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FIXED POINTS AND STABILITY IN NONLINEAR NEUTRAL VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS WITH VARIABLE DELAYS

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Abstract. In this paper we use the contraction mapping theorem to obtain asymptotic stability results of the zero solution of a nonlinear neutral Volterra integro-differential equation with variable delays. Some conditions which allow the coefficient functions to change sign and do not ask the boundedness of delays are given. An asymptotic stability theorem with a necessary and sufficient condition is proved, which improve and extend the results in the literature. Two examples are also given to illustrate this work.

1. INTRODUCTION

Without doubt, the Lyapunov's direct method has been, for more than 100 years, the main tool for investigating the stability properties of a wide variety of ordinary, functional, partial differential and Volterra integro-differential equations. Nevertheless, the application of this method to problems of stability in differential and Volterra integro-differential equations with delay has encountered serious obstacles if the delay is unbounded or if the equation has unbounded terms ([11]–[14]) and it does seem that other ways need to be investigated. In recent years, several investigators such as Burton, Furumochi, Zhang and others began a study in which they noticed that some of these difficulties vanish or might be overcome by means of fixed point theory (see [9]–[19], [21]–[23], [26] and [29]). The fixed point method does not only solve the problem on stability but has a significant advantage over Lyapunov's direct method. The conditions of the former are often averages but those of the latter are usually pointwise see ([12]).

Certain integro-differential equations with variable delays have been of great interest to mathematicians and theoreticians. In 1928 Volterra ([28]) noted that many physical problems were being modeled by integral and integro-differential equations. Today we see that such models have applications in biology, neural networks, viscoelasticity, nuclear reactors, and many other areas (see [1], [10], [11], [21], [26], [20], [24], [25] and the references therein). In this paper we focus on the following nonlinear neutral Volterra integro-differential equation with variable delays

$$x'(t) = -a(t)\, x(t-\tau_1(t)) + c(t)\, x'(t-\tau_2(t))\, Q'(x(t-\tau_2(t))) + \int_{t-\tau_2(t)}^{t} k(t,s)\, G(x(s))\, ds, \quad (1.1)$$

with the initial condition

$$x(t) = \psi(t) \ \text{ for } t \in [m(t_0), t_0],$$

where $\psi \in C([m(t_0), t_0], \mathbb{R})$ and for each $t_0 \geq 0$,

$$m_j(t_0) = \inf\{t - \tau_j(t),\ t \geq t_0\},\ m(t_0) = \min\{m_j(t_0),\ j = 1,2\}.$$

Here $C(S_1, S_2)$ denotes the set of all continuous functions $\varphi : S_1 \to S_2$ with the supremum norm $\|.\|$. Throughout this paper we assume that $a \in C(\mathbb{R}^+, \mathbb{R})$, $c \in C^1(\mathbb{R}^+, \mathbb{R})$, $k \in C([0,\infty) \times [m_2(t_0), \infty), \mathbb{R})$ and $\tau_1, \tau_2 \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $t - \tau_1(t) \to \infty$ and $t - \tau_2(t) \to \infty$ as $t \to \infty$. The functions Q and G are

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locally Lipschitz continuous. That is, there are positive constants L_1 and L_2 so that if $|x|, |y| \leq L$ for some positive constant L then

$$|Q(x) - Q(y)| \leq L_1 \|x - y\| \text{ and } Q(0) = 0, \tag{1.2}$$

and

$$|G(x) - G(y)| \leq L_2 \|x - y\| \text{ and } G(0) = 0. \tag{1.3}$$

Less general forms of equation (1.1) have been previously investigated by many authors. For example, Burton in [14], and Zhang in [29] have studied the equation

$$x'(t) = -a(t)\, x(t - \tau_1(t)), \tag{1.4}$$

and proved the following.

Theorem A (Burton [14]). *Suppose that $\tau_1(t) = \tau$ and there exists a constant $\alpha < 1$ such that*

$$\int_{t-\tau}^{t} |a(s+\tau)|\, ds + \int_0^t |a(s+\tau)|\, e^{-\int_s^t a(u+\tau)du} \left(\int_{s-\tau}^{s} |a(u+\tau)|\, du \right) ds \leq \alpha, \tag{1.5}$$

for all $t \geq 0$ and $\int_0^\infty a(s)\, ds = \infty$. Then, for every continuous initial function $\psi : [-\tau, 0] \to \mathbb{R}$, the solution $x(t) = x(t, 0, \psi)$ of (1.4) is bounded and tends to zero as $t \to \infty$.

Theorem B (Zhang [29]). *Suppose that τ_1 is differentiable, the inverse function g of $t - \tau_1(t)$ exists, and there exists a constant $\alpha \in (0,1)$ such that for $t \geq 0$, $\lim_{t\to\infty} \inf \int_0^t a(g(s))\, ds > -\infty$ and*

$$\int_{t-\tau_1(t)}^{t} |a(g(s))|\, ds + \int_0^t e^{-\int_s^t a(g(u))du} |a(s)|\, |\tau_1'(s)|\, ds$$
$$+ \int_0^t e^{-\int_s^t a(g(u))du} |a(g(s))| \left(\int_{s-\tau_1(s)}^{s} |a(g(u))|\, du \right) ds \leq \alpha. \tag{1.6}$$

Then the zero solution of (1.4) is asymptotically stable if and only if $\int_0^t a(g(s))\, ds \to \infty$, as $t \to \infty$.

Obviously, Theorem B improves Theorem A. On the other hand, Burton and Furumochi in [17] considered the following nonlinear delay Volterra integro-differential equation

$$x'(t) = -a(t)\, x(t) + \int_{t-\tau_2(t)}^{t} k(t,s)\, G(x(s))\, ds, \tag{1.7}$$

where $0 \leq \tau_2(t) \leq \tau_0$ for some constant τ_0, and obtained the following.

