

Nonlocal Boundary Value Problem for Strongly Singular Higher-Order Linear Functional-Differential Equations

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Abstract

For strongly singular higher-order differential equations with deviating arguments, under nonlocal boundary conditions, Agarwal-Kiguradze type theorems are established, which guarantee the presence of the Fredholm property for the problems considered. We also provide easily verifiable conditions that guarantee the existence of a unique solution of the problem.

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1 Statement of the main results

1.1 Statement of the problems and the basic notation

Consider the differential equations with deviating arguments

$$u^{(2m+1)}(t) = \sum_{j=0}^m p_j(t)u^{(j)}(\tau_j(t)) + q(t) \quad \text{for } a < t < b, \quad (1.1)$$

with the boundary conditions

$$\int_a^b u(s)d\varphi(s) = 0 \quad \text{where } \varphi(b) - \varphi(a) \neq 0, \quad (1.2)$$
$$u^{(i)}(a) = 0, \quad u^{(i)}(b) = 0 \quad (i = 1, \dots, m).$$

Here $m \in \mathbb{N}$, $-\infty < a < b < +\infty$, $p_j, q \in L_{loc}([a, b])$ ($j = 0, \dots, m$), $\varphi : [a, b] \rightarrow \mathbb{R}$ is a function of bounded variation, and $\tau_j :]a, b[\rightarrow]a, b[$ are measurable functions. By $u^{(i)}(a)$ (resp., $u^{(i)}(b)$), we denote the right (resp., left) limit of the function $u^{(i)}$ at the point a (resp., b). Problem (1.1), (1.2) is said to be singular if some or all the coefficients of (1.1) are non-integrable on $[a, b]$, having singularities at the end-points of this segment.

The first step in studying the linear ordinary differential equations

$$u^{(n)}(t) = \sum_{j=1}^m p_j(t)u^{(j-1)}(\tau_j(t)) + q(t) \quad \text{for } a < t < b, \quad (1.3)$$

where m is the integer part of $n/2$, under two-point conjugated boundary conditions, in the case when the functions p_j and q have strong singularities at the points a and b , i.e.

$$\begin{aligned} \int_a^b (s-a)^{n-1}(b-s)^{2m-1}[(-1)^{n-m}p_1(s)]_+ ds < +\infty, \\ \int_a^b (s-a)^{n-j}(b-s)^{2m-j}|p_j(s)| ds < +\infty \quad (j = 1, \dots, m), \\ \int_a^b (s-a)^{n-m-1/2}(b-s)^{m-1/2}|q(s)| ds < +\infty, \end{aligned} \quad (1.4)$$

are not fulfilled, was made by R. P. Agarwal and I. Kiguradze in the article [3].

In this paper, Agarwal-Kiguradze type theorems are proved which guarantee the Fredholm property for problem (1.1), (1.2), when for the coefficients p_j ($j = 1, \dots, m$), conditions (1.4), with $n = 2m$, are not satisfied. Throughout the paper we use the following notation.

$$\mathbb{R}^+ = [0, +\infty[;$$

$$[x]_+ \text{ is the positive part of a number } x, \text{ that is } [x]_+ = \frac{x+|x|}{2};$$

$L_{loc}([a, b])$ is the space of functions $y :]a, b[\rightarrow \mathbb{R}$, which are integrable on $[a + \varepsilon, b - \varepsilon]$ for arbitrary small $\varepsilon > 0$;

$L_{\alpha, \beta}([a, b])$ ($L_{\alpha, \beta}^2([a, b])$) is the space of integrable (square integrable) with the weight $(t-a)^\alpha(b-t)^\beta$ functions $y :]a, b[\rightarrow \mathbb{R}$, with the norm

$$\|y\|_{L_{\alpha, \beta}} = \int_a^b (s-a)^\alpha(b-s)^\beta |y(s)| ds \quad \left(\|y\|_{L_{\alpha, \beta}^2} = \left(\int_a^b (s-a)^\alpha(b-s)^\beta y^2(s) ds \right)^{1/2} \right);$$

$L([a, b]) = L_{0,0}([a, b])$, $L^2([a, b]) = L^2_{0,0}([a, b])$;
 $M([a, b])$ is the set of measurable functions $\tau :]a, b[\rightarrow]a, b[$;
 $\tilde{L}^2_{\alpha,\beta}([a, b])$ is the Banach space of functions $y \in L_{loc}([a, b])$ such that

$$\begin{aligned}
 \|y\|_{\tilde{L}^2_{\alpha,\beta}} := & \max \left\{ \left[\int_a^t (s-a)^\alpha \left(\int_s^t y(\xi) d\xi \right)^2 ds \right]^{1/2} : a \leq t \leq \frac{a+b}{2} \right\} + \\
 & + \max \left\{ \left[\int_t^b (b-s)^\beta \left(\int_t^s y(\xi) d\xi \right)^2 ds \right]^{1/2} : \frac{a+b}{2} \leq t \leq b \right\} < +\infty.
 \end{aligned}$$

$\tilde{C}^n_{loc}([a, b])$ is the space of functions $y :]a, b[\rightarrow R$ which are absolutely continuous together with $y', y'', \dots, y^{(n)}$ on $[a + \varepsilon, b - \varepsilon]$ for an arbitrarily small $\varepsilon > 0$.

$\tilde{C}^{m,m}([a, b])$ ($m \leq n$) is the space of functions $y \in \tilde{C}^n_{loc}([a, b])$, satisfying

$$\int_a^b |y^{(m)}(s)|^2 ds < +\infty. \tag{1.5}$$

When problem (1.1), (1.2) is discussed, we assume that the conditions

$$p_j \in L_{loc}([a, b]) \quad (j = 0, \dots, m) \tag{1.6}$$

are fulfilled.

A solution of problem (1.1), (1.2) is sought for in the space $\tilde{C}^{2m, m+1}([a, b])$.

By $h_j :]a, b[\times]a, b[\rightarrow R_+$ and $f_j : R \times M([a, b]) \rightarrow C_{loc}([a, b] \times]a, b[)$ ($j = 1, \dots, m$) we denote the functions and, respectively, the operators defined by the equalities

$$\begin{aligned}
 h_1(t, s) &= \left| \int_s^t [(-1)^m p_1(\xi)]_+ d\xi \right|, \\
 h_j(t, s) &= \left| \int_s^t p_j(\xi) d\xi \right| \quad (j = 2, \dots, m),
 \end{aligned} \tag{1.7}$$

and,

$$f_j(c, \tau_j)(t, s) = \left| \int_s^t |p_j(\xi)| \left| \int_\xi^{\tau_j(\xi)} (\xi_1 - c)^{2(m-j)} d\xi_1 \right|^{1/2} d\xi \right| \quad (j = 1, \dots, m), \tag{1.8}$$

and also we put that

$$f_0(t, s) = \left| \int_s^t |p_0(\xi)| d\xi \right|.$$

Let $m = 2k + 1$, then

$$m!! = \begin{cases} 1 & \text{for } m \leq 0 \\ 1 \cdot 3 \cdot 5 \cdots m & \text{for } m \geq 1 \end{cases}.$$

1.2 Fredholm type theorems

Along with (1.1), we consider the homogeneous equation

$$v^{(2m+1)}(t) = \sum_{j=0}^m p_j(t) v^{(j)}(\tau_j(t)) \quad \text{for } a < t < b. \quad (1.10)$$

Definition 1.1. We will say that problem (1.1), (1.2) has the Fredholm property in the space $\tilde{C}^{2m, m+1}([a, b])$ if the unique solvability of the corresponding homogeneous problem (1.10), (1.2) in that space implies the unique solvability of problem (1.1), (1.2) for every $q \in \tilde{L}_{2m-2, 2m-2}^2([a, b])$.

In the case where conditions (1.4) for $n = 2m$ are violated, the question on the presence of the Fredholm property for problem (1.1), (1.2) in some subspace of the space $\tilde{C}_{loc}^{2m}([a, b])$ remains so far open. This question is answered in Theorem 1.1 formulated below which contains conditions guaranteeing the Fredholm property for problem (1.1), (1.2) in the space $\tilde{C}^{2m, m+1}([a, b])$.

Theorem 1.1. *Let there exist $a_0 \in]a, b[$, $b_0 \in]a_0, b[$, numbers $l_{kj} > 0$, $\gamma_{k0} > 0$, $\gamma_{kj} > 0$ ($k = 0, 1$, $j = 1, \dots, m$) such that*

$$\begin{aligned} (t - a)^{2m-j} h_j(t, s) &\leq l_{0j} \quad (j = 1, \dots, m) \quad \text{for } a < t \leq s \leq a_0, \\ \limsup_{t \rightarrow a} (t - a)^{m-\frac{1}{2}-\gamma_{00}} f_0(t, s) &< +\infty, \end{aligned} \quad (1.9)$$

$$\begin{aligned} \limsup_{t \rightarrow a} (t - a)^{m-\frac{1}{2}-\gamma_{0j}} f_j(a, \tau_j)(t, s) &< +\infty \quad (j = 1, \dots, m), \\ (b - t)^{2m-j} h_j(t, s) &\leq l_{1j} \quad (j = 1, \dots, m) \quad \text{for } b_0 \leq s \leq t < b, \\ \limsup_{t \rightarrow b} (b - t)^{m-\frac{1}{2}-\gamma_{10}} f_0(t, s) &< +\infty, \end{aligned} \quad (1.10)$$

$$\limsup_{t \rightarrow b} (b - t)^{m-\frac{1}{2}-\gamma_{1j}} f_j(b, \tau_j)(t, s) < +\infty \quad (j = 1, \dots, m),$$

and

$$\sum_{j=1}^m \frac{(2m-j)2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!} l_{kj} < 1 \quad (k = 0, 1). \quad (1.11)$$

Let, moreover, the homogeneous problem (1.1₀), (1.2) have only the trivial solution in the space $\tilde{C}^{2m,m+1}([a, b])$. Then problem (1.1), (1.2) has a unique solution u for an arbitrary $q \in \tilde{L}_{2m-2, 2m-2}^2([a, b])$, and there exists a constant r , independent of q , such that

$$\|u^{(m+1)}\|_{L^2} \leq r \|q\|_{\tilde{L}_{2m-2, 2m-2}^2}. \quad (1.12)$$

Corollary 1.1. Let numbers $\kappa_{kj}, \nu_{kj} \in R^+$ be such that

$$\nu_{k1} > 4m + 2, \quad \nu_{kj} > 2 \quad (k = 0, 1; j = 2, \dots, m), \quad (1.13)$$

$$\limsup_{t \rightarrow a} \frac{|\tau_j(t) - t|}{(t-a)^{\nu_{0j}}} < +\infty, \quad \limsup_{t \rightarrow b} \frac{|\tau_j(t) - t|}{(b-t)^{\nu_{1j}}} < +\infty \quad (j = 1, \dots, m), \quad (1.14)$$

and

$$\sum_{j=1}^m \frac{2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!} \kappa_{kj} < 1 \quad (k = 0, 1). \quad (1.15)$$

Moreover, let $\kappa \in R^+$, $p_{00} \in L_{m-1, m-1}([a, b]; R^+)$, $p_{0j} \in L_{2m-j, 2m-j}([a, b]; R^+)$, and

$$-\frac{\kappa}{[(t-a)(b-t)]^{2m}} - p_{01}(t) \leq (-1)^m p_1(t) \leq \frac{\kappa_{01}}{(t-a)^{2m}} + \frac{\kappa_{11}}{(b-t)^{2m}} + p_{01}(t), \quad (1.16)$$

$$\begin{aligned} |p_0(t)| &\leq \frac{\kappa_{00}}{(t-a)^m} + \frac{\kappa_{10}}{(b-t)^m} + p_{00}(t) \\ |p_j(t)| &\leq \frac{\kappa_{0j}}{(t-a)^{2m-j+1}} + \frac{\kappa_{1j}}{(b-t)^{2m-j+1}} + p_{0j}(t) \quad (j = 2, \dots, m). \end{aligned} \quad (1.17)$$

Let, moreover, the homogeneous problem (1.1₀), (1.2) have only the trivial solution in the space $\tilde{C}^{2m,m+1}([a, b])$. Then problem (1.1), (1.2) has a unique solution u for an arbitrary $q \in \tilde{L}_{2m-2, 2m-2}^2([a, b])$, and there exists a constant r , independent of q , such that (1.12) holds.

