

Existence of positive solution for a third-order three-point BVP with sign-changing Green's function*

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Abstract

By using the Guo-Krasnoselskii fixed point theorem, we investigate the following third-order three-point boundary value problem

$$\begin{cases} u'''(t) = f(t, u(t)), & t \in [0, 1], \\ u'(0) = u(1) = 0, & u''(\eta) + \alpha u(0) = 0, \end{cases}$$

where $\alpha \in [0, 2)$ and $\eta \in [\frac{\sqrt{121+24\alpha}-5}{3(4+\alpha)}, 1)$. The emphasis is mainly that although the corresponding Green's function is sign-changing, the solution obtained is still positive.

Keywords: Third-order three-point boundary value problem; Sign-changing Green's function; Positive solution; Existence; Fixed point

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1 Introduction

Third-order differential equations arise from a variety of different areas of applied mathematics and physics, e.g., in the deflection of a curved beam having a constant or varying cross section, a three-layer beam, electromagnetic waves or gravity driven flows and so on [3].

Recently, the existence of single or multiple positive solutions to some third-order three-point boundary value problems (BVPs for short) has received much attention from many authors, see [1, 2, 5, 12, 15, 16] and the references therein.

However, all the above-mentioned papers are achieved when the corresponding Green's functions are positive, which is a very important condition. A natural question is that whether we can obtain the existence of positive solutions to some third-order three-point BVPs when the corresponding Green's functions are sign-changing.

In 2008, Palamides and Smyrlis [11] studied the existence of at least one positive solution to the singular third-order three-point BVP with an indefinitely signed Green's function

$$\begin{cases} u'''(t) = a(t)f(t, u(t)), & t \in (0, 1), \\ u(0) = u(1) = u''(\eta) = 0, \end{cases}$$

where $\eta \in (\frac{17}{24}, 1)$. Their technique was a combination of the Guo-Krasnoselskii fixed point theorem and properties of the corresponding vector field.

In 2012, by using the Guo-Krasnoselskii and Leggett-Williams fixed point theorems, Sun and Zhao [13, 14] discussed the third-order three-point BVP with sign-changing Green's function

$$\begin{cases} u'''(t) = f(t, u(t)), & t \in [0, 1], \\ u'(0) = u(1) = u''(\eta) = 0, \end{cases} \quad (1.1)$$

where $\eta \in (\frac{1}{2}, 1)$. They obtained the existence of single or multiple positive solutions to the BVP (1.1) and proved that the obtained solutions were concave on $[0, \eta]$ and convex on $[\eta, 1]$.

It is worth mentioning that there are other type of works on sign-changing Green's functions which prove the existence of sign-changing solutions, positive in some cases, see Infante and Webb's papers [6–8].

In this paper we study the following third-order three-point BVP

$$\begin{cases} u'''(t) = f(t, u(t)), & t \in [0, 1], \\ u'(0) = u(1) = 0, & u''(\eta) + \alpha u(0) = 0. \end{cases} \quad (1.2)$$

Throughout this paper, we always assume that $\alpha \in [0, 2)$ and $\eta \in [\frac{\sqrt{121+24\alpha}-5}{3(4+\alpha)}, 1)$. Obviously, the BVP (1.1) is a special case of the BVP (1.2). However, it is necessary to point out that this paper is not a simple extension of [13]. In fact, if we let $\alpha = 0$, then $\eta \in [\frac{1}{2}, 1)$, which is different from the restriction in [13]. On the other hand, compared with [13], we can only prove that the obtained solution is concave on $[0, \eta]$.

Our main tool is the following well-known Guo-Krasnoselskii fixed point theorem [4, 9]:

Theorem 1.1 *Let E be a Banach space and K be a cone in E . Assume that Ω_1 and Ω_2 are bounded open subsets of E such that $0 \in \Omega_1$, $\bar{\Omega}_1 \subset \Omega_2$, and let $T : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \rightarrow K$ be a completely continuous operator such that either*

- (1) $\|Tu\| \leq \|u\|$ for $u \in K \cap \partial\Omega_1$ and $\|Tu\| \geq \|u\|$ for $u \in K \cap \partial\Omega_2$, or
 (2) $\|Tu\| \geq \|u\|$ for $u \in K \cap \partial\Omega_1$ and $\|Tu\| \leq \|u\|$ for $u \in K \cap \partial\Omega_2$.

