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Almost periodic solutions of N-th order neutral differential difference equations with piecewise constant arguments*

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Abstract: In this paper, we study the existence of almost periodic solutions of neutral differential difference equations with piecewise constant arguments via difference equation methods.

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1 Introduction

In this paper, consider neutral differential difference equations with piecewise constant argument of the forms

$$(x(t) + px(t-1))^{(N)} = qx([t-1]) + f(t), \tag{1}$$

$$(x(t) + px(t-1))^{(N)} = qx([t-1]) + g(t, x(t), x([t-1])). \tag{2}$$

Here $[\cdot]$ is the greatest integer function, p and q are nonzero constants, N is a positive integer, $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ are continuous. Throughout this paper, we use the following notations: $\mathbb{R}$ is the set of reals; $\mathbb{Z}$ the set of integers; i.e., $\mathbb{Z} = \{0, \pm 1, \pm 2, \cdots\}$; $\mathbb{Z}^+$ the set of positive integers; $\mathbb{C}$ denotes the set of complex numbers. A function $x : \mathbb{R} \to \mathbb{R}$ is called a solution of Eq. (1) (or (2)) if the following conditions are satisfied:

(i) x is continuous on $\mathbb{R}$;

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(ii) the N-th order derivative of $x(t)+p(t)x(t-1)$ exists on $\mathbb{R}$ except possibly at the points $t=n$, $n\in\mathbb{Z}$, where one-sided N-th order derivatives of $x(t)+p(t)x(t-1)$ exist;

(iii) x satisfies Eq. (1) (or (2)) on each interval $(n, n+1)$ with integer $n\in\mathbb{Z}$.

Differential equations with piecewise constant arguments describe hybrid dynamical systems (a combination of continuous and discrete systems) and, therefore, combine the properties of both differential equations and difference equations. For a survey of work on differential equations with piecewise constant arguments we refer to [1], and for some excellent works in this field we refer to [2–5] and references therein.

In paper [2], Yuan studied the existence of almost periodic solutions for second-order equations involving the argument $2[(t+1)/2]$ in the unknown function. In paper [3], Piao studied the existence of almost periodic solutions for second-order equations involving the argument $[t-1]$ in the unknown function. In paper [4], Seifert intensively studied the existence of almost periodic solutions for second-order equations involving the argument $[t]$ in the unknown function by using different methods. However, to the best of our knowledge, there are no results regarding the existence of almost periodic solutions for N-th order neutral differential equations with piecewise constant argument as Eq. (1) (or (2)) up to now. Motivated by the ideas of Yuan [2], Piao[3] and Seifert [4], in this paper we will investigate the existence of almost periodic solutions to Eq. (1) and (2).

Our present paper is organized as follows: in Section 2, we state some definitions and lemmas; in Section 3, we state our main results and prove them.

2 Definitions and Lemmas

Now we start with some definitions.

Definition 2.1 ([6]). A set $K\subset\mathbb{R}$ is said to be relatively dense if there exists $L>0$ such that $[a, a+L]\cap K\neq\emptyset$ for all $a\in\mathbb{R}$.

Definition 2.2 ([6]). A bounded continuous function $f:\mathbb{R}\to\mathbb{R}$ (resp. $\mathbb{C}$) is said to be almost periodic if the $\varepsilon-$translation set of f

$$T(f,\varepsilon)=\{\tau\in\mathbb{R}:|f(t+\tau)-f(t)|<\varepsilon,\quad\forall t\in\mathbb{R}\},$$

is relatively dense for each $\varepsilon>0$. We denote the set of all such function f by $\mathcal{AP}(\mathbb{R},\mathbb{R})$ (resp. $\mathcal{AP}(\mathbb{R},\mathbb{C})$).

Definition 2.3 ([6]). A bounded continuous function $g : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ (resp. $\mathbb{C}$)is said to be almost periodic function for t uniformly on $\mathbb{R} \times \Omega$, where Ω is an open subset of $\mathbb{R}^2$, if, for any compact subset $W \subset \Omega$, the $\varepsilon-$translation set of g,

$$T(g, \varepsilon, W) = \{\tau \in \mathbb{R} : |g(t+\tau, x) - g(t, x)| < \varepsilon, \quad \forall (t, x) \in \mathbb{R} \times W\},$$

is relatively dense in $\mathbb{R}$. τ is called the $\varepsilon-$period for g. Denote by $\mathcal{AP}(\mathbb{R}\times W, \mathbb{R})$ (resp. $\mathcal{AP}(\mathbb{R}\times W, \mathbb{C})$) the set of all such functions.