1.3 Existence and uniqueness theorems

Theorem 1.2. Let there exist numbers $t^* \in]a, b[$, $l_{k0} > 0$, $l_{kj} > 0$, $\bar{l}_{kj} \geq 0$, and $\gamma_{k0} > 0$, $\gamma_{kj} > 0$ ($k = 0, 1$; $j = 1, \dots, m$) such that along with

$$\begin{aligned}
 & B_0 \equiv \\
 \equiv & \bar{l}_{00} \left(\frac{2^{m-1}}{(2m-3)!!} \right)^2 \frac{(b-a)^{m-1/2}}{(2m-1)^{1/2}} \frac{(t^*-a)^{\gamma_{00}}}{\sqrt{2\gamma_{00}}} \int_a^b \frac{|\varphi(\xi) - \varphi(a)| + |\varphi(\xi) - \varphi(b)|}{|\varphi(b) - \varphi(a)|} d\xi + \\
 & + \sum_{j=1}^m \left(\frac{(2m-j)2^{2m-j+1} l_{0j}}{(2m-1)!!(2m-2j+1)!!} + \frac{2^{2m-j-1}(t^*-a)^{\gamma_{0j}} \bar{l}_{0j}}{(2m-2j-1)!!(2m-3)!!\sqrt{2\gamma_{0j}}} \right) < \frac{1}{2},
 \end{aligned} \tag{1.18}$$

$$\begin{aligned}
 & B_1 \equiv \\
 \equiv & \bar{l}_{10} \left(\frac{2^{m-1}}{(2m-3)!!} \right)^2 \frac{(b-a)^{m-1/2}}{(2m-1)^{1/2}} \frac{(b-t^*)^{\gamma_{10}}}{\sqrt{2\gamma_{10}}} \int_a^b \frac{|\varphi(\xi) - \varphi(a)| + |\varphi(\xi) - \varphi(b)|}{|\varphi(b) - \varphi(a)|} d\xi + \\
 & + \sum_{j=1}^m \left(\frac{(2m-j)2^{2m-j+1} l_{1j}}{(2m-1)!!(2m-2j+1)!!} + \frac{2^{2m-j-1}(b-t^*)^{\gamma_{1j}} \bar{l}_{1j}}{(2m-2j-1)!!(2m-3)!!\sqrt{2\gamma_{1j}}} \right) < \frac{1}{2},
 \end{aligned} \tag{1.19}$$

the conditions

$$\begin{aligned}
 & (t-a)^{m-\gamma_{00}-1/2} f_0(t, s) \leq \bar{l}_{00}, \\
 & (t-a)^{2m-j} h_j(t, s) \leq l_{0j}, \quad (t-a)^{m-\gamma_{0j}-1/2} f_j(a, \tau_j)(t, s) \leq \bar{l}_{0j}
 \end{aligned} \tag{1.20}$$

for $a < t \leq s \leq t^*$ and

$$\begin{aligned}
 & (b-t)^{m-\gamma_{10}-1/2} f_0(t, s) \leq \bar{l}_{10}, \\
 & (b-t)^{2m-j} h_j(t, s) \leq l_{1j}, \quad (b-t)^{m-\gamma_{1j}-1/2} f_j(b, \tau_j)(t, s) \leq \bar{l}_{1j}
 \end{aligned} \tag{1.21}$$

for $t^* \leq s \leq t < b$ hold with any $j = 1, \dots, m$. Then problem (1.1), (1.2) is uniquely solvable in the space $\tilde{C}^{2m, m+1}(]a, b[)$ for every $q \in \tilde{L}_{2m-2, 2m-2}^2(]a, b[)$.

Remark 1.1. Let all the conditions of Theorem 1.2 be satisfied. Then the unique solution u of problem (1.1), (1.2) for every $q \in \tilde{L}_{2m-2, 2m-2}^2(]a, b[)$ admits the estimate

$$\|u^{(m+1)}\|_{L^2} \leq r \|q\|_{\tilde{L}_{2m-2, 2m-2}^2}, \tag{1.22}$$

with

$$r = \frac{2^m}{(1 - 2 \max\{B_0, B_1\})(2m-1)!!},$$

and thus the constant $r > 0$ depends only on the numbers $l_{kj}, \bar{l}_{k0}, \bar{l}_{kj}, \gamma_{k0}, \gamma_{kj}$ ($k = 0, 1; j = 0, \dots, m$), and a, b, t^* .

To illustrate this theorem, we consider the third order differential equation with a deviating argument

$$u^{(3)}(t) = p_0(t)u(\tau_0(t)) + p_1(t)u'(\tau_1(t)) + q(t), \quad (1.23)$$

under the boundary conditions

$$\int_a^b u(s)ds = 0, \quad u(a) = 0, \quad u(b) = 0. \quad (1.24)$$

As a corollary of Theorem 1.2 with $m = 1, t^* = (a + b)/2, \gamma_{00} = \gamma_{10} = 1/4, \gamma_{01} = \gamma_{11} = 1/2, \bar{l}_{00} = \bar{l}_{10} = 8 \frac{2^{1/4}\kappa}{(b-a)^{5/4}}, l_{01} = l_{11} = \kappa_0, \bar{l}_{01} = \bar{l}_{11} = \frac{\sqrt{2}\kappa_1}{\sqrt{b-a}}$, we obtain the following statement.

Corollary 1.2. *Let function $\tau_1 \in M(]a, b[)$ be such that*

$$\begin{aligned} 0 \leq \tau_1(t) - t \leq \frac{2^6}{(b-a)^6}(t-a)^7 \quad \text{for } a < t \leq \frac{a+b}{2}, \\ -\frac{2^6}{(b-a)^6}(b-t)^7 \leq t - \tau_1(t) \leq 0 \quad \text{for } \frac{a+b}{2} \leq t < b. \end{aligned} \quad (1.25)$$

Moreover, let function $p :]a, b[\rightarrow R$ and constants κ_0, κ_1 be such that

$$\begin{aligned} |p_0(t)| \leq \frac{\kappa}{[(b-t)(t-a)]^{5/4}} \quad \text{for } a < t < b \\ -\frac{2^{-2}(b-a)^2\kappa_0}{[(b-t)(t-a)]^2} \leq p_1(t) \leq \frac{2^{-7}(b-a)^6\kappa_1}{[(b-t)(t-a)]^4} \quad \text{for } a < t < b \end{aligned} \quad (1.26)$$

and

$$8\kappa\sqrt{2(b-a)} + 4\kappa_0 + \kappa_1 < \frac{1}{2}. \quad (1.27)$$

Then problem (1.23), (1.24) is uniquely solvable in the space $\tilde{C}^{2,2}(]a, b[)$ for every $q \in \tilde{L}_{0,0}^2(]a, b[)$.

2 Auxiliary Propositions

2.1 Lemmas on integral inequalities

Now we formulate two lemmas which are proved in [3].

Lemma 2.1. *Let $u \in \tilde{C}_{loc}^{m-1}([t_0, t_1])$ and*

$$u^{(j-1)}(t_0) = 0 \quad (j = 1, \dots, m), \quad \int_{t_0}^{t_1} |u^{(m)}(s)|^2 ds < +\infty. \quad (2.1)$$

Then

$$\int_{t_0}^t \frac{(u^{(j-1)}(s))^2}{(s - t_0)^{2m-2j+2}} ds \leq \left(\frac{2^{m-j+1}}{(2m - 2j + 1)!!} \right)^2 \int_{t_0}^t |u^{(m)}(s)|^2 ds \quad (2.2)$$

for $t_0 \leq t \leq t_1$.

Lemma 2.2. *Let $u \in \tilde{C}_{loc}^{m-1}([t_0, t_1])$, and*

$$u^{(j-1)}(t_1) = 0 \quad (j = 1, \dots, m), \quad \int_{t_0}^{t_1} |u^{(m)}(s)|^2 ds < +\infty. \quad (2.3)$$

Then

$$\int_t^{t_1} \frac{(u^{(j-1)}(s))^2}{(t_1 - s)^{2m-2j+2}} ds \leq \left(\frac{2^{m-j+1}}{(2m - 2j + 1)!!} \right)^2 \int_t^{t_1} |u^{(m)}(s)|^2 ds \quad (2.4)$$

for $t_0 \leq t \leq t_1$.

Let $t_0, t_1 \in]a, b[$, $u \in \tilde{C}_{loc}^{m-1}([t_0, t_1])$ and $\tau_j \in M([a, b])$ ($j = 0, \dots, m$). Then we define the functions $\mu_j : [a, (a + b)/2] \times [(a + b)/2, b] \times [a, b] \rightarrow [a, b]$, $\rho_k : [t_0, t_1] \rightarrow R_+$ ($k = 0, 1$), $\lambda_j : [a, b] \times]a, (a + b)/2[\times [(a + b)/2, b[\times]a, b[\rightarrow R_+$, and for any $t_0, t_1 \in [a, b]$ the

operator $\chi_{t_0, t_1} : C([t_0, t_1]) \rightarrow C([a, b])$, by the equalities

$$\mu_j(t_0, t_1, t) = \begin{cases} \tau_j(t) & \text{for } \tau_j(t) \in [t_0, t_1] \\ t_0 & \text{for } \tau_j(t) < t_0 \\ t_1 & \text{for } \tau_j(t) > t_1 \end{cases},$$

$$\rho_k(t) = \left| \int_t^{t_k} |u^{(m)}(s)|^2 ds \right|, \quad \lambda_j(c, t_0, t_1, t) = \left| \int_t^{\mu_j(t_0, t_1, t)} (s - c)^{2(m-j)} ds \right|^{\frac{1}{2}}, \quad (2.5)$$

$$\chi_{t_0, t_1}(x)(t) = \begin{cases} x(t_0) & \text{for } a \leq t < t_0 \\ x(t) & \text{for } t_0 \leq t \leq t_1 \\ x(t_1) & \text{for } t_1 < t \leq b \end{cases}.$$

Let also $\alpha_0 : R_+^2 \times [0, 1[\rightarrow R_+$, $\alpha_j : R_+^3 \times [0, 1[\rightarrow R_+$ and $\beta_j \in R_+ \times [0, 1[\rightarrow R_+$ ($j = 0, \dots, m$) be the functions defined by the equalities

$$\alpha_0(x, y, \gamma) = \frac{2^{m-1}(b-a)^{m-1/2}xy^\gamma}{(2m-3)!!(2m-1)^{1/2}} \int_a^b \frac{|\varphi(\xi) - \varphi(a)| + |\varphi(\xi) - \varphi(b)|}{|\varphi(b) - \varphi(a)} d\xi$$

$$\beta_0(x, \gamma) = \left(\frac{2^{m-1}}{(2m-3)!!} \right)^2 \frac{(b-a)^{m-1/2}}{(2m-1)^{1/2}} \frac{x^\gamma}{\sqrt{2}\gamma} \int_a^b \frac{|\varphi(\xi) - \varphi(a)| + |\varphi(\xi) - \varphi(b)|}{|\varphi(b) - \varphi(a)} d\xi, \quad (2.6)$$

$$\alpha_j(x, y, z, \gamma) = x + \frac{2^{m-j} y z^\gamma}{(2m-2j-1)!!},$$

$$\beta_j(y, \gamma) = \frac{2^{2m-j-1}}{(2m-2j-1)!!(2m-3)!!} \frac{y^\gamma}{\sqrt{2}\gamma},$$

and

$$G(t, s) = \frac{1}{\varphi(b) - \varphi(a)} \times \begin{cases} \varphi(s) - \varphi(b) & \text{for } s \geq t \\ \varphi(s) - \varphi(a) & \text{for } s < t \end{cases} \quad (2.7)$$

is the Green function of the problem:

$$w'(t) = 0, \quad \int_a^b w(s) d\varphi(s) = 0, \quad (2.8)$$

where $\varphi : [a, b] \rightarrow R$ is a function of bounded variation and $\varphi(b) - \varphi(a) \neq 0$.

