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# Multiple positive solutions for second order impulsive differential equation*

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**Abstract:** We investigate the existence of positive solutions to a three-point boundary value problem of second order impulsive differential equation. Our analysis rely on the Avery-Peterson fixed point theorem in a cone. An example is given to illustrate our result.
**Keywords:** impulsive differential equation; fixed point theorem; positive solution; completely continuous operator

## 1. Introduction

Impulsive differential equations have very good applications in economics, biology, ecology and other fields(see[1-3]). Many authors are interested in the boundary value problem of impulsive differential equations (see [4-23]). For example, in [6,7], R. P. Agarwal and D. O'Regan studied the existence of solutions for the boundary value problems

$$y''(t) + \phi(t)f(t, y(t)) = 0, \quad t \in (0,1) \setminus \{t_1, t_2, \cdots, t_m\},$$
$$\Delta y(t_k) = I_k(y(t_k^-)), \quad k = 1, 2, \cdots, m,$$
$$\Delta y'(t_k) = J_k(y(t_k^-)), \quad k = 1, 2, \cdots, m,$$
$$y(0) = y(1) = 0,$$

by using Krasnoselskii's fixed point theorem and the Leggett Williams fixed point theorem, respectively. Using the fixed point index theory, T. Jankowski ([23]) obtained the existence of solutions for the boundary value problem

$$x''(t) + \alpha(t)f(x(\alpha(t))) = 0, \quad t \in (0,1) \setminus \{t_1, t_2, \cdots, t_m\},$$

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$$\Delta y'(t_k) = Q_k(x(t_k)), \quad k = 1, 2, \cdots, m,$$

$$x(0) = 0, \quad \beta x(\eta) = x(1).$$

In paper [26], quite general impulsive boundary value problems

$$u''(t) + p(t)u'(t) + q(t)u(t) + g(t)f(t, u(t)) = 0, \quad t \in (0, 1), \ t \neq \tau,$$

$$\Delta u_{(t=\tau)}) = I(u(\tau)),$$

$$\Delta u'_{(t=\tau)}) = N(u(\tau)),$$

$$a_1 u(0) - b_1 u'(0) = \alpha[u], \ a_2 u(1) - b_2 u'(1) = \beta[u].$$

are treated.

Motivated by the excellent results mentioned above and the methods used in [24], in this paper, we examine the second order impulsive equation

$$\begin{cases} u''(t) + \phi(t)f(t, u(t)) = 0, \quad t \in (0, 1) \setminus \{t_1, t_2, \cdots, t_m\}, \\ \Delta u(t_k) = I_k(u(t_k)), \quad k = 1, 2, \cdots, m, \\ \Delta u'(t_k) = J_k(u(t_k)), \quad k = 1, 2, \cdots, m, \\ u(0) = \alpha u(\xi), \quad u'(1) = 0, \end{cases} \tag{1.1}$$

where $\alpha, \xi \in (0, 1)$, $0 < t_1 < t_2 < \cdots < t_m < 1$, $\xi \neq t_k$, $k = 1, 2, \cdots, m$, $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$, $u(t_k^+)$ (respectively $u(t_k^-)$) denotes the right limit (respectively left limit) of $u(t)$ at $t = t_k$. Also $\Delta u'(t_k) = u'(t_k^+) - u'(t_k^-)$. Our result complements the results of [6,7,23] and it can solve the problems which cannot be solved by the results of [26](see example 3.1).

We define the Banach space:

$$PC[0, 1] = \{u : [0, 1] \to R, \text{ there exists } u_k \in C[t_k, t_{k+1}] \text{ such that } u(t) = u_k(t) \text{ for } t \in (t_k, t_{k+1}], \ k = 0, 1, \cdots, m, \ u(0) = u(0 + 0)\},$$

with the norm

$$\|u\| = \sup\{|u(t)| : t \in [0, 1] \setminus \{t_1, \cdots, t_m\}\},$$

where $t_0 = 0$, $t_{m+1} = 1$.

A positive solution of the problem (1.1) means a function $u \in PC[0, 1]$ which satisfies (1.1) with $u(t) > 0$, $t \in [0, 1]$.

In this paper, we will always suppose that the following conditions hold:

$(C_1)$ $\phi \in C(0, 1)$ with $\phi > 0$ on $(0, 1)$ and $\phi \in L^1[0, 1]$.

$(C_2)$ $f : [0, 1] \times [0, \infty) \to [0, \infty)$ is continuous.

