Electronic Journal of Qualitative Theory of Differential Equations
2013, No. **13**, 1-17; **http://www.math.u-szeged.hu/ejqtde/**

THE φ-ORDER OF SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS IN THE UNIT DISC*

LI-PENG XIAO

Abstract. In this paper, some results on the φ-order of solutions of linear differential equations with coefficients in the unit disc are obtained. These results yield a sharp lower bound for the sums of φ-order of functions in the solution bases. The results we obtain are a generalization of a recent result due to I. Chyzhykov, J. Heittokangas and J. Rättyä.

1. Introduction and Main results

A classical result due to H. Wittich [13] states that the coefficients of the linear differential equation

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0 \tag{1.1}$$

are polynomials if and only if all solutions of (1.1) are entire functions of finite order of growth. Later on, more detailed studies on the growth of solutions were done by different authors; see, for instance, [4, 8, 11]. In particular, Gundersen, Steinbart and Wang listed all possible orders of growth of entire solutions of (1.1) in terms of the degrees of the polynomial coefficients [5].

Recently, there has been increasing interest in studying the interaction between the analytic coefficients and solutions of (1.1) in the unit disc. The result of Wittich stated above has a natural analogue in the unit disc, as shown in [1, 6]. For instance, Heittokangas showed that all solutions of (1.1) are finite-order analytic functions in the unit disc if and only if the coefficients are $\mathcal{H}$-functions [6].

A function f, analytic in the unit disc $\mathbb{D} := \{z : |z| < 1\}$, is an $\mathcal{H}$-function if there exists a $q \in [0, \infty)$ such that

$$\sup_{z\in\mathbb{D}} |f(z)|(1 - |z|^2)^q < \infty.$$

2010 *Mathematics Subject Classification.* Primary 30D35, Secondary 34M10.
Key words and phrases. differential equation; φ-order; unit disc.
*This work is supported by the National Natural Science Foundation of China (Nos. 11126144, 11171119), the YFED of Jiangxi (No. GJJ12207) and the Natural Science Foundation of Jiangxi Province in China (No. 20122BAB211005).

The space $A^{-\infty}$, introduced by B. Korenblum [9], coincides with the space of all $\mathcal{H}$-functions. The T-order of a meromorphic function f in $\mathbb{D}$, is defined by

$$\sigma_T(f) = \overline{\lim_{r\to 1^-}} \frac{\log T(r,f)}{\log \frac{1}{1-r}},$$

where $T(r,f)$ denotes the Nevanlinna characteristic of f.

Equation (1.1) with coefficients in the weighted Bergman spaces are studied in [7, 10, 12]. For $0<p<\infty$ and $-1<\alpha<\infty$, the weighted Bergman space B^p_α consists of those functions f, analytic in D, such that

$$\int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\alpha dm(z) < \infty,$$

where $dm(z) = rdrd\theta$ is the usual Euclidean area measure. Moreover, $f \in \mathbb{B}^p_\alpha$ if $\alpha = \inf\{t \geq 0 : f \in \bigcap_{0<s<p} B^s_t\}$. The following result combines Theorems 1 and 2 in [10].

Theorem 1.1. [10] *Let the coefficients $A_0(z), \cdots, A_{k-1}(z)$ of (1.1) be analytic in $\mathbb{D}$.*

(1)*Let $0 \leq \alpha < \infty$. Then all solutions f of (1.1) satisfy $\sigma_T(f) \leq \alpha$ if and only if $A_j \in \bigcap_{0<p<\frac{1}{k-j}} B^p_\alpha$ for all $j = 0, \cdots, k-1$.*

(2)*If $A_j \in \mathbb{B}^{\frac{1}{k-j}}_{\alpha_j}$ for all $j = 0, \cdots, k-1$. Then all non-trivial solutions f of (1.1) satisfy*

$$\min_{1\leq j\leq k} \left\{ \frac{k(\alpha_0-\alpha_j)}{j} + \alpha_j \right\} \leq \sigma_T(f) \leq \max_{0\leq j\leq k-1} \{\alpha_j\}.$$

(3)*If $A_j \in \mathbb{B}^{\frac{1}{k-j}}_{\alpha_j}$ for all $j = 0, \cdots, k-1$ and if $q \in \{0.\cdots, k-1\}$ is the smallest index for which $\alpha_q = \max_{0\leq j\leq k-1}\{\alpha_j\}$, then in every fundamental solution base there are at least $k-q$ linearly independent solutions f of (1.1) such that $\sigma_T(f) = \alpha_q$.*

Later on, Theorem 1.1 is refined in [2].

Theorem 1.2. [2] *Suppose that $A_j(z) \in \mathbb{B}^{\frac{1}{k-j}}_{\alpha_j}$, where $\alpha_j \geq -1$ for $j = 0, \cdots, k-1$, and let $q \in \{0, \cdots, k-1\}$ be the smallest index for which $\alpha_q = \max_{j=0,\cdots,k-1}\{\alpha_j\}$. If $s \in \{0, \cdots, q\}$, then each solution base of (1.1) contains at least $k-s$ linearly independent solutions f such that*

$$\min_{j=s+1,\cdots,k} \left\{ \frac{(k-s)(\alpha_s-\alpha_j)}{j-s} + \alpha_j \right\} \leq \sigma_T(f) \leq \max\{0, \alpha_q\},$$

where $\alpha_k := -1$.

Solutions of (1.1) in terms of the general φ-order have been studied in [3].

Definition 1.1. [3] *Let* $\varphi : [0,1) \to (0,\infty)$ *be a non-decreasing unbounded function. The* φ*-order of* $\psi : [0,1) \to (0,\infty)$ *is defined as*

$$\sigma_\varphi(\psi) = \overline{\lim_{r\to 1^-}} \frac{\log^+ \psi(r)}{\log \varphi(r)}.$$

If f *is meromorphic in* $\mathbb{D}$*, then the* φ*-order of* f *is defined as* $\sigma_\varphi(f) = \sigma_\varphi(T(r,f))$*. The logarithmic order of* f *is defined as*

$$\lambda(f) = \overline{\lim_{r\to 1^-}} \frac{\log^+ T(r,f)}{\log(-\log(1-r))}.$$

Remark 1.1. *The usual order of growth of a meromorphic function* f *in* $\mathbb{D}$ $\sigma_T(f) = \sigma_{\frac{1}{1-r}}(f)$ *and* $\lambda(f) = \sigma_{\log\frac{1}{1-r}}(f)$*. In general, for a function* $\psi : [0,1) \to (0,\infty)$*, the expressions* $\sigma_T(\psi) = \sigma_{\frac{1}{1-r}}(\psi)$ *and* $\lambda(\psi) = \sigma_{\log\frac{1}{1-r}}(\psi)$ *denote the order and the logarithmic order of* ψ*, respectively.*

The following theorem corresponds to Theorem 1.1.