Lemma 2.3. Let $a_0 \in]a, b[, t_0 \in]a, a_0[, t_1 \in]a_0, b[$, and the function $u \in \widetilde{C}_{loc}^{m-1}(]t_0, t_1[)$ be such that conditions (2.1), (2.3) hold. Moreover, let constants $l_{0j} > 0, \bar{l}_{00} \geq 0, \bar{l}_{0j} \geq 0, \gamma_{0j} > 0$, and functions $\bar{p}_j \in L_{loc}(]t_0, t_1[), \tau_j \in M(]a, b[)$ be such that the inequalities

$$(t - t_0)^{2m-1} \int_t^{a_0} [\bar{p}_1(s)]_+ ds \leq l_{01}, \quad (2.9)$$

$$(t - t_0)^{2m-j} \left| \int_t^{a_0} \bar{p}_j(s) ds \right| \leq l_{0j} \quad (j = 2, \dots, m), \quad (2.10)$$

$$(t - t_0)^{m-1/2-\gamma_{00}} \int_t^{a_0} |\bar{p}_0(s)| ds \leq \bar{l}_{00}, \quad (2.11)$$

$$(t - t_0)^{m-\frac{1}{2}-\gamma_{0j}} \int_t^{a_0} |\bar{p}_j(s)| \lambda_j(t_0, t_0, t_1, s) ds \leq \bar{l}_{0j} \quad (j = 1, \dots, m)$$

hold for $t_0 < t \leq a_0$. Then

$$\begin{aligned} & \int_t^{a_0} \bar{p}_j(s) u(s) u^{(j-1)}(\mu_j(t_0, t_1, s)) ds \leq \\ & \leq \alpha_j(l_{0j}, \bar{l}_{0j}, a_0 - a, \gamma_{0j}) \rho_0^{1/2}(\tau^*) \rho_0^{1/2}(t) + \bar{l}_{0j} \beta_j(a_0 - a, \gamma_{0j}) \rho_0^{1/2}(\tau^*) \rho_0^{1/2}(a_0) + \\ & \quad + l_{0j} \frac{(2m-j)2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!} \rho_0(a_0) \quad (j = 1, \dots, m) \end{aligned} \quad (2.12)$$

for $t_0 < t \leq a_0$ and

$$\begin{aligned} & \int_t^{a_0} \bar{p}_0(s) u(s) \left(\int_a^b G(\mu_0(t_0, t_1, s), \xi) \chi_{t_0, t_1}(u)(\xi) d\xi \right) ds \leq \\ & \leq \alpha_0(\bar{l}_{00}, a_0 - a, \gamma_{00}) \rho_0^{1/2}(t_1) \rho_0^{1/2}(t) \\ & \quad + \bar{l}_{00} \beta_0(a_0 - a, \gamma_{00}) \rho_0^{1/2}(t_1) \rho_0^{1/2}(a_0) \end{aligned} \quad (2.13)$$

for $t_0 < t \leq a_0$, where $\tau^* = \sup\{\mu_j(t_0, t_1, t) : t_0 \leq t \leq a_0, j = 1, \dots, m\} \leq t_1$.

Proof. In view of the formula of integration by parts, for $t \in [t_0, a_0]$ we have

$$\begin{aligned} \int_t^{a_0} \bar{p}_j(s)u(s)u^{(j-1)}(\mu_j(t_0, t_1, s))ds &= \int_t^{a_0} \bar{p}_j(s)u(s)u^{(j-1)}(s)ds + \\ &+ \int_t^{a_0} \bar{p}_j(s)u(s) \left(\int_s^{\mu_j(t_0, t_1, s)} u^{(j)}(\xi)d\xi \right) ds = u(t)u^{(j-1)}(t) \int_t^{a_0} \bar{p}_j(s)ds + \\ &+ \sum_{k=0}^1 \int_t^{a_0} \left(\int_s^{a_0} \bar{p}_j(\xi)d\xi \right) u^{(k)}(s)u^{(j-k)}(s)ds + \int_t^{a_0} \bar{p}_j(s)u(s) \left(\int_s^{\mu_j(t_0, t_1, s)} u^{(j)}(\xi)d\xi \right) ds \end{aligned} \quad (2.14)$$

($j = 2, \dots, m$), and

$$\begin{aligned} \int_t^{a_0} \bar{p}_1(s)u(s)u(\mu_1(t_0, t_1, s))ds &\leq \int_t^{a_0} [\bar{p}_1(s)]_+ u^2(s)ds + \\ &+ \int_t^{a_0} |\bar{p}_1(s)u(s)| \left| \int_s^{\mu_1(t_0, t_1, s)} u'(\xi)d\xi \right| ds \leq u^2(t) \int_t^{a_0} [\bar{p}_1(s)]_+ ds + \\ &+ 2 \int_t^{a_0} \left(\int_s^{a_0} [\bar{p}_1(\xi)]_+ d\xi \right) |u(s)u'(s)| ds + \int_t^{a_0} |\bar{p}_1(s)u(s)| \left| \int_s^{\mu_1(t_0, t_1, s)} u'(\xi)d\xi \right| ds. \end{aligned} \quad (2.15)$$

On the other hand, by virtue of conditions (2.1), the Schwartz inequality and Lemma 2.1, we deduce that

$$|u^{(j-1)}(t)| = \frac{1}{(m-j)!} \left| \int_{t_0}^t (t-s)^{m-j} u^{(m)}(s)ds \right| \leq (t-t_0)^{m-j+1/2} \rho_0^{1/2}(t) \quad (2.16)$$

for $t_0 \leq t \leq a_0$ ($j = 1, \dots, m$). If along with this, in the case where $j > 1$, we take inequality (2.10) and Lemma 2.1 into account, for $t \in [t_0, a_0]$, we obtain the estimates

$$\left| u(t)u^{(j-1)}(t) \int_t^{a_0} \bar{p}_j(s)ds \right| \leq (t-t_0)^{2m-j} \left| \int_t^{a_0} \bar{p}_j(s)ds \right| \rho_0(t) \leq l_{0j} \rho_0(t) \quad (2.17)$$

and

$$\begin{aligned}
\sum_{k=0}^1 \int_t^{a_0} \left(\int_s^{a_0} \bar{p}_j(\xi) d\xi \right) u^{(k)}(s) u^{(j-k)}(s) ds &\leq l_{0j} \sum_{k=0}^1 \int_t^{a_0} \frac{|u^{(k)}(s) u^{(j-k)}(s)|}{(s-t)^{2m-j}} ds \leq \\
&\leq l_{0j} \sum_{k=0}^1 \left(\int_t^{a_0} \frac{|u^{(k)}(s)|^2 ds}{(s-t)^{2m-2k}} \right)^{1/2} \left(\int_t^{a_0} \frac{|u^{(j-k)}(s)|^2 ds}{(s-t)^{2m+2k-2j}} \right)^{1/2} \leq \\
&\leq l_{0j} \rho_0(a_0) \sum_{k=0}^1 \frac{2^{2m-j}}{(2m-2k-1)!!(2m+2k-2j-1)!!}. \quad (2.18)
\end{aligned}$$

Analogously, if $j = 1$, by (2.9) we obtain

$$\begin{aligned}
u^2(t) \int_t^{a_0} [\bar{p}_1(s)]_+ ds &\leq l_{01} \rho_0(t), \\
2 \int_t^{a_0} \left(\int_s^{a_0} [\bar{p}_1(\xi)]_+ d\xi \right) |u(s) u'(s)| ds &\leq l_{01} \rho_0(a_0) \frac{(2m-1)2^{2m}}{[(2m-1)!!]^2}
\end{aligned} \quad (2.19)$$

for $t_0 < t \leq a_0$.

By the Schwartz inequality, Lemma 2.1, and the fact that ρ_0 is a nondecreasing function, we get

$$\left| \int_s^{\mu_j(t_0, t_1, s)} u^{(j)}(\xi) d\xi \right| \leq \frac{2^{m-j}}{(2m-2j-1)!!} \lambda_j(t_0, t_0, t_1, s) \rho_0^{1/2}(\tau^*) \quad (2.20)$$

for $t_0 < s \leq a_0$. Also, due to (2.2), (2.11) and (2.16), we have

$$\begin{aligned}
|u(t)| \int_t^{a_0} |\bar{p}_j(s)| \lambda_j(t_0, t_0, t_1, s) ds &= (t-t_0)^{m-1/2} \rho_0^{1/2}(t) \int_t^{a_0} |\bar{p}_j(s)| \lambda_j(t_0, t_0, t_1, s) ds \leq \\
&\leq \bar{l}_{0j} (t-t_0)^{\gamma_{0j}} \rho_0^{1/2}(t)
\end{aligned}$$

and

$$\int_t^{a_0} |u'(s)| \left(\int_s^{a_0} |\bar{p}_j(\xi)| \lambda_j(t_0, t_0, t_1, \xi) d\xi \right) ds \leq \bar{l}_{0j} \int_t^{a_0} \frac{|u'(s)|}{(s-t_0)^{m-\frac{1}{2}-\gamma_{0j}}} ds \leq$$

$$\leq \bar{l}_{0j} \frac{2^{m-1}(a_0-t_0)^{\gamma_{0j}}}{(2m-3)!! \sqrt{2\gamma_{0j}}} \rho_0^{1/2}(a_0)$$

for $t_0 < t \leq a_0$. It is clear from the last three inequalities that

$$\left| \frac{(2m-2j-1)!!}{2^{m-j} \rho_0^{1/2}(\tau^*)} \int_t^{a_0} \bar{p}_j(s) u(s) \left(\int_s^{\mu_j(t_0, t_1, s)} u^{(j)}(\xi) d\xi \right) ds \right| \leq$$

$$\leq \int_t^{a_0} |\bar{p}_j(s) u(s)| \lambda_j(t_0, t_0, t_1, s) ds \leq$$

$$\leq |u(t)| \int_t^{a_0} |\bar{p}_j(s)| \lambda_j(t_0, t_0, t_1, s) ds + \int_t^{a_0} |u'(s)| \left(\int_s^{a_0} |\bar{p}_j(\xi)| \lambda_j(t_0, t_0, t_1, \xi) d\xi \right) ds \leq$$

$$\leq \bar{l}_{0j} (t-t_0)^{\gamma_{0j}} \rho_0^{1/2}(t) + \bar{l}_{0j} \frac{2^{m-1}(a_0-t_0)^{\gamma_{0j}}}{(2m-3)!! \sqrt{2\gamma_{0j}}} \rho_0^{1/2}(a_0) \quad (2.21)$$

for $t_0 < t \leq a_0$. Now we note that, by (2.17)-(2.19) and (2.21), inequality (2.12) follows immediately from (2.14) and (2.15).