Theorem C (Burton and Furumochi [17]). *Suppose (1.3) holds with $L_2 = 1$, and there exists a constant $\alpha \in (0,1)$ such that for $t \geq 0$, $\int_0^t a(s)\, ds \to \infty$ as $t \to \infty$, and*

$$\int_0^t e^{-\int_s^t a(u)du} \left(\int_{s-\tau_2(s)}^{s} |k(s,u)|\, du \right) ds \leq \alpha. \tag{1.8}$$

Then the zero solution of (1.7) is asymptotically stable at $t_0 = 0$.

In [26], Raffoul studied the nonlinear neutral Volterra integro-differential equation

$$x'(t) = -a(t)\, x(t) + c(t)\, x'(t - \tau_2(t)) + \int_{t-\tau_2(t)}^{t} k(t,s)\, G(x(s))\, ds, \tag{1.9}$$

where $0 \leq \tau_2(t) \leq \tau_0$ for some constant τ_0, and obtained the following.

Theorem D (Raffoul [26]). *Let τ_2 be twice differentiable and $\tau_2'(t) \neq 1$ for all $t \in \mathbb{R}^+$. Suppose (1.3) holds with $L_2 = 1$, and there exists a constant $\alpha \in (0,1)$ such that for $t \geq 0$, $\int_0^t a(s)\, ds \to \infty$ as $t \to \infty$, and*

$$\left| \frac{c(t)}{1 - \tau_2'(t)} \right| + \int_0^t e^{-\int_s^t a(u)du} \left(|r_2(s)| + \int_{s-\tau_2(s)}^{s} |k(s,u)|\, du \right) ds \leq \alpha, \tag{1.10}$$

where $r_2(t) = \dfrac{[c(t)a(t) + c'(t)](1 - \tau_2'(t)) + c(t)\tau_2''(t)}{(1 - \tau_2'(t))^2}$. *Then the zero solution of* (1.9) *is asymptotically stable at* $t_0 = 0$.

In [21], the second author with Khemis studied the nonlinear neutral Volterra integro-differential equation

$$x'(t) = -a(t)x(t) + c(t)x'(t - \tau_2(t))x(t - \tau_2(t)) + \int_{t-\tau_2(t)}^{t} k(t,s)x^2(s)\,ds, \tag{1.11}$$

where $0 \le \tau_2(t) \le \tau_0$ for some constant τ_0, and obtained the following.

Theorem E (Djoudi and Khemis [21]). *Let* τ_2 *be twice differentiable and* $\tau_2'(t) \ne 1$ *for all* $t \in \mathbb{R}^+$. *Suppose that there exists a constant* $\alpha \in (0,1)$ *such that for* $t \ge 0$, $\int_0^t a(s)\,ds \to \infty$ *as* $t \to \infty$, *and*

$$L\left\{\left|\frac{c(t)}{1 - \tau_2'(t)}\right| + \int_0^t e^{-\int_s^t a(u)du}\left(|r_2(s)| + 2\int_{s-\tau_2(s)}^{s} |k(s,u)|\,du\right)ds\right\} \le \alpha, \tag{1.12}$$

where r_2 *is as in Theorem D. Then the zero solution of* (1.11) *is asymptotically stable at* $t_0 = 0$.

Remark 1. *The Theorems C, D and E are still true if the delay* τ_2 *is unbounded.*

Our purpose here is to give, by using a fixed point approach, asymptotic stability results of the zero solution of the nonlinear neutral Volterra integro-differential equation with variable delays (1.1). We provide, what we think, minimal conditions to reach these objectives for a such general equation. These conditions allow the coefficient functions to change sign and do not require the boundedness of delays. An asymptotic stability theorem with a necessary and sufficient condition is proved. Two examples are also given to illustrate our results. The results found in this paper contain the main results in [14], [17], [21], [26] and [29].

2. MAIN RESULTS

For each $(t_0, \psi) \in \mathbb{R}^+ \times C([m(t_0), t_0], \mathbb{R})$, a solution of (1.1) through (t_0, ψ) is a continuous function $x : [m(t_0), t_0 + \sigma) \to \mathbb{R}$ for some positive constant $\sigma > 0$ such that x satisfies (1.1) on $[t_0, t_0 + \sigma)$ and $x = \psi$ on $[m(t_0), t_0]$. We denote such a solution by $x(t) = x(t, t_0, \psi)$. For each $(t_0, \psi) \in \mathbb{R}^+ \times C([m(t_0), t_0], \mathbb{R})$, there exists a unique solution $x(t) = x(t, t_0, \psi)$ of (1.1) defined on $[t_0, \infty)$. For fixed t_0, we define $\|\psi\| = \max\{|\psi(t)| : m(t_0) \le t \le t_0\}$.

We need the following stability definitions taken from [12].

Definition 1. *The zero solution of* (1.1) *is sad to be stable at* $t = t_0$ *if, for each* $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that* $\psi : [m(t_0), t_0] \to (-\delta, \delta)$ *implies that* $|x(t)| < \varepsilon$ *for* $t \ge m(t_0)$.

Definition 2. *The zero solution of* (1.1) *is sad to be asymptotically stable if it is stable at* $t = t_0$ *and a* $\delta > 0$ *exists such that for any continuous function* $\psi : [m(t_0), t_0] \to (-\delta, \delta)$ *the solution* $x(t)$ *with* $x(t) = \psi(t)$ *on* $[m(t_0), t_0]$ *tends to zero as* $\to \infty$.

