that

$$\frac{\partial\bar{\xi}^\pm}{\partial\lambda}(0,0,0)=0 \tag{2.17}$$

$$\frac{\partial\bar{\xi}^\pm}{\partial\varepsilon}(0,0,0)=\int_0^{\pm\infty} f(0,y_h(s),0,0)ds \tag{2.18}$$

Then, using (2.14), (2.17), and (2.15), (2.18) we find

$$\begin{aligned}
\frac{\partial\tilde{\xi}^\pm(0,0)}{\partial\lambda} &= \frac{\partial\bar{\xi}^\pm(0,0,0)}{\partial\xi}\frac{\partial\tilde{\xi}(0,0)}{\partial\lambda}+\frac{\partial\bar{\xi}^\pm(0,0,0)}{\partial\lambda}=B_0\\
\frac{\partial\tilde{\xi}^\pm(0,0)}{\partial\varepsilon} &= \frac{\partial\bar{\xi}^\pm(0,0,0)}{\partial\xi}\frac{\partial\tilde{\xi}(0,0)}{\partial\varepsilon}+\frac{\partial\bar{\xi}^\pm(0,0,0)}{\partial\varepsilon}=A_0^\pm \quad \text{where}\\
A_0^\pm &= -\frac{\int_{-\infty}^{\infty}\psi^*(s)\frac{\partial g}{\partial\varepsilon}(s)ds+\int_{-\infty}^{\infty}(\psi^*(s)\frac{\partial g}{\partial x}(s)\int_{\pm\infty}^{s} f(t)dt)ds}{\int_{-\infty}^{\infty}\psi^*(s)\frac{\partial g}{\partial x}(s)ds}.
\end{aligned} \tag{2.19}$$

3 Existence of homoclinic and heteroclinic solutions

In this section we state and prove our main results. Since $(\tilde{x}(t,\varepsilon,\lambda),\tilde{y}(t,\varepsilon,\lambda))$ is constructed via implicit function theorem we have local uniqueness, see the explanation just after theorem 2.2. Our purpose is to divide the parameters space in different subsets in which the solution $(\tilde{x}(t,\varepsilon,\lambda),\tilde{y}(t,\varepsilon,\lambda))$ has a different asymptotic behavior.

Note that $U(\varepsilon,\lambda)$ and $S(\varepsilon,\lambda)$ change their stability properties when they cross the lines lines $\lambda=0$ and $\varepsilon=0$, thus we need to argue separately in the 4 different quadrants. At this point we need to distinguish between f satisfying (1.6) and (1.7).

3.1 Transcritical bifurcation: f of type (1.6)

We observe first that, since f satisfies (1.6) then

$$\frac{\partial u}{\partial\lambda}(0,0)=\sqrt{b(0)}=-\frac{\partial s}{\partial\lambda}(0,0) \quad \text{and} \quad \frac{\partial u}{\partial\varepsilon}(0,0)=0=\frac{\partial s}{\partial\varepsilon}(0,0).$$

Let us start from $\varepsilon > 0$ and $\lambda \geq 0$. The key point to understand the behavior in the future is to establish the mutual positions of $\tilde{\xi}^+(\varepsilon, \lambda)$ and $u(\varepsilon, \lambda)$, while to understand the behavior in the past we need to know the positions of $\tilde{\xi}^-(\varepsilon, \lambda)$ with respect to $s(\varepsilon, \lambda)$. So we define the functions $J_1^{\pm} : [-\varepsilon_0, \varepsilon_0] \times [-\lambda_0, \lambda_0] \to \mathbb{R}$ as follows

$$\begin{aligned} J_1^+(\varepsilon, \lambda) &= \tilde{\xi}^+(\varepsilon, \lambda) - u(\varepsilon, \lambda)\,, \\ J_1^-(\varepsilon, \lambda) &= \tilde{\xi}^-(\varepsilon, \lambda) - s(\varepsilon, \lambda) \end{aligned} \tag{3.1}$$

Note that if (vi) holds

$$\begin{array}{l} \frac{\partial J_1^+}{\partial \lambda}(0,0) = \frac{\partial \tilde{\xi}^+}{\partial \lambda}(0,0) - \frac{\partial u}{\partial \lambda}(0,0) = B_0 - \frac{\partial u}{\partial \lambda}(0,0) \neq 0 \\ \frac{\partial J_1^-}{\partial \lambda}(0,0) = \frac{\partial \tilde{\xi}^-}{\partial \lambda}(0,0) - \frac{\partial s}{\partial \lambda}(0,0) = B_0 + \frac{\partial u}{\partial \lambda}(0,0) \neq 0 \end{array} ; \tag{3.2}$$

so via implicit function theorem we can construct two curves, $\lambda_1^+(\varepsilon)$ and $\lambda_1^-(\varepsilon)$, satisfying $\lambda_1^{\pm}(0) = 0$, and such that $J_1^{\pm}(\varepsilon, \lambda_1^{\pm}(\varepsilon)) = 0$. Then $(\tilde{x}(t, \varepsilon, \lambda_1^+(\varepsilon)),$ $\tilde{y}(t, \varepsilon, \lambda_1^+(\varepsilon))$ converges to $U(\varepsilon, \lambda_1^+(\varepsilon))$ as $t \to +\infty$, while $(\tilde{x}(t, \varepsilon, \lambda_1^-(\varepsilon)),$ $\tilde{y}(t, \varepsilon, \lambda_1^-(\varepsilon)))$ converges to $S(\varepsilon, \lambda_1^-(\varepsilon))$ as $t \to -\infty$. Moreover

$$\begin{aligned} \frac{d}{d\varepsilon}\lambda_1^+(0) &= -\frac{\frac{\partial}{\partial \varepsilon}\tilde{\xi}^+(0,0) - \frac{\partial u}{\partial \varepsilon}(0,0)}{\frac{\partial}{\partial \lambda}\tilde{\xi}^+(0,0) - \frac{\partial u}{\partial \lambda}(0,0)} = -\frac{A_0^+}{B_0 - \frac{\partial u}{\partial \lambda}(0,0)} \\ \frac{d}{d\varepsilon}\lambda_1^-(0) &= -\frac{A_0^-}{B_0 + \frac{\partial u}{\partial \lambda}(0,0)} \end{aligned} \tag{3.3}$$

Remark 3.1. The curves $\lambda_1^+(\varepsilon)$ and $\lambda_1^-(\varepsilon)$ may not intersect the open set $Q_1 = \{(\varepsilon, \lambda) \mid \lambda > 0 \text{ and } \varepsilon > 0\}$. If this is the case for any $(\varepsilon, \lambda) \in Q_1$ the trajectory $(\tilde{x}(t, \varepsilon, \lambda), \tilde{y}(t, \varepsilon, \lambda))$ does not converge respectively to U in the future neither to S in the past.