In view of the definition of the function G , the operator $\chi_{t_0 t_1}$ and condition (2.1), we have

$$\int_t^{a_0} \bar{p}_0(s) u(s) \left(\int_a^b G(\mu_0(t_0, t_1, s), \xi) \chi_{t_0, t_1}(u)(\xi) d\xi \right) ds =$$

$$= \int_t^{a_0} \bar{p}_0(s) u(s) \left(\int_{t_0}^{\mu_0(t_0, t_1, s)} \frac{\varphi(\xi) - \varphi(a)}{\varphi(b) - \varphi(a)} u(\xi) d\xi \right) ds +$$

$$+ \int_t^{a_0} \bar{p}_0(s) u(s) \left(\int_{\mu_0(t_0, t_1, s)}^{t_1} \frac{\varphi(\xi) - \varphi(b)}{\varphi(b) - \varphi(a)} u(\xi) d\xi \right) ds. \quad (2.22)$$

On the other hand, by the carrying out integration by parts and using the Schwartz inequality, we get the inequality

$$\int_{t_0}^{\mu_0(t_0, t_1, s)} \frac{\varphi(\xi) - \varphi(a)}{\varphi(b) - \varphi(a)} u(\xi) d\xi \leq \int_{t_0}^{t_1} \left| \frac{\varphi(\xi) - \varphi(a)}{\varphi(b) - \varphi(a)} \right| d\xi \times \\ \times \left(\int_{t_0}^{t_1} (\xi - t_0)^{2(m-1)} d\xi \right)^{1/2} \left(\int_{t_0}^{t_1} \frac{u'^2(\xi)}{(\xi - t_0)^{2(m-1)}} d\xi \right)^{1/2} \quad (2.23)$$

from which, by Lemma 2.1 and the definition of the function μ_0 , it follows that

$$\int_{t_0}^{t_1} \frac{\varphi(\xi) - \varphi(a)}{\varphi(b) - \varphi(a)} u(\xi) d\xi \leq \frac{2^{m-1}(b-a)^{m-1/2}}{(2m-3)!!(2m-1)^{1/2}} \rho_0^{1/2}(t_1) \int_a^b \left| \frac{\varphi(\xi) - \varphi(a)}{\varphi(b) - \varphi(a)} \right| d\xi \quad (2.24)$$

Analogously, by Lemma 2.2, in view of the fact that $\rho_0(t_1) = \rho_1(t_0)$, we get

$$\int_{\mu_0(t_0, t_1, s)}^{t_1} \frac{\varphi(\xi) - \varphi(b)}{\varphi(b) - \varphi(a)} u(\xi) d\xi \leq \frac{2^{m-1}(b-a)^{m-1/2}}{(2m-3)!!(2m-1)^{1/2}} \rho_0^{1/2}(t_1) \int_a^b \left| \frac{\varphi(\xi) - \varphi(a)}{\varphi(b) - \varphi(a)} \right| d\xi. \quad (2.25)$$

On the other hand by the integration by parts, inequality (2.16), and condition (2.11) we get

$$\int_t^{a_0} |\bar{p}_0(s)u(s)| ds \leq |u(s)| \int_t^{a_0} |\bar{p}_0(s)| ds + \int_t^{a_0} |u'(s)| \int_s^{a_0} |\bar{p}_0(\xi)| d\xi ds \\ \leq (t - t_0)^{\gamma_{00}} \rho_0^{1/2}(t) \bar{l}_{00} + \bar{l}_{00} \int_t^{a_0} \frac{|u'(s)|}{(s - t_0)^{m-1/2-\gamma_{00}}} ds,$$

from which, by the Schwartz inequality and Lemma 2.1, we get

$$\int_t^{a_0} |\bar{p}_0(s)u(s)| ds \leq (t - t_0)^{\gamma_{00}} \rho_0^{1/2}(t) \bar{l}_{00} + \frac{2^{m-1}(a_0 - a)^{\gamma_{00}}}{(2m-3)!!\sqrt{2\gamma_{00}}} \rho_0^{1/2}(a_0) \bar{l}_{00}. \quad (2.26)$$

From (2.22) by (2.24)-(2.26) and notation (2.6), inequality (2.13) follows immediately. \square

The following lemma can be proved similarly to Lemma 2.3.

Lemma 2.4. Let $b_0 \in]a, b[$, $t_1 \in]b_0, b[$, $t_0 \in]a, b_0[$, and the function $u \in \widetilde{C}_{loc}^{m-1}(]t_0, t_1[)$ be such that conditions (2.1), (2.3) hold. Moreover, let constants $l_{1j} > 0$, $\bar{l}_{10} \geq 0$, $\bar{l}_{1j} \geq 0$, $\gamma_{1j} > 0$, and functions $\bar{p}_j \in L_{loc}(]t_0, t_1[)$, $\tau_j \in M(]a, b[)$ be such that the inequalities

$$(t_1 - t)^{2m-1} \int_{b_0}^t [\bar{p}_1(s)]_+ ds \leq l_{11}, \quad (2.27)$$

$$(t_1 - t)^{2m-j} \left| \int_{b_0}^t \bar{p}_j(s) ds \right| \leq l_{1j} \quad (j = 2, \dots, m), \quad (2.28)$$

$$(t_1 - t)^{m-1/2-\gamma_{10}} \int_{b_0}^t |\bar{p}_0(s)| ds \leq \bar{l}_{10}, \quad (2.29)$$

$$(t_1 - t)^{m-\frac{1}{2}-\gamma_{1j}} \left| \int_{b_0}^t \bar{p}_j(s) \lambda_j(t_1, t_0, t_1, s) ds \right| \leq \bar{l}_{1j} \quad (j = 1, \dots, m)$$

hold for $b_0 < t \leq t_1$. Then

$$\begin{aligned} & \int_{b_0}^t \bar{p}_j(s) u(s) u^{(j-1)}(\mu_j(t_0, t_1, s)) ds \leq \\ & \leq \alpha_j(l_{1j}, \bar{l}_{1j}, b - b_0, \gamma_{1j}) \rho_1^{1/2}(\tau_*) \rho_1^{1/2}(t) + \bar{l}_{1j} \beta_j(b - b_0, \gamma_{1j}) \rho_1^{1/2}(\tau_*) \rho_1^{1/2}(b_0) + \\ & \quad + l_{1j} \frac{(2m-j)2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!} \rho_1(b_0) \end{aligned} \quad (2.30)$$

for $b_0 \leq t < t_1$ and

$$\begin{aligned} & \int_{b_0}^t \bar{p}_0(s) u(s) \left(\int_a^b G(\mu_0(t_0, t_1, s), \xi) \chi_{t_0, t_1}(u)(\xi) d\xi \right) ds \leq \\ & \leq \alpha_0(\bar{l}_{10}, b - b_0, \gamma_{10}) \rho_1^{1/2}(t_0) \rho_1^{1/2}(t) + \bar{l}_{10} \beta_0(b - b_0, \gamma_{10}) \rho_1^{1/2}(t_0) \rho_1^{1/2}(b_0), \end{aligned} \quad (2.31)$$

for $b_0 \leq t < t_1$, where $\tau_* = \inf\{\mu_j(t_0, t_1, t) : b_0 \leq t \leq t_1, j = 1, \dots, m\} \geq t_0$.

2.2 Lemma on a property of functions from $\widetilde{C}^{2m,m-1}([a, b])$

Lemma 2.5. *Let*

$$w(t) = \sum_{i=1}^m \sum_{k=i}^m c_{ik}(t) u^{(2m-k)}(t) u^{(i-1)}(t),$$

where $u \in \widetilde{C}^{2m-1,m}([a, b])$, and each $c_{ik} : [a, b] \rightarrow R$ is an $2m - k - i + 1$ times continuously differentiable function. Moreover, if

$$u^{(i-1)}(a) = 0, \quad u^{(i-1)}(b) = 0, \quad \limsup_{t \rightarrow a} |c_{ii}(t)| < +\infty \quad (i = 1, \dots, m),$$

then

$$\liminf_{t \rightarrow a} |w(t)| = 0, \quad \liminf_{t \rightarrow b} |w(t)| = 0.$$

The proof of this Lemma is given in [9].

2.3 Lemmas on the sequences of solutions of auxiliary problems

Remark 2.1. It is easy to verify that the function \tilde{u} is a solution of problem

$$\tilde{u}^{(2m)}(t) = \sum_{j=1}^m p_j(t) \tilde{u}^{(j-1)}(\tau_j(t)) + p_0(t) \int_a^b G(\tau_0(t), s) \tilde{u}(s) ds + q(t) \quad \text{for } a < t < b, \quad (2.32)$$

$$\tilde{u}^{(i-1)}(a) = 0, \quad \tilde{u}^{(i-1)}(b) = 0 \quad (i = 1, \dots, m), \quad (2.33)$$

if and only if the function $u(t) = \int_a^b G(t, s) \tilde{u}(s) ds$ is a solution of the problem (1.1), (1.2), and analogously \tilde{v} is a solution of problem

$$\tilde{v}^{(2m)}(t) = \sum_{j=1}^m p_j(t) \tilde{v}^{(j-1)}(\tau_j(t)) + p_0(t) \int_a^b G(\tau_0(t), s) \tilde{v}(s) ds \quad \text{for } a < t < b, \quad (2.32_0)$$

$$\tilde{v}^{(i-1)}(a) = 0, \quad \tilde{v}^{(i-1)}(b) = 0 \quad (i = 1, \dots, m). \quad (2.33_0)$$

if and only if the function $v(t) = \int_a^b G(t, s) \tilde{v}(s) ds$ is a solution of the problem (1.1₀), (1.2).