Then T has a fixed point in $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

2 Preliminaries

For the BVP

$$\begin{cases} u'''(t) = 0, & t \in [0, 1], \\ u'(0) = u(1) = 0, & u''(\eta) + \alpha u(0) = 0, \end{cases} \quad (2.1)$$

we have the following lemma.

Lemma 2.1 *The BVP (2.1) has only trivial solution.*

Proof. It is simple to check. □

In the remainder of this paper, we always assume that Banach space $C[0, 1]$ is equipped with the norm $\|u\| = \max_{t \in [0, 1]} |u(t)|$.

Now, for any $y \in C[0, 1]$, we consider the BVP

$$\begin{cases} u'''(t) = y(t), & t \in [0, 1], \\ u'(0) = u(1) = 0, & u''(\eta) + \alpha u(0) = 0. \end{cases} \quad (2.2)$$

After a direct computation, one may obtain the expression of Green's function $G(t, s)$ of the BVP (2.2) as follows:

$$G(t, s) = g_1(t, s) + g_2(t, s) + g_3(\eta, t, s),$$

where

$$g_1(t, s) = -\frac{(2 - \alpha t^2)(1 - s)^2}{2(2 - \alpha)}, \quad (t, s) \in [0, 1] \times [0, 1],$$

$$g_2(t, s) = \begin{cases} 0, & 0 \leq t \leq s \leq 1, \\ \frac{(t-s)^2}{2}, & 0 \leq s \leq t \leq 1 \end{cases}$$

and

$$g_3(\eta, t, s) = \begin{cases} 0, & s \geq \eta, \\ \frac{1-t^2}{2-\alpha}, & s < \eta. \end{cases}$$

It is not difficult to verify that the $G(t, s)$ has the following properties:

$$G(t, s) \geq 0 \quad \text{for } 0 \leq s \leq \eta \quad \text{and} \quad G(t, s) \leq 0 \quad \text{for } \eta \leq s \leq 1.$$

Moreover, for $s \geq \eta$,

$$\max\{G(t, s) : t \in [0, 1]\} = G(1, s) = 0,$$

$$\min\{G(t, s) : t \in [0, 1]\} = G(0, s) = -\frac{(1-s)^2}{2-\alpha}$$

and for $s < \eta$,

$$\max\{G(t, s) : t \in [0, 1]\} = G(0, s) = \frac{2s - s^2}{2 - \alpha},$$

$$\min\{G(t, s) : t \in [0, 1]\} = G(1, s) = 0.$$

Let

$$K_0 = \{y \in C[0, 1] : y(t) \text{ is nonnegative and decreasing on } [0, 1]\}.$$

Then K_0 is a cone in $C[0, 1]$.

Lemma 2.2 Let $y \in K_0$ and $u(t) = \int_0^1 G(t, s)y(s)ds$, $t \in [0, 1]$. Then u is the unique solution of the BVP (2.2) and $u \in K_0$. Moreover, $u(t)$ is concave on $[0, \eta]$.

Proof. For $0 \leq t \leq \eta$, we have

$$u(t) = \int_0^t \left[g_1(t, s) + \frac{(t-s)^2}{2} + \frac{1-t^2}{2-\alpha} \right] y(s)ds + \int_t^\eta \left[g_1(t, s) + \frac{1-t^2}{2-\alpha} \right] y(s)ds + \int_\eta^1 g_1(t, s)y(s)ds.$$