Let $\delta > 0$ be sufficiently small, and denote by

$$\begin{array}{ll} \mathfrak{A}^+ = \{x < u(\varepsilon, \lambda) \mid |x| \leq \delta\} & \mathfrak{B}^+ = \{x > u(\varepsilon, \lambda) \mid |x| \leq \delta\} \\ \mathfrak{A}^- = \{x < s(\varepsilon, \lambda) \mid |x| \leq \delta\} & \mathfrak{B}^- = \{x > s(\varepsilon, \lambda) \mid |x| \leq \delta\} \end{array}$$

By construction $s(\varepsilon, \lambda) \in \mathfrak{A}^+$ and $u(\varepsilon, \lambda) \in \mathfrak{B}^-$; hence if $\tilde{\xi}^+(\varepsilon, \lambda) \in \mathfrak{A}^+$ then the trajectory $x_c(t, \tilde{\xi}^+(\varepsilon, \lambda), \varepsilon, \lambda)$ of (1.5) converges to $s(\varepsilon, \lambda)$, while if $\tilde{\xi}^+(\varepsilon, \lambda) \in \mathfrak{B}^+$ there is $T > 0$ such that $x_c(T, \tilde{\xi}^+(\varepsilon, \lambda), \varepsilon, \lambda) > \delta$. So, if $\tilde{\xi}^+(\varepsilon, \lambda) \in \mathfrak{A}^+$ then $(\tilde{x}(t, \varepsilon, \lambda), \tilde{y}(t, \varepsilon, \lambda)) \to S(\varepsilon, \lambda)$ as $t \to +\infty$, while if $\tilde{\xi}^+(\varepsilon, \lambda) \in \mathfrak{B}^+$ then there is $T > 0$ such that $(\tilde{x}(T, \varepsilon, \lambda), \tilde{y}(T, \varepsilon, \lambda))$ is not close to the homoclinic trajectory of the unperturbed problem $(0, h(t))$ (obviously $(\tilde{x}(t, \varepsilon, \lambda_1^+(\varepsilon)), \tilde{y}(t, \varepsilon, \lambda_1^+(\varepsilon))) \to U(\varepsilon, \lambda)$ as $t \to +\infty$). Furthermore

$$\begin{aligned} J_1^+(\varepsilon, \lambda) &= J_1^+(\varepsilon, \lambda_1^+(\varepsilon)) + \frac{\partial J_1^+}{\partial \lambda}(\varepsilon, \lambda_1^+(\varepsilon))(\lambda - \lambda_1^+(\varepsilon)) + O((\lambda - \lambda_1^+(\varepsilon))^2) \\ J_1^-(\varepsilon, \lambda) &= J_1^-(\varepsilon, \lambda_1^-(\varepsilon)) + \frac{\partial J_1^-}{\partial \lambda}(\varepsilon, \lambda_1^-(\varepsilon))(\lambda - \lambda_1^-(\varepsilon)) + O((\lambda - \lambda_1^-(\varepsilon))^2) \end{aligned} \tag{3.4}$$

From (3.2) we know the signs of $\frac{\partial}{\partial\lambda} J_1^{\pm}(\varepsilon, \lambda_1^{\pm}(\varepsilon))$; thus, exploiting these two elementary observations we deduce for which values of the nonnegative parameters ε, λ the point $\tilde{\xi}^+(\varepsilon, \lambda)$ belongs to $\mathfrak{A}^+$, $\mathfrak{B}^+$ or coincides with $u(\varepsilon, \lambda)$, and we obtain a detailed bifurcation diagram (we give some examples in figures 1, 2, 3).

To complete the picture we need to repeat the analysis in the other quadrants. When $\lambda \leq 0 < \varepsilon$ the critical point u is stable and s is unstable with respect to the flow of (1.5). So we define

$$\begin{aligned} J_4^+(\varepsilon, \lambda) &= \tilde{\xi}^+(\varepsilon, \lambda) - s(\varepsilon, \lambda)\,, \\ J_4^-(\varepsilon, \lambda) &= \tilde{\xi}^-(\varepsilon, \lambda) - u(\varepsilon, \lambda) \end{aligned} \tag{3.5}$$

Thus, if (vi) holds, we can apply the implicit function theorem and construct the curves $\lambda_4^{\pm}(\varepsilon)$ such that $J_4^{\pm}(\varepsilon, \lambda_4^{\pm}(\varepsilon)) = 0$. Moreover we find

$$\begin{aligned} \frac{d}{d\varepsilon}\lambda_4^+(0) &= -\frac{\frac{\partial}{\partial\varepsilon} J_4^+(0,0)}{\frac{\partial}{\partial\lambda} J_4^+(0,0)} = -\frac{A_0^+}{B_0 + \frac{\partial u}{\partial\lambda}(0,0)} \\ \frac{d}{d\varepsilon}\lambda_4^-(0) &= -\frac{\frac{\partial}{\partial\varepsilon} J_4^-(0,0)}{\frac{\partial}{\partial\lambda} J_4^-(0,0)} = -\frac{A_0^-}{B_0 - \frac{\partial u}{\partial\lambda}(0,0)} \end{aligned} \tag{3.6}$$

Obviously Remark 3.1 holds also in this setting with trivial modifications (and when $\varepsilon < 0$ as well, see below). Reasoning as above and using a Taylor expansion analogous to (3.4), we can draw a detailed bifurcation diagram (see again figures 1, 2, 3).

When $\varepsilon < 0$ we have an inversion in the stability properties of the critical points of (1.1) with respect to the stability properties of (1.5). Once again we assume (vi) and we distinguish between negative and positive values of λ. When $\lambda > 0$ we use again the functions $J_4^{\pm}$ defined in (3.1) and we extend the curves $\lambda_4^{\pm}(\varepsilon)$ to $\varepsilon < 0$; similarly for $\lambda < 0$ we use $J_1^{\pm}$ defined in (3.5) and we extend the curves $\lambda_1^{\pm}(\varepsilon)$. Note that also in these cases the derivatives of $\lambda_1^{\pm}$ and $\lambda_4^{\pm}$ are the ones given in (3.3) and in (3.6) so the curves are C^1 in the origin.