Now for every natural k we consider the auxiliary equation

$$\begin{aligned} \tilde{u}^{(2m)}(t) = \sum_{j=1}^m p_j(t) \tilde{u}^{(j-1)}(\mu_j(t_{0k}, t_{1k}, t)) + \\ + p_0(t) \int_a^b G(\mu_0(t_{0k}, t_{1k}, t), s) \chi_{t_{0k}t_{1k}}(\tilde{u})(s) ds + q_k(t) \end{aligned} \quad (2.34)$$

for $t_{0k} \leq t \leq t_{1k}$, with the corresponding homogenous equation

$$\tilde{u}^{(2m)}(t) = \sum_{j=1}^m p_j(t) \tilde{u}^{(j-1)}(\mu_j(t_{0k}, t_{1k}, t)) + p_0(t) \int_a^b G(\mu_0(t_{0k}, t_{1k}, t), s) \chi_{t_{0k}t_{1k}}(\tilde{u})(s) ds \quad (2.34_0)$$

for $t_{0k} \leq t \leq t_{1k}$, under the boundary conditions

$$\tilde{u}^{(i-1)}(t_{0k}) = 0, \quad \tilde{u}^{(j-1)}(t_{1k}) = 0 \quad (i = 1, \dots, m), \quad (2.35)$$

where

$$a < t_{0k} < t_{1k} < b \quad (k \in N), \quad \lim_{k \rightarrow +\infty} t_{0k} = a, \quad \lim_{k \rightarrow +\infty} t_{1k} = b. \quad (2.36)$$

Throughout this section, when problems (2.32), (2.33) and (2.34), (2.35) are discussed we assume that

$$p_j \in L_{loc}(]a, b[) \quad (j = 0, \dots, m), \quad q, q_k \in \tilde{L}_{2m-2, 2m-2}^2(]a, b[), \quad (2.37)$$

and for an arbitrary $m - 1$ -times continuously differentiable function $x :]a, b[\rightarrow R$, we set

$$\begin{aligned} \Lambda_k(x)(t) = \sum_{j=1}^m p_j(t) x^{(j-1)}(\mu_j(t_{0k}, t_{1k}, t)) \\ + p_0(t) \int_a^b G(\mu_0(t_{0k}, t_{1k}, t), s) \chi_{t_{0k}t_{1k}}(x)(s) ds, \end{aligned} \quad (2.38)$$

$$\Lambda(x)(t) = \sum_{j=1}^m p_j(t) x^{(j-1)}(\tau_j(t)) + p_0(t) \int_a^b G(\tau_0(t), s) x(s) ds.$$

Remark 2.2. From the definition of the functions μ_j ($j = 0, \dots, m$), the estimate

$$|\mu_j(t_{0k}, t_{1k}, t) - \tau_j(t)| \leq \begin{cases} 0 & \text{for } \tau_j(t) \in]t_{0k}, t_{1k}[\\ \max\{b - t_{1k}, t_{0k} - a\} & \text{for } \tau_j(t) \notin]t_{0k}, t_{1k}[\end{cases}$$

follows and thus, if conditions (2.36) hold, then

$$\lim_{k \rightarrow +\infty} \mu_j(t_{0k}, t_{1k}, t) = \tau_j(t) \quad (j = 0, \dots, m) \quad \text{uniformly in }]a, b[. \quad (2.39)$$

Let now the sequence of the $m - 1$ times continuously differentiable functions $x_k :]t_{0k}, t_{1k}[\rightarrow R$, and functions $x^{(j-1)} \in C([a, b])$ ($j = 1, \dots, m$) be such that

$$\lim_{k \rightarrow +\infty} x_k^{(j-1)}(t) = x^{(j-1)}(t) \quad (j = 1, \dots, m) \quad \text{uniformly in }]a, b[. \quad (2.40)$$

Remark 2.3. Let the functions $x_k :]t_{0k}, t_{1k}[\rightarrow R$, and $x \in C([a, b])$ be such that (2.40) with $j = 1$ holds. Then from the definition of the operators $\chi_{t_{0k}t_{1k}}$ and (2.40) it is clear that

$$\lim_{k \rightarrow +\infty} \chi_{t_{0k}t_{1k}}(x_k)(t) = \chi_{t_{0k}t_{1k}}(x)(t), \quad \lim_{k \rightarrow +\infty} \chi_{t_{0k}t_{1k}}(x)(t) = x(t) \quad (2.41)$$

uniformly in $]a, b[$.

Lemma 2.6. *Let conditions (2.36) hold and the sequence of the $m - 1$ -times continuously differentiable functions $x_k :]t_{0k}, t_{1k}[\rightarrow R$, and functions $x^{(j-1)} \in C([a, b])$ ($j = 1, \dots, m$) be such that (2.40) holds. Then for any nonnegative function $w \in C([a, b])$ and $t^* \in]a, b[$,*

$$\lim_{k \rightarrow +\infty} \int_{t^*}^t w(s) \Lambda_k(x_k)(s) ds = \int_{t^*}^t w(s) \Lambda(x)(s) ds \quad (2.42)$$

uniformly in $]a, b[$, where Λ_k and Λ are defined by equalities (2.38).

Proof. We have to prove that for any $\delta \in]0, \min\{b - t^*, t^* - a\}[$, and $\varepsilon > 0$, there exists a constant $n_0 \in N$ such that

$$\left| \int_{t^*}^t w(s) (\Lambda_k(x_k)(s) - \Lambda(x)(s)) ds \right| \leq \varepsilon \quad \text{for } t \in [a + \delta, b - \delta], \quad k > n_0. \quad (2.43)$$

Let now $w(t_*) = \max_{a \leq t \leq b} w(t)$ and $\varepsilon_1 = \varepsilon \left(2w(t_*) \sum_{j=0}^m \int_{a+\delta}^{b-\delta} |p_j(s)| ds \right)^{-1}$. Then from the inclusions $x_k^{(j-1)} \in C([a + \delta, b - \delta])$, $x^{(j-1)} \in C([a, b])$ ($j = 1, \dots, m$), conditions (2.39) and

(2.40), it follows the existence of such constant $n_{01} \in N$ that

$$\begin{aligned} |x_k^{(j-1)}(\mu_j(t_{0k}, t_{1k}, s)) - x^{(j-1)}(\mu_j(t_{0k}, t_{1k}, s))| &\leq \varepsilon_1, \\ |x^{(j-1)}(\mu_j(t_{0k}, t_{1k}, s)) - x^{(j-1)}(\tau_j(s))| &\leq \varepsilon_1 \end{aligned} \quad (2.44)$$

for $t \in [a+\delta, b-\delta]$, $k > n_{01}$, $j = 1, \dots, m$. Furthermore, (2.39)-(2.41) imply the existence of such constant $n_{02} \in N$ that

$$\begin{aligned} \left| \int_a^b G(\mu_0(t_{0k}, t_{1k}, t), s) \chi_{t_{0k}t_{1k}}(x_k)(s) ds - \int_a^b G(\mu_0(t_{0k}, t_{1k}, t), s) \chi_{t_{0k}t_{1k}}(x)(s) ds \right| &\leq \\ &\leq \alpha \int_a^b |\chi_{t_{0k}t_{1k}}(x_k)(s) - \chi_{t_{0k}t_{1k}}(x)(s)| ds \leq \varepsilon_1, \end{aligned} \quad (2.45)$$

if $k > n_{02}$, and

$$\begin{aligned} \left| \int_a^b G(\mu_0(t_{0k}, t_{1k}, t), s) \chi_{t_{0k}t_{1k}}(x)(s) ds - \int_a^b G(\tau_0(t), s) x(s) ds \right| &= \\ = \left| \int_a^{\mu_0(t_{0k}, t_{1k}, t)} \frac{\varphi(s) - \varphi(a)}{\varphi(b) - \varphi(a)} \chi_{t_{0k}t_{1k}}(x)(s) ds - \int_a^{\tau_0(t)} \frac{\varphi(s) - \varphi(a)}{\varphi(b) - \varphi(a)} x(s) ds \right| &+ \\ + \left| \int_{\mu_0(t_{0k}, t_{1k}, t)}^b \frac{\varphi(s) - \varphi(b)}{\varphi(b) - \varphi(a)} \chi_{t_{0k}t_{1k}}(x)(s) ds - \int_{\tau_0(t)}^b \frac{\varphi(s) - \varphi(b)}{\varphi(b) - \varphi(a)} x(s) ds \right| &\leq \\ \leq \alpha \int_a^b |\chi_{t_{0k}t_{1k}}(x)(s) - x(s)| ds + 2\alpha \left| \int_{\tau_0(t)}^{\mu_0(t_{0k}, t_{1k}, t)} x(s) ds \right| &\leq \varepsilon_1, \end{aligned} \quad (2.46)$$

if $k > n_{02}$, where $\alpha = \max_{a \leq s \leq t \leq b} \left\{ \frac{|\varphi(s) - \varphi(t)|}{|\varphi(b) - \varphi(a)|} \right\}$. Thus from (2.43)-(2.46) it is clear that

$$|\Lambda_k(x_k)(s) - \Lambda(x)(s)| \leq |\Lambda_k(x_k)(s) - \Lambda_k(x)(s)| + |\Lambda_k(x)(s) - \Lambda(x)(s)| \leq 2\varepsilon_1 \sum_{j=0}^m |p_j(t)|,$$

if $k > n_0$, with $n_0 = \max\{n_{01}, n_{02}\}$, and (2.43) follows immediately from the last inequality. \square

Lemma 2.7. *Let condition (2.36) hold, and for every natural k , problem (2.34), (2.35) have a solution $\tilde{u}_k \in \tilde{C}_{loc}^{2m-1}(]a, b[)$, and there exist a constant $r_0 > 0$ such that*

$$\int_{t_{0k}}^{t_{1k}} |\tilde{u}_k^{(m)}(s)|^2 ds \leq r_0^2 \quad (k \in N) \quad (2.47)$$

holds. Moreover, let

$$\lim_{k \rightarrow +\infty} \|q_k - q\|_{\tilde{L}_{2m-2, 2m-2}^2} = 0, \quad (2.48)$$

and the homogeneous problem (2.32₀), (2.33₀) have only the trivial solution in the space $\tilde{C}^{2m-1, m}(]a, b[)$. Then the inhomogeneous problem (2.32), (2.33) has a unique solution \tilde{u} such that

$$\|\tilde{u}^{(m)}\|_{L^2} \leq r_0, \quad (2.49)$$

and

$$\lim_{k \rightarrow +\infty} \tilde{u}_k^{(j-1)}(t) = \tilde{u}^{(j-1)}(t) \quad (j = 1, \dots, 2m) \quad \text{uniformly in }]a, b[\quad (2.50)$$

(that is, uniformly on $[a + \delta, b - \delta]$ for an arbitrarily small $\delta > 0$).