Since $\eta \geq \frac{\sqrt{121+24\alpha}-5}{3(4+\alpha)}$ implies that $\eta \geq \frac{2\alpha}{3\alpha+6}$, we get

$$\begin{aligned} u'(t) &= -\frac{\alpha t}{2-\alpha} \int_0^\eta (2s-s^2)y(s)ds - \int_0^t sy(s)ds - t \int_t^\eta y(s)ds + \frac{\alpha t}{2-\alpha} \int_\eta^1 (1-s)^2y(s)ds \\ &\leq y(\eta) \left[-\frac{\alpha t}{2-\alpha} \int_0^\eta (2s-s^2)ds - \int_0^t sds - t \int_t^\eta ds + \frac{\alpha t}{2-\alpha} \int_\eta^1 (1-s)^2ds \right] \\ &= ty(\eta) \left[\frac{\alpha(1-3\eta)}{3(2-\alpha)} - \eta + \frac{t}{2} \right] \\ &\leq ty(\eta) \left[\frac{\alpha(1-3\eta)}{3(2-\alpha)} - \frac{\eta}{2} \right] \\ &\leq 0. \end{aligned}$$

At the same time, $\eta \geq \frac{\sqrt{121+24\alpha}-5}{3(4+\alpha)} > \frac{1}{3}$ shows that

$$\begin{aligned} u''(t) &= -\frac{\alpha}{2-\alpha} \int_0^\eta (2s-s^2)y(s)ds - \int_t^\eta y(s)ds + \frac{\alpha}{2-\alpha} \int_\eta^1 (1-s)^2y(s)ds \\ &\leq -\frac{\alpha y(\eta)}{2-\alpha} \int_0^\eta (2s-s^2)ds - y(\eta) \int_t^\eta ds + \frac{\alpha y(\eta)}{2-\alpha} \int_\eta^1 (1-s)^2ds \\ &\leq \frac{\alpha y(\eta)(1-3\eta)}{3(2-\alpha)} \\ &\leq 0. \end{aligned}$$

For $\eta < t \leq 1$, we have

$$u(t) = \int_0^\eta \left[g_1(t, s) + \frac{(t-s)^2}{2} + \frac{1-t^2}{2-\alpha} \right] y(s)ds + \int_\eta^t \left[g_1(t, s) + \frac{(t-s)^2}{2} \right] y(s)ds + \int_t^1 g_1(t, s)y(s)ds.$$

Since $\eta \geq \frac{\sqrt{121+24\alpha}-5}{3(4+\alpha)}$ implies that $\eta \geq \frac{6-\alpha}{12}$, we get

$$\begin{aligned} u'(t) &= -\frac{\alpha t}{2-\alpha} \int_0^\eta (2s-s^2)y(s)ds + \int_\eta^t (t-s)y(s)ds - \int_0^\eta sy(s)ds + \frac{\alpha t}{2-\alpha} \int_\eta^1 (1-s)^2y(s)ds \\ &\leq -\frac{\alpha ty(\eta)}{2-\alpha} \int_0^\eta (2s-s^2)ds + \frac{y(\eta)(\eta-t)^2}{2} - y(\eta) \int_0^\eta sds + \frac{\alpha ty(\eta)(1-\eta)^3}{3(2-\alpha)} \\ &= ty(\eta) \left[\frac{\alpha(1-3\eta)}{3(2-\alpha)} + \frac{t-2\eta}{2} \right] \\ &\leq 0. \end{aligned}$$

Obviously, $u'''(t) = y(t)$ for $t \in [0, 1]$, $u'(0) = u(1) = 0$ and $u''(\eta) + \alpha u(0) = 0$. This shows that u is a solution of the BVP (2.2). The uniqueness follows immediately from Lemma 2.1. Since $u'(t) \leq 0$ for $t \in [0, 1]$ and $u(1) = 0$, we have $u(t) \geq 0$ for $t \in [0, 1]$. So, $u \in K_0$. In view of $u''(t) \leq 0$ for $t \in [0, \eta]$, we know that $u(t)$ is concave on $[0, \eta]$. \square

Lemma 2.3 *Let $y \in K_0$. Then the unique solution u of the BVP (2.2) satisfies*

$$\min_{t \in [0, \theta]} u(t) \geq \theta^* \|u\|,$$

where $\theta \in (0, \frac{1}{3}]$ and $\theta^* = \frac{\eta-\theta}{\eta}$.