$(C_3)$ $I_k$ , $J_k$ :$[0, \infty) \to R$ are continuous for $k = 1, 2, \cdots, m$.

$(C_4)$ There exists a function $\Omega: \ \{u : u \in PC[0,1], \ u \geq 0\} \to [0,+\infty)$ and a constant $0 < c_0 < 1$ such that

$$c_0\Omega(u) \leq \omega_0(t,u) \leq \Omega(u), \quad (t,u) \in [0,1] \times \{u : u \in PC[0,1], \ u \geq 0\},$$

where

$$\omega_0(t,u) = \frac{\alpha}{1-\alpha} \sum_{t_k<\xi} [I_k(u(t_k)) + (\xi - t_k)J_k(u(t_k))] + \sum_{t_k<t} \left[ I_k(u(t_k)) - \frac{\alpha\xi + (1-\alpha)t_k}{1-\alpha} J_k(u(t_k)) \right] - \sum_{t\leq t_k} \frac{\alpha\xi + (1-\alpha)t}{1-\alpha} J_k(u(t_k)).$$

## 2. Preliminaries

For $y \in L[0,1]$, let's consider the following problem:

$$\begin{cases} u''(t) + y(t) = 0, \quad t \in (0,1) \setminus \{t_1, t_2, \cdots, t_m\}, \\ \Delta u(t_k) = I_k(u(t_k)), \quad k = 1,2,\cdots,m, \\ \Delta u'(t_k) = J_k(u(t_k)), \quad k = 1,2,\cdots,m, \\ u(0) = \alpha u(\xi), \quad u'(1) = 0. \end{cases} \tag{2.1}$$

**Lemma 2.1** Let $u \geq 0$. Then $u$ is a solution of the problem (2.1) if and only if it satisfies

$$u(t) = \int_0^1 G(t,s)y(s)ds + \omega_0(t,u), \tag{2.2}$$

where

$$G(t,s) = \frac{1}{1-\alpha} \begin{cases} s, & s < \xi, s < t, \\ \alpha s + (1-\alpha)t, & t \leq s \leq \xi, \\ \alpha\xi + (1-\alpha)s, & \xi \leq s \leq t, \\ \alpha\xi + (1-\alpha)t, & \xi < s, t < s, \end{cases}$$

$\omega_0(t,u)$ is the same as in condition $(C_4)$.

**Proof.** Let $u$ be a solution of the problem (2.1), then

$$u''(t) = -y(t). \tag{2.3}$$

For $t \in (0,t_1]$, integrating (2.3) from 0 to $t$, we have

$$u'(t) = c_1 - \int_0^t y(s)ds,$$

$$u(t) = c_2 + c_1 t - \int_0^t (t-s)y(s)ds.$$

So, we have

$$u(t_1^-) = c_1 t_1 - \int_0^{t_1} (t_1 - s)y(s)ds + c_2, \tag{2.4}$$

$$u'(t_1^-) = c_1 - \int_0^{t_1} y(s))ds. \tag{2.5}$$

For $t \in (t_1, t_2]$, integrating (2.3) from $t_1$ to $t$, we have

$$u(t) = b_2 + b_1(t - t_1) - \int_{t_1}^{t} (t - s)y(s)ds. \tag{2.6}$$

By (2.1), (2.4), (2.5) and (2.6), we have

$$b_2 = I_1(u(t_1)) + c_1 t_1 - \int_0^{t_1} (t_1 - s)y(s)ds + c_2,$$

$$b_1 = J_1(u(t_1)) + c_1 - \int_0^{t_1} y(s)ds.$$

Thus,

$$u(t) = I_1(u(t_1)) + c_1 t - \int_0^{t} (t - s)y(s)ds + J_1(u(t_1))(t - t_1) + c_2.$$

For $t \in (t_k, t_{k+1}]$, by the same way, we can get

$$u(t) = c_1 t + c_2 - \int_0^t (t - s)y(s)ds + \sum_{i=1}^{k} (t - t_i)J_i(u(t_i)) + \sum_{i=1}^{k} I_i(u(t_i)). \tag{2.7}$$