Proof. Suppose that t_1, \dots, t_{2m} are the numbers such that

$$\frac{a+b}{2} = t_1 < \dots < t_{2m} < b, \quad (2.51)$$

and $g_i(t)$ are the polynomials of $(2m - 1)$ th degree satisfying the conditions

$$g_j(t_j) = 1, \quad g_j(t_i) = 0 \quad (i \neq j; \quad i, j = 1, \dots, 2m). \quad (2.52)$$

Then, for every natural k , the solution \tilde{u}_k of problem (2.34), (2.35) admits the representation

$$\begin{aligned} \tilde{u}_k(t) = & \sum_{j=1}^{2m} \left(\tilde{u}_k(t_j) - \frac{1}{(2m-1)!} \int_{t_1}^{t_j} (t_j - s)^{2m-1} (\Lambda_k(\tilde{u}_k)(s) + q_k(s)) ds \right) g_j(t) + \\ & + \frac{1}{(2m-1)!} \int_{t_1}^t (t - s)^{2m-1} (\Lambda_k(\tilde{u}_k)(s) + q_k(s)) ds. \quad (2.53) \end{aligned}$$

For an arbitrary $\delta \in]0, \frac{a+b}{2}[$, we have

$$\begin{aligned}
 \left| \int_t^{t_1} (s-t)^{2m-j} (q_k(s) - q(s)) ds \right| &= (2m-j) \left| \int_t^{t_1} (s-t)^{2m-j-1} \left(\int_s^{t_1} (q_k(\xi) - q(\xi)) d\xi \right) ds \right| \leq \\
 &\leq 2m \left(\int_t^{t_1} (s-a)^{2m-2j} ds \right)^{1/2} \left(\int_t^{t_1} (s-a)^{2m-2} \left(\int_s^{t_1} (q_k(\xi) - q(\xi)) d\xi \right)^2 ds \right)^{1/2} \leq \\
 &\leq n \left| (t_1 - a)^{2m-2j+1} - \delta^{2m-2j+1} \right|^{1/2} \|q_k - q\|_{\tilde{L}_{2m-2, 2m-2}^2} \quad \text{for } a + \delta \leq t \leq t_1, \\
 \left| \int_{t_1}^t (t-s)^{2m-j} (q_k(s) - q(s)) ds \right| &\leq 2m \left| (b - t_1)^{2m-2j+1} - \delta^{2m-2j+1} \right|^{1/2} \times \\
 &\quad \times \|q_k - q\|_{\tilde{L}_{2m-2, 2m-2}^2} \quad \text{for } t_1 \leq t \leq b - \delta \quad (j = 1, \dots, 2m-1).
 \end{aligned} \tag{2.54}$$

Hence, by condition (2.48), we find

$$\lim_{k \rightarrow +\infty} \int_t^{t_1} (s-t)^{2m-j} (q_k(s) - q(s)) ds = 0 \quad \text{uniformly in }]a, b[, \tag{2.55}$$

for $(j = 1, \dots, 2m-1)$. Analogously, one can show that if $t_0 \in]a, b[$, then

$$\lim_{k \rightarrow +\infty} \int_{t_0}^t (s-t_0)(q_k(s) - q(s)) ds = 0 \quad \text{uniformly on } I(t_0), \tag{2.56}$$

where $I(t_0) = [t_0, (a+b)/2]$ for $t_0 < (a+b)/2$ and $I(t_0) = [(a+b)/2, t_0]$ for $t_0 > (a+b)/2$.

In view of inequalities (2.47), the identities

$$\tilde{u}_k^{(j-1)}(t) = \frac{1}{(m-j)!} \int_{t_{ik}}^t (t-s)^{m-j} \tilde{u}_k^{(m)}(s) ds \tag{2.57}$$

for $i = 0, 1; j = 1, \dots, m; k \in N$, yield

$$|\tilde{u}_k^{(j-1)}(t)| \leq r_j [(t-a)(b-t)]^{m-j+1/2} \tag{2.58}$$

for $t_{0k} \leq t \leq t_{1k}$ $j = 1, \dots, m$; $k \in N$, where

$$r_j = \frac{r_0}{(m-j)!} (2m-2j+1)^{-1/2} \left(\frac{2}{b-a}\right)^{m-j+1/2}. \quad (2.59)$$

By virtue of the Arzela-Ascoli Lemma and conditions (2.47) and (2.58), the sequence $\{\tilde{u}_k\}_{k=1}^{+\infty}$ contains a subsequence $\{\tilde{u}_{k_l}\}_{l=1}^{+\infty}$ such that $\{\tilde{u}_{k_l}^{(j-1)}\}_{l=1}^{+\infty}$ ($j = 1, \dots, m$) are uniformly convergent in $]a, b[$. Suppose that

$$\lim_{l \rightarrow +\infty} \tilde{u}_{k_l}(t) = \tilde{u}(t). \quad (2.60)$$

Then, in view of (2.58), $\tilde{u}^{(j-1)} \in C([a, b])$ ($j = 1, \dots, m$), and

$$\lim_{l \rightarrow +\infty} \tilde{u}_{k_l}^{(j-1)}(t) = \tilde{u}^{(j-1)}(t) \quad (j = 1, \dots, m) \quad \text{uniformly in }]a, b[. \quad (2.61)$$

If, along with this, we take conditions (2.36) and (2.55) into account, from (2.53) by Lemma 2.6 we find

$$\begin{aligned} \tilde{u}(t) = & \sum_{j=1}^{2m} \left(\tilde{u}(t_j) - \frac{1}{(2m-1)!} \int_{t_1}^{t_j} (t_j-s)^{2m-1} (\Lambda(\tilde{u})(s) + q(s)) ds \right) g_j(t) + \\ & + \frac{1}{(2m-1)!} \int_{t_1}^t (t-s)^{2m-1} (\Lambda(\tilde{u})(s) + q(s)) ds \quad \text{for } a < t < b, \end{aligned} \quad (2.62)$$

$$|\tilde{u}^{(j-1)}(t)| \leq r_j [(t-a)(b-t)]^{m-j+1/2} \quad \text{for } a < t < b \quad (j = 1, \dots, m), \quad (2.63)$$

$\tilde{u} \in \tilde{C}_{loc}^{2m-1}(]a, b[)$, and

$$\lim_{l \rightarrow +\infty} \tilde{u}_{k_l}^{(j-1)}(t) = \tilde{u}^{(j-1)}(t) \quad (j = 1, \dots, 2m-1) \quad \text{uniformly in }]a, b[. \quad (2.64)$$

On the other hand, for any $t_0 \in]a, b[$ and natural l , we have

$$(t-t_0)\tilde{u}_{k_l}^{(2m-1)}(t) = \tilde{u}_{k_l}^{(2m-2)}(t) - \tilde{u}_{k_l}^{(2m-2)}(t_0) + \int_{t_0}^t (s-t_0)(\Lambda_k(\tilde{u}_{k_l})(s) + q_{k_l}(s)) ds. \quad (2.65)$$

Hence, due to (2.36), (2.56), (2.64), and Lemma 2.6 we get

$$\lim_{l \rightarrow +\infty} \tilde{u}_{k_l}^{(2m-1)}(t) = \tilde{u}^{(2m-1)}(t) \quad \text{uniformly in }]a, b[. \quad (2.66)$$

Now it is clear that relations (2.64), (2.66), and (2.47) result in (2.49). Consequently, $\tilde{u} \in \tilde{C}^{2m-1, m}(]a, b[)$. On the other hand, from (2.62) it is obvious that \tilde{u} is a solution of (2.32), and from (2.63) equalities (2.33) follow, that is, \tilde{u} is a solution of problem (2.32), (2.33).

To complete the proof of the Lemma, it remains to show that equality (2.50) is satisfied. First note that in the space $\tilde{C}^{2m-1, m}(]a, b[)$ problem (2.32), (2.33) does not have another solution since in that space the homogeneous problem (2.32₀), (2.33₀) has only the trivial solution. Now let assume the contrary. Then there exist $\delta \in]0, \frac{b-a}{2}[$, $\varepsilon > 0$, and an increasing sequence of natural numbers $\{k_l\}_{l=1}^{+\infty}$ such that

$$\max \left\{ \sum_{j=1}^{2m} |\tilde{u}_{k_l}^{(j-1)}(t) - \tilde{u}^{(j-1)}(t)| : a + \delta \leq t \leq b - \delta \right\} > \varepsilon \quad (l \in N). \quad (2.67)$$

By virtue of the Arzela-Ascoli Lemma and condition (2.47), the sequence $\{\tilde{u}_{k_l}^{(j-1)}\}_{l=1}^{+\infty}$ ($j = 1, \dots, m$), without loss of generality, can be assumed to be uniformly converging in $]a, b[$. Then, in view of what we have shown above, conditions (2.64) and (2.66) hold. However, this contradicts condition (2.67). The obtained contradiction proves the validity of the lemma. \square

Lemma 2.8. *Let $a_0 \in]a, b[$, $b_0 \in]a_0, b[$, the functions h_j and the operators f_j be given by equalities (1.7) and (1.8). Let, moreover, $\tau_j \in M(]a, b[)$, and the constants $l_{k,j} > 0$, $\gamma_{kj} > 0$ ($k = 0, 1$; $j = 1, \dots, m$) be such that conditions (1.9)-(1.11) are fulfilled. Then there exists positive constants δ and r_1 such that if $a_0 \in]a, a + \delta[$, $b_0 \in]b - \delta, b[$, $t_0 \in]a, a_0[$, $t_1 \in]b_0, b[$, and $q \in \tilde{L}_{2m-2, 2m-2}^2(]a, b[)$, an arbitrary solution $\tilde{u} \in C_{loc}^{2m-1}(]a, b[)$ of the problem*

$$\begin{aligned} \tilde{u}^{(2m)}(t) &= \sum_{j=1}^m p_j(t) \tilde{u}^{(j-1)}(\mu_j(t_0, t_1, t)) + \\ &+ p_0(t) \int_a^b G(\mu_j(t_0, t_1, t), s) \chi_{t_0 t_1}(\tilde{u})(s) ds + q(t) \quad \text{for } t_0 \leq t \leq t_1, \end{aligned} \quad (2.68)$$

$$\tilde{u}^{(j-1)}(t_0) = 0, \quad \tilde{u}^{(j-1)}(t_1) = 0 \quad (j = 1, \dots, m) \quad (2.69)$$

satisfies the inequality

$$\int_{t_0}^{t_1} |\tilde{u}^{(m)}(s)|^2 ds \leq r_1 \left(\left| \sum_{j=1}^m \int_{a_0}^{b_0} p_j(s) \tilde{u}(s) \tilde{u}^{(j-1)}(\mu_j(t_0, t_1, s)) ds \right| + \left| \int_{a_0}^{b_0} p_0(s) \tilde{u}(s) \int_a^b G(\mu_j(t_0, t_1, s), \xi) \chi_{t_0 t_1}(\tilde{u})(\xi) d\xi ds \right| + \|q\|_{\tilde{L}_{2m-2, 2m-2}^2}^2 \right). \quad (2.70)$$

Proof. Conditions (1.9) and (1.10) imply the existence of constants $\bar{l}_{kj} \geq 0$ ($k = 0, 1$) such that

$$\begin{aligned} (t-a)^{m-\frac{1}{2}-\gamma_{0j}} f_j(a, \tau_j)(t, s) &\leq \bar{l}_{0j} \quad \text{for } a < t \leq s \leq a_0, \\ (b-t)^{m-\frac{1}{2}-\gamma_{1j}} f_j(b, \tau_j)(t, s) &\leq \bar{l}_{1j} \quad \text{for } b_0 \leq s \leq t < b. \end{aligned}$$

Consequently, all the requirements of Lemma 2.3 with $\bar{p}_j(t) = (-1)^m p_j(t)$, $a < t_0 < a_0$, and Lemma 2.4 with $\bar{p}_j(t) = (-1)^m p_j(t)$, $b_0 < t_1 < b$, are fulfilled. Condition (1.11) also guarantees the existence of a $\nu \in]0, 1[$ such that

$$\sum_{j=1}^m \frac{(2m-j)2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!} l_{kj} < 1 - 2\nu \quad (k = 0, 1). \quad (2.71)$$