Definition 2.4 ([6]). A sequence $x : \mathbb{Z} \to \mathbb{R}^k$ (resp. $\mathbb{C}^k$), $k \in \mathbb{Z}, k > 0$, denoted by $\{x_n\}$, is called an almost periodic sequence if the ε-translation set of $\{x_n\}$

$$T(\{x_n\}, \varepsilon) = \{\tau \in \mathbb{Z} : |x_{n+\tau} - x_n| < \varepsilon, \quad \forall n \in \mathbb{Z}\},$$

is relatively dense for each $\varepsilon > 0$, here $|\cdot|$ is any convenient norm in $\mathbb{R}^k$ (resp. $\mathbb{C}^k$). We denote the set of all such sequences $\{x_n\}$ by $\mathcal{APS}(\mathbb{Z}, \mathbb{R}^k)$ (resp. $\mathcal{APS}(\mathbb{Z}, \mathbb{C}^k)$).

Proposition 2.5. $\{x_n\} = \{(x_{n1}, x_{n2}, \cdots, x_{nk})\} \in \mathcal{APS}(\mathbb{Z}, \mathbb{R}^k)$ (resp. $\mathcal{APS}(\mathbb{Z}, \mathbb{C}^k)$) if and only if $\{x_{ni}\} \in \mathcal{APS}(\mathbb{Z}, \mathbb{R})$ (resp. $\mathcal{APS}(\mathbb{Z}, \mathbb{C})$), $i = 1, 2, \cdots, k$.

Proposition 2.6. Suppose that $\{x_n\} \in \mathcal{APS}(\mathbb{Z}, \mathbb{R})$, $f \in \mathcal{AP}(\mathbb{R}, \mathbb{R})$. Then the sets $T(f, \varepsilon) \cap \mathbb{Z}$ and $T(\{x_n\}, \varepsilon) \cap T(f, \varepsilon)$ are relatively dense.

Lemma 2.7 ([2, 3]). Let $x : \mathbb{R} \to \mathbb{R}$ is a continuous function, and $w(t) = x(t) + px(t-1)$. Then

$$|x(t)| \le e^{-a(t-t_0)} \sup_{-1\le\theta\le 0} |x(t_0+\theta)| + b \sup_{t_0\le u\le t} |w(u)|, \quad t \ge t_0,$$

where $|p| < 1, a = \log 1/|p|, b = 1/(1-|p|)$, or

$$|x(t)| \le e^{\log|p|(t-t_0)} \sup_{0\le\theta\le 1} |x(t_0+\theta)| + b \sup_{t\le u\le t_0} |w(u+1)|, \quad t \le t_0,$$

where $|p| > 1, b = 1/(|p|-1)$.

3 Main Results

Now we rewrite Eq. (1) as the following equivalent system

$$\begin{cases} (x(t) + px(t-1))' = y_1(t), & (3_1) \\ y_1'(t) = y_2(t), & (3_2) \\ \quad\vdots & \quad\vdots \\ y_{N-2}'(t) = y_{N-1}(t), & (3_{N-1}) \\ y_{N-1}'(t) = qx([t-1]) + f(t). & (3_N) \end{cases} \qquad (3)$$

Lemma 3.1. If $(x(t), y_1(t), \cdots, y_{N-1}(t))$ be a solutions of system (3) on $\mathbb{R}$, (defined similarly as above for (1)), then for $n \in \mathbb{Z}$,

$$\begin{cases} x(n+1) = (1-p)x(n) + y_1(n) + \frac{1}{2!}y_2(n) + \cdots + \frac{1}{(N-1)!}y_{N-1}(n) + (p + \frac{q}{N!})x(n-1) \\ \qquad\qquad + f_n^{(1)}, & (4_1) \\ y_1(n+1) = y_1(n) + y_2(n) + \frac{1}{2!}y_3(n) + \cdots + \frac{1}{(N-2)!}y_{N-1}(n) + \frac{q}{(N-1)!}x(n-1) + f_n^{(2)}, & (4_2) \\ \quad \vdots & \vdots \\ y_{N-2}(n+1) = y_{N-2}(n) + y_{N-1}(n) + \frac{q}{2}x(n-1) + f_n^{(N-1)}, & (4_{N-1}) \\ y_{N-1}(n+1) = y_{N-1}(n) + qx(n-1) + f_n^{(N)}, & (4_N) \end{cases} \tag{4}$$

where

$$f_n^{(1)} = \int_n^{n+1} \int_n^{t_N} \cdots \int_n^{t_2} f(t_1)dt_1dt_2\cdots dt_N, \quad \cdots, \quad f_n^{(N-1)} = \int_n^{n+1} \int_n^{t_2} f(t_1)dt_1dt_2,$$
$$f_n^{(N)} = \int_n^{n+1} f(t_1)dt_1. \tag{5}$$