Proof. By Lemma 2.2, we know that $u(t)$ is concave on $[0, \eta]$, thus for $t \in [0, \eta]$,

$$u(t) \geq (1 - \frac{t}{\eta})u(0) + \frac{t}{\eta}u(\eta). \tag{2.3}$$

In view of $u \in K_0$, we know that $\|u\| = u(0)$, which together with (2.3) implies that

$$u(t) \geq \frac{\eta-t}{\eta} \|u\|, \quad 0 \leq t \leq \eta.$$

Consequently,

$$\min_{t \in [0, \theta]} u(t) = u(\theta) \geq \frac{\eta-\theta}{\eta} \|u\| = \theta^* \|u\|.$$

\square

3 Main results

For convenience, we denote

$$A = \int_0^\eta G(0, s)ds \text{ and } B = \int_0^\theta G(\eta, s)ds.$$

Then it is obvious that $0 < B < A$.

Theorem 3.1 *Assume that $f : [0, 1] \times [0, +\infty) \rightarrow [0, +\infty)$ is continuous and satisfies the following conditions:*

(H1) *For each $u \in [0, +\infty)$, the mapping $t \mapsto f(t, u)$ is decreasing;*

(H2) *For each $t \in [0, 1]$, the mapping $u \mapsto f(t, u)$ is increasing;*

(H3) *There exist two positive constants r and R with $r \neq R$ such that*

$$f(0, r) \leq \frac{r}{A} \text{ and } f(\theta, \theta^* R) \geq \frac{R}{B}.$$

Then the BVP (1.2) has a positive and decreasing solution u satisfying $\min\{r, R\} \leq \|u\| \leq \max\{r, R\}$. Moreover, the obtained solution $u(t)$ is concave on $[0, \eta]$.

Proof. Let

$$K = \left\{ u \in K_0 : \min_{t \in [0, \theta]} u(t) \geq \theta^* \|u\| \right\}.$$

Then it is easy to see that K is a cone in $C[0, 1]$. Now, we define an operator T on K by

$$(Tu)(t) = \int_0^1 G(t, s)f(s, u(s))ds, \quad t \in [0, 1].$$

Obviously, if u is a fixed point of T in K , then u is a nonnegative and decreasing solution of the BVP (1.2). In what follows, we will seek a fixed point of T in K by using Theorem 1.1.

First, by Lemma 2.2 and Lemma 2.3, we know that $T : K \rightarrow K$. Furthermore, although $G(t, s)$ is not continuous, it follows from known textbook results, for example see [10], that $T : K \rightarrow K$ is completely continuous.

Next, for any $u \in K$, we claim that

$$\int_\theta^\eta G(\eta, s)f(s, u(s))ds + \int_\eta^1 G(\eta, s)f(s, u(s))ds \geq 0. \tag{3.1}$$

In fact, if $u \in K$, recall that $G(t, s) \geq 0$ for $0 \leq s \leq \eta$ and $G(t, s) \leq 0$ for $\eta \leq s \leq 1$, then it follows from $\eta \geq \frac{\sqrt{121+24\alpha}-5}{3(4+\alpha)}$ that

$$\begin{aligned} & \int_{\theta}^{\eta} G(\eta, s)f(s, u(s))ds + \int_{\eta}^1 G(\eta, s)f(s, u(s))ds \\ & \geq f(\eta, u(\eta)) \left[\int_{\theta}^{\eta} G(\eta, s)ds + \int_{\eta}^1 G(\eta, s)ds \right] \\ & = f(\eta, u(\eta)) \left[\int_{\theta}^{\eta} \left(g_1(\eta, s) + \frac{(\eta-s)^2}{2} + \frac{1-\eta^2}{2-\alpha} \right) ds + \int_{\eta}^1 g_1(\eta, s)ds \right] \\ & = \frac{(1-\eta)f(\eta, u(\eta))}{6(2-\alpha)} [(4+\alpha)\eta^2 + (4+\alpha\theta^3 - 3\alpha\theta^2)\eta - 6\theta^2 + \alpha\theta^3 - 2] \\ & \geq \frac{(1-\eta)f(\eta, u(\eta))}{6(2-\alpha)} \left[(4+\alpha)\eta^2 + \frac{10}{3}\eta - \frac{8}{3} \right] \\ & \geq 0. \end{aligned}$$