Theorem 1.3. [3] *Let the coefficients* $A_0(z), \cdots, A_{k-1}(z)$ *of (1.1) be analytic in* $\mathbb{D}$*. Let* $\varphi : [0,1) \to (0,\infty)$ *be a non-decreasing function such that* $\lambda(\varphi) = \infty$ *and*

$$\overline{\lim_{r\to 1^-}} \frac{\varphi(\frac{1+r}{2})}{\varphi(r)} = C \in [1,\infty). \tag{1.2}$$

Denote $\alpha_j = \sigma_\varphi(M_{\frac{1}{k-j}}(r,A_j)^{\frac{1}{k-j}}(1-r))$ *for* $j = 0, \cdots, k-1$*, where*

$$M_p(r,g) = \left(\frac{1}{2\pi}\int_0^{2\pi} |g(re^{i\theta})|^p d\theta\right)^{\frac{1}{p}}, \quad 0 < p < \infty,$$

is the standard L^p*-mean of the restriction of an analytic function* g *on the circle* $\{z : |z| = r\}$*.*

(1) *Let* $0 \le \alpha < \infty$*. Then all solutions* f *of (1.1) satisfy* $\sigma_\varphi(f) \le \alpha$ *if and only if* $\max_{0\le j\le k-1}\{\alpha_j\} \le \alpha$*.*

(2) *Then all non-trivial solutions* f *of (1.1) satisfy*

$$\min_{1\le j\le k}\left\{\frac{k(\alpha_0-\alpha_j)}{j} + \alpha_j\right\} \le \sigma_\varphi(f) \le \max_{0\le j\le k-1}\{\alpha_j\}.$$

(3)*If* $q \in \{0, \cdots, k-1\}$ *is the smallest index for which* $\alpha_q = \max_{0\le j\le k-1}\{\alpha_j\}$, *then in every fundamental solution base there are at least* $k-q$ *linearly independent solutions* f *of (1.1) such that* $\sigma_\varphi(f) = \alpha_q$.

The purpose of this paper is to refine Theorem 1.3. We obtain a result analogous to Theorem 1.2 in terms of the general φ-order. In fact, we obtain the following theorem.

Theorem 1.4. *Let the coefficients* $A_0(z), \cdots, A_{k-1}(z)$ *of (1.1) be analytic in* $\mathbb{D}$. *Let* $\varphi : [0,1) \to (0,\infty)$ *be a non-decreasing function such that* $\lambda(\varphi) = \infty$, *and that (1.2) is satisfied for some constant* $C \in [1,\infty)$. *Denote* $\alpha_j = \sigma_\varphi(M_{\frac{1}{k-j}}(r, A_j)^{\frac{1}{k-j}}(1-r))$ *for* $j = 0, \cdots, k-1$. *Let* $q \in \{0, \cdots, k-1\}$ *be the smallest index for which* $\alpha_q = \max_{j=0,\cdots,k-1}\{\alpha_j\}$. *If* $s \in \{0, \cdots, q\}$, *then each solution base of (1.1) contains at least* $k-s$ *linearly independent solutions* f *such that*

$$\min_{j=s+1,\cdots,k}\left\{\frac{(k-s)(\alpha_s-\alpha_j)}{j-s}+\alpha_j\right\} \le \sigma_\varphi(f) \le \alpha_q, \tag{1.3}$$

where $\alpha_k := -1$.

Remark 1.2. *The case* $s = 0$ *of Theorem 1.4 clearly reduces to Theorem 1.3* (2), *and the assertion of Theorem 1.4 for* $s = q$ *is contained in Theorem 1.3* (3).

In order to state the following corollaries of Theorem 1.4, we denote

$$\beta(s) := \min_{j=s+1,\cdots,k}\left\{\frac{(k-s)(\alpha_s-\alpha_j)}{j-s}+\alpha_j\right\}, s = 0, \cdots, q,$$

where $\alpha_k := -1$. Moreover, we define

$$s^\star := \min\{s \in \{0, \cdots, q\} : \beta(s) > 0\}.$$

Corollary 1.1. *Let the coefficients* $A_0(z), \cdots, A_{k-1}(z)$ *of (1.1) be analytic in* $\mathbb{D}$. *Let* $\varphi : [0,1) \to (0,\infty)$ *be a non-decreasing function such that both* $\lambda(\varphi) = \infty$ *and (1.2) is satisfied for some constant* $C \in [1,\infty)$. *Denote* $\alpha_j = \sigma_\varphi(M_{\frac{1}{k-j}}(r, A_j)^{\frac{1}{k-j}}(1-r))$ *for* $j = 0, \cdots, k-1$. *Let* $q \in \{0, \cdots, k-1\}$ *be the smallest index for which* $\alpha_q = \max_{j=0,\cdots,k-1}\{\alpha_j\}$. *Then each solution base of (1.1) admits at most* $s^\star \le q$ *solutions* f *satisfying* $\sigma_\varphi(f) < \beta(s^\star)$. *In particular, there are at most* $s^\star \le q$ *solutions* f *satisfying* $\sigma_\varphi(f) = 0$.

To estimate the quantity $\sum_{j=1}^{k}\sigma_\varphi(f_j)$ by using Theorem 1.4, we set

$$\gamma(j) := \max\{\beta(0), \cdots, \beta(j)\}, \quad j = 0, \cdots, q.$$

Corollary 1.2. *Let the coefficients $A_0(z), \cdots, A_{k-1}(z)$ of (1.1) be analytic in $\mathbb{D}$. Let $\varphi : [0,1) \to (0,\infty)$ be a non-decreasing function such that both $\lambda(\varphi) = \infty$ and (1.2) is satisfied for some constant $C \in [1,\infty)$. Denote $\alpha_j = \sigma_\varphi(M_{\frac{1}{k-j}}(r, A_j)^{\frac{1}{k-j}}(1-r))$ for $j = 0, \cdots, k-1$. Let $q \in \{0, \cdots, k-1\}$ be the smallest index for which $\alpha_q = \max_{j=0,\cdots,k-1}\{\alpha_j\}$. Let $f_1, \cdots, f_k$ be a solution base of (1.1). If $q = 0$, then $\sum_{j=1}^{k} \sigma_\varphi(f_j) = k\alpha_0$, while if $q \geq 1$, then*

$$(k-q)\alpha_q + \sum_{j=s^\star}^{q-1} \gamma(j) \leq \sum_{j=1}^{k} \sigma_\varphi(f_j) \leq k\alpha_q. \tag{1.4}$$

Note that the sum in (1.4) is considered to be empty, if $s^\star = q$. Corollary 1.2 is sharp. This is illustrated by an example in Section 5.