The bifurcation diagram changes according to the signs of the nonzero computable constants $\frac{\partial J_i^{\pm}}{\partial\lambda}(0,0)$ and of the following computable constants which may be zero

$$\frac{d}{d\varepsilon}\lambda_i^+(0), \qquad \frac{d}{d\varepsilon}\lambda_i^-(0), \qquad \frac{d}{d\varepsilon}\lambda_i^+(0) - \frac{d}{d\varepsilon}\lambda_i^-(0) \tag{3.7}$$

for $i = 1, 4$. To illustrate the meaning of Theorem 3.2 we draw some pictures for specific nonzero values of the constants given in (3.7), the other

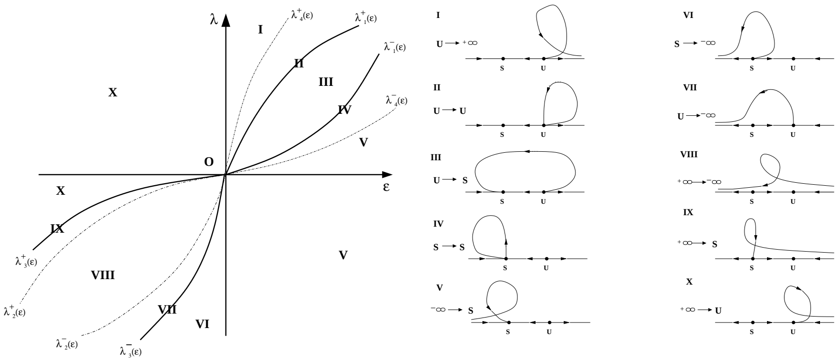

Figure 1: Bifurcation diagram for (1.6). Here we assume $\frac{\partial J_1^\pm}{\partial \lambda} > 0$, and $\frac{d\lambda_i^+}{d\varepsilon} > \frac{d\lambda_i^-}{d\varepsilon} > 0$) for $i = 1, 4$.

possibilities can be obtained similarly (not all the combinations are effectively possible). In section 4 we construct a differential equation for which the values of these constants are explicitly computed.

Theorem 3.2. *Assume that Hypotheses (i)–(vi) of the Introduction hold and that f satisfies (1.6). Then we can draw the bifurcation diagram for system (1.1), see figures 1, 2, 3)*

Remark 3.3. We think it is worthwhile to observe that generically, when $(\tilde{x}(t,\varepsilon,\lambda), \tilde{y}(t,\varepsilon,\lambda))$ tends to a critical point it has a slow rate of convergence. Namely if it converges to $S(\varepsilon,\lambda)$ as $t \to +\infty$ and $(\varepsilon,\lambda) \in Q_1$ there is $C_1 > 0$ (independent of ε and λ), such that $\|(\tilde{x}(t,\varepsilon,\lambda), \tilde{y}(t,\varepsilon,\lambda)) - S(\varepsilon,\lambda)\| \sim \exp(-C_1|\varepsilon\lambda|t)$ as $t \to +\infty$. However when $\lambda = \lambda_1^+(\varepsilon)$ and λ and ε are both positive, we have faster convergence i. e. there is $C_2 > 0$ (independent of ε and λ), such that $\|(\tilde{x}(t,\varepsilon,\lambda), \tilde{y}(t,\varepsilon,\lambda)) - U(\varepsilon, \lambda_1^+(\varepsilon))\| \sim \exp(-\varepsilon C_2 t)$ as $t \to +\infty$.

For completeness we observe that, when $\varepsilon = 0$ (1.1) reduces to

$$\dot{y} = g(\xi, y, 0, \lambda) \qquad y(\xi, t)\lfloor_{t=0} = \zeta \in \mathbb{R}^n \qquad \xi \in \mathbb{R} \tag{3.8}$$

From Hypothesis (v) it follows that there are $\delta > 0$ and a unique $\bar{\xi} \in (-\delta, \delta)$ (which is not necessarily a critical point for (1.5)) such that $y(\bar{\xi}, t)$ is a homoclinic trajectory.

When the computable constants given in (3.7) are null we cannot draw the bifurcation diagram in all details. However, using the expansion (3.4), we obtain the asymptotic behavior of $(\tilde{x}(t,\varepsilon,\lambda), \tilde{y}(t,\varepsilon,\lambda))$, far from the $\lambda = 0$ axis. When $\frac{d\lambda_i^+}{d\varepsilon}(0) = \frac{d\lambda_i^-}{d\varepsilon}(0)$ for either $i = 1, 4$, we cannot exactly determine the behavior of $(\tilde{x}(t,\varepsilon,\lambda), \tilde{y}(t,\varepsilon,\lambda))$ for (ε,λ) close to the curves $\lambda = \lambda_i^\pm(\varepsilon)$.

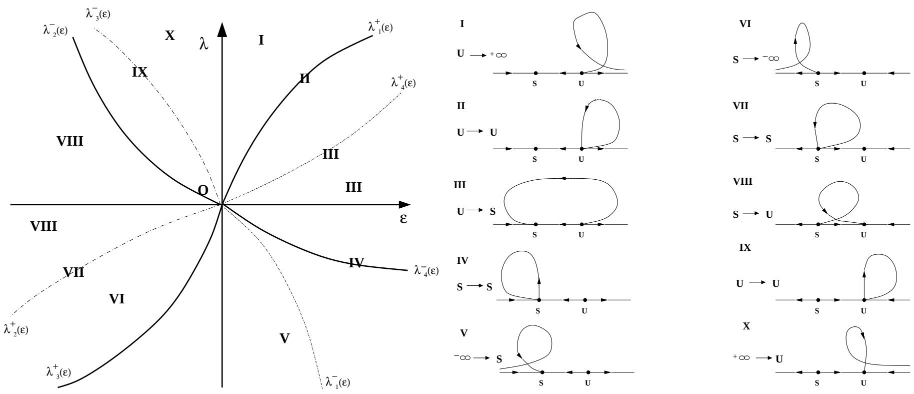

Figure 2: Bifurcation diagram for (1.6): $\frac{\partial J_1^{\pm}}{\partial \lambda} > 0$, $\frac{d}{d\varepsilon}\lambda_i^- < 0 < \frac{d}{d\varepsilon}\lambda_i^+ < 0$ for $i = 1, 4$ (and $\frac{\partial u}{\partial \lambda} > 0$).