Our aim here is to improve and generalize Theorems $A-$ E to (1.1) by giving a necessary and sufficient condition for asymptotic stability of the zero solution of equation (1.1). One crucial step in the investigation of the stability of an equation using fixed point technic involves the construction of a suitable fixed point mapping. This can, in so many cases, be an arduous task. So, to construct our mapping, we begin by inverting (1.1) to have a more tractable equation, with the same structure and properties as the initial one, from which we derive a fixed point mapping P. After then, we, prudently, choose a suitable complete space depending on the initial condition ψ and with elements that tend to zero as $t \to \infty$ on which P is a contraction mapping. Using Banach's contraction mapping principle, we obtain a solution for P, and hence a solution for (1.1), which is asymptotically stable. This procedure has been been used by investigators to overcome the difficulties of stability in delay equations. This can be seen in the works of Azbelev et al. (see [1]–[8]).

Theorem 1. *Suppose* (1.2) *and* (1.3) *hold. Let* τ_1 *be differentiable and* τ_2 *be twice differentiable with* $\tau_2'(t) \neq 1$ *for all* $t \in \mathbb{R}^+$. *Suppose that there exist continuous functions* $h_j : [m_j(t_0), \infty) \to \mathbb{R}$ *for* $j = 1, 2$ *and a constant* $\alpha \in (0, 1)$ *such that for* $t \geq 0$

$$\liminf_{t \to \infty} \int_0^t H(s)\, ds > -\infty, \tag{2.1}$$

and

$$\begin{aligned} L_1 \left| \frac{c(t)}{1 - \tau_2'(t)} \right| + \sum_{j=1}^{2} \int_{t-\tau_j(t)}^{t} |h_j(s)|\, ds + \int_0^t e^{-\int_s^t H(u)du} \{ |-a(s) + h_1(s - \tau_1(s))(1 - \tau_1'(s))| \\ + |h_2(s - \tau_2(s))(1 - \tau_2'(s))| + L_1 |r(s)| + L_2 \int_{s-\tau_2(s)}^{s} |k(s, u)|\, du \Big\}\, ds \\ + \sum_{j=1}^{2} \int_0^t e^{-\int_s^t H(u)du} |H(s)| \left(\int_{s-\tau_j(s)}^{s} |h_j(u)|\, du \right) ds \leq \alpha, \end{aligned} \tag{2.2}$$

where $H(t) = \sum_{j=1}^{2} h_j(t)$ *and* $r(t) = \dfrac{[c(t) H(t) + c'(t)](1 - \tau_2'(t)) + c(t) \tau_2''(t)}{(1 - \tau_2'(t))^2}$. *Then the zero solution of* (1.1) *is asymptotically stable if and only if*

$$\int_0^t H(s)\, ds \to \infty \text{ as } t \to \infty. \tag{2.3}$$

Proof. First, suppose that (2.3) holds. For each $t_0 \geq 0$, we set

$$K = \sup_{t \geq 0} \left\{ e^{-\int_0^t H(s)ds} \right\}. \tag{2.4}$$

Let $\psi \in C([m(t_0), t_0], \mathbb{R})$ be fixed and define

$$S_\psi = \{ \varphi \in C([m(t_0), \infty), \mathbb{R}) : \varphi(t) \to 0 \text{ as } t \to \infty,\ \varphi(t) = \psi(t) \text{ for } t \in [m(t_0), t_0] \}.$$

Then S_ψ is a complete metric space with metric $\rho(x, y) = \sup_{t \geq t_0} \{ |x(t) - y(t)| \}$.

Multiply both sides of (1.1) by $e^{\int_{t_0}^t H(u)du}$ and then integrate from t_0 to t to obtain

$$\begin{aligned} x(t) = \psi(t_0) e^{-\int_{t_0}^t H(u)du} + \sum_{j=1}^{2} \int_{t_0}^{t} e^{-\int_s^t H(u)du} h_j(s) x(s)\, ds \\ + \int_{t_0}^{t} e^{-\int_s^t H(u)du} \{ -a(s) x(s - \tau_1(s)) + c(s) x'(s - \tau_2(s)) Q'(x(s - \tau_2(s))) \\ + \int_{s-\tau_2(s)}^{s} k(s, u) G(x(u))\, du \Big\}\, ds. \end{aligned}$$

Performing an integration by parts, we have

$$
\begin{aligned}
x(t) =& \left(\psi(t_0) - \frac{c(t_0)}{1-\tau_2'(t_0)} Q(\psi(t_0 - \tau_2(t_0)))\right) e^{-\int_{t_0}^t H(u)du} \\
&+ \frac{c(t)}{1-\tau_2'(t)} Q(x(t-\tau_2(t))) + \sum_{j=1}^{2} \int_{t_0}^{t} e^{-\int_s^t H(u)du} d\left(\int_{s-\tau_j(s)}^{s} h_j(u)x(u)\,du\right) \\
&+ \sum_{j=1}^{2} \int_{t_0}^{t} e^{-\int_s^t H(u)du} h_j(s-\tau_j(s))\left(1-\tau_j'(s)\right) x(s-\tau_j(s))\,ds \\
&+ \int_{t_0}^{t} e^{-\int_s^t H(u)du} \left\{-a(s)x(s-\tau_1(s)) - r(s)Q(x(s-\tau_2(s))) + \int_{s-\tau_2(s)}^{s} k(s,u)G(x(u))\,du\right\} ds \\
=& \left(\psi(t_0) - \frac{c(t_0)}{1-\tau_2'(t_0)} Q(\psi(t_0-\tau_2(t_0))) - \sum_{j=1}^{2}\int_{t_0-\tau_j(t_0)}^{t_0} h_j(s)\psi(s)\,ds\right) e^{-\int_{t_0}^t H(u)du} \\
&+ \frac{c(t)}{1-\tau_2'(t)} Q(x(t-\tau_2(t))) + \sum_{j=1}^{2}\int_{t-\tau_j(t)}^{t} h_j(s)x(s)\,ds \\
&+ \int_{t_0}^{t} e^{-\int_s^t H(u)du} \{(-a(s) + h_1(s-\tau_1(s))(1-\tau_1'(s)))x(s-\tau_1(s)) \\
&+ h_2(s-\tau_2(s))(1-\tau_2'(s))x(s-\tau_2(s))\}\,ds \\
&+ \int_{t_0}^{t} e^{-\int_s^t H(u)du} \left\{-r(s)Q(x(s-\tau_2(s))) + \int_{s-\tau_2(s)}^{s} k(s,u)G(x(u))\,du\right\} ds \\
&- \sum_{j=1}^{2}\int_{t_0}^{t} e^{-\int_s^t H(u)du} H(s)\left(\int_{s-\tau_j(s)}^{s} h_j(u)x(u)\,du\right) ds. \qquad (2.5)
\end{aligned}
$$