By $u'(1) = 0$ and (2.7), we have

$$c_1 = \int_0^1 y(s)ds - \sum_{i=1}^{m} J_i(u(t_i)).$$

It follows from (2.7) and $u(0) = \alpha u(\xi)$ that

$$c_2 = \frac{\alpha}{1-\alpha}[\xi \int_0^1 y(s)ds - \int_0^{\xi} (\xi - s)y(s)ds - \sum_{k=1}^{m} \xi J_k(u(t_k)) + \sum_{t_k<\xi} (\xi - t_k)J_k(u(t_k))$$
$$+ \sum_{t_k<\xi} I_k(u(t_k))].$$

So, we get

$$u(t) = \int_0^1 ty(s)ds + \frac{\alpha\xi}{1-\alpha}\int_0^1 y(s)ds - \frac{\alpha}{1-\alpha}\int_0^{\xi} (\xi - s)y(s)ds - \int_0^t (t - s)y(s)ds$$
$$+ \frac{\alpha}{1-\alpha}\sum_{t_k<\xi} [I_k(u(t_k)) + (\xi - t_k)J_k(u(t_k))] + \sum_{t_k<t}\left[I_k(u(t_k)) - \frac{\alpha\xi + (1-\alpha)t_k}{1-\alpha}J_k(u(t_k))\right]$$
$$- \sum_{t\le t_k} \frac{\alpha\xi + (1-\alpha)t}{1-\alpha}J_k(u(t_k))$$
$$= \int_0^1 ty(s)ds + \frac{\alpha\xi}{1-\alpha}\int_0^1 y(s)ds - \frac{\alpha}{1-\alpha}\int_0^{\xi} (\xi - s)y(s)ds - \int_0^t (t - s)y(s)ds + \omega_0(t, u).$$

For $t \le \xi$, we obtain

$$u(t) = \int_0^t \frac{s}{1-\alpha}y(s)ds + \int_t^{\xi} \frac{\alpha s + (1-\alpha)t}{1-\alpha}y(s)ds + \int_{\xi}^1 \frac{\alpha\xi + (1-\alpha)t}{1-\alpha}y(s)ds + \omega_0(t, u).$$

For $t \geq \xi$, we have

$$u(t) = \int_0^\xi \frac{s}{1-\alpha} y(s)ds + \int_\xi^t \frac{\alpha\xi + (1-\alpha)s}{1-\alpha} y(s)ds + \int_t^1 \frac{\alpha\xi + (1-\alpha)t}{1-\alpha} y(s)ds + \omega_0(t,u).$$

So, we get

$$u(t) = \int_0^1 G(t,s)y(s)ds + \omega_0(t,u).$$

Conversely, if $u(t)$ satisfies (2.2), it's easy to get that $u(t)$ is a solution of (2.1). □

**Lemma 2.2.** The function $G(t,s)$ is continuous on $[0,1] \times [0,1]$ and it satisfies

$$\rho_0 g(s) \leq G(t,s) \leq g(s), \quad t, s \in [0,1],$$

where $g(s) = \dfrac{s}{1-\alpha}, \quad \rho_0 = \alpha\xi.$

**Proof.** The proof of this lemma is easy. So, we omit it. □

Now we define a cone $P$ on $PC[0,1]$ and an operator $T : P \to PC[0,1]$ as follows:

$$P = \{u \in PC[0,1] : u(t) \geq 0, \inf_{t\in[0,1]} u(t) \geq \rho\|u\|\}, \text{ where } \rho = \min\{c_0,\ \rho_0\}.$$

$$Tu(t) = \int_0^1 G(t,s)\phi(s)f(s,u(s))ds + \omega_0(t,u).$$

Obviously, if $u \in P$ is a fixed point of $T$, it is a solution of the problem (1.1).

**Lemma 2.3.** Assume $(C_1) - (C_4)$ hold. Then $T : P \to P$ is a completely continuous operator.

**Proof.** By $(C_1)$, $(C_2)$ and $(C_4)$, we have $Tu(t) \geq 0,\ u \in P$. By $(C_4)$ and Lemma 2.2, we can get

$$\begin{aligned} |Tu(t)| &= |\int_0^1 G(t,s)\phi(s)f(s,u(s))ds + \omega_0(t,u)| \\ &\leq \int_0^1 g(s)\phi(s)f(s,u(s))ds + \Omega(u), \end{aligned}$$

and

$$\begin{aligned} \inf_{t\in[0,1]} Tu(t) &= \inf_{t\in[0,1]} \left[\int_0^1 G(t,s)\phi(s)f(s,u(s))ds + \omega_0(t,u)\right] \\ &\geq \rho_0 \int_0^1 g(s)\phi(s)f(s,u(s))ds + c_0\Omega(u) \\ &\geq \rho\|Tu\|. \end{aligned}$$