Proof. For $n \le t < n+1, n \in \mathbb{Z}$, using (3_N) we obtain

$$y_{N-1}(t) = y_{N-1}(n) + qx(n-1)(t-n) + \int_n^t f(t_1)dt_1, \tag{6}$$

and using this with (3_{N-1}) we obtain

$$y_{N-2}(t) = y_{N-2}(n) + y_{N-1}(n)(t-n) + \frac{1}{2}qx(n-1)(t-n)^2 + \int_n^t \int_n^{t_2} f(t_1)dt_1dt_2. \tag{7}$$

Continuing this way, and at last we get

$$\begin{aligned} x(t) + px(t-1) &= x(n) + px(n-1) + y_1(n)(t-n) + \frac{1}{2}y_2(n)(t-n)^2 + \cdots \\ &\quad + \frac{1}{(N-1)!}y_{N-1}(n)(t-n)^{N-1} + \frac{1}{N!}qx(n-1)(t-n)^N \\ &\quad + \int_n^t \int_n^{t_N} \cdots \int_n^{t_2} f(t_1)dt_1dt_2\cdots dt_N. \end{aligned} \tag{8}$$

Since $x(t)$ must be continuous at $n+1$, these equations yield system (4).

Lemma 3.2. If $f \in \mathcal{AP}(\mathbb{R}, \mathbb{R})$, then sequences $\{f_n^{(i)}\} \in \mathcal{APS}(\mathbb{Z}, \mathbb{R}), i = 1, 2, \cdots, N$.

Proof. We typically consider $\{f_n^{(1)}\}$. $\forall \varepsilon > 0$ and $\tau \in T(f, \varepsilon) \cap \mathbb{Z}$, we have

$$\begin{aligned} \left|f_{n+\tau}^{(1)} - f_n^{(1)}\right| &= \left|\int_{n+\tau}^{n+\tau+1} \int_{n+\tau}^{t_N} \cdots \int_{n+\tau}^{t_2} f(t_1)dt_1dt_2\cdots dt_N - \int_n^{n+1} \int_n^{t_N} \cdots \int_n^{t_2} f(t_1)dt_1dt_2\cdots dt_N\right| \\ &\le \int_n^{n+1} \int_n^{t_N} \cdots \int_n^{t_2} |f(t_1+\tau) - f(t_1)|dt_1dt_2\cdots dt_N \\ &\le \frac{\varepsilon}{N!}. \end{aligned}$$

From Definition 2.4, it follows that $\{f_n^{(1)}\}$ is an almost periodic sequence. In a manner similar to the proof just completed, we know that $\{f_n^{(2)}\}, \{f_n^{(3)}\}, \cdots, \{f_n^{(N)}\}$ are also almost periodic sequences. This completes the proof of lemma. □

Next we express system (4) in terms of an equivalent system in $\mathbb{R}^{N+1}$ given by

$$v_{n+1} = Av_n + h_n, \tag{9}$$

where

$$A = \begin{pmatrix} 1-p & 1 & \frac{1}{2!} & \cdots & \frac{1}{(N-1)!} & p + \frac{q}{N!} \\ 0 & 1 & 1 & \cdots & \frac{1}{(N-2)!} & \frac{q}{(N-1)!} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1 & \frac{q}{2!} \\ 0 & 0 & 0 & \cdots & 1 & q \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

and $v_n = (x(n), y_1(n), y_2(n), \cdots, y_{N-1}, x(n-1))^T, h_n = (f_n^{(1)}, f_n^{(2)}, \cdots, f_n^{(N)}, 0)^T$.

Lemma 3.3. Suppose that all eigenvalues of A are simple (denoted by $\lambda_1, \lambda_2, \cdots, \lambda_{N+1}$) and $|\lambda_i| \neq 1, 1 \leq i \leq N+1$. Then system (9) has a unique almost periodic solution.