Use (2.5) to define the operator $P : S_\psi \to S_\psi$ by $(P\varphi)(t) = \psi(t)$ for $t \in [m(t_0), t_0]$ and

$$
\begin{aligned}
(P\varphi)(t) =& \left(\psi(t_0) - \frac{c(t_0)}{1-\tau_2'(t_0)} Q(\psi(t_0-\tau_2(t_0))) - \sum_{j=1}^{2}\int_{t_0-\tau_j(t_0)}^{t_0} h_j(s)\psi(s)\,ds\right) e^{-\int_{t_0}^t H(u)du} \\
&+ \frac{c(t)}{1-\tau_2'(t)} Q(\varphi(t-\tau_2(t))) + \sum_{j=1}^{2}\int_{t-\tau_j(t)}^{t} h_j(s)\varphi(s)\,ds \\
&+ \int_{t_0}^{t} e^{-\int_s^t H(u)du} \{(-a(s) + h_1(s-\tau_1(s))(1-\tau_1'(s)))\varphi(s-\tau_1(s)) \\
&+ h_2(s-\tau_2(s))(1-\tau_2'(s))\varphi(s-\tau_2(s))\}\,ds \\
&+ \int_{t_0}^{t} e^{-\int_s^t H(u)du} \left\{-r(s)Q(\varphi(s-\tau_2(s))) + \int_{s-\tau_2(s)}^{s} k(s,u)G(\varphi(u))\,du\right\} ds \\
&- \sum_{j=1}^{2}\int_{t_0}^{t} e^{-\int_s^t H(u)du} H(s)\left(\int_{s-\tau_j(s)}^{s} h_j(u)\varphi(u)\,du\right) ds. \qquad (2.6)
\end{aligned}
$$

for $t \geq t_0$. It is clear that $(P\varphi) \in C([m(t_0), \infty), \mathbb{R})$. We now show that $(P\varphi)(t) \to 0$ as $t \to \infty$. Since $\varphi(t) \to 0$ and $t - \tau_j(t) \to \infty$ as $t \to \infty$, for each $\varepsilon > 0$, there exists a $T_1 > t_0$ such that $s \geq T_1$

implies that $|\varphi(s-\tau_j(s))| < \varepsilon$ for $j = 1, 2$. Thus, for $t \geq T_1$, the last term I_6 in (2.6) satisfies

$$
\begin{aligned}
|I_6| &= \left| \sum_{j=1}^{2} \int_{t_0}^{t} e^{-\int_s^t H(u)du} H(s) \left(\int_{s-\tau_j(s)}^{s} h_j(u)\,\varphi(u)\,du \right) ds \right| \\
&\leq \sum_{j=1}^{2} \int_{t_0}^{T_1} e^{-\int_s^t H(u)du} |H(s)| \left(\int_{s-\tau_j(s)}^{s} |h_j(u)|\,|\varphi(u)|\,du \right) ds \\
&+ \sum_{j=1}^{2} \int_{T_1}^{t} e^{-\int_s^t H(u)du} |H(s)| \left(\int_{s-\tau_j(s)}^{s} |h_j(u)|\,|\varphi(u)|\,du \right) ds \\
&\leq \sup_{\sigma \geq m(t_0)} |\varphi(\sigma)| \sum_{j=1}^{2} \int_{t_0}^{T_1} e^{-\int_s^t H(u)du} |H(s)| \left(\int_{s-\tau_j(s)}^{s} |h_j(u)|\,du \right) ds \\
&+ \varepsilon \sum_{j=1}^{2} \int_{T_1}^{t} e^{-\int_s^t H(u)du} |H(s)| \left(\int_{s-\tau_j(s)}^{s} |h_j(u)|\,du \right) ds.
\end{aligned}
$$

By (2.3), there exists $T_2 > T_1$ such that $t \geq T_2$ implies

$$
\begin{aligned}
&\sup_{\sigma \geq m(t_0)} |\varphi(\sigma)| \sum_{j=1}^{2} \int_{t_0}^{T_1} e^{-\int_s^t H(u)du} |H(s)| \left(\int_{s-\tau_j(s)}^{s} |h_j(u)|\,du \right) ds \\
&= \sup_{\sigma \geq m(t_0)} |\varphi(\sigma)|\, e^{-\int_{T_1}^{t} H(u)du} \sum_{j=1}^{2} \int_{t_0}^{T_1} e^{-\int_s^{T_1} H(u)du} |H(s)| \left(\int_{s-\tau_j(s)}^{s} |h_j(u)|\,du \right) ds < \varepsilon.
\end{aligned}
$$

Apply (2.2) to obtain $|I_6| < \varepsilon + \alpha\epsilon < 2\varepsilon$. Thus, $I_6 \to 0$ as $t \to \infty$. Similarly, we can show that the rest of the terms in (2.6) approach zero as $t \to \infty$. This yields $(P\varphi)(t) \to 0$ as $t \to \infty$, and hence $P\varphi \in S_\psi$. Also, by (2.2), P is a contraction mapping with contraction constant α. By the contraction mapping principle ([27], p. 2), P has a unique fixed point x in S_ψ which is a solution of (1.1) with $x(t) = \psi(t)$ on $[m(t_0), t_0]$ and $x(t) = x(t, t_0, \psi) \to 0$ as $t \to \infty$.