Now, without loss of generality, we assume that $r < R$. Let

$$\Omega_1 = \{u \in C[0, 1] : \|u\| < r\} \text{ and } \Omega_2 = \{u \in C[0, 1] : \|u\| < R\}.$$

For any $u \in K \cap \partial\Omega_1$, we get $0 \leq u(s) \leq r$ for $s \in [0, 1]$, which together with (H3) implies that

$$\begin{aligned} 0 \leq (Tu)(t) & \leq \int_0^{\eta} \max_{t \in [0,1]} G(t, s)f(s, u(s))ds + \int_{\eta}^1 \max_{t \in [0,1]} G(t, s)f(s, u(s))ds \\ & = \int_0^{\eta} G(0, s)f(s, u(s))ds \\ & \leq \int_0^{\eta} G(0, s)f(0, r)ds \\ & \leq r = \|u\|, \quad t \in [0, 1]. \end{aligned}$$

This shows that

$$\|Tu\| \leq \|u\| \text{ for } u \in K \cap \partial\Omega_1. \tag{3.2}$$

For any $u \in K \cap \partial\Omega_2$, we get $\theta^*R \leq u(s) \leq R$ for $s \in [0, \theta]$, which together with (3.1) and

(H3) implies that

$$\begin{aligned}
 Tu(\eta) &= \int_0^1 G(\eta, s)f(s, u(s))ds \\
 &= \int_0^\theta G(\eta, s)f(s, u(s))ds + \int_\theta^\eta G(\eta, s)f(s, u(s))ds + \int_\eta^1 G(\eta, s)f(s, u(s))ds \\
 &\geq \int_0^\theta G(\eta, s)f(s, u(s))ds \\
 &\geq \int_0^\theta G(\eta, s)f(\theta, \theta^*R)ds \\
 &\geq R = \|u\|,
 \end{aligned}$$

This indicates that

$$\|Tu\| \geq \|u\| \text{ for } u \in K \cap \partial\Omega_2. \quad (3.3)$$

Therefore, it follows from Theorem 1.1, (3.2) and (3.3) that the operator T has a fixed point $u \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$, which is a desired positive and decreasing solution of the BVP (1.2) with $r \leq \|u\| \leq R$. Moreover, similar to the proof of Lemma 2.2, we can prove that the obtained solution $u(t)$ is concave on $[0, \eta]$. \square

Example 3.2 We consider the BVP

$$\begin{cases} u'''(t) = \frac{u^2(t)}{4} + \frac{9(1-t^2)}{2}, & t \in [0, 1], \\ u'(0) = u(1) = 0, & u''(\frac{1}{2}) + u(0) = 0. \end{cases} \quad (3.4)$$

Since $\alpha = 1$ and $\eta = \frac{1}{2}$, if we choose $\theta = \frac{1}{3}$, then a simple calculation shows that

$$\theta^* = \frac{1}{3}, \quad A = \frac{5}{24} \text{ and } B = \frac{7}{108}.$$

Let $f(t, u) = \frac{u^2}{4} + \frac{9(1-t^2)}{2}$, $(t, u) \in [0, 1] \times [0, +\infty)$. Then (H1) and (H2) are satisfied. Moreover, it is easy to verify that

$$f(\theta, \frac{\theta^*}{4}) \geq \frac{1}{4B}, \quad f(0, 1) \leq \frac{1}{A}$$

and

$$f(0, 18) \leq \frac{18}{A}, \quad f(\theta, 556\theta^*) \geq \frac{556}{B}.$$

Therefore, it follows from Theorem 3.1 that the BVP (3.4) has positive and decreasing solutions u_1 and u_2 satisfying

$$\frac{1}{4} \leq \|u_1\| \leq 1 < 18 \leq \|u_2\| \leq 556.$$

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