2. Lemmas for the proof of Theorem

The following lemma on the order reduction procedure originates from $\mathbb{C}$.

Lemma 2.1. ([5]) *Let $f_{0,1}, f_{0,2}, \cdots, f_{0,m}$ be $m \geq 2$ linearly independent meromorphic solutions of*

$$y^{(k)} + A_{0,k-1}(z)y^{(k-1)} + \cdots + A_{0,0}(z)y = 0, \quad k \geq m,$$

where $A_{0,0}(z), \cdots, A_{0,k-1}(z)$ are meromorphic functions in $\mathbb{D}$. For $1 \leq p \leq m-1$, set

$$f_{p,j} = \left(\frac{f_{p-1,j+1}}{f_{p-1,1}}\right)', \quad j = 1, \cdots, m-p.$$

Then $f_{p,1}, f_{p,2}, \cdots, f_{p,m-p}$ are linearly independent meromorphic solutions of

$$y^{(k-p)} + A_{p,k-p-1}(z)y^{(k-p-1)} + \cdots + A_{p,0}(z)y = 0, \tag{2.1}$$

where

$$A_{p,j}(z) = \sum_{n=j+1}^{k-p+1} \binom{n}{j+1} A_{p-1,n}(z)\frac{f_{p-1,1}^{(n-j-1)}(z)}{f_{p-1,1}(z)}$$

for $j = 0, \cdots, k-p-1$. Here, $\binom{n}{j+1}$ denotes the binomial coefficient, and $A_{n,k-n}(z) \equiv 1$ for all $n = 0, \cdots, p$.

Lemma 2.2. ([3]) *Let k and j be integers satisfying $k > j \geq 0$, and let $0 < \delta < 1$ and $\varepsilon > 0$. Let f be a meromorphic function in $\mathbb{D}$ such that $f^{(j)}$ does not vanish identically.*

(1) *If $\sigma_\varphi(f) < \infty$, where $\varphi : [0,1) \to (0,\infty)$ is a non-decreasing function such that $\lambda(\varphi) = \infty$, and that (1.2) is satisfied for some $C \in [1,\infty)$, then there exists a measurable set $E \subset [0,1)$ with $\overline{D}(E) \leq \delta$ such that*

$$\int_0^{2\pi} \left| \frac{f^{(k)}(re^{i\theta})}{f^{(j)}(re^{i\theta})} \right|^{\frac{1}{k-j}} d\theta \leq \frac{\varphi(r)^{\sigma_\varphi(f)+\varepsilon}}{1-r}, \quad r \notin E.$$

(2) *If $\lambda(f) < \infty$, then there exists a measurable set $E \subset [0,1)$ with $\overline{D}(E) \leq \delta$ such that*

$$\int_0^{2\pi} \left| \frac{f^{(k)}(re^{i\theta})}{f^{(j)}(re^{i\theta})} \right|^{\frac{1}{k-j}} d\theta \leq \frac{1}{1-r} \left(\log \frac{1}{1-r} \right)^{\max\{\lambda(f),1\}+\varepsilon}, \quad r \notin E.$$

where

$$\overline{D}(E) = \overline{\lim_{r\to 1^-}} \frac{m(E \cap [r,1))}{1-r}.$$

Lemma 2.3. ([3]) *Let $0 \leq q < \infty$, $0 \leq \alpha \leq \infty$, $0 < p, \varepsilon < \infty$ and $0 < \eta < 1$. Let $\varphi : [0,1) \to (0,\infty)$ be a non-decreasing unbounded function such that (1.2) is satisfied for some $C \in [1,\infty)$. If f is an analytic function in $\mathbb{D}$ such that*

$$\overline{\lim_{r\to 1^-}} \frac{\log^+(M_p(r,f)^p(1-r)^q)}{\log \varphi(r)} = \alpha,$$

then there is a set $F \subset [0,1)$ with $\overline{D}(F) \geq \eta$ such that

$$\underline{\lim}_{r\to 1^-, r\in F} \frac{\log^+(M_p(r,f)^p(1-r)^q)}{\log \varphi(r)} \geq \alpha - \varepsilon, \quad \alpha < \infty,$$

$$\lim_{r\to 1^-, r\in F} \frac{\log^+(M_p(r,f)^p(1-r)^q)}{\log \varphi(r)} = \infty, \quad \alpha = \infty.$$

3. Proof of Theorem 1.4

Proof. We only need to prove the first inequality in (1.3) for $s \in \{1, \cdots, q-1\}$. We consider two separate cases.

Case (i). $s = 1$.

Let $k \geq 3$, $q \geq 2$, $s = 1$, and $\beta(1) > 0$, since otherwise there is nothing to prove. Let $\{f_{0,1}, f_{0,2}, \cdots, f_{0,k}\}$ be a solution base of

(1.1), and assume on the contrary to the assertion that there exist $s+1=2$ linearly independent solutions $f_{0,1}$ and $f_{0,2}$ such that $\max\{\sigma_\varphi(f_{0,1}),\, \sigma_\varphi(f_{0,2})\} =: \sigma < \beta(1)$. Then the meromorphic function $g := (\frac{f_{0,2}}{f_{0,1}})'$ satisfies $\sigma_\varphi(g) \le \sigma$. Moreover, Lemma 2.1 implies that g satisfies

$$g^{(k-1)} + A_{1,k-2}(z)g^{(k-2)} + \cdots + A_{1,0}(z)g = 0, \tag{3.1}$$

where

$$A_{1,j}(z) = A_{0,j+1}(z) + \sum_{n=j+2}^{k} \binom{n}{j+1} A_{0,n}(z)\frac{f_{0,1}^{(n-j-1)}(z)}{f_{0,1}(z)} \tag{3.2}$$