Proof. From our hypotheses, there exists a $(N+1) \times (N+1)$ nonsingular matrix P such that $PAP^{-1} = \Lambda$, where $\Lambda = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_{N+1})$ and $\lambda_1, \lambda_2, \cdots, \lambda_{N+1}$ are the distinct eigenvalues of A. Define $\bar{v}_n = Pv_n$, then (9) becomes

$$\bar{v}_{n+1} = \Lambda \bar{v}_n + \bar{h}_n, \tag{10}$$

where $\bar{h}_n = Ph_n$.

For the sake of simplicity, we consider first the case $|\lambda_1| < 1$. Define

$$\bar{v}_{n1} = \sum_{m \leq n} \lambda_1^{n-m} \bar{h}_{(m-1)1}$$

where $\bar{h}_n = (\bar{h}_{n1}, \bar{h}_{n2}, \cdots, \bar{h}_{n(N+1)})^T, n \in \mathbb{Z}$. Clearly $\{\bar{h}_{n1}\}$ is almost periodic, since $\bar{h}_n = Ph_n$, and

$\{h_n\}$ is. For $\tau \in T(\{\bar{h}_{n1}\}, \varepsilon)$, we have

$$\begin{aligned}
&\left|\bar{v}_{(n+\tau)1} - \bar{v}_{n1}\right| \\
&= \left|\sum_{m \le n+\tau} \lambda_1^{n+\tau-m} \bar{h}_{(m-1)1} - \sum_{m \le n} \lambda_1^{n-m} \bar{h}_{(m-1)1}\right| \\
&\quad (\text{letting} \quad m = m' + \tau, \text{then replacing} \quad m' \quad \text{by} \quad m) \\
&= \left|\sum_{m \le n} \lambda_1^{n-m} \bar{h}_{(m+\tau-1)1} - \sum_{m \le n} \lambda_1^{n-m} \bar{h}_{(m-1)1}\right| \\
&= \left|\sum_{m \le n} \lambda_1^{n-m} \left(\bar{h}_{(m+\tau-1)1} - \bar{h}_{(m-1)1}\right)\right| \\
&\le \frac{\varepsilon}{1 - |\lambda_1|}
\end{aligned}$$

this shows that $\{\bar{v}_{n1}\} \in \mathcal{APS}(\mathbb{Z}, \mathbb{C})$.

If $|\lambda_i| < 1, 2 \le i \le N+1$, in a manner similar to the proof just completed for λ_1, we know that $\{\bar{v}_{ni}\} \in \mathcal{APS}(\mathbb{Z}, \mathbb{C}), 2 \le i \le N+1$, and so $\{\bar{v}_n\} \in \mathcal{APS}(\mathbb{Z}, \mathbb{C}^{N+1})$. It follows easily that then $\{P^{-1}\bar{v}_n\} = \{v_n\} \in \mathcal{APS}(\mathbb{Z}, \mathbb{R}^{N+1})$ and our lemma follows.

Assume now $|\lambda_1| > 1$. Now define

$$\bar{v}_{n1} = \sum_{m \le n} \lambda_1^{m-n} \bar{h}_{(m-1)1}, n \in \mathbb{Z}.$$

As before, the fact that $\{\bar{v}_{n1}\} \in \mathcal{APS}(\mathbb{Z}, \mathbb{C})$ follows easily from the fact that $\{\bar{h}_{n1}\} \in \mathcal{APS}(\mathbb{Z}, \mathbb{C})$. So in every possible case, we see that each component $v_{ni}, i = 1, 2, \cdots, N+1$, of v_n is almost periodic and so $\{v_n\} \in \mathcal{APS}(\mathbb{Z}, \mathbb{R}^{N+1})$.

The uniqueness of this almost periodic solution $\{v_n\}$ of (9) follows from the uniqueness of the solution $\bar{v}_n$ of (10) since $P^{-1}\bar{v}_n = v_n$, and the uniqueness of $\bar{v}_n$ of (10) follows, since if $\tilde{v}_n$ were a solution of (10) distinct from $\bar{v}_n, u_n = \bar{v}_n - \tilde{v}_n$ would also be almost periodic and solve $u_{n+1} = \Lambda u_n, n \in \mathbb{Z}$. But by our condition on Λ, it follows that each component of u_n must become unbounded either as $n \to \infty$ or as $n \to -\infty$, and that is impossible, since it must be almost periodic. This proves the lemma. □

Lemma 3.4. Let $(c_n, d_n^{(1)}, d_n^{(2)}, \cdots, d_n^{(N-1)})$ the unique first N components of the almost periodic solution of (9) given by Lemma 3.3, then there exists a solution $(x(t), y_1(t), y_2(t), \cdots, y_{N-1}(t)), t \in R$, of (3) such that $x(n) = c_n, y_1(n) = d_n^{(1)}, \cdots, y_{N-1}(n) = d_n^{(N-1)}, n \in \mathbb{Z}$.