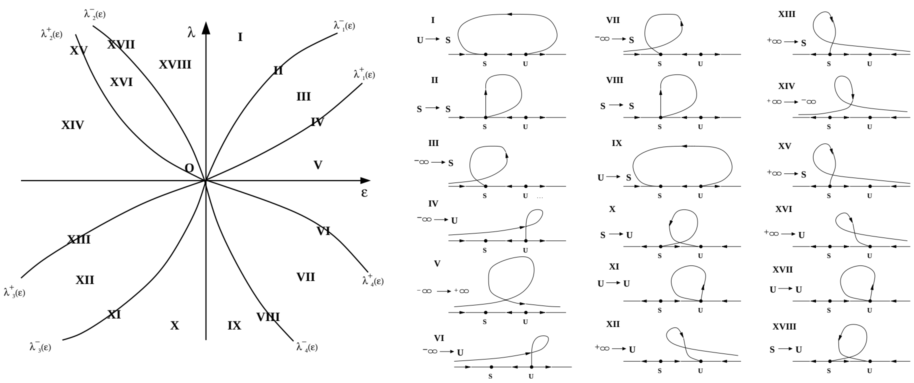

Figure 3: Bifurcation diagram for (1.6): $\frac{\partial J_1^+}{\partial \lambda} < 0 < \frac{\partial J_1^-}{\partial \lambda}$, $0 < \frac{\partial \lambda_1^+}{\partial \varepsilon} < \frac{\partial \lambda_1^-}{\partial \varepsilon}$, and $\frac{\partial \lambda_4^-}{\partial \varepsilon} < \frac{\partial \lambda_4^+}{\partial \varepsilon} < 0$, (and $\frac{\partial u}{\partial \lambda} > 0$).

Similarly when either $\frac{d\lambda_i^+}{d\varepsilon}(0) = 0$ or $\frac{d\lambda_i^-}{d\varepsilon} = 0$ for $i = 1, 4$, we cannot say whether the curves $\lambda_i^{\pm}$ are above or below the line $\lambda = 0$.
We think it is worth observing that in the previous case a new scenario may arise. In fact a priori we could have uncountably many intersections between λ_i^+ and λ_i^-. These intersections would correspond to heteroclinic trajectories with fast convergence and following the unusual direction: when ε and λ are positive the trajectory tends to S in the past and to U in the future. So $(\tilde{x}(t, \varepsilon, \lambda), \tilde{y}(t, \varepsilon, \lambda))$, together with the heteroclinic connection between U and S contained in $\mathcal{M}^c(\varepsilon, \lambda)$, form a heteroclinic cycle.

Remark 3.4. Observe that the classification result can be developed also when Hyp. (vi) is not satisfied. In such a case we should replace condition (vi) with the following, which is more difficult to handle:

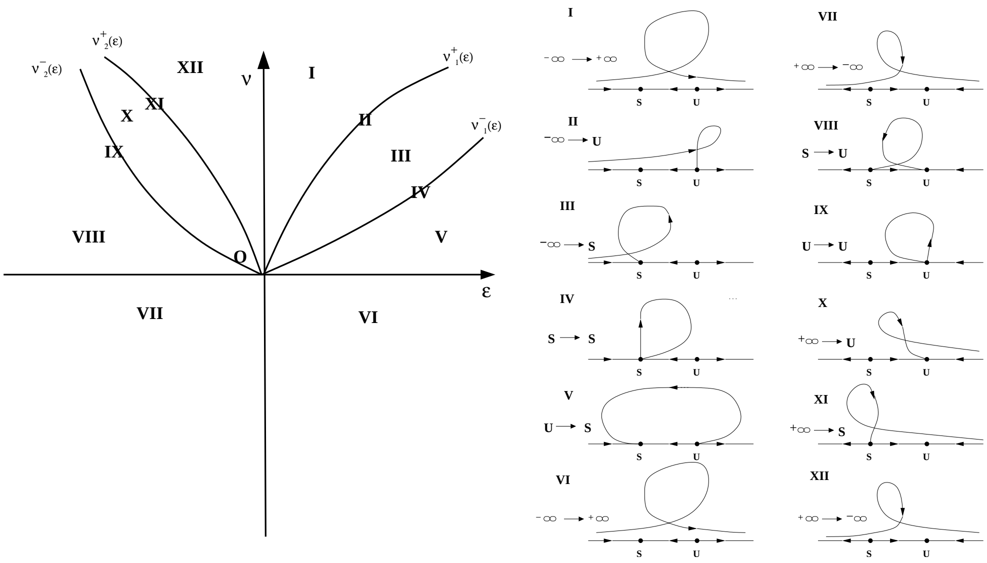

Figure 4: Bifurcation diagram in the saddle node case (1.7). We have assumed, $\frac{\partial\tilde{\xi}^+}{\partial\varepsilon}(0,0) < 0 < \frac{\partial\tilde{\xi}^-}{\partial\varepsilon}(0,0)$, $\frac{\partial\tilde{\xi}^+}{\partial\varepsilon}(0,0) < 0 < \frac{\partial\tilde{\xi}^-}{\partial\varepsilon}(0,0)$ (and $\frac{\partial u}{\partial\nu}(0) > 0$).

(vi$_\varepsilon$) $\frac{\partial\tilde{\xi}^+}{\partial\varepsilon}(0,0) \neq 0 = \frac{du}{d\varepsilon}(0)$ and $\frac{\partial\tilde{\xi}^-}{\partial\varepsilon}(0,0) \neq 0 = \frac{ds}{d\varepsilon}(0)$,

where $\frac{\partial\tilde{\xi}^\pm}{\partial\varepsilon}(0,0) = A_0^\pm$ is explicitly computed in (2.19).

In fact we may use the implicit function Theorem to prove the existence of curves $\varepsilon_i^\pm(\lambda)$ in the i^{th} quadrant, such that $(\tilde{x}(t,\varepsilon_i^\pm(\lambda),\lambda), \tilde{y}(t,\varepsilon_i^\pm(\lambda),\lambda))$ converges respectively to the unstable point of (1.5) as $t \to +\infty$ and to the stable point as $t \to -\infty$. However if $\frac{\partial\tilde{\xi}^+}{\partial\lambda}(0,0) = \frac{\partial u}{\partial\lambda}(0,0)$ the curve $\varepsilon_i^+(\varepsilon,\lambda)$ would be tangent to the $\varepsilon = 0$ axis, so once again we could not decide the behavior of the trajectory $(\tilde{x}(t,\varepsilon,\lambda), \tilde{y}(t,\varepsilon,\lambda))$ for (ε,λ) close to the $\varepsilon = 0$ axis.

3.2 Saddle-node bifurcation: f of type (1.7)

We briefly consider the case where f satisfies (1.7) so that the origin of (1.5) undergoes a saddle-node bifurcation. We need to introduce the auxiliary variable $\nu = \sqrt{|\lambda|}$ and we observe that $u(\varepsilon,\nu^2)$ and $s(\varepsilon,\nu^2)$ are smooth functions (while they are just Hölder functions of λ). Then both (1.5) and (1.1) admit two critical points in a neighborhood of the origin for $\lambda > 0$ and no critical points for $\lambda < 0$.