for $j = 0, 1, \cdots, k-2$, and $A_{0,k}(z) \equiv 1$. Therefore

$$|A_{0,1}(z)| \le |A_{1,0}(z)| + \sum_{n=2}^{k} \binom{n}{1} |A_{0,n}(z)| \left|\frac{f_{0,1}^{(n-1)}(z)}{f_{0,1}(z)}\right|,$$

where

$$|A_{1,0}(z)| \le \left|\frac{g^{(k-1)}(z)}{g(z)}\right| + |A_{1,k-2}(z)| \left|\frac{g^{(k-2)}(z)}{g(z)}\right| + \cdots + |A_{1,1}(z)| \left|\frac{g'(z)}{g(z)}\right|,$$

since g satisfies (3.1). Putting the last two inequalities together, we obtain

$$|A_{0,1}(z)| \lesssim \sum_{j=1}^{k-1} |A_{1,j}(z)| \left|\frac{g^{(j)}(z)}{g(z)}\right| + \sum_{n=2}^{k} |A_{0,n}(z)| \left|\frac{f_{0,1}^{(n-1)}(z)}{f_{0,1}(z)}\right|.$$

Here $|f(z)| \lesssim |g(z)|$ if there exists a constant $C > 0$ independent of z such that $|f(z)| \le C|g(z)|$. Raising both sides to the power $\frac{1}{k-1}$ and integrating θ from 0 to 2π, we obtain,

$$\begin{aligned}\int_0^{2\pi} |A_{0,1}(re^{i\theta})|^{1/(k-1)} d\theta \lesssim &\sum_{j=1}^{k-1} \int_0^{2\pi} |A_{1,j}(re^{i\theta})|^{1/(k-1)} \left|\frac{g^{(j)}(re^{i\theta})}{g(re^{i\theta})}\right|^{1/(k-1)} d\theta \\ &+ \sum_{n=2}^{k} \int_0^{2\pi} |A_{0,n}(re^{i\theta})|^{1/(k-1)} \left|\frac{f_{0,1}^{(n-1)}(re^{i\theta})}{f_{0,1}(re^{i\theta})}\right|^{1/(k-1)} d\theta.\end{aligned} \tag{3.3}$$

To deal with the second sum in (3.3), consider

$$I_n := \int_0^{2\pi} |A_{0,n}(re^{i\theta})|^{1/(k-1)} \left|\frac{f_{0,1}^{(n-1)}(re^{i\theta})}{f_{0,1}(re^{i\theta})}\right|^{1/(k-1)} d\theta, \quad n = 2, \cdots, k.$$

Let $\varepsilon > 0$ be a small constant. Then by Lemma 2.2 (1),

(3.4)
$$I_k = \int_0^{2\pi} \left| \frac{f_{0,1}^{(k-1)}(re^{i\theta})}{f_{0,1}(re^{i\theta})} \right|^{1/(k-1)} d\theta \le \frac{\varphi(r)^{\sigma+\varepsilon}}{1-r} \le \frac{\varphi(r)^{\alpha_1 - 2\varepsilon}}{1-r}, \quad r \notin E,$$

holds, since $\sigma_\varphi(f_{0,1}) \le \sigma < \beta(1) \le \alpha_1$. Moreover, by the Hölder inequality (with the indices $(k-1)/(k-n)$ and $(k-1)/(n-1)$) and the definition of α_n, we have

$$\begin{aligned} I_n &= \int_0^{2\pi} |A_{0,n}(re^{i\theta})|^{1/(k-1)} \left| \frac{f_{0,1}^{(n-1)}(re^{i\theta})}{f_{0,1}(re^{i\theta})} \right|^{1/(k-1)} d\theta \\ &\le \left(\int_0^{2\pi} |A_{0,n}(re^{i\theta})|^{\frac{1}{k-n}} d\theta \right)^{\frac{k-n}{k-1}} \left(\int_0^{2\pi} \left| \frac{f_{0,1}^{(n-1)}(re^{i\theta})}{f_{0,1}(re^{i\theta})} \right|^{\frac{1}{n-1}} d\theta \right)^{\frac{n-1}{k-1}} \\ &\le \left(\frac{\varphi(r)^{\alpha_n+\varepsilon}}{1-r} \right)^{\frac{k-n}{k-1}} \left(\frac{\varphi(r)^{\sigma+\varepsilon}}{1-r} \right)^{\frac{n-1}{k-1}} \end{aligned}$$

(3.5)
$$= \frac{\varphi(r)^{\alpha_n \frac{k-n}{k-1} + \sigma \frac{n-1}{k-1} + \varepsilon}}{1-r}, \quad r \notin E,$$

for all $n = 2, \cdots, k-1$. Since

$$\sigma_\varphi(f_{0,1}) \le \sigma < \beta(1) \le \frac{(k-1)(\alpha_1 - \alpha_n)}{n-1} + \alpha_n, \quad n = 2, \cdots, k-1,$$

we have

$$\begin{aligned} &\alpha_n \frac{k-n}{k-1} + \sigma \frac{n-1}{k-1} + \varepsilon \\ \le &\alpha_n \frac{k-n}{k-1} + \left(\frac{(k-1)(\alpha_1 - \alpha_n)}{n-1} + \alpha_n - \frac{3(k-1)}{n-1}\varepsilon \right) \frac{n-1}{k-1} + \varepsilon \end{aligned}$$

(3.6)
$$= \alpha_1 - 2\varepsilon, \quad n = 2, \cdots, k-1.$$

Inequalities (3.4)-(3.6) show that

(3.7)
$$I_n \le \frac{\varphi(r)^{\alpha_1 - 2\varepsilon}}{1-r}, \quad r \notin E,$$

for $n = 2, \cdots, k$.