This shows that $T : P \to P$. By the continuity of $f,\ I_k,\ J_k,\ k = 1, 2, \cdots, m$, we can easily obtain that $T : P \to P$ is continuous. Let $S \subset P$ be bounded. Obviously, $T(S) \subset P$ is bounded. For $u \in S,\ t, t' \in (t_k, t_{k+1}]$, we have

$$\begin{aligned} &|Tu(t) - Tu(t')| \leq \textstyle\int_0^1 |G(t,s) - G(t',s)|\phi(s)f(s,u(s))ds + |\omega_0(t,u) - \omega_0(t',u)| \\ &\leq \textstyle\int_0^1 |G(t,s) - G(t',s)|\phi(s)f(s,u(s))ds + |t-t'| \sum\limits_{k=1}^m |J_k(u(t_k))|. \end{aligned}$$

By $(C_1)$, the uniform continuity of $G$ on $[0,1]\times[0,1]$, the boundedness of $f$ on $[0,1]\times S$ and the boundedness of $J_k$ on $S$, we obtain that $T(S)$ is quasi-equicontinuous on [0,1]. By [1], $T$ is a compact map. So, $T: P \to P$ is completely continuous. □

In order to obtain our main results, we need the following definitions and theorem.

**Definition 2.1.** A map $\phi$ is said to be a non-negative, continuous and concave functional on a cone $P$ of a real Banach space $E$ iff $\phi: P \to R_+$ is continuous and

$$\phi(tx+(1-t)y) \geq t\phi(x)+(1-t)\phi(y),$$

for all $x, y \in P$ and $t \in [0,1]$.

**Definition 2.2.** A map $\Phi$ is said to be a non-negative, continuous and convex functional on a cone $P$ of a real Banach space $E$ iff $\Phi: P \to R_+$is continuous and

$$\Phi(tx+(1-t)y) \leq t\Phi(x)+(1-t)\Phi(y),$$

for all $x, y \in P$ and $t \in [0,1]$.

Let $\varphi$ and $\Theta$ be non-negative, continuous and convex functional on $P$, $\Phi$ be a non-negative, continuous and concave functional on $P$, and $\Psi$ be a non-negative continuous functional on $P$. Then, for positive numbers $a$, $b$, $c$ and $d$, we define the following sets:

$$P(\varphi, d) = \{x \in P : \varphi(x) < d\},$$

$$P(\varphi, \Phi, b, d) = \{x \in P : b \leq \Phi(x), \varphi(x) \leq d\},$$

$$P(\varphi, \Theta, \Phi, b, c, d) = \{x \in P : b \leq \Phi(x), \Theta(x) \leq c, \varphi(x) \leq d\},$$

$$R(\varphi, \Psi, a, d) = \{x \in P : a \leq \Psi(x), \varphi(x) \leq d\}.$$

We will use the following fixed point theorem of Avery and Peterson to study the problem (1.1), (2.1).

**Theorem 2.1[25].** Let $P$ be a cone in a real Banach space $E$. Let $\varphi$ and $\Theta$ be non-negative, continuous and convex functionals on $P$, $\Phi$ be a non-negative, continuous and concave functional on $P$, and $\Psi$ be a non-negative continuous functional on $P$ satisfying $\Psi(kx) \leq k\Psi(x)$ for $0 \leq k \leq 1$, such that for some positive numbers $M$ and $d$,

$$\Phi(x) \leq \Psi(x) \text{ and } \|x\| \leq M\varphi(x)$$

for all $x \in \overline{P(\varphi, d)}$. Suppose that

$$T : \overline{P(\varphi, d)} \to \overline{P(\varphi, d)}$$

is completely continuous and there exist positive numbers $a$, $b$, $c$ with $a < b$, such that the following conditions are satisfied:

$(S_1)$ $\{x \in P(\varphi, \Theta, \Phi, b, c, d) : \Phi(x) > b\} \neq \emptyset$ and $\Phi(Tx) > b$ for $x \in P(\varphi, \Theta, \Phi, b, c, d)$;
$(S_2)$ $\Phi(Tx) > b$ for $x \in P(\varphi, \Phi, b, d)$ with $\Theta(Tx) > c$;
$(S_3)$ $0 \notin R(\varphi, \Psi, a, d)$ and $\Psi(Tx) < a$ for $x \in R(\varphi, \Psi, a, d)$ with $\Psi(x) = a$.