On the other hand, without loss of generality we can assume that $a_0 \in]a, a + \delta[$ and $b_0 \in]b - \delta, b[$, where δ is a constant such that

$$\sum_{j=0}^m (\bar{l}_{0j} \beta_j(\delta, \gamma_{0j}) + \bar{l}_{1j} \beta_j(\delta, \gamma_{1j})) < \nu, \quad (2.72)$$

where the functions β_j are defined by (2.6). Let now $q \in \tilde{L}_{2m-2, 2m-2}^2(]a, b[)$, u be a solution of problem (2.68), (2.69), and

$$r_1 = \frac{2^{2m}}{(\nu(2m-3)!!)^2}. \quad (2.73)$$

Multiplying both sides of (2.68) by $(-1)^m \tilde{u}(t)$ and then integrating by parts from t_0 to

t_1 , in view of conditions (2.69), we obtain

$$\begin{aligned} \int_{t_0}^{t_1} |\tilde{u}^{(m)}(s)|^2 ds &= (-1)^m \sum_{j=1}^m \int_{t_0}^{t_1} p_j(s) \tilde{u}(s) \tilde{u}^{(j-1)}(\mu_j(t_0, t_1, s)) ds + \\ &+ (-1)^m \int_{t_0}^{t_1} p_0(s) \tilde{u}(s) \int_a^b G(\mu_j(t_0, t_1, s), \xi) \chi_{t_0 t_1}(\tilde{u})(\xi) d\xi ds + \\ &+ (-1)^m \int_{t_0}^{t_1} q(s) \tilde{u}(s) ds. \end{aligned} \quad (2.74)$$

Applying Lemmas 2.3 and 2.4 with $\bar{p}_j(t) = (-1)^m p_j(t)$, and using equalities $\rho_0(t_0) = \rho_1(t_1) = 0$, by virtue of (2.71), we get

$$\begin{aligned} (-1)^m \sum_{j=1}^m \int_{t_0}^{a_0} p_j(s) \tilde{u}(s) \tilde{u}^{(j-1)}(\mu_j(t_0, t_1, s)) ds + \\ + (-1)^m \int_{t_0}^{a_0} p_0(s) \tilde{u}(s) \int_a^b G(\mu_j(t_0, t_1, s), \xi) \chi_{t_0 t_1}(\tilde{u})(\xi) d\xi ds \leq \\ \leq \sum_{j=1}^m \frac{(2m-j)2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!} l_{0j} \rho_0(a_0) + \sum_{j=0}^m \bar{l}_{0j} \beta_j(a - a_0, \gamma_{0j}) \rho_0(t_1) \leq \\ \leq (1 - 2\nu) \rho_0(a_0) + \sum_{j=0}^m \bar{l}_{0j} \beta_j(\delta, \gamma_{0j}) \int_{t_0}^{t_1} |\tilde{u}^{(m)}(s)|^2 ds, \end{aligned} \quad (2.75)$$

and

$$\begin{aligned} (-1)^m \sum_{j=1}^m \int_{b_0}^{t_1} p_j(s) \tilde{u}(s) \tilde{u}^{(j-1)}(\mu_j(t_0, t_1, s)) ds + \\ + (-1)^m \int_{b_0}^{t_1} p_0(s) \tilde{u}(s) \int_a^b G(\mu_j(t_0, t_1, s), \xi) \chi_{t_0 t_1}(\tilde{u})(\xi) d\xi ds \leq \end{aligned}$$

$$\begin{aligned} &\leq \sum_{j=1}^m \frac{(2m-j)2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!} l_{1j} \rho_1(b_0) + \sum_{j=0}^m \bar{l}_{1j} \beta_j(b_0 - b, \gamma_{1j}) \rho_1(t_0) \leq \\ &\leq (1 - 2\nu) \rho_1(b_0) + \sum_{j=0}^m \bar{l}_{1j} \beta_j(\delta, \gamma_{1j}) \int_{t_0}^{t_1} |\tilde{u}^{(m)}(s)|^2 ds. \quad (2.76) \end{aligned}$$

If along with this we take into account inequalities (2.72) and $a_0 \leq b_0$, we find

$$\begin{aligned} &(-1)^m \sum_{j=1}^m \int_{t_0}^{t_1} p_j(s) \tilde{u}(s) \tilde{u}^{(j-1)}(\mu_j(t_0, t_1, s)) ds + \\ &\quad + (-1)^m \int_{t_0}^{t_1} p_0(s) \tilde{u}(s) \int_a^b G(\mu_j(t_0, t_1, s), \xi) \chi_{t_0 t_1}(\tilde{u})(\xi) d\xi ds \leq \\ &\quad \leq \left| \sum_{j=1}^m \int_{a_0}^{b_0} p_j(s) \tilde{u}(s) \tilde{u}^{(j-1)}(\mu_j(t_0, t_1, s)) ds \right| + \\ &\quad + \left| \int_{a_0}^{b_0} p_0(s) \tilde{u}(s) \int_a^b G(\mu_j(t_0, t_1, s), \xi) \chi_{t_0 t_1}(\tilde{u})(\xi) d\xi ds \right| + \\ &\quad + (1 - 2\nu) (\rho_0(a_0) + \rho_1(b_0)) + \nu \int_{t_0}^{t_1} |\tilde{u}^{(m)}(s)|^2 ds \leq (1 - \nu) \int_{t_0}^{t_1} |\tilde{u}^{(m)}(s)|^2 ds + \\ &\quad + \left| \sum_{j=1}^m \int_{a_0}^{b_0} p_j(s) \tilde{u}(s) \tilde{u}^{(j-1)}(\mu_j(t_0, t_1, s)) ds \right| + \\ &\quad + \left| \int_{a_0}^{b_0} p_0(s) \tilde{u}(s) \int_a^b G(\mu_j(t_0, t_1, s), \xi) \chi_{t_0 t_1}(\tilde{u})(\xi) d\xi ds \right|. \quad (2.77) \end{aligned}$$

On the other hand, if we put $c = (a + b)/2$, then, again on the basis of Lemmas 2.1, 2.2, and the Young inequality, we get

$$\left| \int_{t_0}^{t_1} q(s) \tilde{u}(s) ds \right| \leq \left| \int_{t_0}^c \tilde{u}'(s) \left(\int_s^c q(\xi) d\xi \right) ds \right| + \left| \int_c^{t_1} \tilde{u}'(s) \left(\int_c^s q(\xi) d\xi \right) ds \right| \leq$$

$$\begin{aligned}
&\leq \left(\int_{t_0}^c \frac{\tilde{u}'^2(s)}{(s-a)^{2m-2}} ds \right)^{1/2} \left(\int_{t_0}^c (s-a)^{2m-2} \left(\int_s^c q(\xi) d\xi \right)^2 ds \right)^{1/2} + \\
&+ \left(\int_c^{t_1} \frac{\tilde{u}'^2(s)}{(b-s)^{2m-2}} ds \right)^{1/2} \left(\int_c^{t_1} (b-s)^{2m-2} \left(\int_c^s q(\xi) d\xi \right)^2 ds \right)^{1/2} \leq \\
&\leq \frac{2^m}{(2m-3)!!} \left(\int_{t_0}^{t_1} |\tilde{u}^{(m)}(s)|^2 ds \right)^{1/2} \|q\|_{\tilde{L}_{2m-2, 2m-2}^2} \leq \\
&\leq \frac{\nu}{2} \int_{t_0}^{t_1} |\tilde{u}^{(m)}(s)|^2 ds + \frac{2^{2m}}{\nu((2m-3)!!)^2} \|q\|_{\tilde{L}_{2m-2, 2m-2}^2}^2 \quad (2.78)
\end{aligned}$$

and without loss of generality we can assume that $\frac{2^{2m}}{\nu((2m-3)!!)^2} \geq 1$. In view of inequalities (2.77), (2.78) and notation (2.73), equality (2.74) results in estimate (2.70). \square

Lemma 2.9. *Let $\tau_j \in M(]a, b[)$, $a_0 \in]a, b[$, $b_0 \in]a_0, b[$, conditions (1.6), (1.9)- (1.11), hold, where the functions h_j , β_j and the operators f_j are given by equalities (1.7), (1.8), and l_{kj} , \bar{l}_{kj} , γ_{kj} ($k = 0, 1$; $j = 1, \dots, m$) are nonnegative numbers. Moreover, let the homogeneous problem (2.32₀), (2.33₀) have only the trivial solution in the space $\tilde{C}^{2m-1, m}(]a, b[)$. Then there exist $\delta \in]0, \frac{b-a}{2}[$ and $r > 0$ such that for any $t_0 \in]a, a + \delta[$, $t_1 \in]b + \delta, b[$, and $q \in \tilde{L}_{2m-2, 2m-2}^2(]a, b[)$ problem (2.68), (2.69) is uniquely solvable in the space $\tilde{C}^{2m-1}(]a, b[)$, and its solution admits the estimate*

$$\left(\int_{t_0}^{t_1} |\tilde{u}^{(m)}(s)|^2 ds \right)^{1/2} \leq r \|q\|_{\tilde{L}_{2m-2, 2m-2}^2}. \quad (2.79)$$

Proof. We first note that all the requirements of Lemmas 2.7 and 2.8 are fulfilled.

Let now $\delta \in]0, \min\{b - b_0, a_0 - a\}[$ be such as in Lemma 2.8 and assume that estimate (2.79) is invalid. Then, for an arbitrary natural k , there exist

$$t_{0k} \in]a, a + \delta/k[, \quad t_{1k} \in]b - \delta/k, b[, \quad (2.80)$$

and a function $q_k \in \tilde{L}_{2m-2, 2m-2}^2(]a, b[)$ such that problem (2.34), (2.35) has a solution $\tilde{u}_k \in \tilde{C}^{2m-1}(]a, b[)$ satisfying the inequality

$$\left(\int_{t_{0k}}^{t_{1k}} |\tilde{u}_k^{(m)}(s)|^2 ds \right)^{1/2} > k \|q_k\|_{\tilde{L}_{2m-2, 2m-2}^2}. \quad (2.81)$$

In the case when the homogeneous problem (2.34₀), (2.35) has a nontrivial solution, in (2.34) we put that $q_k(t) \equiv 0$ and assume that \tilde{u}_k is that nontrivial solution of problem (2.34₀), (2.35).

