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PARAMETRIZATION FOR NON-LINEAR PROBLEMS WITH INTEGRAL BOUNDARY CONDITIONS

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ABSTRACT. We consider the integral boundary-value problem for a certain class of non-linear systems of ordinary differential equations of the form

$$x'(t) = f\left(t, x\left(t\right)\right), \qquad t \in [0, T],$$
$$Ax(0) + \int_0^T P(s)x(s)ds + Cx(T) = d,$$

where $f : [0, T] \times D \to \mathbb{R}^n$ is continuous vector function, $D \subset \mathbb{R}^n$ is a closed and bounded domain.

By using an appropriate parametrization technique, the given problem is reduced to an equivalent parametrized family of two-point boundary-value problems with linear boundary conditions without integral terms. To study the transformed problem, we use a method based upon a special type of successive approximations which are constructed analytically. We establish sufficient conditions for the uniform convergence of that sequence and introduce a certain finite-dimensional determining system whose solutions give all the initial values of the solutions of the given boundary-value problem. Based upon properties of the functions of the constructed sequence and of the determining equations, we give efficient conditions for the solvability of the original integral boundary-value problem.

1. Introduction

Recently, boundary-value problems with integral conditions for non-linear differential equations have attracted much attention, see, e. g. [3, 17]. However, mainly scalar non-linear differential equations of special kinds have been studied. According our best knowledge, there are only a few works dealing with a constructive investigation of systems of non-linear differential equations of a general form with integral boundary restrictions (see, e. g., [2, 6, 15, 16]).

The aim of this paper is to extend the numerical-analytic technique, which had been used earlier successfully in relation to different types of boundary-value problems with two-point and multipoint linear and non-linear boundary conditions [4, 5, 7, 13], for a class of non-linear differential systems of the form

$$x'(t) = f\left(t, x\left(t\right)\right), \qquad t \in [0, T],$$

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under the integral boundary conditions

$$Ax(0) + \int_0^T P(s)x(s)ds + Cx(T) = d.$$

We use an approach based on an appropriate parametrization technique [12,13], which allows us to reduce the given problem to an equivalent family of parametrized two-point boundary-value problems with linear boundary conditions without integral terms. To study the transformed problem, we use a method based upon a special type of successive approximations constructed analytically. We give conditions sufficient for the uniform convergence of this sequence and introduce a certain finite-dimensional "determining" system of algebraic or transcendental equations whose solutions give all the initial values of the solutions of the given boundary-value problem. Using properties of the functions of the sequence and determining equations and applying an argument based on the Brouwer degree, we give efficient conditions ensuring the solvability of the original integral boundary-value problem.

2. Notation

(1) In the sequel, the operations $|\cdot|$, $\geq$, $\leq$, max, min between matrices and vectors are understood componentwise.
(2) $\mathscr{L}(\mathbb{R}^n)$ is the algebra of n-dimensional square matrices with real elements.
(3) $\mathbf{1}_m$ and $\mathbf{0}_m$ stand, respectively, for the unit and zero matrix of dimension $m \leq n$.
(4) For any $u \in \mathbb{R}^n$ and any non-negative vector $r \in \mathbb{R}^n$, we put

$$B(u,r) := \{\xi \in \mathbb{R}^n : |\xi - u| \leq r\}. \tag{2.1}$$

(5) $r(K)$ is the spectral radius of a matrix K.
(6) $\partial\Omega$ is the boundary of Ω.
(7) $\deg(\Phi, \Omega, 0)$ is the Brouwer degree of Φ over Ω with respect to zero.

3. Problem setting

We consider the non-linear system of differential equations subjected to the integral boundary conditions

$$x'(t) = f(t, x(t)), \qquad t \in [0,T], \tag{3.1}$$

$$Ax(0) + \int_0^T P(s)x(s)ds + Cx(T) = d, \tag{3.2}$$

where A is arbitrary and C is a given singular $n \times n$ matrix of the form

$$C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & \mathbf{0}_{n-p} \end{pmatrix},$$

where C_{11} is a $p \times p$ matrix, $\det C_{11} \neq 0$, C_{12} is a $p \times (n-p)$ matrix, C_{21} is a $(n-p) \times p$ matrix, and $P : [0,T] \to \mathscr{L}(\mathbb{R}^n)$ is a continuous $n \times n$

matrix-valued function. We also assume that

$$\det(\mathbf{1}_{n-p} - C_{21}C_{11}^{-1}C_{12}) \neq 0. \tag{3.3}$$

Here, we suppose that the vector function $f : [0,T] \times D \to \mathbb{R}^n$ is continuous, where $D \subset \mathbb{R}^n$ is a closed and bounded domain.

The problem is to find a solution of the system of differential equations (3.1) with property (3.2) in the class of continuously differentiable vector functions $x : [0,T] \to D$.

4. Parametrization of the integral boundary conditions

To replace (3.2) by certain linear two-point boundary conditions, similarly to [5, 7, 12], we apply a "freezing" technique. Namely, we introduce the vectors of parameters $z = \mathrm{col}\,(z_1, z_2, \dots, z_n)$, $\lambda = \mathrm{col}\,(\lambda_1, \lambda_2, \dots, \lambda_n)$, and

$$\eta = \mathrm{col}\Big(\underbrace{0, 0, \dots, 0,}_{p} \eta_{p+1}, \eta_{p+2}, \dots, \eta_n\Big)$$

by formally putting

$$\begin{aligned} z &:= x(0), \\ \lambda &:= \int_0^T P(s)x(s)ds, \\ \eta_i &:= x_i(T), \qquad i = p+1, p+2, \dots, n, \end{aligned} \tag{4.1}$$

in (3.2). Using parametrization (4.1), the integral boundary restrictions (3.2) can be written as the linear ones:

$$Ax(0) + C_1x(T) = d - \lambda + \eta, \tag{4.2}$$

where

$$C_1 = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & \mathbf{1}_{n-p} \end{pmatrix},$$

$\mathbf{1}_{n-p}$ is a $(n-p) \times (n-p)$ unit matrix, and λ and η are the parameters with meaning (4.1).

Remark 4.1. In view of assumption (3.3), the matrix C_1 is non-singular in condition (4.4).

Let us put

$$d(\lambda, \eta) := d - \lambda + \eta. \tag{4.3}$$

Taking (4.3) into account, one can rewrite the parametrized boundary conditions (4.2) in the form

$$Ax(0) + C_1x(T) = d(\lambda, \eta). \tag{4.4}$$

The parametrization technique that we are going to use suggests that, instead of the original boundary-value problem with the integral boundary conditions (3.1), (3.2), we study the family of parametrized boundary value problems (3.1), (4.4), where the boundary restrictions are linear. We then go back to the original problem by choosing the values of the parameters appropriately.

Remark 4.2. The set of the solutions of the non-linear boundary-value problem with integral boundary conditions (3.1), (3.2) coincides with the set of the solutions of the parametrized problem (3.1) with linear boundary restrictions (4.4), satisfying additional conditions (4.1).

5. Construction of the successive approximations

Assume that the function f in the right hand-side of (3.1) satisfies the Lipschitz condition of the form

$$|f(t,u) - f(t,v)| \leq K\,|u - v|\,, \tag{5.1}$$

for all $t \in [0,T]\,,\ \{u,v\} \subset D$, where $K = (k_{ij})_{i,j=1}^{n}$ is a certain non-negative constant matrix.