Then $T$ has at least three fixed points $x_1,\ x_2,\ x_3 \in \overline{P(\varphi, d)}$, such that

$$\varphi(x_i) \le d, \text{ for } i = 1, 2, 3,$$

and

$$b < \Phi(x_1),\ a < \Psi(x_2),\ \Phi(x_2) < b,$$
$$\Psi(x_3) < a.$$

## 3. Main results

We define a concave function $\Phi(x) = \inf\limits_{t\in[0,1]} |x(t)|$ and convex functions $\Psi(x) = \Theta(x) = \varphi(x) = \|x\|$.

**Theorem 3.1.** Suppose $(C_1) - (C_4)$ hold. In additions, we assume that there exist positive constants $\mu,\ L,\ a,\ b,\ c,\ d$ with $a < b < \dfrac{b}{\rho} = c < d$, $\mu > D_1 + D_2$, $0 < L < \rho(D_1 + D_3)$, where $D_1 = \int_0^1 g(s)\phi(s)ds$, $D_2$, $D_3 \ge 0$, such that the following conditions hold:

$(A_1)$ $f(t, u) \le \dfrac{d}{\mu}$, for $(t, u) \in [0, 1]\times[0, d]$, and $\omega_0(t, u) \le \dfrac{D_2}{\mu}d$, for $u \in P$, $\|u\| \le d$;

$(A_2)$ $f(t, u) \ge \dfrac{b}{L}$, for $(t, u) \in [0, 1] \times \left[b, \dfrac{b}{\rho}\right]$, and $\omega_0(t, u) \ge \dfrac{D_3}{L}b$, for $u \in P$, $b \le u(t) \le \dfrac{b}{\rho}$, $t \in [0, 1]$;

$(A_3)$ $f(t, u) \le \dfrac{a}{\mu}$, for $(t, u) \in [0, 1]\times[0, a]$, and $\omega_0(t, u) \le \dfrac{D_2}{\mu}a$, for $u \in P$, $\|u\| \le a$.

Then the problem (1.1) has at least two positive solutions when $f(t, 0) \equiv 0$, $t \in [0, 1]$ and at least three positive solutions when $f(t, 0) \not\equiv 0$, $t \in [0, 1]$.

**Proof.** Take $u \in \overline{P(\varphi, d)}$. By assumption $(A_1)$, we have

$$\begin{aligned}\varphi(Tu) = \|Tu\| &\le \int_0^1 g(s)\phi(s)f(s, u(s))ds + \frac{D_2}{\mu}d \\ &\le \frac{d}{\mu}\int_0^1 g(s)\phi(s)ds + \frac{D_2}{\mu}d = \frac{D_1}{\mu}d + \frac{D_2}{\mu}d < d.\end{aligned}$$

Thus, $T : \overline{P(\varphi, d)} \to \overline{P(\varphi, d)}$.

Let's prove that condition $S_1$ holds.

Take $u(t) = \dfrac{b(\rho + 1)}{2\rho}$, $t \in [0, 1]$. By simple calculation, we can get that

$$\|u\| = \frac{b(\rho + 1)}{2\rho} < \frac{b}{\rho} = c,$$

and

$$\Phi(u) = \inf_{t\in[0,1]} |u(t)| = \frac{b(\rho+1)}{2\rho} > b.$$

Therefore,

$$\{u \in P(\varphi,\Theta,\Phi,b,c,d) : b < \Phi(u)\} \neq \emptyset.$$

$u \in P(\varphi,\Theta,\Phi,b,c,d)$ means that $b \le u(t) \le \dfrac{b}{\rho},\ t\in[0,1]$. By $(A_2)$, we get

$$\Phi(Tu) = \inf_{t\in[0,1]} |Tu(t)| \ge \rho\left[\int_0^1 g(s)\phi(s)f(s,u(s))ds + \frac{b}{L}D_3\right] \ge \rho\frac{b}{L}(D_1+D_3) > b.$$

So, condition $S_1$ holds.

Now we will show that condition $S_2$ holds.

Take $u \in P(\varphi,\Phi,b,d)$ and $\|Tu\| > \dfrac{b}{\rho} = c$. Considering $Tu \in P$, we get

$$\Phi(Tu) = \inf_{t\in[0,1]} |Tu(t)| \ge \rho\|Tu\| > \rho\cdot\frac{b}{\rho} = b,$$

This shows that condition $S_2$ is satisfied.