To obtain the asymptotic stability, we need to show that the zero solution of (1.1) is stable. Let $\varepsilon > 0$ be given and choose $\delta > 0$ $(\delta < \varepsilon)$ satisfying $2\delta K e^{\int_0^{t_0} H(u)du} + \alpha\varepsilon < \varepsilon$. If $x(t) = x(t, t_0, \psi)$ is a solution of (1.1) with $\|\psi\| < \delta$, then $x(t) = (Px)(t)$ defined in (2.6). We claim that $|x(t)| < \varepsilon$ for all $t \geq t_0$. Notice that $|x(s)| < \varepsilon$ on $[m(t_0), t_0]$. If there exists $t^* > t_0$ such that $|x(t^*)| = \varepsilon$ and

$|x(s)| < \varepsilon$ for $m(t_0) \le s < t^*$, then it follows from (2.6) that

$$\begin{aligned}
|x(t^*)| &\le \|\psi\| \left(1 + L_1 \left|\frac{c(t_0)}{1-\tau_2'(t_0)}\right| + \sum_{j=1}^{2} \int_{t_0-\tau_j(t_0)}^{t_0} |h_j(s)|\, ds\right) e^{-\int_{t_0}^{t^*} H(u)du} \\
&+ \epsilon L_1 \left|\frac{c(t^*)}{1-\tau_2'(t^*)}\right| + \epsilon \sum_{j=1}^{2} \int_{t^*-\tau_j(t^*)}^{t^*} |h_j(s)|\, ds \\
&+ \epsilon \int_{t_0}^{t^*} e^{-\int_s^{t^*} H(u)du} \left\{ |-a(s) + h_1(s-\tau_1(s))(1-\tau_1'(s))| \right. \\
&\left. + |h_2(s-\tau_2(s))(1-\tau_2'(s))| + L_1|r(s)| + L_2 \int_{s-\tau_j(s)}^{s} |k(s,u)|\, du \right\} ds \\
&+ \epsilon \sum_{j=1}^{2} \int_{t_0}^{t^*} e^{-\int_s^{t^*} H(u)du} |H(s)| \left(\int_{s-\tau_j(s)}^{s} |h_j(u)|\, du\right) ds \\
&\le 2\delta K e^{\int_0^{t_0} H(u)du} + \alpha\varepsilon < \epsilon,
\end{aligned}$$

which contradicts the definition of t^*. Thus, $|x(t)| < \varepsilon$ for all $t \ge t_0$, and the zero solution of (1.1) is stable. This shows that the zero solution of (1.1) is asymptotically stable if (2.3) holds.

Conversely, suppose (2.3) fails. Then by (2.1) there exists a sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$ such that $\lim_{n\to\infty} \int_0^{t_n} H(u)\, du = l$ for some $l \in \mathbb{R}$. We may also choose a positive constant J satisfying

$$-J \le \int_0^{t_n} H(u)\, du \le J,$$

for all $n \ge 1$. To simplify our expressions, we define

$$\begin{aligned}
\omega(s) &= |-a(s) + h_1(s-\tau_1(s))(1-\tau_1'(s))| + |h_2(s-\tau_2(s))(1-\tau_2'(s))| \\
&+ L_1|r(s)| + L_2 \int_{s-\tau_2(s)}^{s} |k(s,u)|\, du + |H(s)| \sum_{j=1}^{2} \int_{s-\tau_j(s)}^{s} |h_j(u)|\, du,
\end{aligned}$$

for all $s \ge 0$. By (2.2), we have

$$\int_0^{t_n} e^{-\int_s^{t_n} H(u)du} \omega(s)\, ds \le \alpha.$$

This yields

$$\int_0^{t_n} e^{\int_0^s H(u)du} \omega(s)\, ds \le \alpha e^{\int_0^{t_n} H(u)du} \le e^J.$$

The sequence $\left\{\int_0^{t_n} e^{\int_0^s H(u)du} \omega(s)\, ds\right\}$ is bounded, so there exists a convergent subsequence. For brevity of notation, we may assume that

$$\lim_{n\to\infty} \int_0^{t_n} e^{\int_0^s H(u)du} \omega(s)\, ds = \gamma,$$

for some $\gamma \in \mathbb{R}^+$ and choose a positive integer m so large that

$$\int_{t_m}^{t_n} e^{\int_0^s H(u)du} \omega(s)\, ds < \delta_0/4K,$$

for all $n \ge m$, where $\delta_0 > 0$ satisfies $2\delta_0 K e^J + \alpha \le 1$.