Let us put

$$\mathscr{P} := \left\{ \int_0^T P(s)x(s)ds : x \in C\left([0,T],D\right) \right\}.$$

Furthermore, introduce the vector

$$\delta_D(f) := \frac{1}{2}\left[\max_{(t,x)\in[0,T]\times D} f(t,x) - \min_{(t,x)\in[0,T]\times D} f(t,x) \right], \tag{5.2}$$

and assume that the set D_* defined according to the formula

$$D_* := \Big\{ z \in D : B\Big(z + tT^{-1}C_1^{-1}\left[d(\lambda,\eta) - (A + C_1)\,z\right], \frac{T}{2}\delta_D(f)\Big) \subset D \\ \text{for all } \lambda \in \mathscr{P},\ \eta \in D,\ t \in [0,T] \Big\}$$

is non-empty:

$$D_* \neq \emptyset. \tag{5.3}$$

Recalling notation (2.1), we see that the inclusion $z \in D_*$ holds if and only if the vector $[(1 - tT^{-1})\mathbf{1}_n - tT^{-1}C_1^{-1}A]z + tT^{-1}C_1^{-1}d(\lambda,\eta)$ belongs to D together with its "vector" $\frac{T}{2}\delta_D(f)$-neighbourhood for any $\lambda \in \mathscr{P}$, $\eta \in D$, and $t \in [0,T]$.

Remark 5.1. The technical assumption (5.3) means that the domain D, where the right-hand side of the differential equation is assumed to satisfy the Lipschitz condition, is wide enough.

Let us associate with the parametrized boundary-value problem (3.1), (4.4) the sequence of functions defined recurrently by the formula

$$\begin{aligned} x_m(t,z,\lambda,\eta) := z &+ \int_0^t f(s, x_{m-1}(s,z,\lambda,\eta))ds \\ &- \frac{t}{T}\int_0^T f(s, x_{m-1}(s,z,\lambda,\eta))ds \\ &+ \frac{t}{T}C_1^{-1}\left[d(\lambda,\eta) - (A + C_1)\,z\right] \end{aligned} \tag{5.4}$$

for $t \in [0, T]$, $m = 1, 2, 3, \ldots$, where

$$x_0(t, z, \lambda, \eta) := z + \frac{t}{T} C_1^{-1} \left[d(\lambda, \eta) - (A + C_1) z \right], \qquad t \in [0, T], \tag{5.5}$$

and the vectors z, λ, and η are considered as parameters.

It is easy to check that the functions $x_m(\cdot, z, \lambda, \eta)$ satisfy linear parametrized boundary conditions (4.4) for all $m \geq 1$, z, η, $\lambda \in \mathbb{R}^n$.

Remark 5.2. It follows from the definition of the set D_* that the values of function (5.5) do not escape from D for any $z \in D_*$, $\lambda \in \mathscr{P}$, and $\eta \in D$.

The following statement establishes the convergence of sequence (5.4).

Theorem 5.1. *Let condition* (5.3) *be fulfilled, and moreover, assume that the matrix K appearing in the Lipschitz condition* (5.1) *satisfies the relation*

$$r(K) < \frac{10}{3T}. \tag{5.6}$$

Then, for all fixed $z \in D_*$, $\lambda \in \mathscr{P}$, and $\eta \in D$:

(1) *The functions of sequence* (5.4) *are continuously differentiable and satisfy the parametrized boundary conditions* (4.4)*:*

$$A x_m(0, z, \lambda, \eta) + C_1 x_m(T, z, \lambda, \eta) = d(\lambda, \eta),$$

for all $m = 1, 2, 3, \ldots$.

(2) *Sequence* (5.4) *converges uniformly in $t \in [0, T]$ as $m \to \infty$ to a limit function*

$$x^*(t, z, \lambda, \eta) := \lim_{m \to \infty} x_m(t, z, \lambda, \eta). \tag{5.7}$$

(3) *The limit function $x^*(\cdot, z, \lambda, \eta)$ satisfies the parametrized linear two-point boundary conditions:*

$$A x^*(0, z, \lambda, \eta) + C_1 x^*(T, z, \lambda, \eta) = d(\lambda, \eta).$$

(4) *Function* (5.7) *is a unique continuously differentiable solution of the integral equation*

$$x(t) = z + \int_0^t f(s, x(s)) ds - \frac{t}{T} \int_0^T f(s, x(s)) ds + \frac{t}{T} C_1^{-1} \left[d(\lambda, \eta) - (A + C_1) z \right], \qquad t \in [0, T], \tag{5.8}$$

or, which is the same, a solution of the Cauchy problem

$$x'(t) = f(t, x) + \Delta(z, \lambda, \eta), \qquad t \in [0, T], \tag{5.9}$$

$$x(0) = z, \tag{5.10}$$

where

$$\Delta(z, \lambda, \eta) := \frac{1}{T} C_1^{-1} \left[d(\lambda, \eta) - (A + C_1) \right] z - \frac{1}{T} \int_0^T f(s, x^*(s, z, \lambda, \eta)) ds. \tag{5.11}$$

(5) *The error estimate*

$$|x^*(t,z,\lambda,\eta) - x_m(t,z,\lambda,\eta)| \\ \le \frac{20}{9} t\left(1-\frac{t}{T}\right) Q^m (\mathbf{1}_n - Q)^{-1}\delta_D(f), \qquad t\in[0,T],\ m\ge 1, \quad (5.12)$$

holds, where

$$Q := \frac{3T}{10}K. \quad (5.13)$$

Proof. We will prove that the sequence of functions (5.4) is a Cauchy sequence in the Banach space $C([0,T],\mathbb{R}^n)$ of continuous vector functions on $[0,T]$. We first show that $x_m(t,z,\lambda,\eta)\in D$ for arbitrary $(t,z,\lambda,\eta)\in[0,T]\times D_*\times\mathscr{P}\times D$ and $m\ge 0$.

Indeed, it follows from Remark 5.2 that all the values of the function $x_0(\cdot,z,\lambda,\eta)$ lie in D. Let us now use the estimate of [14, Lemma 2.3] (see also [8, Lemma 3] and [11, Lemma 2]) valid for all $y\in C([0,T],\mathbb{R}^n)$:

$$\left|\int_0^t \left[y(\tau) - \frac{1}{T}\int_0^T y(s)ds\right]d\tau\right| \le \frac{1}{2}\alpha_1(t)\left[\max_{t\in[0,T]} y(t) - \min_{t\in[0,T]} y(t)\right], \quad (5.14)$$

where

$$\alpha_1(t) = 2t\left(1-\frac{t}{T}\right), \qquad t\in[0,T]. \quad (5.15)$$

Considering relation (5.4) for $m=0$ and applying estimate (5.14) with $y(t) := f(t, x_0(t,z,\lambda,\eta))$, $t\in[0,T]$, we get

$$|x_1(t,z,\lambda,\eta) - x_0(t,z,\lambda,\eta)| \le \\ \le \left|\int_0^t \left[f(t,x_0(s,z,\lambda,\eta)) - \frac{1}{T}\int_0^T f(s,x_0(s,z,\lambda,\eta))ds\right]dt\right| \\ \le \alpha_1(t)\delta_D(f) \le \frac{T}{2}\delta_D(f), \qquad t\in[0,T]. \quad (5.16)$$

In (5.16), we have used the inequality

$$|\alpha_1(t)| \le \frac{T}{2}, \qquad t\in[0,T],$$

which is obtained directly from (5.15), and the fact that $x_0(\cdot,z,\lambda,\eta)$ has values in D. Recall that the vector $\delta_D(f)$ is given by formula (5.2).

Therefore, by virtue of (5.16), we conclude that $x_1(t,z,\lambda,\eta)\in D$ whenever $(t,z,\lambda,\eta)\in[0,T]\times D_*\times\mathscr{P}\times D$. Using this and arguing by induction, we can easily establish that all functions (5.4) are also contained in the domain D for all $m=1,2,3\ldots$, $t\in[0,T]$, $z\in D_*$, $\lambda\in\mathscr{P}$, and $\eta\in D$.