Theorem 3.2 works also in this setting, with some minor changes, but condition (vi) is not needed anymore. Once again we have to argue separately in

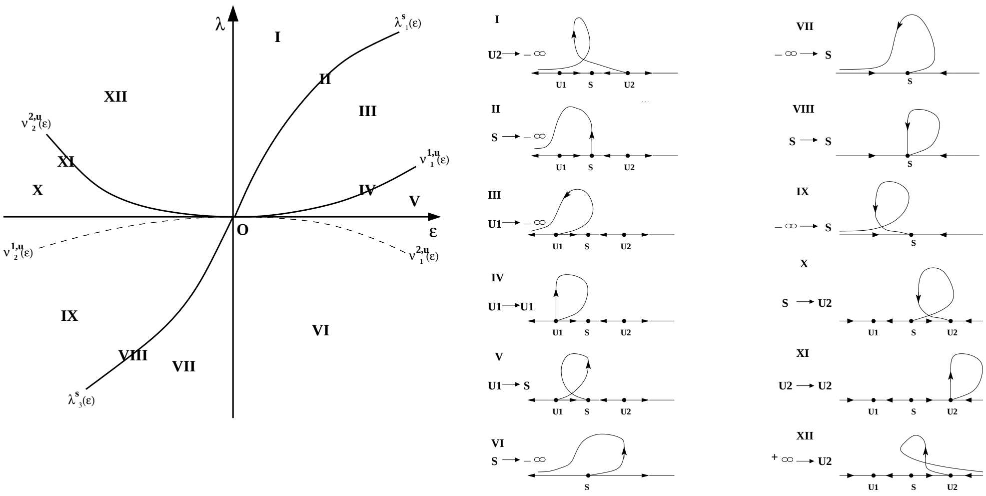

Figure 5: An example of bifurcation diagram in the pitchfork case (1.8). We have assumed, $A_0^\pm > 0$, $A_0^\pm > 0$, $\frac{\partial u^1}{\partial \nu} > 0$ and $B_0 - \frac{\partial s}{\partial \lambda} > 0$.

each quadrant of the parameter plane; we start from ε and λ positive, and we define

$$\tilde{J}_1^+(\varepsilon,\nu) = \tilde{\xi}^+(\varepsilon,\nu^2) - u(\varepsilon,\nu^2) \qquad \text{and}$$
$$\tilde{J}_1^-(\varepsilon,\nu) = \tilde{\xi}^-(\varepsilon,\nu^2) - s(\varepsilon,\nu^2)$$

and we repeat the analysis made in the previous subsection. So the solution defined by (2.16) converges to U as $t \to +\infty$ if $\tilde{J}_1^+(\varepsilon,\nu) = 0$ and to S as $t \to -\infty$ if $\tilde{J}_1^-(\varepsilon,\nu) = 0$. We stress that $\frac{\partial \tilde{\xi}^\pm}{\partial \nu}(0,0) = 0$ since $\frac{\partial \lambda}{\partial \nu}(0) = 0$, therefore

$$\frac{\partial \tilde{J}_1^-}{\partial \nu}(0,0) = \frac{\partial u}{\partial \nu}(0,0) = -\frac{\partial \tilde{J}_1^+}{\partial \nu}(0,0)\,.$$

So we can apply the implicit function Theorem and construct smooth curves $\nu_1^\pm(\varepsilon)$ such that $\nu_1^\pm(0) = 0$, $\tilde{J}_1^+(\varepsilon,\nu_1^+(\varepsilon)) = 0$ and $\tilde{J}_1^-(\varepsilon,\nu_1^-(\varepsilon)) = 0$ respectively. Furthermore

$$\frac{d}{d\varepsilon}\nu_1^+(0) = \frac{A_0^+}{\frac{\partial}{\partial \nu}u(0,0)}\,, \qquad \frac{d}{d\varepsilon}\nu_1^-(0) = -\frac{A_0^-}{\frac{\partial}{\partial \nu}u(0,0)} \tag{3.9}$$

When $\varepsilon < 0 \le \lambda$ we define

$$\tilde{J}_2^+(\varepsilon,\nu) = \tilde{\xi}^+(\varepsilon,\nu^2) - s(\varepsilon,\nu^2)\,,$$
$$\tilde{J}_2^-(\varepsilon,\nu) = \tilde{\xi}^-(\varepsilon,\nu^2) - u(\varepsilon,\nu^2)$$

and we find again curves $\nu_2^{\pm}(\varepsilon)$ such that $\nu_2^{\pm}(0) = 0$, $\tilde{J}_2^{\pm}(\varepsilon, \nu_2^{\pm}(\varepsilon)) = 0$ respectively, and

$$\frac{d}{d\varepsilon}\nu_2^+(0) = -\frac{A_0^+}{\frac{\partial}{\partial\nu}u(0,0)}\,, \quad \frac{d}{d\varepsilon}\nu_2^-(0) = \frac{A_0^-}{\frac{\partial}{\partial\nu}u(0,0)} \tag{3.10}$$

As usual the solution defined by (2.16) converges to S as $t \to +\infty$ if $\tilde{J}_2^+(\varepsilon,\nu) = 0$ and to U as $t \to -\infty$ if $\tilde{J}_2^-(\varepsilon,\nu) = 0$.

Obviously in both the cases for $\lambda < 0$ there are no critical points and hence no bounded trajectories close to the unperturbed homoclinic. Arguing as in the previous subsection we obtain a result analogous to Theorem 3.2 (see Remark 1.1 for the assumptions on the smoothness of f and g).

Theorem 3.5. *Assume Hypotheses (i)–(v) of the Introduction hold and that f satisfies (1.7). Then we can draw the bifurcation diagram for system (1.1).*

If $\frac{\partial \tilde{J}_1^+}{\partial\nu}(0,0) = \frac{\partial u}{\partial\nu}(0,0) > 0$, the bifurcation diagram of (1.1) described in Theorem 3.5 depends on the signs of the following computable constants:

$$\frac{d\nu_i^+}{d\varepsilon}(0)\,, \qquad \frac{d\nu_i^-}{d\varepsilon}(0)\,, \qquad \frac{d}{d\varepsilon}\nu_i^+(0) - \frac{d}{d\varepsilon}\nu_i^-(0). \tag{3.11}$$

We give again one example for illustrative purposes, see figure 4.