Proof. Define

$$
\begin{aligned}
w(t) &= c_n + pc_{n-1} + d_n^{(1)}(t-n) + \frac{1}{2!}d_n^{(2)}(t-n)^2 + \cdots + \frac{1}{(N-1)!}d_n^{(N-1)}(t-n)^{N-1} \\
&\quad + \frac{1}{N!}qc_{n-1}(t-n)^N + \int_n^t \int_n^{t_N} \cdots \int_n^{t_2} f(t_1)dt_1dt_1\cdots dt_N, \qquad (11)
\end{aligned}
$$

for $n \le t < n+1, n \in \mathbb{Z}$. It can easily be verified that $w(t)$ is continuous on $\mathbb{R}$. we omit the details.

Define $x(t) = \phi(t), -1 \le t \le 0$, where $\phi(t)$ is continuous, and $\phi(0) = c_0, \phi(-1) = c_{-1}$;

$$x(t) = (w(t+1) - \phi(t+1))/p, \quad -2 \le t < -1,$$

$$x(t) = (w(t+1) - x(t+1))/p, \quad -3 \le t \le -2.$$

Continuing this way, we can define $x(t)$ for $t < 0$. Similarly, define

$$x(t) = -p\phi(t-1) + w(t), \quad 0 \le t < 1,$$

$$x(t) = -px(t-1) + w(t), \quad 1 \le t < 2;$$

continuing in this way $x(t)$ is defined for $t \ge 0$, and so $x(t)$ is defined for all $t \in \mathbb{R}$.

Next, define $y_1(t) = w'(t), y_2(t) = w''(t), \cdots, y_{N-1}(t) = w^{(N-1)}(t), t \ne n \in \mathbb{Z}$, and by the appropriate one sided derivative of $w'(t), w''(t), \cdots, w^{(N-1)}(t)$ at $n \in \mathbb{Z}$. It is easy to see that $y_1(t), y_2(t), \cdots, y_{N-1}(t)$ are continuous on $\mathbb{R}$, and $(x(n), y_1(n), y_2(n), \cdots, y_{N-1}(n)) = (c_n, d_n^{(1)}, d_n^{(2)}, \cdots, d_n^{(N-1)})$ for $n \in \mathbb{Z}$; we omit the details. □

It is easy to see that $x(t)$ is continuous on $\mathbb{R}, x(n) = c_n$ and satisfies Eq. (1). We don't know if $x(t)$ is an almost periodic one, but we can show that $w(t) := x(t) + px(t-1)$ is an almost periodic function.

Lemma 3.5. Suppose that conditions of Lemma 3.3 hold, $x(t)$ is as defined as above with $(c_n, d_n^{(1)}, d_n^{(2)}, \cdots, d_n^{(N-1)})$ the unique first N components of the almost periodic solution of (9) given by Lemma 3.3, then $w(t) := x(t) + px(t-1)$ is almost periodic.

Proof. Indeed, for $\tau \in T(\{c_n\},\varepsilon) \cap T(\{d_n^{(1)}\},\varepsilon) \cap T(\{d_n^{(2)}\},\varepsilon) \cap \cdots \cap T(\{d_n^{(N-1)}\},\varepsilon) \cap T(f,\varepsilon)$,

$$\begin{aligned}
& |w(t+\tau) - w(t)| \\
= \ & \big|(c_{n+\tau} - c_n) + p(c_{n+\tau-1} - c_{n-1}) + (d_{n+\tau}^{(1)} - d_n^{(1)})(t-n) + \frac{1}{2!}(d_{n+\tau}^{(2)} - d_n^{(2)})(t-n)^2 + \cdots \\
& + \frac{1}{(N-1)!}(d_{n+\tau}^{(N-1)} - d_n^{(N-1)})(t-n)^{N-1} + \frac{q}{N!}(c_{n+\tau-1} - c_{n-1})(t-n)^N \\
& + \int_{n+\tau}^{t+\tau}\int_{n+\tau}^{t_N}\cdots\int_{n+\tau}^{t_2} f(t_1)dt_1dt_2\cdots dt_N - \int_n^t\int_n^{t_N}\cdots\int_n^{t_2} f(t_1)dt_1dt_2\cdots dt_N\big| \\
\le \ & \left(|p| + \frac{|q|}{N!} + \sum_{i=0}^{N}\frac{1}{i!}\right)\varepsilon. \qquad (12)
\end{aligned}$$

It follows from definition that $w(t)$ is almost periodic. □

Theorem 3.6. Suppose that $|p| \neq 1$ and all eigenvalues of A in (9) are simple (denoted by $\lambda_1, \lambda_2, \cdots, \lambda_{N+1}$) and satisfy $|\lambda_i| \neq 1, 1 \le i \le N+1$. Then Eq. (1) has a unique almost periodic solution $\bar{x}(t)$.