Furthermore, (5.4) gives

$$x_{m+1}(t,z,\lambda,\eta) - x_m(t,z,\lambda,\eta) =$$
$$= \int_0^t \big[f(s, x_m(s,z,\lambda,\eta)) - f(s, x_{m-1}(s,z,\lambda,\eta))\big]ds -$$
$$- \frac{t}{T}\int_0^T \big[f\left(s, x_m(s,z,\lambda,\eta)\right) - f\left(s, x_{m-1}(s,z,\lambda,\eta)\right)\big]ds \quad (5.17)$$

for all $m = 1,2,3,\ldots$. Introduce the notation:

$$r_m(t,z,\lambda,\eta) := |x_m(t,z,\lambda,\eta) - x_{m-1}(t,z,\lambda,\eta)|$$

for all $m = 1,2,3\ldots$, $t \in [0,T]$, $z \in D_*$, $\lambda \in \mathscr{P}$, and $\eta \in D$. By virtue of equality (5.17), estimate (5.14) and the Lipschitz condition (5.1), we have

$$r_{m+1}(t,z,\lambda,\eta) \le K\Big[\Big(1-\frac{t}{T}\Big)\int_0^t r_m(s,z,\lambda,\eta)ds +$$
$$+ \frac{t}{T}\int_t^T r_m(s,z,\lambda,\eta)ds\Big], \quad (5.18)$$

for any $m = 0,1,2,\ldots$. According to (5.16),

$$r_1(t,z,\lambda,\eta) = |x_1(t,z,\lambda,\eta) - x_0(t,z,\lambda,\eta)| \le \alpha_1(t)\delta_D(f).$$

For $m = 1$, it follows from (5.18) that

$$r_2(t,z,\lambda,\eta) \le K\delta_D(f)\left[\Big(1-\frac{t}{T}\Big)\int_0^t \alpha_1(s)ds + \frac{t}{T}\int_t^T \alpha_1(s)ds\right]$$
$$\le K\alpha_2\left(t\right)\delta_D(f).$$

Using (5.20), we can easily obtain by induction that

$$r_{m+1}(t,z,\lambda,\eta) \le K^m \alpha_{m+1}(t)\delta_D(f), \quad (5.19)$$

for all $m = 0,1,2,\ldots$, where $\delta_D(f)$ is given by (5.2) and $\alpha_m(\cdot)$, $m = 1,2,\ldots$, are defined by the formula

$$\alpha_{m+1}(t) := \Big(1-\frac{t}{T}\Big)\int_0^t \alpha_m(s)ds + \frac{t}{T}\int_t^T \alpha_m(s)ds \quad (5.20)$$

where $m = 0,1,2,\ldots$ and

$$\alpha_0(t) := 1$$

for all $t \in [0,T]$. Clearly, α_1 is given by (5.15).

Let us now recall the estimate of [11, Lemma 3]

$$\alpha_{m+1}(t) \le \frac{10}{9}\left(\frac{3}{10}T\right)^m \alpha_1(t), \qquad t \in [0,T],\ m = 0,1,2,\ldots, \quad (5.21)$$

obtained for the sequence of functions (5.20). By virtue of the estimate (5.21), from (5.19) we get

$$r_{m+1}(t,z,\lambda,\eta) \le \frac{10}{9}\alpha_1(t)Q^m\delta_D(f) \quad (5.22)$$

for all $t \in [0, T]$ and $m = 0, 1, 2, \dots$, where the matrix Q is given by (5.13). Therefore, in view of (5.22),

$$
\begin{aligned}
|x_{m+j}(t, z, \lambda, \eta) - x_m(t, z, \lambda, \eta)| \le \\
\le |x_{m+j}(t, z, \lambda, \eta) - x_{m+j-1}(t, z, \lambda, \eta)| + \\
+ |x_{m+j-1}(t, z, \lambda, \eta) - x_{m+j-2}(t, z, \lambda, \eta)| + \ldots + \\
+ |x_{m+1}(t, z, \lambda, \eta) - x_m(t, z, \lambda, \eta)| = \\
= \sum_{i=1}^{j} r_{m+i}(t, z, \lambda, \eta) \le \frac{10}{9}\alpha_1(t) \sum_{i=1}^{j} Q^{m+i}\delta_D(f) = \\
= \frac{10}{9}\alpha_1(t) Q^m \sum_{i=0}^{j-1} Q^i \delta_D(f). \quad (5.23)
\end{aligned}
$$

Since, due to the condition (5.6), the maximal eigenvalue of the matrix Q of the form (5.13) does not exceed 1, we have

$$\sum_{i=0}^{j-1} Q^i \le (\mathbf{1}_n - Q)^{-1}$$

and $\lim_{m\to\infty} Q^m = \mathbf{0}_n$, where $\mathbf{0}_n$ is the $n \times n$ zero matrix. Therefore, we conclude from (5.23) that, according to the Cauchy criterion, the sequence $\{x_m(\cdot, z, \lambda, \eta) : m \ge 1\}$ of the form (5.4) uniformly converges in the domain $[0, T] \times D_* \times \mathscr{P} \times D$ to a limit function $x^*(\cdot, z, \lambda, \eta)$. Since all the functions $x_m(\cdot, z, \lambda, \eta)$ of the sequence (5.4) satisfy the boundary conditions (4.4) for all values of the introduced parameters, we conclude that the limit function $x^*(\cdot, z, \lambda, \eta)$ also satisfies these conditions. Passing to the limit as $m \to \infty$ in equality (5.4), we show that the limit function satisfies both the integral equation (5.8) and the Cauchy problem (5.9), (5.10), where $\Delta(z, \lambda, \eta)$ is given by (5.11). □

Consider the Cauchy problem

$$x'(t) = f(t, x) + \mu, \qquad t \in [0, T], \tag{5.24}$$

$$x(0) = z, \tag{5.25}$$

where $\mu = \operatorname{col}(\mu_1, \dots, \mu_n)$ is a control parameter.

Theorem 5.2. *Let $z \in D_*$, $\lambda \in \mathscr{P}$, $\eta \in D$ and $\mu \in \mathbb{R}^n$ be fixed. Suppose that for the system of differential equations (3.1) all conditions of Theorem 5.1 hold.*

Then, for the solution $x(\cdot, z, \lambda, \eta, \mu)$ of the initial-value problem (5.24), (5.25) *to satisfy the parametrized boundary conditions* (4.4), *it is necessary and sufficient that μ be given by the formula $\mu = \mu_{z,\lambda,\eta}$, where*

$$\mu_{z,\lambda,\eta} := \frac{1}{T}\Big[C_1^{-1}[d(\lambda, \eta) - (A + C_1)z] - \int_0^T f(s, x^*(s, z, \lambda, \eta))\Big] ds. \tag{5.26}$$

In that case,

$$x(t, z, \lambda, \eta, \mu) = x^*(t, z, \lambda, \eta), \qquad t \in [0, T], \tag{5.27}$$

where $x^*(\cdot, z, \lambda, \eta)$ *is the function* (5.7).

Proof. Sufficiency. Let us suppose that

$$\mu = \mu_{z,\lambda,\eta} \tag{5.28}$$

in the right-hand side of the system of differential equations (5.24). By virtue of Theorem 5.1, the limit function (5.7) of the sequence (5.4) is the unique solution of the problem (5.24), (4.4) for the fixed values of parameters z, λ and η and μ of form (5.28). Furthermore, the function $x^*(\cdot, z, \lambda, \eta)$ satisfies the initial conditions (5.25), i. e., it is a solution of the Cauchy problem (5.24), (5.25) for that μ. Thus, we have found the value of the parameter μ given by (5.26), for which (5.27) holds.

Necessity. Now we show that the parameter value (5.26) is unique because for any other value $\mu = \bar{\mu}$, $\bar{\mu} \neq \mu_{z,\lambda,\eta}$, the corresponding solution $x(\cdot, z, \lambda, \eta, \bar{\mu})$ of the initial value problem (5.29), (5.25),

$$x'(t) = f(t, x(t)) + \bar{\mu}, \qquad t \in [0, T], \tag{5.29}$$

does not satisfy the boundary conditions (4.4).