To deal with the first sum in (3.3), denote

$$J_j := \int_0^{2\pi} |A_{1,j}(re^{i\theta})|^{\frac{1}{k-1}} \left| \frac{g^{(j)}(re^{i\theta})}{g(re^{i\theta})} \right|^{\frac{1}{k-1}} d\theta, \quad j = 1, \cdots, k-1.$$

Lemma 2.2 (1) implies that
(3.8)

$$J_{k-1} = \int_0^{2\pi} \left| \frac{g^{(k-1)}(re^{i\theta})}{g(re^{i\theta})} \right|^{\frac{1}{k-1}} d\theta \leq \frac{\varphi(r)^{\sigma+\varepsilon}}{1-r} \leq \frac{\varphi(r)^{\alpha_1-2\varepsilon}}{1-r}, \quad r \notin E,$$

for $\varepsilon > 0$ being small enough since $\sigma_\varphi(g) \leq \sigma < \beta(1) \leq \alpha_1$. Moreover, by (3.2) we have

$$\begin{aligned} J_j &= \int_0^{2\pi} |A_{1,j}(re^{i\theta})|^{\frac{1}{k-1}} \left| \frac{g^{(j)}(re^{i\theta})}{g(re^{i\theta})} \right|^{\frac{1}{k-1}} d\theta \\ &\lesssim \int_0^{2\pi} |A_{0,j+1}(re^{i\theta})|^{\frac{1}{k-1}} \left| \frac{g^{(j)}(re^{i\theta})}{g(re^{i\theta})} \right|^{\frac{1}{k-1}} d\theta \\ &\quad + \sum_{n=j+2}^{k} \int_0^{2\pi} |A_{0,n}(re^{i\theta})|^{\frac{1}{k-1}} \left| \frac{f_{0,1}^{(n-j-1)}(re^{i\theta})}{f_{0,1}(re^{i\theta})} \right|^{\frac{1}{k-1}} \left| \frac{g^{(j)}(re^{i\theta})}{g(re^{i\theta})} \right|^{\frac{1}{k-1}} d\theta \end{aligned}$$

(3.9)

$$=: K_j + L_{j,k} + \sum_{n=j+2}^{k-1} L_{j,n}$$

for all $j = 1, \cdots, k-2$. Since $\max\{\sigma_\varphi(g), \sigma_\varphi(f_{0,1})\} \leq \sigma < \beta(1) \leq \alpha_1$, we deduce that K_j behaves like I_{j+1} and hence

$$K_j \leq \frac{\varphi(r)^{\alpha_1-2\varepsilon}}{1-r}, \quad r \notin E, \tag{3.10}$$

for $\varepsilon > 0$ being small enough. Moreover, by the Hölder inequality, with the indices $\frac{k-1}{k-j-1}$ and $\frac{k-1}{j}$, and Lemma 2.2 (1), we have

$$\begin{aligned}
L_{j,k} &= \int_0^{2\pi} \left| \frac{f_{0,1}^{(k-j-1)}(re^{i\theta})}{f_{0,1}(re^{i\theta})} \right|^{\frac{1}{k-1}} \left| \frac{g^{(j)}(re^{i\theta})}{g(re^{i\theta})} \right|^{\frac{1}{k-1}} d\theta \\
&\leq \left(\int_0^{2\pi} \left| \frac{f_{0,1}^{(k-j-1)}(re^{i\theta})}{f_{0,1}(re^{i\theta})} \right|^{\frac{1}{k-j-1}} d\theta \right)^{\frac{k-j-1}{k-1}} \left(\int_0^{2\pi} \left| \frac{g^{(j)}(re^{i\theta})}{g(re^{i\theta})} \right|^{\frac{1}{j}} d\theta \right)^{\frac{j}{k-1}} \\
&\leq \left(\frac{\varphi(r)^{\sigma+\varepsilon}}{1-r} \right)^{\frac{k-j-1}{k-1}} \left(\frac{\varphi(r)^{\sigma+\varepsilon}}{1-r} \right)^{\frac{j}{k-1}} \\
&= \frac{\varphi(r)^{\sigma+\varepsilon}}{1-r} \\
&\leq \frac{\varphi(r)^{\alpha_1-2\varepsilon}}{1-r}, \quad r \notin E, \qquad (3.11)
\end{aligned}$$

for all $j = 1, \cdots, k-2$ when $\varepsilon > 0$ is sufficiently small. It remains to consider $L_{j,n}$. By general form of the Hölder inequality with the indices $\frac{k-1}{k-n}$, $\frac{k-1}{n-j-1}$, $\frac{k-1}{j}$ and (3.6), we have

$$\begin{aligned}
L_{j,n} &= \int_0^{2\pi} |A_{0,n}(re^{i\theta})|^{\frac{1}{k-1}} \left| \frac{f_{0,1}^{(n-j-1)}(re^{i\theta})}{f_{0,1}(re^{i\theta})} \right|^{\frac{1}{k-1}} \left| \frac{g^{(j)}(re^{i\theta})}{g(re^{i\theta})} \right|^{\frac{1}{k-1}} d\theta \\
&\leq \left(\int_0^{2\pi} |A_{0,n}(re^{i\theta})|^{\frac{1}{k-n}} d\theta \right)^{\frac{k-n}{k-1}} \left(\int_0^{2\pi} \left| \frac{f_{0,1}^{(n-j-1)}(re^{i\theta})}{f_{0,1}(re^{i\theta})} \right|^{\frac{1}{n-j-1}} d\theta \right)^{\frac{n-j-1}{k-1}} \\
&\quad \left(\int_0^{2\pi} \left| \frac{g^{(j)}(re^{i\theta})}{g(re^{i\theta})} \right|^{\frac{1}{j}} d\theta \right)^{\frac{j}{k-1}} \\
&\leq \left(\frac{\varphi(r)^{\alpha_n+\varepsilon}}{1-r} \right)^{\frac{k-n}{k-1}} \left(\frac{\varphi(r)^{\sigma+\varepsilon}}{1-r} \right)^{\frac{n-j-1}{k-1}} \left(\frac{\varphi(r)^{\sigma+\varepsilon}}{1-r} \right)^{\frac{j}{k-1}} \\
&= \frac{1}{1-r} \varphi(r)^{\alpha_n \frac{k-n}{k-1} + \sigma \frac{n-1}{k-1} + \varepsilon} \\
&\leq \frac{\varphi(r)^{\alpha_1-2\varepsilon}}{1-r}, \quad r \notin E. \qquad (3.12)
\end{aligned}$$

Inequalities (3.8)-(3.12) show that

$$J_j \lesssim \frac{\varphi(r)^{\alpha_1-2\varepsilon}}{1-r}, \quad r \notin E, \tag{3.13}$$

for $j = 1, \cdots, k-1$.