Proof . Case I: $|p| < 1$. For each $m \in \mathbb{Z}^+$ define $x_m(t)$ as follows:

$$x_m(t) = w(t) - px_m(t-1), \quad t > -m, \qquad (13)$$

$$x_m(t) = \phi(t), \quad t \le -m, \qquad (14)$$

here $w(t)$ is as defined in the proof of Lemma 3.4, and

$$\phi(t) = c_n + (c_{n+1} - c_n)(t-n), \quad n \le t < n+1, n \in \mathbb{Z},$$

where c_n is the first component of the solution v_n of (9) given by Lemma 3.3. Let $l \in \mathbb{Z}^+$, then from (13) we get

$$(-p)^l x_m(t-l) = (-p)^l w(t-l) + (-p)^{l+1} x_m(t-l-1), \quad t > -m. \qquad (15)$$

It follows

$$x_m(t) = \sum_{j=0}^{l-1}(-p)^j w(t-j) + (-p)^l x_m(t-l), \qquad t > -m.$$

If $l > t+m, x_m(t-l) = \phi(t-l)$, and so for such l,

$$\left|x_m(t) - \sum_{j=0}^{l-1}(-p)^j w(t-j)\right| \le |p|^l |\phi(t-l)|.$$

Let $l \to \infty$, we get

$$x_m(t) = \begin{cases} \sum_{j=0}^{\infty}(-p)^j w(t-j), & t > -m, \\ \phi(t), & t \le -m. \end{cases} \tag{16}$$

Since $w(t)$ and $\phi(t)$ are uniformly continuous on $\mathbb{R}$, it follows that $\{x_m(t) : m \in \mathbb{Z}^+\}$ is equicontinuous on each interval $[-L, L], L \in \mathbb{Z}^+$, and by the Ascoli-Arzelá Theorem, there exists a subsequence, which we again denote by $x_m(t)$, and a function $\bar{x}(t)$ such that $x_m(t) \to \bar{x}(t)$ uniformly on $[-L, L]$, and by a familiar diagonalization procedure, can find a subsequence, again denoted by $x_m(t)$ which is such that $x_m(t) \to \bar{x}(t)$ for each $t \in \mathbb{R}$. From (16) it follows that

$$\bar{x}(t) = \sum_{j=0}^{\infty}(-p)^j w(t-j), \tag{17}$$

and so $\bar{x}(t)$ is almost periodic since $w(t-j)$ is almost periodic in t for each $j \ge 0$, and $|p| < 1$. From (13), letting $m \to \infty$, we get $\bar{x}(t) + p\bar{x}(t-1) = w(t), t \in \mathbb{R}$, and since $w(t)$ solves Eq. (1), $\bar{x}(t)$ does also.

The uniqueness of $\bar{x}(t)$ as an almost periodic solution of (1) follows from the uniqueness of the almost periodic solution $v_n : \mathbb{Z} \to \mathbb{R}^{N+1}$ of (9) given by Lemma 3.3, which determines the uniqueness of $w(t)$, and therefore from (17) the uniqueness of $\bar{x}(t)$.

Case II: $|p| > 1$. Rewriting (15) as

$$\left(\frac{-1}{p}\right)^l x_m(t-l) = \left(\frac{-1}{p}\right)^l w(t-l) + \left(\frac{-1}{p}\right)^{l+1} x_m(t-l-1), \quad t > -m, \tag{18}$$

we deduce in a similar manner that

$$x_m(t) = \begin{cases} \sum_{j=0}^{\infty}\left(\frac{-1}{p}\right)^j w(t-j), & t > -m, \\ \phi(t), & t \le -m. \end{cases} \tag{19}$$

The remainder of the proof is similar to that of Case I, we omit the details. □

Theorem 3.7. Suppose that the conditions of Theorem 3.6 are satisfied, and $g : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is almost periodic for t uniformly on $\mathbb{R} \times \mathbb{R}$, then there exists $\eta^* > 0$ such that if g satisfies

$$|g(t, x_1, y_1) - g(t, x_2, y_2)| \le \eta[|x_1 - x_2| + |y_1 - y_2|], 0 \le \eta < \eta^*,$$

for $(t, x_i, y_i) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}, i = 1, 2$. Eq. (2) has a unique almost periodic solution.