Indeed, assume that there exists a $\bar{\mu}$ such that $\bar{\mu} \neq \mu_{z,\lambda,\eta}$ and the solution

$$\bar{x}(t) := x(\cdot, z, \lambda, \eta, \bar{\mu}), \qquad t \in [0, T],$$

of the Cauchy problem (5.29), (5.25) satisfies the two-point parametrized boundary conditions (4.4). Let

$$x_{z,\lambda,\eta}(t) := x(t, z, \lambda, \eta, \mu_{z,\lambda,\eta}), \qquad t \in [0, T].$$

It is obvious that the functions $x_{z,\lambda,\eta}(\cdot)$ and $\bar{x}(\cdot)$ satisfy the integral equations

$$x_{z,\lambda,\eta}(t) = z + \int_0^t f(s, x_{z,\lambda,\eta}(s))ds + \mu_{z,\lambda,\eta}t \tag{5.30}$$

and

$$\bar{x}(t) = z + \int_0^t f(s, \bar{x}(s))ds + \bar{\mu}t. \tag{5.31}$$

By assumption, the functions $x_{z,\lambda,\eta}(\cdot)$ and $\bar{x}(\cdot)$ satisfy the parametrized boundary conditions (4.4) and the initial conditions (5.25). Hence,

$$Ax_{z,\lambda,\eta}(0) + C_1 x_{z,\lambda,\eta}(T) = d(\lambda, \eta), \tag{5.32}$$

$$x_{z,\lambda,\eta}(0) = z, \tag{5.33}$$

$$A\bar{x}(0) + C_1\bar{x}(T) = d(\lambda, \eta), \tag{5.34}$$

$$\bar{x}(0) = z. \tag{5.35}$$

Taking (5.32)–(5.35) into account, we get

$$x_{z,\lambda,\eta}(T) = C_1^{-1}[d(\lambda, \eta) - Az], \tag{5.36}$$

$$\bar{x}(T) = C_1^{-1}[d(\lambda, \eta) - Az]. \tag{5.37}$$

Relations (5.30), (5.31) for $t = T$ give

$$\mu_{z,\lambda,\eta} = \frac{1}{T} C_1^{-1}[d(\lambda, \eta) - (A + C_1)z] - \frac{1}{T} \int_0^T f(s, x_{z,\lambda,\eta}(s))ds, \tag{5.38}$$

$$\bar{\mu} = \frac{1}{T} C_1^{-1}[d(\lambda, \eta) - (A + C_1)z] - \frac{1}{T} \int_0^T f(s, \bar{x}(s))ds. \tag{5.39}$$

Substituting (5.38), (5.39) into the integral equations (5.30), (5.31), we get that for all $t \in [0, T]$

$$x_{z,\lambda,\eta}(t) = z + \int_0^t f(s, x_{z,\lambda,\eta}(s))ds + \\ + \frac{t}{T}\Big[C_1^{-1}[d(\lambda, \eta) - (A + C_1)z] - \int_0^T f(s, x_{z,\lambda,\eta}(s))ds\Big] \tag{5.40}$$

and

$$\bar{x}(t) = z + \int_0^t f(s, \bar{x}(s))ds + \\ + \frac{t}{T}\Big[C_1^{-1}[d(\lambda, \eta) - (A + C_1)z] - \int_0^T f(s, \bar{x}(s))ds\Big]. \tag{5.41}$$

As $z \in D_*$ and $\lambda \in \mathscr{P}$, by analogy to the proof of Theorem 5.1, according to the form of equations (5.40), (5.41) and the definition of the set D_*, it can be shown that all the values of the functions $x_{z,\lambda,\eta}(\cdot)$ and $\bar{x}(\cdot)$ are contained in D.

It is clear from (5.40), (5.41) that

$$x_{z,\lambda,\eta}(t) - \bar{x}(t) = \int_0^t \big[f(s, x_{z,\lambda,\eta}(s)) - f(s, \bar{x}(s))\big]ds - \\ - \frac{t}{T}\int_0^T \big[f(s, x_{z,\lambda,\eta}(s)) - f(s, \bar{x}(s))\big]ds, \qquad t \in [0, T]. \tag{5.42}$$

By virtue of the Lipschitz condition (5.1), from the relation (5.42) we get that the function

$$\omega(t) := |x_{z,\lambda,\eta}(t) - \bar{x}(t)|, \qquad t \in [0, T], \tag{5.43}$$

satisfies integral inequalities

$$\omega(t) \le K\Big(\int_0^t \omega(s)ds + \frac{t}{T}\int_0^T \omega(s)ds\Big) \\ \le K\alpha_1(t) \max_{s\in[0,T]} \omega(s), \qquad t \in [0, T], \tag{5.44}$$

where $\alpha_1(\cdot)$ is given by (5.15). Using (5.44) recursively, we arrive at the inequality

$$\omega(t) \le K^m \alpha_m(t) \max_{s\in[0,T]} \omega(s), \qquad t \in [0, T], \tag{5.45}$$

where $m \in \mathbb{N}$ is arbitrary and the functions α_m, $m \ge 1$, are given by the formula (5.20).

Taking (5.21) into account, from (5.45) we get the following estimate for every $m \in \mathbb{N}$:

$$\omega(t) \le K\alpha_1(t)\frac{10}{9}\left(\frac{3T}{10}K\right)^{m-1}\max_{s\in[0,T]}\omega(s), \qquad t\in[0,T].$$

By passing to the limit as $m \to \infty$ in the last inequality and by virtue of (5.6), we come to the conclusion that

$$\max_{s\in[0,T]}\omega(s) \le Q^m \max_{s\in[0,T]}\omega(s) \to 0.$$

According to (5.43), this means that the function $x_{z,\lambda,\eta}(\cdot)$ coincides with $\bar{x}(\cdot)$. Using (5.38) and (5.39), we get that $\mu_{z,\lambda,\eta} = \bar{\mu}$. This contradiction proves the theorem. □

Let us find out the relation of the limit function $x^*(\cdot, z, \lambda, \eta)$ of the sequence (5.4) to the solution of the parametrized two-point boundary-value problem (3.1) with linear boundary conditions (4.4) or the equivalent nonlinear problem (3.1) with integral conditions (3.2).

Theorem 5.3. *Under the conditions stated above, $x^*(\cdot, z^*, \lambda^*, \eta^*)$ is a solution of the integral boundaryvalue problem* (3.1), (3.2) *if and only if the components of the vectors*

$$\begin{aligned} z^* &= \mathrm{col}(z_1^*, z_2^*, \dots, z_n^*),\\ \eta^* &= \mathrm{col}(\underbrace{0,0,\dots,0}_{p}, \eta_{p+1}^*, \eta_{p+2}^*, \dots, \eta_{p+n}^*),\\ \lambda^* &= \mathrm{col}(\lambda_1^*, \lambda_2^*, \dots, \lambda_n^*) \end{aligned}$$

satisfy the determining system of algebraic or transcendental equations

$$\Delta(z,\lambda,\eta) = 0, \tag{5.46}$$

$$V(z,\lambda,\eta) = 0, \tag{5.47}$$

$$x_i^*(T, z, \lambda, \eta) - \eta_i = 0, \qquad i = p+1, \dots, n, \tag{5.48}$$

where

$$\Delta(z,\lambda,\eta) := C_1^{-1}\left[d(\lambda,\eta) - (A + C_1)z\right] - \frac{1}{T}\int_0^T f(s, x^*(s,z,\lambda,\eta))ds,$$

$$V(z,\lambda,\eta) := \int_0^T P(s)x^*(s,z,\lambda,\eta)\,ds - \lambda.$$

Proof. It suffices to apply Theorem 5.2 and notice that the differential equation (5.9) coincides with (3.1) if and only if (z^*, λ^*, η^*) satisfies the equation

$$\Delta(z^*, \lambda^*, \eta^*) = 0.$$

Moreover, from (4.1) it is clear that $x^*(\cdot, z^*, \lambda^*, \eta^*)$ coincides with the solution of the integral boundary-value problem (3.1), (3.2), if and only if

$x^*(\cdot, z^*, \lambda^*, \eta^*)$ satisfies the equations

$$\int_0^T P(s)x^*(s, z, \lambda, \eta)ds = \lambda,$$
$$x_i^*(T, z, \lambda, \eta) = \eta_i, \qquad i = p+1, \dots, n.$$

This means that $x^*(\cdot, z^*, \lambda^*, \eta^*)$ is the solution of the integral boundary-value problem (3.1), (3.2) if and only if (5.46)–(5.48) hold. □

The next statement proves that the system of determining equations (5.46)–(5.48) defines all possible solutions of the original non-linear boundary-value problem (3.1) with integral boundary restrictions (3.2).