By (2.1), K in (2.4) is well defined. We now consider the solution $x(t) = x(t, t_m, \psi)$ of (1.1) with $\psi(t_m) = \delta_0$ and $|\psi(s)| \le \delta_0$ for $s \le t_m$. We may choose ψ so that $|x(t)| \le 1$ for $t \ge t_m$ and

$$\psi(t_m) - \frac{c(t_m)}{1 - \tau_2'(t_m)} Q(\psi(t_m - \tau_2(t_m))) - \sum_{j=1}^{2} \int_{t_m - \tau_j(t_m)}^{t_m} h_j(s)\psi(s)\,ds \ge \frac{1}{2}\delta_0.$$

It follows from (2.6) with $x(t) = (Px)(t)$ that for $n \ge m$

$$\begin{aligned}
&\left| x(t_n) - \frac{c(t_n)}{1 - \tau_2'(t_n)} Q(x(t_n - \tau_2(t_n))) - \sum_{j=1}^{2} \int_{t_n - \tau_j(t_n)}^{t_n} h_j(s)x(s)\,ds \right| \\
&\ge \frac{1}{2}\delta_0 e^{-\int_{t_m}^{t_n} H(u)du} - \int_{t_m}^{t_n} e^{-\int_s^{t_n} H(u)du}\omega(s)\,ds \\
&= \frac{1}{2}\delta_0 e^{-\int_{t_m}^{t_n} H(u)du} - e^{-\int_0^{t_n} H(u)du} \int_{t_m}^{t_n} e^{\int_0^s H(u)du}\omega(s)\,ds \\
&= e^{-\int_{t_m}^{t_n} H(u)du} \left(\frac{1}{2}\delta_0 - e^{-\int_0^{t_m} H(u)du} \int_{t_m}^{t_n} e^{\int_0^s H(u)du}\omega(s)\,ds \right) \\
&\ge e^{-\int_{t_m}^{t_n} H(u)du} \left(\frac{1}{2}\delta_0 - K \int_{t_m}^{t_n} e^{\int_0^s H(u)du}\omega(s)\,ds \right) \\
&\ge \frac{1}{4}\delta_0 e^{-\int_{t_m}^{t_n} H(u)du} \ge \frac{1}{4}\delta_0 e^{-2J} > 0. \qquad (2.7)
\end{aligned}$$

On the other hand, if the zero solution of (1.1) is asymptotically stable, then $x(t) = x(t, t_m, \psi) \to 0$ as $t \to \infty$. Since $t_n - \tau_j(t_n) \to \infty$ as $n \to \infty$ and (2.2) holds, we have

$$x(t_n) - \frac{c(t_n)}{1 - \tau_2'(t_n)} Q(x(t_n - \tau_2(t_n))) - \sum_{j=1}^{2} \int_{t_n - \tau_j(t_n)}^{t_n} h_j(s)x(s)\,ds \to 0 \text{ as } n \to \infty,$$

which contradicts (2.7). Hence condition (2.3) is necessary for the asymptotic stability of the zero solution of (1.1). The proof is complete. □

Remark 2. *It follows from the first part of the proof of Theorem 1 that the zero solution of* (1.1) *is stable under* (2.1) *and* (2.2). *Moreover, Theorem 1 still holds if* (2.2) *is satisfied for* $t \ge t_\sigma$ *for some* $t_\sigma \in \mathbb{R}^+$.

For the special case $c = 0$ and $k = 0$, we can get

Corollary 1. *Let* τ_1 *be differentiable, and suppose that there exist continuous function* $h_1 : [m_1(t_0), \infty) \to \mathbb{R}$ *for and a constant* $\alpha \in (0,1)$ *such that for* $t \ge 0$

$$\liminf_{t \to \infty} \int_0^t h_1(s)\,ds > -\infty,$$

and

$$\begin{aligned}
\int_{t-\tau_1(t)}^{t} |h_1(s)|\,ds &+ \int_0^t e^{-\int_s^t h_1(u)du} |-a(s) + h_1(s - \tau_1(s))(1 - \tau_1'(s))|\,ds \\
&+ \int_0^t e^{-\int_s^t h_1(u)du} |h_1(s)| \left(\int_{s-\tau_1(s)}^{s} |h_1(u)|\,du \right) ds \le \alpha. \qquad (2.8)
\end{aligned}$$

Then the zero solution of (1.4) *is asymptotically stable if and only if*

$$\int_0^t h_1(s)\,ds \to \infty \text{ as } t \to \infty.$$

Remark 3. *When* $\tau_1(s) = \tau$, *a constant,* $h_1(s) = a(s+\tau)$, *Corollary* 1 *contains Theorem A. When* $h_1(s) = a(g(s))$, *where* $g(s)$ *is the inverse function of* $s - \tau_1(s)$, *Corollary* 1 *reduces to Theorem B.*

For the special case $\tau_1 = 0$ and $c = 0$, we can get the following corollary.

Corollary 2. *Suppose* (1.3) *hold with* $L_2 = 1$. *Let* τ_2 *be differentiable, and suppose that there exist continuous functions* $h_j : [m_j(t_0), \infty) \to \mathbb{R}$ *for* $j = 1, 2$ *and a constant* $\alpha \in (0, 1)$ *such that for* $t \geq 0$

$$\liminf_{t\to\infty} \int_0^t H(s)\,ds > -\infty,$$

and

$$\begin{aligned}\int_{t-\tau_2(t)}^t |h_2(s)|\,ds + \int_0^t e^{-\int_s^t H(u)du} \Big\{ |-a(s) + h_1(s)| \\ + |h_2(s-\tau_2(s))(1-\tau_2'(s))| + \int_{s-\tau_2(s)}^s |k(s,u)|\,du \Big\} ds \\ + \int_0^t e^{-\int_s^t H(u)du} |H(s)| \left(\int_{s-\tau_2(s)}^s |h_2(u)|\,du \right) ds \leq \alpha, \quad (2.9)\end{aligned}$$

where $H(t) = \sum_{j=1}^{2} h_j(t)$. *Then the zero solution of* (1.7) *is asymptotically stable if and only if*

$$\int_0^t H(s)\,ds \to \infty \text{ as } t \to \infty.$$

For the special case $\tau_1 = 0$ and $Q(x) = x$, we can get

Corollary 3. *Suppose* (1.3) *hold with* $L_2 = 1$. *Let* τ_2 *be twice differentiable with* $\tau_2'(t) \neq 1$ *for all* $t \in \mathbb{R}^+$. *Suppose that there exist continuous functions* $h_j : [m_j(t_0), \infty) \to \mathbb{R}$ *for* $j = 1, 2$ *and a constant* $\alpha \in (0, 1)$ *such that for* $t \geq 0$