3.3 Degree 3 or more: f of type (1.8)

In this subsection we show briefly how our methods can be applied to study more degenerate singularities. We just sketch the case where f satisfies (1.8), stressing that the construction can be easily generalized to describe other singularities, even of higher order. We denote by u^1, s and u^2 the critical points of (1.5), and we set $U^1(\varepsilon,\lambda) = (u^1(\varepsilon,\lambda), v(u^1(\varepsilon,\lambda),\varepsilon,\lambda))$, $U^2(\varepsilon,\lambda) = (u^2(\varepsilon,\lambda), v(u^2(\varepsilon,\lambda),\varepsilon,\lambda))$, $S(\varepsilon,\lambda) = (s(\varepsilon,\lambda), v(s(\varepsilon,\lambda),\varepsilon,\lambda)))$. We recall that to achieve a complete unfolding of the singularity one more parameter is needed. We assume w.l.o.g. that $u^1(\varepsilon,\lambda) > u^2(\varepsilon,\lambda)$ are unstable for (1.5) (when $\lambda > 0$, they do not exist for $\lambda < 0$) while $s(\varepsilon,\lambda)$ is stable for $\lambda > 0$ and unstable for $\lambda < 0$.

Similarly to the saddle-node case the functions $u^1(\varepsilon,\lambda)$ and $u^2(\varepsilon,\lambda)$ are not smooth at the origin, so we need to introduce the parameter $\nu = \sqrt{\lambda}$. On the other hand the function $s(\varepsilon,\lambda)$ is smooth and its derivative with respect to ν is null; so, in order to apply the implicit function theorem, we have to work with $u^1(\varepsilon,\nu^2)$, $u^2(\nu^2)$ and $s(\varepsilon,\lambda)$.

Assume $\lambda \geq 0$ and $\varepsilon > 0$, we define the functions

$$J_1^{i,u}(\varepsilon,\nu) = \tilde{\xi}^+(\varepsilon,\nu^2) - u^i(\varepsilon,\nu)\,, \qquad J_1^s(\varepsilon,\lambda) = \tilde{\xi}^-(\varepsilon,\lambda) - s(\varepsilon,\lambda)$$

for $i = 1, 2$; obviously $J_1^{i,u}(0,0) = 0$ for $i = 1, 2$ and $J_1^s(0,0) = 0$. We stress that $\frac{\partial}{\partial \nu}\tilde{\xi}^+(\varepsilon, \nu^2) = 0$ for $(\varepsilon, \nu) = (0,0)$. To apply the implicit function theorem we just need to assume

(vi') $B_0 \neq \frac{\partial}{\partial \lambda} s(0,0)$

So we prove the existence of curves $\nu_1^{i,u}(\varepsilon)$, $\lambda_1^s(\varepsilon)$ such that $(\tilde{x}(t,\varepsilon,\lambda), \tilde{y}(t,\varepsilon,\lambda))$ converges to U^i as $t \to +\infty$ when $\lambda = [\nu_1^{i,u}(\varepsilon)]^2$ for $i = 1, 2$, and to S as $t \to -\infty$ when $\lambda = \lambda_1^s(\varepsilon)$. If f satisfies (1.8) then we have $\frac{\partial u}{\partial \nu}(0,0) := \frac{\partial u^1}{\partial \nu}(0,0) = -\frac{\partial u^2}{\partial \nu}(0,0)$ and $\frac{\partial u^1}{\partial \varepsilon}(0,0) = 0 = \frac{\partial u^2}{\partial \varepsilon}(0,0)$, and we get:

$$
\begin{aligned}
\frac{d}{d\varepsilon}\nu_1^{1,u}(0) &= \frac{\frac{\partial}{\partial \varepsilon}\tilde{\xi}^+(0,0) - \frac{\partial u}{\partial \varepsilon}(0,0)}{\frac{\partial u}{\partial \nu}(0,0)} = \frac{A_0^+}{\frac{\partial u}{\partial \nu}(0,0)} \\
\frac{d}{d\varepsilon}\nu_1^{2,u}(0) &= \frac{\frac{\partial}{\partial \varepsilon}\tilde{\xi}^+(0,0) + \frac{\partial u}{\partial \varepsilon}(0,0)}{\frac{\partial u}{\partial \nu}(0,0)} = -\frac{A_0^+}{\frac{\partial u}{\partial \nu}(0,0)} \\
\frac{d}{d\varepsilon}\lambda_1^s(0) &= -\frac{\frac{\partial}{\partial \varepsilon}\tilde{\xi}^-(0,0) - \frac{\partial s}{\partial \varepsilon}(0,0)}{\frac{\partial}{\partial \lambda}\tilde{\xi}(0,0) - \frac{\partial s}{\partial \lambda}(0,0)} = -\frac{A_0^-}{B_0 - \frac{\partial s}{\partial \lambda}(0,0)} ,
\end{aligned}
\tag{3.12}
$$

When $\lambda \leq 0$ the only critical point of (1.5) in a neighborhood of the origin is $s(\varepsilon, \lambda)$, which is unstable. So we define the function

$$J_4^s(\varepsilon, \lambda) = \tilde{\xi}^+(\varepsilon, \lambda) - s(\varepsilon, \lambda) .$$

Then via implicit function theorem we construct the curve $\lambda_4^s(\varepsilon)$ such that $(\tilde{x}(t,\varepsilon,\lambda_4^s(\varepsilon)), \tilde{y}(t,\varepsilon,\lambda_4^s(\varepsilon)))$ converges to S as $t \to +\infty$; moreover

$$\frac{d}{d\varepsilon}\lambda_4^s(0) = -\frac{\frac{\partial}{\partial \varepsilon}\tilde{\xi}^+(0,0) - \frac{\partial s}{\partial \varepsilon}(0,0)}{\frac{\partial}{\partial \lambda}\tilde{\xi}_0(0,0) - \frac{\partial s}{\partial \lambda}(0,0)} = -\frac{A_0^+}{B_0 - \frac{\partial s}{\partial \lambda}(0,0)} . \tag{3.13}$$

If $\lambda < 0 < \varepsilon$ and $\lambda \neq \lambda_4^s(\varepsilon)$ then $\tilde{x}(t,\varepsilon,\lambda)$ gets out from a neighborhood of the origin for $t >> 0$. When $\lambda < 0 < \varepsilon$ the trajectory $(\tilde{x}(t,\varepsilon,\lambda), \tilde{y}(t,\varepsilon,\lambda)) \to S$ as $t \to -\infty$.