Lemma 5.1. *Let all conditions of Theorem 5.1 be satisfied. Furthermore there exist some vectors $z \in D_*$, $\lambda \in \mathscr{P}$ and $\eta \in D$ that satisfy the system of determining equations* (5.46)*–*(5.48)*.*

Then:

(1) *The non-linear boundary-value problem* (3.1)*,* (3.2) *with integral boundary conditions has a solution $x(\cdot)$ such that*

$$x(0) = z,$$
$$\int_0^T P(s)x(s)ds = \lambda,$$
$$x_i(T) = \eta_i, \qquad i = p+1, \dots n.$$

Moreover, this solution is given by the formula

$$x(t) = x^*(t, z, \lambda, \eta), \qquad t \in [0, T], \tag{5.49}$$

where $x^*(\cdot, z, \lambda, \eta)$ is the limit function of sequence (5.4)*.*

(2) *If the boundary-value problem* (3.1)*,* (3.2) *has a solution $x(\cdot)$, then this solution is given by* (5.49)*, and the system of determining equations* (5.46)*–*(5.48) *is satisfied with*

$$z = x(0),$$
$$\lambda = \int_0^T P(s)x(s)ds, \tag{5.50}$$
$$\eta_i = x_i(T), \qquad i = p+1, \dots n.$$

Proof. We will apply Theorems 5.2 and 5.3. If there exist some $z \in D_*$, $\lambda \in \mathscr{P}$ and $\eta \in D$ that satisfy determining system (5.46)–(5.48), then according to Theorem 5.3 function (5.49) is a solution of the given boundary-value problem (3.1), (3.2). On the other hand, if $x(\cdot)$ is the solution of the original boundary-value problem (3.1), (3.2), then this function is a solution of the Cauchy problem (5.24), (5.25) with

$$\mu = 0,$$
$$z = x(0).$$

As $x(\cdot)$ satisfies integral boundary restrictions (3.2) and the corresponding conditions (4.4), by virtue of Theorem 5.2, equality (5.49) holds. Moreover,

$$\mu = \mu_{z,\lambda,\eta} = 0,$$

where the vectors λ, η are defined by (5.50).However, $\mu_{z,\lambda,\eta}$ is given by formula (5.26), and hence the first equation (5.46) of the determining system is satisfied, if z, λ, and η are given by (5.50). Using (4.4), we obtain that the other two equations (5.47), (5.48) of the determining system also hold. So, we have specified values (z, λ, η) that satisfy the system of determining equations (5.46)–(5.48), which proves the lemma. □

6. Remarks on the constructive applications of the method

Although Theorem 5.3 gives sufficient and necessary conditions for the solvability and construction of the solution of the given problem, its application faces with difficulties due the fact that the explicit form of the functions $\Delta : D_* \times \mathscr{P} \times D \to \mathbb{R}^n$, $V : D_* \times \mathscr{P} \times D \to \mathbb{R}^n$, and $x^*(\cdot, z, \lambda, \eta)$ in (5.46)–(5.48) is usually unknown. This complication can be overcome by using the properties of the function $x_m(\cdot, z, \lambda, \eta)$ of the form (5.4) for a fixed m, which will lead one, instead of the exact determining system (5.46)–(5.48), to the mth approximate system of determining equations

$$\Delta_m(z, \lambda, \eta) = 0, \tag{6.1}$$

$$V_m(z, \lambda, \eta) = 0, \tag{6.2}$$

$$x_{m,i}(T, z, \lambda, \eta) - \eta_i = 0, \qquad i = p+1, \ldots n, \tag{6.3}$$

where $\Delta_m : D_* \times \mathscr{P} \times D \to \mathbb{R}^n$ and $V_m : D_* \times \mathscr{P} \times D \to \mathbb{R}^n$ are given by the formulas

$$\begin{aligned}\Delta_m(z, \lambda, \eta) &:= \frac{1}{T} C_1^{-1} \left[d(\lambda, \eta) - (A + C_1) z \right] \\ &\quad - \frac{1}{T} \int_0^T f(s, x_m(s, z, \lambda, \eta)) ds,\end{aligned} \tag{6.4}$$

$$V_m(z, \lambda, \eta) := \int_0^T P(s) x_m(s, z, \lambda, \eta)\, ds - \lambda, \tag{6.5}$$

and $x_m(\cdot, z, \lambda, \eta)$ is the vector-function defined according to relation (5.4).

It is important to note that, unlike to system (5.46)–(5.48) the mth approximate determining system (6.1)–(6.3) contains only terms involving the function $x_m(\cdot, z, \lambda, \eta)$ and, therefore, constructed explicitly.

In the next section we will show how, under certain natural assumptions, the approximate determining system can be used in solvability analysis.

7. Existence of solutions of the integral boundary-value problem

In the sequel, we need a lemma providing an estimate for functions (5.11) and (6.4).

Lemma 7.1. *Let conditions of Theorem 5.1 be satisfied.*

Then, for an arbitrary $m \geq 1$, the exact and approximate determining functions $\Delta : D_* \times \mathscr{P} \times D \to \mathbb{R}^n$ and $\Delta_m : D_* \times \mathscr{P} \times D \to \mathbb{R}^n$ defined by (5.11) *and* (6.4) *satisfy the estimate*

$$|\Delta(z, \lambda, \eta) - \Delta_m(z, \lambda, \eta)| \leq \frac{10T}{27} K Q^m (\mathbf{1}_n - Q)^{-1} \delta_D(f), \tag{7.1}$$

where $(z, \lambda, \eta) \in D_* \times \mathscr{P} \times D$ of the form (4.1) *are arbitrary and K, Q, $\delta_D(f)$ are given respectively by* (5.1), (5.13), *and* (5.2).

Proof. Let us fix arbitrary z, λ, η of the form (4.1). Using the Lipschitz condition (5.1), estimate (5.12), and the equality

$$\int_0^T \alpha_1(t)dt = \frac{T^2}{3},$$

we have

$$\begin{aligned}
&|\Delta(z, \lambda, \eta) - \Delta_m(z, \lambda, \eta)| = \\
&= \left| \frac{1}{T} \int_0^T f\left(s, x_m(s, z, \lambda, \eta)\right) ds - \frac{1}{T} \int_0^T f\left(s, x^*(s, z, \lambda, \eta)\right) ds \right| \leq \\
&\leq \frac{1}{T} \int_0^T K \left| x^*(s, z, \lambda, \eta) - x_m(s, z, \lambda, \eta) \right| ds \leq \\
&\leq \frac{1}{T} K \int_0^T \frac{10}{9} \alpha_1(s) Q^m \left(\mathbf{1}_n - Q\right)^{-1} \delta_D(f) ds = \\
&= \frac{10}{9T} K Q^m \left(\mathbf{1}_n - Q\right)^{-1} \delta_D(f) \int_0^T \alpha_1(s) ds = \\
&= \frac{10T}{27} K Q^m \left(\mathbf{1}_n - Q\right)^{-1} \delta_D(f),
\end{aligned}$$

which completes the proof. □

Lemma 7.2. *Let conditions of Theorem 5.1 be satisfied. Then for arbitrary $m \geq 1$ and $(z, \lambda, \eta) \in D_* \times \mathscr{P} \times D$ of the form* (4.1), *the functions $x^*(\cdot, z, \lambda, \eta)$ and $x_m(\cdot, z, \lambda, \eta)$ defined by* (5.7), (5.4) *satisfy the estimate*