Proof. It is easy to see that the space $\mathcal{AP}(\mathbb{R}, \mathbb{R})$ is a Banach space with supremum norm $||\phi|| = \sup_{t\in\mathbb{R}} |\phi(t)|$ (see [6, 7]). For any $\phi \in \mathcal{AP}(\mathbb{R}, \mathbb{R})$, $g(t, \phi(t), \phi([t-1]))$ is an almost periodic function (see [6]). We consider the following equation:

$$(x(t) + px(t-1))^{(N)} = qx([t-1]) + g(t, \phi(t), \phi([t-1])). \tag{20}$$

From Theorem 3.6, it follows that Eq. (20) has a unique almost periodic solution, denote by $T\phi$. Thus, we obtain a mapping $T : \mathcal{AP}(\mathbb{R}, \mathbb{R}) \to \mathcal{AP}(\mathbb{R}, \mathbb{R})$. For any $\varphi, \psi \in \mathcal{AP}(\mathbb{R}, \mathbb{R}), T\varphi - T\psi$ satisfies the following equation:

$$(z(t) + pz(t-1))^{(N)} = qz([t-1]) + g(t, \varphi(t), \varphi([t-1])) - g(t, \psi(t), \psi([t-1])). \tag{21}$$

Since $T\varphi$ and $T\psi$ are almost periodic, there exists a constant $B > 0$ such that $|T\varphi|, |T\psi| \le B$ and there exists a sequence $\{\alpha_k\}, \alpha \to +\infty$ as $k \to +\infty$, such that $(T\varphi)(t + \alpha_k) \to (T\varphi)(t)$ and $(T\psi)(t + \alpha_k) \to (T\psi)(t)$ as $k \to \infty$. Setting $c_n = (T\varphi)(n) - (T\psi)(n)$, and repeating the preceding steps of Lemma 3.1, we have

$$\begin{cases} c_{n+1} = (1-p)c_n + d_n^{(1)} + \frac{1}{2!}d_n^{(2)} + \cdots + \frac{1}{(N-1)!}d_n^{(N-1)} + (p + \frac{q}{N!})c_{n-1} + \widetilde{f}_n^{(1)}, \\ d_{n+1}^{(1)} = d_n^{(1)} + d_n^{(2)} + \frac{1}{2!}d_n^{(3)} + \cdots + \frac{1}{(N-2)!}d_n^{(N-1)} + \frac{q}{(N-1)!}c_{n-1} + \widetilde{f}_n^{(2)}, \\ \quad\vdots \\ d_{n+1}^{(N-2)} = d_n^{(N-2)} + d_n^{(N-1)} + \frac{q}{2!}c_{n-1} + \widetilde{f}_n^{(N-1)}, \\ d_{n+1}^{(N-1)} = d_n^{(N-1)} + qc_{n-1} + \widetilde{f}_n^{(N)}, \end{cases} \tag{22}$$

where

$$\widetilde{f}_n^{(1)} = \int_n^{n+1}\int_n^{t_N}\cdots\int_n^{t_2}[g(t_1, \varphi(t_1), \varphi([t_1-1])) - g(t_1, \psi(t_1), \psi([t_1-1]))]dt_1dt_2\cdots dt_N,$$

$\cdots$

$$\widetilde{f}_n^{(N-1)} = \int_n^{n+1}\int_n^{t_2}[g(t_1, \varphi(t_1), \varphi([t_1-1])) - g(t_1, \psi(t_1), \psi([t_1-1]))]dt_1dt_2,$$

$$\widetilde{f}_n^{(N)} = \int_n^{n+1}[g(t_1, \varphi(t_1), \varphi([t_1-1])) - g(t_1, \psi(t_1), \psi([t_1-1]))]dt_1.$$