$$\liminf_{t\to\infty} \int_0^t H(s)\,ds > -\infty,$$

and

$$\begin{aligned}\left| \frac{c(t)}{1-\tau_2'(t)} \right| + \int_{t-\tau_2(t)}^t |h_2(s)|\,ds + \int_0^t e^{-\int_s^t H(u)du} \Big\{ |-a(s) + h_1(s)| \\ + |h_2(s-\tau_2(s))(1-\tau_2'(s)) - r(s)| + \int_{s-\tau_2(s)}^s |k(s,u)|\,du \Big\} ds \\ \int_0^t e^{-\int_s^t H(u)du} |H(s)| \left(\int_{s-\tau_2(s)}^s |h_2(u)|\,du \right) ds \leq \alpha, \quad (2.10)\end{aligned}$$

where $H(t) = \sum_{j=1}^{2} h_j(t)$ *and* $r(t) = \dfrac{[c(t)H(t) + c'(t)](1-\tau_2'(t)) + c(t)\tau_2''(t)}{(1-\tau_2'(t))^2}$. *Then the zero solution of* (1.9) *is asymptotically stable if and only if*

$$\int_0^t H(s)\,ds \to \infty \text{ as } t \to \infty.$$

For the special case $\tau_1 = 0$, $Q(x) = \frac{1}{2}x^2$ and $G(x) = x^2$, we can get

Corollary 4. *Let τ_2 be twice differentiable with $\tau_2'(t) \neq 1$ for all $t \in \mathbb{R}^+$. Suppose that there exist continuous functions $h_j : [m_j(t_0), \infty) \to \mathbb{R}$ for $j = 1, 2$ and a constant $\alpha \in (0,1)$ such that for $t \geq 0$*

$$\liminf_{t\to\infty} \int_0^t H(s)\, ds > -\infty,$$

and

$$\begin{aligned} L\left\{\left|\frac{c(t)}{1-\tau_2'(t)}\right| + \int_0^t e^{-\int_s^t H(u)du}\left(|r(s)| + 2\int_{s-\tau_2(s)}^s |k(s,u)|\, du\right) ds\right\} \\ + \int_0^t e^{-\int_s^t H(u)du}\left(|-a(s) + h_1(s)| + |h_2(s-\tau_2(s))(1-\tau_2'(s))|\right) ds \\ + \int_0^t e^{-\int_s^t H(u)du}|H(s)|\left(\int_{s-\tau_2(s)}^s |h_2(u)|\, du\right) ds \leq \alpha, \quad (2.11) \end{aligned}$$

where $H(t) = \sum_{j=1}^{2} h_j(t)$ and $r(t) = \dfrac{[c(t)H(t) + c'(t)](1-\tau_2'(t)) + c(t)\tau_2''(t)}{(1-\tau_2'(t))^2}$. Then the zero solution of (1.11) is asymptotically stable if and only if

$$\int_0^t H(s)\, ds \to \infty \text{ as } t \to \infty.$$

Remark 4. *When $h_1(s) = a(s)$ and $h_2(s) = 0$, then Corollaries 2, 3 and 4 contain Theorems C, D and E, respectively.*

3. TWO EXAMPLES

In this section, we give two examples to illustrate the applications of Corollary 3 and Theorem 1.

Example 1. *Consider the following nonlinear neutral Volterra integro-differential equation*

$$x'(t) = -a(t)x(t) + c(t)x'(t-\tau_2(t)) + \int_{t-\tau_2(t)}^t k(t,s)G(x(s))\, ds, \quad (3.1)$$

where $\tau_2(t) = 0.063t$, $a(t) = 1/(t+1)$, $c(t) = 0.33$, $k(t,s) = 8.2/[(t+1)(s+1)]$ and $G(x) = \sin x$. Then the zero solution of (3.1) is asymptotically stable.

Proof. Choosing $h_1(t) = 1/(t+1)$ and $h_2(t) = 0.22/(t+1)$ in Corollary 3, we have $H(t) = 1.22/(t+1)$,

$$\left|\frac{c(t)}{1-\tau_2'(t)}\right| = \frac{0.33}{0.937} < 0.3522,$$

$$\int_{t-\tau_2(t)}^t |h_2(s)|\, ds = \int_{0.937t}^t \frac{0.22}{s+1} ds = 0.22\ln\left(\frac{t+1}{0.937t+1}\right) < 0.0144,$$

$$\int_0^t e^{-\int_s^t H(u)du}|H(s)|\left(\int_{s-\tau_2(s)}^s |h_2(u)|\, du\right) ds < \int_0^t e^{-\int_s^t (1.22/(u+1))du}\frac{1.22}{s+1} \times 0.0144 < 0.0144,$$

$$\begin{aligned} &\int_0^t e^{-\int_s^t H(u)du}\{|-a(s) + h_1(s)| + |h_2(s-\tau_2(s))(1-\tau_2'(s)) - r(s)|\} ds \\ &= \int_0^t e^{-\int_s^t (1.22/(u+1))du}\left|\frac{0.22 \times 0.937}{0.937s+1} - \frac{1.22 \times 0.33}{0.937(s+1)}\right| ds < \frac{0.33}{0.937} - \frac{0.22}{1.22} < 0.1719, \end{aligned}$$

and

$$\int_0^t e^{-\int_s^t H(u)du} \left(\int_{s-\tau_2(s)}^s |k(s,u)|\, du \right) ds = \int_0^t e^{-\int_s^t (1.22/(u+1))du} \left(\frac{8.2}{s+1} \int_{0.937s}^s \frac{1}{u+1} du \right) ds$$
$$< \frac{0.0651 \times 8.2}{1.22} \int_0^t e^{-\int_s^t (1.22/(u+1))du} \frac{1.22}{s+1} ds < 0.4376.$$

It is easy to see that all the conditions of Corollary 3 hold for $\alpha = 0.3522 + 0.0144 + 0.1719 + 0.4376 + 0.0144 = 0.9905 < 1$. Thus, Corollary 3 implies that the zero solution of (3.1) is asymptotically stable.