$$\begin{aligned}
&\left| \int_0^T P(s)[x^*\left(s, z, \lambda, \eta\right) ds - x_m\left(s, z, \lambda, \eta\right)] ds \right| \leq \\
&\qquad \leq \frac{10}{9} \bar{B} Q^m \left(\mathbf{1}_n - Q\right)^{-1} \delta_D(f)
\end{aligned} \tag{7.2}$$

where Q, $\delta_D(f)$ are given by (5.13), (5.2) *and*

$$\bar{B} := \int_0^T |P(s)| \alpha_1(s) ds.$$

Proof. Let us fix arbitrary z, λ, η of the form (4.1). By virtue of the estimate (5.12), we have:

$$\begin{aligned}\left|\int_0^T P(s)x^*(s,z,\lambda,\eta)\,ds-\int_0^T P(s)x_m(s,z,\lambda,\eta)\,ds\right|\\ \le\int_0^T |P(s)|\,|x^*(s,z,\lambda,\eta)-x_m(s,z,\lambda,\eta)|\,ds\\ \le\int_0^T |P(s)|\,\frac{10}{9}\alpha_1(s)Q^m\left(\mathbf{1}_n-Q\right)^{-1}\delta_D(f)ds\\ =\frac{10}{9}\int_0^T |P(s)|\,\alpha_1(s)dsQ^m\left(\mathbf{1}_n-Q\right)^{-1}\delta_D(f)\\ =\frac{10}{9}\bar{B}Q^m\left(\mathbf{1}_n-Q\right)^{-1}\delta_D(f).\end{aligned}$$

The last estimate completes the proof. □

On the base of equations (5.46)–(5.48) and (6.1)–(6.3) let us introduce the mappings $\Phi : D_*\times\mathscr{P}\times D\to\mathbb{R}^{3n}$ and $\Phi_m : D_*\times\mathscr{P}\times D\to\mathbb{R}^{3n}$ by setting

$$\Phi(z,\lambda,\eta):=\begin{pmatrix}\Delta(z,\lambda,\eta)\\ V(z,\lambda,\eta)\\ x^*_{p+1}(T,z,\lambda,\eta)-\eta_{p+1}\\ x^*_{p+2}(T,z,\lambda,\eta)-\eta_{p+2}\\ \vdots\\ x^*_n(T,z,\lambda,\eta)-\eta_n\end{pmatrix}, \tag{7.3}$$

and

$$\Phi_m(z,\lambda,\eta):=\begin{pmatrix}\Delta_m(z,\lambda,\eta)\\ V_m(z,\lambda,\eta)\\ x_{m,p+1}(T,z,\lambda,\eta)-\eta_{p+1}\\ x_{m,p+2}(T,z,\lambda,\eta)-\eta_{p+2}\\ \vdots\\ x_{m,n}(T,z,\lambda,\eta)-\eta_n\end{pmatrix} \tag{7.4}$$

for all $(z,\lambda,\eta)\in D_*\times\mathscr{P}\times D$ of the form (4.1).

Definition 7.1 ([10])**.** Let $H\subset\mathbb{R}^{3n}$ be an arbitrary non-empty set. For any pair of functions $f_j=\operatorname{col}(f_{j1},\dots,f_{j,3n}) : H\to\mathbb{R}^{3n}$, $j=1,2$, we write

$$f_1 \rhd_H f_2 \tag{7.5}$$

if and only if there exist a function

$$k : H\to\{1,2,\dots,3n\}$$

such that

$$f_{1,k(x)}(x)>f_{2,k(x)}(x)$$

for all $x\in H$.

Remark 7.1. Relation (7.5) means that at every point $x \in H$ at least one of the components of the vector $f_1(x)$ is greater then the corresponding component of the vector $f_2(x)$.

Let us consider the set

$$\Omega = D_1 \times \Lambda_1 \times D_2, \tag{7.6}$$

where $D_1 \subset D_*$, $\Lambda_1 \subset \mathscr{P}$, $D_2 \subset D$ are certain bounded open sets.

Theorem 7.1. *Assume that conditions of Theorem 5.1 hold and, moreover, one can specify an $m \geq 1$ and a set $\Omega \subset \mathbb{R}^{3n}$ of the form* (7.6) *such that*

$$|\Phi_m| \triangleright_{\partial\Omega} \begin{pmatrix} \frac{10T}{27} K Q^m \left(\mathbf{1}_n - Q\right)^{-1} \delta_D(f) \\ \frac{10}{9} \bar{B} Q^m \left(\mathbf{1}_n - Q\right)^{-1} \delta_D(f) \\ \frac{5T}{9} Q^m \left(\mathbf{1}_n - Q\right)^{-1} \delta_D(f) \end{pmatrix}. \tag{7.7}$$

If, in addition,

$$\deg\left(\Phi_m, \Omega, 0\right) \neq 0, \tag{7.8}$$

then there exist some $(z^*, \lambda^*, \eta^*) \in \Omega$ such that the function

$$x^*(t) := x^*(t, z^*, \lambda^*, \eta^*), \qquad t \in [0, T], \tag{7.9}$$

is a solution of the boundary-value problem (3.1) (3.2) *with the initial condition*

$$x^*(0) = z^*. \tag{7.10}$$

Proof. Let us prove that the vector fields Φ and Φ_m are homotopic. For this purpose, following [12], we consider the "linear deformation"

$$P(\theta, z, \lambda, \eta) := \Phi_m(z, \lambda, \eta) + \theta\left[\Phi(z, \lambda, \eta) - \Phi_m(z, \lambda, \eta)\right], \tag{7.11}$$

where $(z, \lambda, \eta) \in \partial\Omega$, $\theta \in [0, 1]$.

Obviously, $P(\theta, \cdot, \cdot, \cdot)$ is continuous mapping on $\partial\Omega$ for every $\theta \in [0, 1]$ and, furthermore,

$$P(0, z, \lambda, \eta) = \Phi_m(z, \lambda, \eta), \qquad P(1, z, \lambda, \eta) = \Phi(z, \lambda, \eta)$$

for all $(z, \lambda, \eta) \in \partial\Omega$.

For an arbitrary $(z, \lambda, \eta) \in \partial\Omega$, in view of (7.11), we have

$$\begin{aligned} |P\left(\theta, z, \lambda, \eta\right)| = |\Phi_m(z, \lambda, \eta) + \theta\left[\Phi(z, \lambda, \eta) - \Phi_m(z, \lambda, \eta)\right]| \geq \\ \geq |\Phi_m(z, \lambda, \eta)| - |\Phi(z, \lambda, \eta) - \Phi_m(z, \lambda, \eta)|. \end{aligned} \tag{7.12}$$

On the other hand, recalling equalities (7.3), (7.4) and using approximation (5.4) and estimate (7.2), we obtain the inequalities

$$|\Phi(z, \lambda, \eta) - \Phi_m(z, \lambda, \eta)| \leq \begin{pmatrix} \frac{10T}{27} K Q^m \left(\mathbf{1}_n - Q\right)^{-1} \delta_D(f) \\ \frac{10}{9} \bar{B} Q^m \left(\mathbf{1}_n - Q\right)^{-1} \delta_D(f) \\ \frac{5T}{9} Q^m \left(\mathbf{1}_n - Q\right)^{-1} \delta_D(f) \end{pmatrix}, \tag{7.13}$$

whence, in view of (7.7), (7.12), (7.13), it follows that

$$|P\left(\theta, \cdot, \cdot, \cdot\right)| \triangleright_{\partial\Omega} 0, \qquad \theta \in [0, 1]. \tag{7.14}$$

The relation (7.14) implies, in particular, that $P\left(\theta, \cdot, \cdot, \cdot\right)$ does not vanish on $\partial\Omega$ for any value of $\theta \in [0, 1]$, i. e., deformation (7.11) is non-degenerate and, thus, Φ_m is homotopic to Φ. Using (7.8) and the property of invariance of Brouwer degree under homotopy, we conclude that

$$\deg(\Phi, \Omega, 0) = \deg(\Phi, \Omega, 0) \neq 0.$$

The classical topological result (see, e. g., [1, Theorem A.2.4]) then guaranties the existence of vectors $(z^*, \lambda^*, \eta^*) \in \Omega$ such that

$$\Phi(z^*, \lambda^*, \eta^*) = 0.$$

Therefore, the vector (z^*, λ^*, η^*) satisfies the system of determining equations (5.46)–(5.48).