Let $\eta \in (\delta, 1)$, and let F be the set in Lemma 2.3 with $\overline{D}(F) \geq \eta$. Then $\overline{D}(F \setminus E) \geq \eta - \delta > 0$, and Lemma 2.3 yield

$$M_{\frac{1}{k-1}}(r, A_{0,1})^{\frac{1}{k-1}} \geq \frac{\varphi(r)^{\alpha_1-\varepsilon}}{1-r}, \quad r \in F \setminus E. \tag{3.14}$$

This, with the aid of (3.3), (3.7) and (3.13), results in $\alpha_1 - \varepsilon \leq \alpha_1 - 2\varepsilon$, a contradiction. It follows that (1.1) has at least $k-1$ linearly independent solutions f such that $\sigma_\varphi(f) \geq \beta(1)$.

Case (ii). $s > 1$.

Let $k \geq 3$, $q \geq 2$, $s > 1$, and $\beta(s) > 0$, since otherwise there is nothing to prove. In particular, it follows that $\alpha_s > 0$. Let $\{f_{0,1}, f_{0,2}, \cdots, f_{0,k}\}$ be a solution base of (1.1), and assume on the contrary to the assertion that there exist $s+1$ linearly independent solutions $f_{0,1}, \cdots, f_{0,s+1}$ such that

$$\sigma := \max\{\sigma_\varphi(f_{0,1}), \cdots, \sigma_\varphi(f_{0,s+1})\} < \beta(s) \leq \alpha_s.$$

Set

$$f_{p,j} = \left(\frac{f_{p-1,j+1}}{f_{p-1,1}}\right)', \quad j = 1, 2, \cdots, s+1-p. \tag{3.15}$$

From Lemma 2.1, $f_{p,1}, \cdots, f_{p,s+1-p}$ are linearly independent meromorphic solutions of Eq.(2.1) and $\sigma_\varphi(f_{p,j}) \leq \sigma$. Taking $p = s$ and using (3.15) and Lemma 2.1, we obtain that $f_{s,1}$ is a nontrivial solution of an equation of the form

$$f^{(k-s)} + A_{s,k-s-1}(z) f^{(k-s-1)} + \cdots + A_{s,0}(z) f = 0.$$

Moreover, as in the case $s = 1$, Lemma 2.1 implies

$$|A_{0,s}(z)| \leq |A_{s,0}(z)| + \sum_{m=0}^{s-1} \sum_{n=s+1-m}^{k-m} \left(\begin{array}{c} n \\ s-m \end{array}\right) |A_{m,n}(z)| \left|\frac{f_{m,1}^{(n-s+m)}(z)}{f_{m,1}(z)}\right|,$$

where

$$|A_{s,0}(z)| \leq \left|\frac{f_{s,1}^{(k-s)}(z)}{f_{s,1}(z)}\right| + \sum_{m=1}^{k-s-1} |A_{s,m}(z)| \left|\frac{f_{s,1}^{(m)}(z)}{f_{s,1}(z)}\right|.$$

Putting these inequalities together, we obtain,

$$|A_{0,s}(z)| \lesssim \sum_{m=0}^{s} \sum_{n=s+1-m}^{k-m-1} |A_{m,n}(z)| \left| \frac{f_{m,1}^{(n-s+m)}(z)}{f_{m,1}(z)} \right| + \sum_{m=0}^{s} \left| \frac{f_{m,1}^{(k-s)}(z)}{f_{m,1}(z)} \right|.$$

Raising both sides to the power $\frac{1}{k-s}$ and integrating θ from 0 to 2π, we obtain

$$\begin{aligned} \int_0^{2\pi} |A_{0,s}(re^{i\theta})|^{\frac{1}{k-s}} d\theta \lesssim & \sum_{m=0}^{s} \sum_{n=s+1-m}^{k-m-1} \int_0^{2\pi} |A_{m,n}(re^{i\theta})|^{\frac{1}{k-s}} \left| \frac{f_{m,1}^{(n-s+m)}(re^{i\theta})}{f_{m,1}(re^{i\theta})} \right|^{\frac{1}{k-s}} d\theta \\ & + \sum_{m=0}^{s} \int_0^{2\pi} \left| \frac{f_{m,1}^{(k-s)}(re^{i\theta})}{f_{m,1}(re^{i\theta})} \right|^{\frac{1}{k-s}} d\theta \\ =: & \sum_{m=0}^{s} \sum_{n=s+1-m}^{k-m-1} I_{m,n} + \sum_{m=0}^{s} J_m. \end{aligned} \tag{3.16}$$

Lemma 2.2 (1) implies that

$$J_m = \int_0^{2\pi} \left| \frac{f_{m,1}^{(k-s)}(re^{i\theta})}{f_{m,1}(re^{i\theta})} \right|^{\frac{1}{k-s}} d\theta \leq \frac{\varphi(r)^{\sigma+\varepsilon}}{1-r} \leq \frac{\varphi(r)^{\alpha_s-\varepsilon}}{1-r}, \quad r \notin E \tag{3.17}$$

for $m = 0, \cdots, s$ and $\varepsilon > 0$ being small enough. It remains ro consider $I_{m,n}$ for $m = 0, \cdots, s$ and $n = s+1-m, \cdots, k-m-1$.

By the Hölder inequality (with the indices $\frac{k-s}{k-n}$ and $\frac{k-s}{n-s}$) and the fact

$$\sigma_\varphi(f_{0,1}) \leq \sigma < \beta(s) \leq \frac{(k-s)(\alpha_s - \alpha_n)}{n-s} + \alpha_n, \tag{3.18}$$

for $n = s+1, \cdots, k-1$, we have

$$
\begin{aligned}
I_{0,n} &= \int_0^{2\pi} |A_{0,n}(re^{i\theta})|^{\frac{1}{k-s}} \left| \frac{f_{0,1}^{(n-s)}(re^{i\theta})}{f_{0,1}(re^{i\theta})} \right|^{\frac{1}{k-s}} d\theta \\
&\leq \left(\int_0^{2\pi} |A_{0,n}(re^{i\theta})|^{\frac{1}{k-n}} d\theta \right)^{\frac{k-n}{k-s}} \left(\int_0^{2\pi} \left| \frac{f_{0,1}^{(n-s)}(re^{i\theta})}{f_{0,1}(re^{i\theta})} \right|^{\frac{1}{n-s}} d\theta \right)^{\frac{n-s}{k-s}} \\
&\leq \left(\frac{\varphi(r)^{\alpha_n+\varepsilon}}{1-r} \right)^{\frac{k-n}{k-s}} \left(\frac{\varphi(r)^{\sigma+\varepsilon}}{1-r} \right)^{\frac{n-s}{k-s}} \\
&= \frac{1}{1-r} \varphi(r)^{\alpha_n \frac{k-n}{k-s} + \sigma \frac{n-s}{k-s} + \varepsilon} \\
&\leq \frac{1}{1-r} \varphi(r)^{\alpha_s - 2\varepsilon},
\end{aligned} \tag{3.19}
$$