When $\varepsilon < 0$ as usual the critical points of (1.1) reverse their stability behavior. When $\varepsilon < 0 \leq \lambda$ we construct via implicit function theorem the curves $\nu_2^{1,u}(\varepsilon)$, $\nu_2^{2,u}(\varepsilon)$ and $\lambda_2^s(\varepsilon)$ with the following properties: $(\tilde{x}(t,\varepsilon,\lambda), \tilde{y}(t,\varepsilon,\lambda))$ converges to U^i as $t \to -\infty$ when $\sqrt{\lambda}$ equals $\nu_2^{i,u}(\varepsilon)$ for $i = 1, 2$, and to S as $t \to +\infty$ when $\lambda = \lambda_2^s(\varepsilon)$. Moreover $\frac{d}{d\varepsilon}\lambda_2^s(0) = \frac{d}{d\varepsilon}\lambda_4^s(0)$ so $\lambda_4^s(\varepsilon)$ can be continued for $\varepsilon \leq 0$ and it is smooth, and

$$\frac{d}{d\varepsilon}\nu_2^{1,u}(0) = -\frac{A_0^-}{\frac{\partial u}{\partial \nu}(0,0)} , \qquad \frac{d}{d\varepsilon}\nu_2^{2,u}(0) = \frac{A_0^-}{\frac{\partial u}{\partial \nu}(0,0)} . \tag{3.14}$$

Similarly when both ε and λ are negative, we construct the curve $\lambda_3^s(\varepsilon)$, such that $(\tilde{x}(t,\varepsilon,\lambda),\tilde{y}(t,\varepsilon,\lambda)) \to S$ as $t \to -\infty$; moreover $\frac{d}{d\varepsilon}\lambda_3^s(0) = \frac{d}{d\varepsilon}\lambda_1^s(0)$. Furthermore $(\tilde{x}(t,\varepsilon,\lambda),\tilde{y}(t,\varepsilon,\lambda)) \to S(\varepsilon,\lambda)$ as $t \to +\infty$. Now, similarly to the previous subsections, using a Taylor expansion analogous to (3.4), we can draw the bifurcation diagram for (1.1). Once again the bifurcation diagram depends on the sign of some computable constants, i. e. $B_0 - \frac{\partial s}{\partial \lambda}(0,0)$, $\frac{\partial \nu_i^{j,u}}{\partial \varepsilon}$, for $i,j = 1,2$, $\frac{\partial \lambda_i^s}{\partial \varepsilon}$ for $i = 1,4$, see figure 5.

Remark 3.6. When f does not depend on λ, or anyway ε is the only parameter involved in the bifurcation, we can still perform our analysis, with some trivial (and simplifying) changes. When both f and g do not depend on λ, we cannot unfold completely the singularity. However the behavior of the solution $(\tilde{x}(t,\varepsilon),\tilde{y}(t,\varepsilon))$ defined by (2.16) is determined in the transcritical case by the sign of the following constants:

$$K^+ = A_0^+ - \frac{\partial u}{\partial \varepsilon}(0)\,, \qquad K^- = A_0^- - \frac{\partial s}{\partial \varepsilon}(0)\,, \tag{3.15}$$

see (2.19). E.g. if $K^\pm$ are positive and $\frac{\partial s}{\partial \varepsilon}(0) < \frac{\partial u}{\partial \varepsilon}(0)$, using a Taylor expansion we find that $\tilde{\xi}^+(\varepsilon) - u(\varepsilon)$ and $\tilde{\xi}^-(\varepsilon) - s(\varepsilon)$ are positive for $\varepsilon > 0$; thus $(\tilde{x}(t,\varepsilon),\tilde{y}(t,\varepsilon))$ converges to $U(\varepsilon)$ as $t \to -\infty$ and leaves a neighborhood of the origin for t large. Similarly when $\varepsilon < 0$ we find that $\tilde{\xi}^+(\varepsilon) - u(\varepsilon)$ and $\tilde{\xi}^-(\varepsilon) - s(\varepsilon)$ are both negative, so again $(\tilde{x}(t,\varepsilon),\tilde{y}(t,\varepsilon))$ converges to $U(\varepsilon)$ as $t \to -\infty$ and leaves a neighborhood of the origin for t large.

Reasoning in the same way it is easy to see that when (1.5) exhibits a saddle-node bifurcation, then $(\tilde{x}(t,\varepsilon),\tilde{y}(t,\varepsilon))$ is a heteroclinic connection between U and S and converges to the former in the past and to the latter in the future, since $s(\varepsilon) < \tilde{\xi}^\pm(\varepsilon) < u(\varepsilon)$ for $\varepsilon > 0$; in fact $\frac{\partial s}{\partial \varepsilon}(0) = -\infty$ and $\frac{\partial u}{\partial \varepsilon}(0) = +\infty$.

4 Examples

In this section we construct examples for which the conditions of Theorems 3.2, 3.5 are fulfilled and the derivatives of the bifurcation curves can be explicitly computed. Let us consider the following system:

$$\begin{cases} \dot{x} &= \varepsilon\big[x^2 - (\sigma\lambda)^2 + \alpha y_1 y_2 + \omega(x,y,\varepsilon,\lambda)\big] := \varepsilon f(x,y,\varepsilon,\lambda) \\ \dot{y}_1 &= y_2 + a' x y_1 + b\lambda y_1^2 + n_1(x,y,\varepsilon,\lambda) \\ \dot{y}_2 &= y_1 - (y_1)^3 + a'' x y_2 + \varepsilon c y_1^2 y_2 + n_2(x,y,\varepsilon,\lambda) \end{cases} \tag{4.1}$$

where a', a'', b, c are constants, $\omega(x,y,\varepsilon,\lambda) = O(x|y|) + o(x^2 + \varepsilon^2 + \lambda^2)$ and the functions n_1 and n_2 satisfy:

$$\begin{cases} n_1(x,y,\varepsilon,\lambda) &= d' x y_2 + \varepsilon y_2 k(y_1) + O((\lambda+\varepsilon)|x||y|) \\ n_2(x,y,\varepsilon,\lambda) &= d'' x y_1 + \lambda h(y_1) + \varepsilon r(y_1) e(y_2) + O((\lambda+\varepsilon)|x||y|) \end{cases}$$

where d', d'' constants, and h, k, r and e are smooth functions such that $h(0) = 0 = k(0)$, and $e(y_2)$ is even. We stress that the functions n_1 and n_2 play no role in the forthcoming computation. To simplify matters we have assumed $g(x, 0, \lambda, \varepsilon) = 0$, so that $v(x, \varepsilon, \lambda) \equiv 0$.