Applying now Theorem 5.3, we find that the function (7.9) is a solution of the original nonlinear boundary-value problem (3.1), (3.2) with the initial value satisfying (7.10). □

8. Notes on proving the solvability

According to the approach developed here, the proof of the solvability of the original boundary-value problem (3.1), (3.2) is based on Theorems 5.1 and 7.1. Theorem 5.1 ensures the convergence of the iteration method and, in particular, justifies the further argument that involves functions of sequence (5.4) and their limit (5.7). On the other hand, applying Theorem 7.1, one can use properties of finitely many functions of sequence (5.4) to establish that the solution of (3.1), (3.2) exists.

Remark 8.1. In order to apply Theorem 7.1, one has to:

- compute the vector $\delta_D(f)$ according to (5.2) (or estimate it from above)
- construct the function $x_m(\cdot, z, \lambda, \eta)$ analytically for a certain *fixed* value $m = m_0$, keeping z, λ, and η as parameters
- select a suitable set Ω and verify conditions (7.7), (7.8) for $m = m_0$.

Remark 8.2. To verify condition (7.7) of Theorem 7.1 in concrete cases, one has to use the recurrence formula (5.4) to compute the function $x_m(\cdot, z, \lambda, \eta)$ depending on $z \in D_*$, $\lambda \in \mathscr{P}$, $\eta \in D$ as parameters and verify whether at least one of the components of the vector $|\Phi_m(z, \lambda, \eta)|$ is strictly greater than the corresponding component of the appropriate vector in the right-hand side at every point (z, λ, η) of $\partial\Omega$.

After that, we need verify in (7.8) whether the topological degree of Φ_m is not zero. This is rather difficult problem in general. However, there are sufficient conditions applicable in a number of important cases. In particular, when Φ_m is an odd mapping, i. e.,

$$\Phi_m(-z, -\lambda, -\eta) = -\Phi_m(z, \lambda, \eta)$$

for all (z, λ, η), then, according to the Borsuk theorem (see [1, Theorem A2.12]), its Brouwer degree is an odd number and therefore, is not equal to zero.

Alternatively, it follows directly from the definition of the topological degree (see [1, Definition A2.1]) that if the Jacobian matrix of the function Φ_m in (7.4) is non-singular at its isolated zero $(z_{m,0}, \lambda_{m,0}, \eta_{m,0})$, i. e.,

$$\det \frac{\partial}{\partial z\, \partial\lambda\, \partial\eta} \Phi_m(z_{m,0}, \lambda_{m,0}, \eta_{m,0}) \neq 0,$$

then inequality (7.8) holds.

9. An illustrative example

Let us apply the numerical-analytic scheme described above to the system of differential equations

$$\begin{aligned} x_1'(t) &= 0.05x_2 + x_1x_2 - 0.005t^2 - 0.01t^3 + 0.1, \\ x_2'(t) &= 0.5x_1 - x_2^2 + 0.01t^4 + 0.15t, \end{aligned} \tag{9.1}$$

considered for $t \in \left[0, \frac{1}{2}\right]$ with the two-point integral boundary conditions

$$Ax(0) + \int_0^{\frac{1}{2}} P(s)x(s)ds + Cx\left(\frac{1}{2}\right) = d, \tag{9.2}$$

where

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad d = \begin{pmatrix} 13/256 \\ 7/960 \end{pmatrix},$$

and

$$P(t) = \begin{pmatrix} 0 & t/2 \\ 1/2 & 1/4 \end{pmatrix}, \qquad t \in [0, \tfrac{1}{2}].$$

It is easy to check that the pair of functions

$$x_1^*(t) = 0.1t, \qquad x_2^*(t) = 0.1t^2$$

is an exact solution of the problem (9.1), (9.2).

Suppose that the boundary-value problem (9.1), (9.2) is considered in the domain

$$D = \{(x_1, x_2) : |x_1| \le 0.42, |x_2| \le 0.4\}.$$

Following (4.1), introduce the parameters:

$$\begin{gathered} \operatorname{col}(x_1(0), x_2(0)) =: \operatorname{col}(z_1, z_2), \\ \int_0^T P(s)x(s)ds =: \operatorname{col}(\lambda_1, \lambda_2), \\ x_2\left(\frac{1}{2}\right) =: \eta_2. \end{gathered} \tag{9.3}$$

The formal substitution (9.3) transforms the boundary restrictions (9.2) to the linear conditions

$$Ax(0) + C_1x\left(\frac{1}{2}\right) = d(\lambda, \eta), \tag{9.4}$$

where $\eta = \operatorname{col}(0, \eta_2)$, $C_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $d(\lambda, \eta) := d - \lambda + \eta$. The matrix C_1 is, of course, non-singular.

Put

$$f_1(t, x_1, x_2) := 0.05x_2 + x_1x_2 - 0.005t^2 - 0.01t^3 + 0.1,$$
$$f_2(t, x_1, x_2)) := 0.5x_1 - x_2^2 + 0.01t^4 + 0.15t.$$

Then (9.1) takes form (3.1) with $T = \frac{1}{2}$, $n = 2$, and it is then easy to check that the matrix K from the Lipschitz condition (5.1) can be taken as

$$K = \begin{pmatrix} 0 & 0.05 \\ 0.5 & 0.8 \end{pmatrix},$$

and

$$r(K) < 0.84 < \frac{10}{3T}.$$

The vector $\delta_D(f)$ can be estmated as

$$\delta_D(f) \leq \begin{pmatrix} 0.18925 \\ 0.3278125 \end{pmatrix}.$$

The role of D_* is played by the domain defined by inequalities:

$$z_1 + 2t(0.05078125000 - \lambda_1 - z_1) \leq 0.0473125,$$
$$z_2 + 2t(0.007291666667 - \lambda_2 + \eta_2 - 2z_2) \leq 0.081953125.$$

The domain $\mathscr{P}$ is such that

$$\mathscr{P} = \{(\lambda_1, \lambda_2) : |\lambda_1| \leq 0.105, |\lambda_2| \leq 0.31\}.$$

One can verify that, for the parametrized boundary-value problem (9.1), (9.4), all the needed conditions are fulfilled, and we can proceed with application of the numerical-analytic scheme described above. As a result, we construct the sequence of approximate solutions.

The components of the iteration sequence (5.4) for the boundary-value problem (9.1) under the linear parametrized two-point boundary conditions (9.4) have the form

$$\begin{aligned} x_{m,1}(t, z, \lambda, \eta) := z_1 + \int_0^t & f_1(s, x_{m-1,1}(s, z, \lambda, \eta), x_{m-1,2}(s, z, \lambda, \eta))\, ds \\ -2t \int_0^{\frac{1}{2}} & f_1(s, x_{m-1,1}(s, z, \eta, \lambda), x_{m-1,2}(s, z, \eta, \lambda))\, ds \\ & +2t(0.05078125 - \lambda_1 - z_1), \end{aligned} \tag{9.5}$$

$$\begin{aligned} x_{m,2}(t, z, \lambda, \eta) := z_2 + \int_0^t & f_2(s, x_{m-1,1}(s, z, \lambda, \eta), x_{m-1,2}(s, z, \lambda, \eta))\, ds \\ -2t \int_0^{\frac{1}{2}} & f_2(s, x_{m-1,1}(s, z, \lambda, \eta), x_{m-1,2}(s, z, \lambda, \eta)))ds \\ & +2t(0.007291666667 - \lambda_2 + \eta_2 - 2z_2), \end{aligned} \tag{9.6}$$

for $m = 1, 2, 3, \dots$, where

$$x_{0,1}(t, z, \eta, \lambda) = z_1 + 2t(0.05078125 - \lambda_1 - z_1), \tag{9.7}$$
$$x_{0,2}(t, z, \eta, \lambda) = z_2 + 2t(0.007291666667 - \lambda_2 + \eta_2 - 2z_2). \tag{9.8}$$