for $\varepsilon > 0$ being small enough. In the general case Lemma 2.1 gives

$$
\begin{aligned}
I_{m,n} &= \int_0^{2\pi} |A_{m,n}(re^{i\theta})|^{\frac{1}{k-s}} \left| \frac{f_{m,1}^{(n-s+m)}(re^{i\theta})}{f_{m,1}(re^{i\theta})} \right|^{\frac{1}{k-s}} d\theta \\
&\lesssim \sum_{n_1=n+1}^{k-m+1} \int_0^{2\pi} |A_{m-1,n_1}(re^{i\theta})|^{\frac{1}{k-s}} \left| \frac{f_{m-1,1}^{(n_1-n-1)}(re^{i\theta})}{f_{m-1,1}(re^{i\theta})} \right|^{\frac{1}{k-s}} \left| \frac{f_{m,1}^{(n-s+m)}(re^{i\theta})}{f_{m,1}(re^{i\theta})} \right|^{\frac{1}{k-s}} d\theta \\
&\lesssim \sum_{n_1=n+1}^{k-m+1} \sum_{n_2=n_1+1}^{k-m+2} \int_0^{2\pi} |A_{m-2,n_2}(re^{i\theta})|^{\frac{1}{k-s}} \left| \frac{f_{m-2,1}^{(n_2-n_1-1)}(re^{i\theta})}{f_{m-2,1}(re^{i\theta})} \right|^{\frac{1}{k-s}} \\
&\quad \left| \frac{f_{m-1,1}^{(n_1-n-1)}(re^{i\theta})}{f_{m-1,1}(re^{i\theta})} \right|^{\frac{1}{k-s}} \left| \frac{f_{m,1}^{(n-s+m)}(re^{i\theta})}{f_{m,1}(re^{i\theta})} \right|^{\frac{1}{k-s}} d\theta \\
&\lesssim \sum_{n_1=n+1}^{k-m+1} \sum_{n_2=n_1+1}^{k-m+2} \cdots \sum_{n_m=n_{m-1}+1}^{k} K(n, n_1, \cdots, n_m),
\end{aligned} \tag{3.20}
$$

where

$$K(n,n_1,\cdots,n_m)=\int_0^{2\pi}|A_{0,n_m}(re^{i\theta})|^{\frac{1}{k-s}}\left|\frac{f_{0,1}^{(n_m-n_{m-1}-1)}(re^{i\theta})}{f_{0,1}(re^{i\theta})}\right|^{\frac{1}{k-s}}\cdots$$
$$\left|\frac{f_{m-2,1}^{(n_2-n_1-1)}(re^{i\theta})}{f_{m-2,1}(re^{i\theta})}\right|^{\frac{1}{k-s}}\left|\frac{f_{m-1,1}^{(n_1-n-1)}(re^{i\theta})}{f_{m-1,1}(re^{i\theta})}\right|^{\frac{1}{k-s}}\left|\frac{f_{m,1}^{(n-s+m)}(re^{i\theta})}{f_{m,1}(re^{i\theta})}\right|^{\frac{1}{k-s}}d\theta.$$

If $n_m=k$, then $A_{0,k}(z)\equiv 1$, and general form of the Hölder inequality with the indices

$$\frac{n_m-s}{n_m-n_{m-1}-1},\frac{n_m-s}{n_{m-1}-n_{m-2}-1},\cdots,\frac{n_m-s}{n_1-n-1},\frac{n_m-s}{n-s+m},$$

together with Lemma 2.2 (1) shows that

$$K(n,n_1,\cdots,n_m)\le\frac{\varphi(r)^{\sigma+\varepsilon}}{1-r}\le\frac{\varphi(r)^{\alpha_s-2\varepsilon}}{1-r},\quad r\notin E. \tag{3.21}$$

If $n_m<k$, then general form of the Hölder inequality with the indices

$$\frac{k-s}{k-n_m},\frac{k-s}{n_m-n_{m-1}-1},\frac{k-s}{n_{m-1}-n_{m-2}-1},\cdots,\frac{k-s}{n_1-n-1},\frac{k-s}{n-s+m},$$

together with Lemma 2.2 (1) and (3.18) shows that

$$\begin{aligned}K(n,n_1,\cdots,n_m)&\le\left(\frac{\varphi(r)^{\alpha_{n_m}+\varepsilon}}{1-r}\right)^{\frac{k-n_m}{k-s}}\left(\frac{\varphi(r)^{\sigma+\varepsilon}}{1-r}\right)^{\frac{n_m-n_{m-1}-1}{k-s}}\cdots\left(\frac{\varphi(r)^{\sigma+\varepsilon}}{1-r}\right)^{\frac{n-s+m}{k-s}}\\&=\frac{1}{1-r}\varphi(r)^{\alpha_{n_m}\frac{k-n_m}{k-s}+\sigma\frac{n_m-s}{k-s}+\varepsilon}\\&\le\frac{1}{1-r}\varphi(r)^{\alpha_s-2\varepsilon},r\notin E.\end{aligned}\tag{3.22}$$

Inequalities (3.19)-(3.22) show that

$$I_{m,n}\lesssim\frac{\varphi(r)^{\alpha_s-2\varepsilon}}{1-r},\quad r\notin E, \tag{3.23}$$

for $m=0,\cdots,s$ and $n=s+1-m,\cdots,k-m-1$.