We stress that the y component of (4.1) is constructed on the unperturbed problem

$$\begin{cases} \dot{y}_1 &= g_1(0, y, 0, 0) := y_2 \\ \dot{y}_2 &= g_2(0, y, 0, 0) := y_1 - (y_1)^3 \end{cases} \tag{4.2}$$

which admits two homoclinic trajectories $\pm(\chi_1(t), \chi_2(t))$ where

$$\chi_1(t) = \frac{2\sqrt{2}}{e^t + e^{-t}}, \quad \chi_2(t) = -2\sqrt{2}\frac{e^t - e^{-t}}{(e^t + e^{-t})^2}$$

and $\chi_1^4/2 - \chi_1^2 + \chi_2^2 = 0$. So $\chi(t) = (0, \chi_1(t), \chi_2(t))$ and $-\chi(t)$ are homoclinic trajectories for (4.1) for $\varepsilon = \lambda = 0$. Note that the adjoint variational system $\dot{y} = -[\partial g/\partial y]^*(0, \pm\chi(t), 0, 0)y$ admits the unique (up to multiplicative constant) solution $\pm\psi(t) = \pm(\{\chi_1(t) - [\chi_1(t)]^3\}, -\chi_2(t))$.
From a straightforward computation we find $\frac{\partial u}{\partial \lambda}(0, 0) = \sigma = -\frac{\partial s}{\partial \lambda}(0, 0)$, and $\frac{\partial u}{\partial \varepsilon}(0, 0) = \frac{\partial s}{\partial \varepsilon}(0, 0) = 0$.

From further computations we get $\chi_1(0) = \sqrt{2}$, $\chi_2(0) = 0$, $\int_{\mathbb{R}} \chi_1^4 = \int_{\mathbb{R}} \chi_2^2 = \frac{16}{3}$, $\int_{\mathbb{R}} \chi_1^2 = 4$, $\int_{\mathbb{R}} \chi_1^6 = \frac{128}{15}$, $\int_{\mathbb{R}} \chi_1^2\chi_2^2 = \frac{16}{15}$, $\int_{\mathbb{R}} \chi_1^3 = \pi\sqrt{2}$, $\int_{\mathbb{R}} \chi_1^5 = \frac{3\pi}{\sqrt{2}}$.

$$X = \int_{-\infty}^{\infty} \pm\psi^*(t)\frac{\partial g}{\partial x}(\pm\chi(t), 0, 0)dt = \int_{-\infty}^{\infty} \left[a'(\chi_1^2 - \chi_1^4(t)) - a''\chi_2^2(t)\right]dt = \\ = -\frac{4}{3}a' - \frac{16}{3}a''$$

Moreover

$$\int_0^t f(\pm\chi(s), 0, 0)ds = \frac{\alpha}{2}[\chi_1^2(t) - \chi_1^2(0)],$$

and

$$K = \int_{-\infty}^{\infty} \pm\psi^*(t)\frac{\partial g}{\partial x}(\pm\chi(t), 0, 0)\Big[\int_{\pm\infty}^t f(\pm\chi(s), 0, 0)ds\Big]dt = -\frac{8\alpha}{15}(3a' + a''),$$
$$L = \int_{-\infty}^{\infty} \pm\psi^*(t)\frac{\partial g}{\partial \lambda}(\pm\chi(t), 0, 0)dt = \mp\frac{\pi}{\sqrt{2}}b$$
$$E = \int_{-\infty}^{\infty} \pm\psi^*(t)\frac{\partial g}{\partial \varepsilon}(\pm\chi(t), 0, 0)dt = -\frac{16}{15}c$$

We stress that conditions (i-iv) are always satisfied and (v) holds whenever $X \neq 0$, so it is in force for both $\pm\chi$ when $a' \neq -4a''$. Condition (vi) is

satisfied whenever $\sigma X \neq \pm L$. Moreover f is of type (1.6).
We just consider the bifurcation of the trajectory close to $\chi(t)$, the case of $-\chi(t)$ being analogous. So when (v) and (vi) hold, using (2.19) we find:

$$A_0^{\pm} = -\frac{K+E}{X}, \quad B_0 = -\frac{L}{X},$$

Thus, from (3.3), (3.6), we get the following:

$$\frac{\partial \lambda_1^{\pm}}{\partial \varepsilon}(0) = \frac{\partial \lambda_4^{\mp}}{\partial \varepsilon}(0) = -\frac{K+E}{L \pm \sigma X}. \tag{4.3}$$

So we can draw explicitly the bifurcation diagram of (1.1), and our description is accurate at least at the first order.

If we replace f in (4.1) by

$$f(x,y,\varepsilon,\lambda) := x^2 - (\sigma\lambda) + \alpha y_1 y_2 + \omega_{sn}(x,y,\varepsilon,\lambda)$$

where $\omega_{sn}(x,y,\varepsilon,\lambda) = O(|y||x|) + o(\varepsilon^2 + \lambda + |x|^2)$ and $\sigma > 0$ we have a saddle-node bifurcation, i.e. f satisfies (1.7). Once again condition (v) is satisfied whenever $X \neq 0$, and using (3.9) we find

$$\frac{\partial \nu_1^{\pm}}{\partial \varepsilon}(0) = \mp\frac{K}{\sqrt{\sigma}X} = \frac{\partial \nu_2^{\mp}}{\partial \varepsilon}(0) \tag{4.4}$$

So we can draw the bifurcation diagram of (1.1), also in this case.

If we replace f in (4.1) by

$$f(x,y,\varepsilon,\lambda) := (x - \tilde{\sigma}\lambda)(x^2 - \sigma\lambda) + \alpha y_1 y_2 + \omega_p(x,y,\varepsilon,\lambda)$$

where $\omega_p(x,y,\varepsilon,\lambda) = o(\varepsilon^2 + \lambda^2 + x|y| + |x|^3)$ and $\sigma > 0$, we have a pitchfork bifurcation, i.e. f satisfies (1.8). So we find

$$\begin{aligned} &\frac{\partial \nu_1^{1,u}}{\partial \varepsilon}(0) = \frac{\partial \nu_2^{2,u}}{\partial \varepsilon}(0) = -\frac{K}{\sqrt{\sigma}X} = -\frac{\partial \nu_1^{2,u}}{\partial \varepsilon}(0) = -\frac{\partial \nu_2^{1,u}}{\partial \varepsilon}(0) \\ &\frac{\partial \lambda_i^s}{\partial \varepsilon}(0) = -\frac{K+E}{L - \tilde{\sigma}X} \qquad \text{for } i = 1,4 \end{aligned} \tag{4.5}$$

Thus we obtain the bifurcation diagram of (1.1) in this case, too.
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