In the following we will show that the condition $S_3$ is satisfied. Since $\Psi(0) = 0,\ 0 < a,\ 0 \notin R(\varphi,\Psi,a,d)$. Assume that $u \in R(\varphi,\Psi,a,d)$ with $\Psi(u) = \|u\| = a$. Then, by $(A_3)$, we have

$$\Psi(Tu) = \|Tu(t)\| \le \int_0^1 g(s)\phi(s)f(s,u(s))ds + \frac{a}{\mu}D_2 \le \frac{a}{\mu}(D_1+D_2) < a.$$

Thus, condition $S_3$ is satisfied. By Theorem 2.1, we get that the problem (1.1) has at least three solutions $u_1,u_2,u_3 \in P$ satisfying

$$\|u_i\| \le d,\ i=1,2,3\ ,\ \text{and}\ b < \inf_{t\in[0,1]} |u_1(t)|,$$

$$a \le \|u_2\|,\ \inf_{t\in[0,1]} |u_2(t)| < b,\ \|u_3\| < a.$$

Obviously, $u_1(t) > 0,\ u_2(t) > 0,\ t\in[0,1]$. If $f(t,0) \not\equiv 0,\ t\in[0,1]$, then $u=0$ is not a solution of (1.1). So, $u_3 \neq 0$. This, together with $u_3 \in P$, means that $u_3(t) > 0,\ t\in[0,1]$. □

**Example 3.1.** Consider the following boundary value problem

$$\begin{cases} u''(t) + f(t,u(t)) = 0,\ t\in(0,1)\setminus\{\frac{1}{8}\}, \\ \Delta u(\frac{1}{8}) = I_1(u(\frac{1}{8})), \\ \Delta u'(\frac{1}{8}) = J_1(u(\frac{1}{8})), \\ u(0) = \frac{1}{4}u(\frac{1}{4}),\quad u'(1) = 0, \end{cases} \tag{3.1}$$

where

$$f(t,u)=\begin{cases}\frac{1}{4}u^2t, & t\in[0,1],\ u\in\left[0,\frac{1}{2}\right],\\ \frac{1}{2}u^2t(1-u)+(60+2\sqrt{u}t)(u-\frac{1}{2}), & t\in[0,1],\ u\in[\frac{1}{2},1],\\ 30+\sqrt{u}t, & t\in[0,1],\ u\in[1,16],\\ 30+4t, & t\in[0,1],\ u\in[16,\infty).\end{cases}$$

Corresponding to Theorem 3.1, we take $\alpha=\xi=\frac{1}{4}, c_0=\frac{1}{6}, \rho=\frac{1}{16}, \mu=2, D_1=\int_0^1 g(s)ds=\frac{2}{3}, D_2=\frac{1}{3}, D_3=0, L=\frac{1}{30}, I_1(\omega)=\frac{1}{64}\sqrt{\omega}, J_1(\omega)=\frac{-\sqrt{\omega}}{64}, \Omega(u)=\frac{3\sqrt{u(\frac{1}{8})}}{128}$, and

$$\omega_0(t,u)=\begin{cases}\frac{3\sqrt{u(\frac{1}{8})}}{128}, & t>\frac{1}{8},\\ (\frac{3}{8}+t)\frac{1}{64}\sqrt{u(\frac{1}{8})}, & t\le\frac{1}{8}.\end{cases}$$

It is easy to check that $\frac{1}{6}\Omega(u)\le\omega_0(t,u)\le\Omega(u)$. Let $a=\frac{1}{2}, b=1, d=68$. By simple calculation, we can get that the conditions of Theorem 3.1 are satisfied. So, the problem (3.1) has at least three solutions $u_1,u_2,u_3\in P$ satisfying

$$\|u_i\|\le 68,\ i=1,2,3\ ,$$

and

$$1<\Phi(u_1),\ \frac{1}{2}<\|u_2\|,\ \Phi(u_2)<1,\ \|u_3\|<\frac{1}{2},$$

where $u_1,u_2$ are positive solutions of (3.1).

**Remark.** Corresponding to the condition $(C_3)$ in [26], we get $(d_1I+e_1N)(\omega)=\frac{9}{512}\sqrt{\omega},\ (d_2I+e_2N)(\omega)=\frac{1}{64}\sqrt{\omega}$. The problem (3.1) cannot be solved by the Theorems in [26] because the condition $(C_3)$ in [26] is not satisfied. So, our result may be considered as a complementary result of [26].

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