Let $\eta\in(\delta,1)$, and let F be the set in Lemma 2.3 with $\overline{D}(F)\ge\eta$. Then $\overline{D}(F\setminus E)\ge\eta-\delta>0$ and Lemma 2.3 yield

$$M_{\frac{1}{k-s}}(r,A_{0,s})^{\frac{1}{k-s}}\ge\frac{\varphi(r)^{\alpha_s-\varepsilon}}{1-r},\quad r\in F\setminus E. \tag{3.24}$$

This, with the aid of (3.16), (3.17) and (3.23) results in $\alpha_s - \varepsilon \leq \alpha_s - 2\varepsilon$, a contradiction. It follows that (1.1) has at least $k-s$ linearly independent solutions f such that $\sigma_\varphi(f) \geq \beta(s)$. This completes the proof of Theorem 1.4. □

4. Proof of Corollary 1.2

Proof. The upper bound in (1.4) follows directly from Theorem 1.4. To conclude the lower bound in (1.4), assume that solutions $f_1, f_2, \cdots, f_k$ are given in increasing order with respect to φ-order of growth; that is,

$$\sigma_\varphi(f_1) \leq \cdots \leq \sigma_\varphi(f_k). \tag{4.1}$$

By applying Theorem 1.4 with $s = 0, \cdots, q$, we can get

$$\sigma_\varphi(f_1) \geq \beta(0),\ \sigma_\varphi(f_2) \geq \beta(1),\ \cdots, \sigma_\varphi(f_{q+1}) \geq \beta(q),\ \cdots,\ \sigma_\varphi(f_k) \geq \beta(q). \tag{4.2}$$

(4.1) and (4.2) show that

$$\begin{aligned} \sigma_\varphi(f_1) &\geq \beta(0) = \gamma(0), \\ \sigma_\varphi(f_2) &\geq \max\{\beta(0), \beta(1)\} = \gamma(1), \\ &\cdots \\ \sigma_\varphi(f_q) &\geq \max\{\beta(0), \beta(1), \cdots, \beta(q-1)\} = \gamma(q-1), \\ \sigma_\varphi(f_{q+1}) &\geq \max\{\beta(0), \beta(1), \cdots, \beta(q)\} = \gamma(q) = \alpha_q, \\ &\cdots \\ \sigma_\varphi(f_k) &\geq \beta(q) = \alpha_q. \end{aligned}$$

Hence the assertion follows by noting that if $j \in \{0, \cdots, s^\star - 1\}$, then $\gamma(j) \leq 0$. This completes the proof of Corollary 1.2. □

5. Example

The sharpness of Corollary 1.2 in the case $\varphi(r) = \frac{1}{1-r}$ is discussed as follows. For $\beta \geq 1$, the functions $f_{1,2}(z) = \exp\{\pm i(\frac{1+z}{1-z})^\beta\}$, and $f_3(z) = (\frac{1+z}{1-z})^\beta$ are linearly independent solutions of

$$f''' + A_2(z)f'' + A_1(z)f' + A_0(z)f = 0,$$

where

$$\begin{aligned} A_0(z) &= -8\beta^3 \frac{(1+z)^{2\beta-3}}{(1-z)^{2\beta+3}}, \\ A_1(z) &= 4\beta^2 \frac{(1+z)^{2\beta-2}}{(1-z)^{2\beta+2}} + 2\frac{3z^2 + 8\beta z + 6\beta^2 - 1}{(1+z)^2(1-z)^2}, \end{aligned}$$

and

$$A_2(z) = -2\frac{3z+4\beta}{(1+z)(1-z)}$$

are analytic in $\mathbb{D}$; see [10]. Clearly, $\sigma_T(f_{1,2}) = \beta - 1$, and $\sigma_T(f_3) = 0$. On the other hand, a routine computation shows that $\alpha_0 = \sigma_T(M_{\frac{1}{3}}(r, A_0)^{\frac{1}{3}}(1-r)) = \frac{2}{3}\beta - 1$, $\alpha_1 = \sigma_T(M_{\frac{1}{2}}(r, A_1)^{\frac{1}{2}}(1-r)) = \beta - 1$, $\alpha_2 = \sigma_T(M_1(r, A_2)(1-r)) = 0$, $\gamma(0) = \gamma(1) = \gamma(2) = \beta(0) = -1$. It follows that for the solution base $\{f_1, f_2, f_3\}$ equality holds in the first inequality in (1.4), and for the solution base $\{f_1, f_2, f_1 + f_3\}$ equality holds in the last inequality in (1.4). This shows the sharpness of Corollary 1.2 in the case $\varphi(r) = \frac{1}{1-r}$.

6. Acknowledgement

The author would like to thank the referees and editor for their thoughtful comments and helpful suggestions which greatly improved the final version of this paper.

References

[1] D. Benbourenane and L. Sons, On global solutions of complex differential equations in the unit disk, Complex Var. Theory Appl. 49 (2004), 913-925.

[2] M. Chuaqui, J. Gröhn, J. Heittokangas and J. Rättyä, Possible intervals for T- and M-orders of solutions of linear differential equations in the unit disc, Abstract and Applied Analysis, Volume 2011, Article ID 928194, 25 pages.

[3] I. Chyzhykov, J. Heittokangas and J. Rättyä, Finiteness of φ-order of solutions of linear differential equations in the unit disc, J. Anal. Math. 109 (2009), 163-198.

[4] G. G. Gundersen, Finite order solutions of second order linear differential equations, Trans. Amer. Math. Soc. 305 (1988), no. 1, 415-429.

[5] G. G. Gundersen, E. M. Steinbart and S. Wang, The possible orders of solutions of linear differential equations with polynomial coefficients, Trans. Amer. Math. Soc. 350 (1998), no. 3, 1225-1247.

[6] J. Heittokangas, On complex differential equations in the unit disc, Ann. Acad. Sci. Fenn. Math. Diss. 122 (2000), 1-54.

[7] J. Heittokangas, R. Korhonen and J. Rättyä, Linear differential equations with coefficients in weighted Bergman and Hardy spaces, Trans. Amer. Math. Soc. 360 (2008), 1035-1055.

[8] S. Hellerstein, J. Miles and J. Rossi, On the growth of solutions of $f'' + gf' + hf = 0$, Trans. Amer. Math. Soc. 324 (1991), no. 2, 693-706.

[9] B. Korenblum, An extension of the Nevanlinna theory, Acta. Math. 135 (1975), no. 3-4, 187-219.

[10] R. Korhonen and J. Rättyä, Finite order solutions of linear differential equations in the unit disc, J. Math. Anal. Appl. 349 (2009), 43-54.

[11] I. Laine, Nevanlinna Theory and Complex Differential Equations, Walter de Gruyter, Berlin, 1993.
[12] J. A. Peláez, J. Rättyä, Weighted Bergman spaces induced by rapidly increasing weights, to appear in Mem. Amer. Math. Soc.
[13] H. Wittich, Zur Theorie linear Differentialgleichungen im Komplexen, Ann. Acad. Sci. Fenn. Ser. A. I. 379 (1966).

(Received September 26, 2012)

Institute of Mathematics and Informations, Jiangxi Normal University, Nanchang, 330022, China
E-mail address: lipeng_xiao@yahoo.com.cn