However, Theorem D cannot be used to verify that the zero solution of (3.1) is asymptotically stable. Obviously,

$$\left| \frac{c(t)}{1-\tau_2'(t)} \right| + \int_0^t e^{-\int_s^t a(u)du} |r_2(s)|\, ds = \frac{0.33\,(2t+1)}{0.937\,(t+1)} \to \frac{0.66}{0.937},$$

and

$$\int_0^t e^{-\int_s^t a(u)du} \left(\int_{s-\tau_2(s)}^s |k(s,u)|\, du \right) ds = \frac{8.2}{t+1} \int_0^t \left[\ln(s+1) - \ln(0.937s+1) \right] ds$$
$$= 8.2 \left[\ln(t+1) - \frac{t+1/0.937}{t+1} \ln(0.937t+1) \right] \to -8.2 \ln(0.937).$$

Thus, we have

$$\limsup_{t\geq 0} \left\{ \left| \frac{c(t)}{1-\tau_2'(t)} \right| + \int_0^t e^{-\int_s^t a(u)du} \left(|r_2(s)| + \int_{s-\tau_2(s)}^s |k(s,u)|\, du \right) ds \right\}$$
$$= \frac{0.66}{0.937} - 8.2 \ln(0.937) \simeq 1.238.$$

In addition, the left-hand side of the following inequality is increasing in $t > 0$, then there exists some $t_0 > 0$ such that for $t \geq t_0$,

$$\left| \frac{c(t)}{1-\tau_2'(t)} \right| + \int_0^t e^{-\int_s^t a(u)du} \left(|r_2(s)| + \int_{s-\tau_2(s)}^s |k(s,u)|\, du \right) ds > 1.23.$$

This implies that condition (1.10) does not hold. Thus, Theorem D cannot be applied to equation (3.1). □

Example 2. *Consider the following nonlinear neutral Volterra integro-differential equation*

$$x'(t) = -a(t)\, x(t-\tau_1(t)) + c(t)\, x'(t-\tau_2(t))\, Q'(x(t-\tau_2(t))) + \int_{t-\tau_2(t)}^t k(t,s)\, G(x(s))\, ds, \quad (3.2)$$

where $\tau_1(t) = 0.068t$, $\tau_2(t) = 0.074t$, $a(t) = 0.932/(0.932t+1)$, $c(t) = 0.44$, $Q(x) = 0.52(1-\cos(x))$, $G(x) = 1.22\sin(x)$, *and* $k(t,s) = 1/[(t+1)(s+1)]$. *Then the zero solution of* (3.2) *is asymptotically stable.*

Proof. Choosing $h_1(t) = 1/(t+1)$ and $h_2(t) = 0.31/(t+1)$ in Theorem 1, we have $H(t) = 1.31/(t+1)$,

$$L_1 = 0.52,\ L_2 = 1.22,$$

$$L_1 \left| \frac{c(t)}{1-\tau_2'(t)} \right| = 0.52 \times \frac{0.44}{0.926} < 0.2471,$$

$$\sum_{j=1}^{2}\int_{t-\tau_j(t)}^{t}|h_j(s)|\,ds=\int_{0.932t}^{t}\frac{1}{s+1}ds+\int_{0.926t}^{t}\frac{0.31}{s+1}ds$$
$$=\ln\left(\frac{t+1}{0.932t+1}\right)+0.31\ln\left(\frac{t+1}{0.926t+1}\right)<0.0943,$$

$$\sum_{j=1}^{2}\int_{0}^{t}e^{-\int_s^t H(u)du}|H(s)|\left(\int_{s-\tau_j(s)}^{s}|h_j(u)|\,du\right)ds<\int_0^t e^{-\int_s^t(1.31/(u+1))du}\frac{1.31}{s+1}\times 0.0943<0.0943,$$

$$\int_0^t e^{-\int_s^t H(u)du}\left\{|-a(s)+h_1(s-\tau_1(s))(1-\tau_1'(s))|+|h_2(s-\tau_2(s))(1-\tau_2'(s))|+L_1|r(s)|\right\}ds$$
$$=\int_0^t e^{-\int_s^t(1.31/(u+1))du}\left(\frac{0.31\times 0.926}{0.926s+1}+\frac{0.52\times 1.31\times 0.44}{0.926(s+1)}\right)ds<\frac{0.31}{1.31}+\frac{0.52\times 0.44}{0.926}<0.4838,$$

and

$$\int_0^t e^{-\int_s^t H(u)du}\left(L_2\int_{s-\tau_2(s)}^{s}|k(s,u)|\,du\right)ds=\int_0^t e^{-\int_s^t(1.31/(u+1))du}\left(\frac{1.22}{s+1}\int_{0.926s}^{s}\frac{1}{u+1}du\right)ds$$
$$<\frac{0.0769\times 1.22}{1.31}\int_0^t e^{-\int_s^t(1.31/(u+1))du}\frac{1.31}{s+1}ds<0.0717.$$

It is easy to see that all the conditions of Theorem 1 hold for $\alpha=0.2471+0.0943+0.4838+0.0717+0.0943=0.9912<1$. Thus, Theorem 1 implies that the zero solution of (3.2) is asymptotically stable. □

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