Systems (22) equivalent to

$$v_{n+1} = Av_n + \widetilde{h}_n, \tag{23}$$

where $v_n = (c_n, d_n^{(1)}, d_n^{(2)}, \cdots, d_n^{(N-1)}, c_{n-1})^T, \widetilde{h}_n = (\widetilde{f}_n^{(1)}, \widetilde{f}_n^{(2)}, \cdots, \widetilde{f}_n^{(N)}, 0)^T$. Let P be the nonsingular matrix such that $PAP^{-1} = \Lambda$, where $\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \cdots, \lambda_{N+1})$ and $\lambda_1, \lambda_2, \cdots, \lambda_{N+1}$ are the distinct eigenvalues of A. Define $\bar{v}_n = Pv_n$, then (23) becomes

$$\bar{v}_{n+1} = \Lambda\bar{v}_n + \bar{h}_n, \tag{24}$$

where $\bar{h}_n = P\widetilde{h}_n$. Repeating the steps of lemma 3.3, we can prove that Eq. (23) has a unique almost periodic solution v_n. Let the elements of the first column of matrix P^{-1} be $k_1, k_2, \cdots, k_{N+1}$, then we have

$$c_n = v_{n1} = k_1\overline{v}_{n1} + k_2\overline{v}_{n2} + \cdots + k_{N+1}\overline{v}_{N+1}.$$

Let $L=\{l||\lambda_l|<1, 1\le l\le N+1\}, L'=\{l||\lambda_l|>1, 1\le l\le N+1\}$. Then $L\cap L'=\emptyset, L\cup L'=\{1,2,\cdots,N+1\}$, and

$$c_n=\sum_{l\in L}k_l\sum_{m\le n}\lambda_l^{n-m}\bar{h}_{(m-1)l}+\sum_{l\in L'}k_l\sum_{m\le n}\lambda_l^{m-n}\bar{h}_{(m-1)l}.$$

So, there exists $K_0>0, K_1>0$ such that

$$|(T\varphi)(n)-(T\psi)(n)|\le K_0\sup_{n\in\mathbb{Z}}|\bar{h}_n|=K_0\sup_{n\in\mathbb{Z}}|P\widetilde{h}_n|\le K_1\eta|\varphi-\psi|, n\in\mathbb{Z}.$$

Denote

$$\begin{aligned} W(t) &= c_n+pc_{n-1}+d_n^{(1)}(t-n)+\frac{1}{2!}d_n^{(2)}(t-n)^2+\cdots \\ &\quad+\frac{1}{(N-1)!}d_n^{(N-1)}(t-n)^{N-1}+\frac{1}{N!}qc_{n-1}(t-n)^N \\ &\quad+\int_n^t\int_n^{t_N}\cdots\int_n^{t_2}[g(t_1,\varphi(t_1),\varphi([t_1-1]))-g(t_1,\psi(t_1),\psi([t_1-1]))]dt_1dt_1\cdots dt_N, \end{aligned}$$

Then there exists a sufficiently large $K_2>0$ such that $|W(t)|\le K_2\eta|\varphi-\psi|$. We easily conclude that

$$(T\varphi)(t)-(T\psi)(t)+p[(T\varphi)(t-1)-(T\psi)(t-1)]=W(t).$$

We typically consider the case when $|p|<1$. Using the first inequality in Lemma 2.7, we have

$$\begin{aligned} |(T\varphi)(t)-(T\psi)(t)| &\le e^{-a(t-t_0)}\sup_{-1\le\theta\le0}|(T\varphi)(t_0+\theta)-(T\psi)(t_0+\theta)|+b\sup_{t_0\le u\le t}|W(u)| \\ &\le 2e^{-a(t-t_0)}B+bK_2\eta|\varphi-\psi|, t\ge t_0. \end{aligned}$$

Furthermore, we have

$$|(T\varphi)(t+\alpha_k)-(T\psi)(t+\alpha_k)|\le 2e^{-a(t+\alpha_k-t_0)}B+bK_2\eta|\varphi-\psi|, t+\alpha_k\ge t_0.$$

Note $(T\varphi)(t+\alpha_k)\to(T\varphi)(t)$ and $(T\psi)(t+\alpha_k)\to(T\psi)(t)$ as $k\to\infty$. we have

$$|(T\varphi)(t)-(T\psi)(t)|\le bK_2\eta|\varphi-\psi|.$$

Let $\eta^*=(bK_2)^{-1}$. Then $0<\eta<\eta^*$ implies that $T:\mathcal{AP}(\mathbb{R},\mathbb{R})\to\mathcal{AP}(\mathbb{R},\mathbb{R})$ is a contraction mapping. It follows that there exists a unique $\phi\in\mathcal{AP}(\mathbb{R},\mathbb{R})$ such that $T\phi=\phi$, that is, Eq. (2) has a unique almost periodic solution.

For the case when $|p|>1$, we use the second inequality in Lemma 2.7, the rest of the proof is virtually the same as the above.

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