The system of approximate determining equations of the form (6.1)–(6.3) for the given example at the mth step is

$$\Delta_{m,1}(z,\lambda,\eta) = 0, \tag{9.9}$$
$$\Delta_{m,2}(z,\lambda,\eta) = 0, \tag{9.10}$$
$$\int_0^{\frac{1}{2}} P(s)x_m(s,z,\lambda,\eta)ds = \lambda, \tag{9.11}$$
$$x_{m,2}\left(\frac{1}{2},z,\lambda,\eta\right) = \eta_2, \tag{9.12}$$
$$\tag{9.13}$$

where

$$\begin{aligned}\Delta_{m,1}(z,\lambda,\eta) := & -2\int_0^{\frac{1}{2}} f_1\left(s, x_{m-1,1}(s,z,\lambda,\eta), x_{m-1,2}(s,z,\lambda,\eta)\right)ds \\ & + 2(0.05078125 - \lambda_1 - z_1), \\ \Delta_{m,2}(z,\lambda,\eta) = & -2\int_0^{\frac{1}{2}} f_2\left(s, x_{m-1,1}(s,z,\lambda,\eta), x_{m-1,2}(s,z,\lambda,\eta)\right)ds \\ & + 2(0.007291666667 - \lambda_2 + \eta_2 - 2z_2).\end{aligned}$$

Using (9.5)–(9.8) at the first iteration ($m = 1$) and applying Maple 13, we get

$$\begin{aligned}x_{11} = & -0.0025t^4 + 0.1019859484t + 1.333333333t^3\lambda_1\lambda_2 - \\ & - 1.333333333t^3\lambda_1\eta_2 + 2.666666666t^3\lambda_1 z_2 + 1.333333333t^3 z_1\lambda_2 - \\ & - 1.333333333t^3 z_1\eta_2 + 2.666666666t^3 z_1 z_2 + t^2 z_1\eta_2 - t^2 z_1\lambda_2 - \\ & - 3t^2 z_1 z_2 - t^2\lambda_1 z_2 - 0.3333333334t\lambda_1\lambda_2 + \\ & + 0.3333333334t\lambda_1\eta_2 - 0.1666666667t\lambda_1 z_2 + 0.1666666666tz_1\lambda_2 - \\ & - 0.1666666666tz_1\eta_2 - 2.001215278tz_1 - 0.6770833333t^3\lambda_2 + \\ & + 0.06770833333t^3\eta_2 - 0.1354166667t^3 z_2 - 0.009722222219t^3\lambda_1 - \\ & - 0.009722222219t^3 z_1 - 0.05t^2\lambda_2 + 0.05t^2\eta_2 - 0.04921875t^2 z_2 + \\ & + 0.007291666665t^2 z_1 + 0.04192708333t\lambda_2 - 0.04192708333t\eta_2 - \\ & - 1.997569444t\lambda_1 + 0.0003645833334t^2 - 0.001172960069t^3 + z_1\end{aligned}$$

and

$$
\begin{aligned}
x_{12} = {} & -0.03571925636t - 1.333333333t^3\lambda_2^2 - 1.333333333t^3\eta_2^2 - \\
& - 5.333333333t^3 z_2^2 - 0.5t^2\lambda_1 + 4t^2 z_2^2 + 0.3333333334t\lambda_2^2 + \\
& + 0.3333333334t\eta_2^2 + 0.25tz_1 + 0.01944444444t^3\lambda_2 - \\
& - 0.01944444444t^3\eta_2 + 0.03888888888t^3 z_2 - 0.01458333333t^2 z_2 - 0.5t^2 z_1 - \\
& - 2.004861111t\lambda_2 + 2.004861111t\eta_2 + 0.25t\lambda_1 + 0.002t^5 + \\
& + 2.666666666t^3\lambda_2\eta_2 - 5.333333333t^3\lambda_2 z_2 + \\
& + 5.333333333t^3\eta_2 z_2 + 2t^2\lambda_2 z_2 - 2t^2\eta_2 z_2 - 0.6666666667t\lambda_2\eta_2 + \\
& + 0.3333333334t\lambda_2 z_2 - 0.3333333334t\eta_2 z_2 + \\
& + 0.100390625t^2 - 0.00007089120366t^3 + z_2
\end{aligned}
$$

for all $t \in \left[0, \frac{1}{2}\right]$. Here and below, we omit the obvious arguments reflecting the dependence on z_1, z_2, λ_1, λ_2, and η_2.

The computation shows that the approximate solutions of the determining system (9.9)–(9.12) for $m = 1$ are

$$
\begin{aligned}
z_1 &\approx z_{11} = -4.253290711 \cdot 10^{-7}, \\
z_2 &\approx z_{12} = 7.295492706 \cdot 10^{-7}, \\
\lambda_1 &\approx \lambda_{11} = 0.0007814848293, \\
\lambda_2 &\approx \lambda_{12} = 0.007290937121, \\
\eta_2 &\approx \eta_{12} = 0.0249993271.
\end{aligned}
$$

Hence, the components of the first approximation to the first and second components of solution are

$$
\begin{aligned}
x_{11} = {} & -0.0025t^4 + 0.09968792498t - 4.253290711 \cdot 10^{-7} + \\
& + 0.001249955722t^2 - 8.714713042 \cdot 10^{-8}t^3
\end{aligned}
$$

and

$$
\begin{aligned}
x_{12} = {} & 0.00008047566353t + 0.002t^5 + 7.295492706 \cdot 10^{-7} + \\
& + 0.1000000588t^2 - 0.0008332398387t^3.
\end{aligned}
$$

The graphs of the first approximation and the exact solution of the original boundary-value problem are shown on Figure 1.

The error of the first approximation is

$$
\begin{aligned}
\max_{t \in [0, \frac{1}{2}]} |x_1^*(t) - x_{11}(t)| &\le 2.1 \cdot 10^{-5}, \\
\max_{t \in [0, \frac{1}{2}]} |x_2^*(t) - x_{12}(t)| &\le 2.2 \cdot 10^{-6}.
\end{aligned}
$$

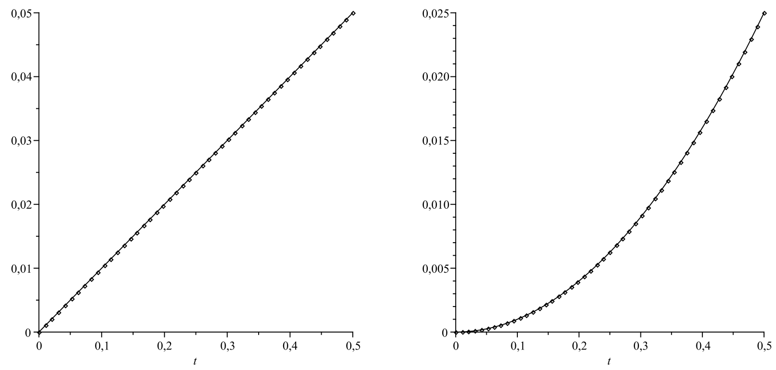

Figure 1: The first components of the exact solution (solid line) and its first approximation (drawn with dots)

Similarly, the error of the second approximation is

$$\max_{t\in\left[0,\frac{1}{2}\right]} |x_1^*(t) - x_{21}(t)| \le -4.03 \cdot 10^{-8},$$

$$\max_{t\in\left[0,\frac{1}{2}\right]} |x_2^*(t) - x_{22}(t)| \le 1.2 \cdot 10^{-6}.$$

Continuing calculations, one can get approximate solutions of the original boundary-value problem with higher precision.

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