Let now

$$\tilde{v}_k(t) = \left(\int_{t_{0k}}^{t_{1k}} |\tilde{u}_k^{(m)}(s)|^2 ds \right)^{-1/2} \tilde{u}_k(t), \quad q_{0k}(t) = \left(\int_{t_{0k}}^{t_{1k}} |\tilde{u}_k^{(m)}(s)|^2 ds \right)^{-1/2} q_k(t). \quad (2.82)$$

Then \tilde{v}_k is a solution of the problem

$$\begin{aligned} \tilde{v}^{(2m)}(t) &= \sum_{j=1}^m p_j(t) \tilde{v}^{(j-1)}(\tau_j(t)) + \\ &+ p_0(t) \int_a^b G(\mu_0(t_{0k}, t_{1k}, t), s) \chi_{t_{0k}t_{1k}}(\tilde{v})(s) ds + q_{0k}(t) \quad \text{for } t_{0k} \leq t \leq t_{1k}, \\ \tilde{v}^{(i-1)}(t_{0k}) &= 0, \quad \tilde{v}^{(i-1)}(t_{1k}) = 0 \quad (i = 1, \dots, m). \end{aligned} \quad (2.83)$$

Moreover, in view of (2.81), it is clear that

$$\int_{t_{0k}}^{t_{1k}} |\tilde{v}_k^{(m)}(s)|^2 ds = 1, \quad \|q_{0k}\|_{\tilde{L}_{2m-2, 2m-2}^2} < \frac{1}{k} \quad (k \in N). \quad (2.84)$$

On the other hand, in view of the fact that problem (2.32₀), (2.33₀) has only the trivial solution in the space $\tilde{C}^{2m-1, m}([a, b])$, by Lemmas 2.7, 2.8, and (2.84) we have

$$\begin{aligned} \lim_{t \rightarrow +\infty} \tilde{v}_k^{(j-1)}(t) &= 0 \quad \text{uniformly in }]a, b[\quad (j = 1, \dots, n), \\ 1 &< r_0 \left(\left| \int_{a_0}^{b_0} \Lambda_k(\tilde{v}_k)(s) ds \right| + k^{-2} \right) \quad (k \in N), \end{aligned} \quad (2.85)$$

where r_0 is a positive constant independent of k . Now, if we pass to the limit in (2.85) as $k \rightarrow +\infty$, by Lemma 2.6 we obtain the contradiction $1 < 0$. Consequently, for any solution of problem (2.68), (2.69), with arbitrary $q \in \tilde{L}_{2m-2, 2m-2}^2([a, b])$, estimate (2.79) holds. Thus, under conditions (2.69), the homogeneous equation

$$\tilde{u}^{(2m)}(t) = \sum_{j=1}^m p_j(t) \tilde{u}^{(j-1)}(\mu_j(t_0, t_1, t)) + p_0(t) \int_a^b G(\mu_j(t_0, t_1, t), s) \chi_{t_0 t_1}(\tilde{u})(s) ds \quad (2.82_0)$$

has only the trivial solution. However, for arbitrarily fixed $t_0 \in]a, a + \delta[$, $t_1 \in]b - \delta, b[$, and $q \in L([t_0, t_1])$ problem (2.68), (2.69) is regular and has the Fredholm property in the space $\widetilde{C}^{2m-1}([t_0, t_1])$. Thus, problem (2.68), (2.69) is uniquely solvable. \square

Lemma 2.10. *Let $\tau \in M(]a, b[)$, $\alpha \geq 0$, $\beta \geq 0$, and let there exist $\delta \in]0, b - a[$ such that*

$$|\tau(t) - t| \leq k_1(t - a)^\beta \quad \text{for } a < t \leq a + \delta. \quad (2.86)$$

Then

$$\left| \int_t^{\tau(t)} (s - a)^\alpha ds \right| \leq \begin{cases} k_1[1 + k_1\delta^{\beta-1}]^\alpha (t - a)^{\alpha+\beta} & \text{for } \beta \geq 1 \\ k_1[\delta^{1-\beta} + k_1]^\alpha (t - a)^{\alpha+\beta} & \text{for } 0 \leq \beta < 1 \end{cases},$$

for $a < t \leq a + \delta$.

Proof. We first note that

$$\left| \int_t^{\tau(t)} (s - a)^\alpha ds \right| \leq (\max\{\tau(t), t\} - a)^\alpha |\tau(t) - t| \quad \text{for } a \leq t \leq a + \delta,$$

and $\max\{\tau(t), t\} \leq t + |\tau(t) - t|$ for $a \leq t \leq a + \delta$. Then, in view of condition (2.86), we get

$$\left| \int_t^{\tau(t)} (s - a)^\alpha ds \right| \leq k_1[(t - a) + k_1(t - a)^\beta]^\alpha (t - a)^\beta \quad \text{for } a \leq t \leq a + \delta.$$

This inequality proves the validity of the lemma. \square

Analogously, one can prove

Lemma 2.11. *Let $\tau_j \in M(]a, b[)$, $\alpha \geq 0$, $\beta \geq 0$ and let there exist $\delta \in]0, b - a[$ such that*

$$|\tau_j(t) - t| \leq k_1(b - t)^\beta \quad \text{for } b - \delta \leq t < b. \quad (2.87)$$

Then

$$\left| \int_t^{\tau(t)} (b - t)^\alpha ds \right| \leq \begin{cases} k_1[1 + k_1\delta^{\beta-1}]^\alpha (b - t)^{\alpha+\beta} & \text{for } \beta \geq 1 \\ k_1[\delta^{1-\beta} + k_1]^\alpha (b - t)^{\alpha+\beta} & \text{for } 0 \leq \beta < 1 \end{cases},$$

for $b - \delta \leq t < b$.

3 Proofs

Proof of Theorem 1.1. Suppose that problem (1.1₀), (1.2) has only the trivial solution. Then, in view of Remark 2.1, it follows that problem (2.32₀), (2.33₀) also has only the trivial solution. Let now r and δ be the numbers appearing in Lemma 2.9 and

$$t_{0k} = a + \delta/k \quad t_{1k} = b - \delta/k \quad (k \in N). \quad (3.1)$$

By Lemma 2.9, for every natural k , problem (2.34), (2.35) with $q_k = q$, has a unique solution \tilde{u}_k in the space $\tilde{C}_{loc}^{2m-1}(]a, b[)$ and

$$\left(\int_{t_{0k}}^{t_{1k}} |\tilde{u}_k^{(m)}(s)|^2 ds \right)^{1/2} \leq r \|q\|_{\tilde{L}_{2m-2, 2m-2}^2}, \quad (3.2)$$

where the constant r does not depend on q . by Lemma 2.7 with $r_0 = r \|q\|_{\tilde{L}_{2m-2, 2m-2}^2}$, it follows from (3.2) that problem (2.32), (2.33) has a unique solution $\tilde{u} \in \tilde{C}_{loc}^{2m-1}(]a, b[)$ for an arbitrary $q \in \tilde{L}_{2m-2, 2m-2}^2(]a, b[)$, where

$$\lim_{k \rightarrow +\infty} \tilde{u}_k^{(j-1)}(t) = \tilde{u}^{(j-1)}(t) \quad (j = 1, \dots, 2m) \quad \text{uniformly in }]a, b[, \quad (3.3)$$

and

$$\|\tilde{u}^{(m)}\|_{L^2} \leq r \|q\|_{\tilde{L}_{2m-2, 2m-2}^2}.$$

Thus problem (2.32), (2.33) has the Fredholm property and $\tilde{u} \in \tilde{C}^{2m-1, m}(]a, b[)$ for any $q \in \tilde{L}_{2m-2, 2m-2}^2(]a, b[)$.

Consequently, it follows from Remark 2.1 that problem (1.1), (1.2) has the Fredholm property in the space $\tilde{C}^{2m, m+1}(]a, b[)$, and its solution u , where $u(t) = \int_a^b G(t, s) \tilde{u}(s) ds$, i.e. $u'(t) = \tilde{u}(t)$, admits estimate (1.12). \square

Proof of Corollary 1.1. In view of conditions (1.15), there exists a number $\varepsilon > 0$ such that

$$\sum_{j=1}^m \frac{(2m-j)2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!} \left(\frac{\kappa_{kj}}{2m-j} + \varepsilon \right) < 1 \quad (k = 0, 1). \quad (3.4)$$

On the other hand, in view of conditions (1.16) and (1.17), we have

$$\begin{aligned}
 (t-a)^{2m-j}h_j(t,s) &\leq \frac{\kappa_{0j}}{2m-j} + \kappa_{1j} \int_a^{a_0} \frac{(\xi-a)^{2m-j}}{(b-\xi)^{2m+1-j}} d\xi + \int_a^{a_0} (\xi-a)^{2m-j} p_{0j}(\xi) d\xi \\
 &\quad \text{for } a < t \leq s \leq a_0, \\
 (b-t)^{2m-j}h_j(t,s) &\leq \frac{\kappa_{1j}}{2m-j} + \kappa_{0j} \int_{b_0}^b \frac{(b-\xi)^{2m-j}}{(\xi-a)^{2m-j+1}} d\xi + \int_{b_0}^b (b-\xi)^{2m-j} p_{0j}(\xi) d\xi \\
 &\quad \text{for } b_0 \leq s \leq t < b.
 \end{aligned} \tag{3.5}$$

Let δ be the constant defined in Lemmas 2.10 and 2.11. Relation (1.16) implies the existence of $a_0 \in]a, a + \delta[$ and $b_0 \in]b - \delta, b[$ such that

$$|p_1(t)| \leq \frac{\kappa}{[(t-a)(b-t)]^{4m}} + p_{01}(t) \quad \text{for } t \in [a, a_0] \cup [b_0, b]. \tag{3.6}$$

On the other hand, by condition (1.14), it follows from Lemmas 2.10 and 2.11 that there exists a constant k_0 such that

$$\begin{aligned}
 \left| \int_t^{\tau_j(t)} (s-a)^{2(m-j)} ds \right|^{1/2} &\leq k_0^{1/2} (s-a)^{m-j+\nu_{0j}/2} \quad \text{for } a \leq t \leq a_0, \\
 \left| \int_t^{\tau_j(t)} (b-s)^{2(m-j)} ds \right|^{1/2} &\leq k_0^{1/2} (b-s)^{m-j+\nu_{1j}/2} \quad \text{for } b_0 \leq t \leq b.
 \end{aligned} \tag{3.7}$$

Consequently, if $p_{01} \in L_{n-j, 2m-j}(]a, b[)$, then, by (1.13) and (3.7), relations (1.16) and (1.17) imply the existence of a nonnegative constant k_2 such that

$$\begin{aligned}
 (t-a)^{m-1}f_0(a, \tau_0)(t,s) &\leq \int_a^{a_0} (\xi-a)^{m-1} |p_{00}(\xi)| d\xi + \\
 &\quad + \frac{1}{m-1} + \frac{(a_0-a)^m}{(b_0-a_0)^m} \quad \text{for } a \leq t < s \leq a_0 \\
 (b-t)^{m-1}f_0(b, \tau_0)(t,s) &\leq \int_{b_0}^b (b-\xi)^{m-1} |p_{00}(\xi)| d\xi + \\
 &\quad + \frac{1}{m-1} + \frac{(b-b_0)^m}{(b_0-a_0)^m} \quad \text{for } b_0 \leq s < t \leq b
 \end{aligned} \tag{3.8}$$

$$\begin{aligned} (t-a)^{m-1}f_j(a, \tau_1)(t, s) &\leq k_2(a_0 - a)^{\varepsilon_0} \quad \text{for } a \leq t < s \leq a_0, \\ (b-t)^{m-1}f_j(b, \tau_1)(t, s) &\leq k_2(b - b_0)^{\varepsilon_0} \quad \text{for } b_0 \leq s < t \leq b, \end{aligned} \quad (3.9)$$

where $0 < \varepsilon_0 = \min\{\nu_{k_1} - 4m - 2, \nu_{k_j} - 2 : k = 0, 1; j = 1, \dots, m\}$. Now, from (3.5), (3.8) and (3.9) it is clear that we can choose $\delta_1 \leq \delta$ so that if $\max\{b - b_0, a_0 - a\} \leq \delta_1$, then

$$\begin{aligned} (t-a)^{2m-j}h_j(t, s) &\leq \frac{\kappa_{0j}}{2m-j} + \varepsilon \quad \text{for } a < t \leq s \leq a_0, \\ (b-t)^{2m-j}h_j(t, s) &\leq \frac{\kappa_{1j}}{2m-j} + \varepsilon \quad \text{for } b_0 \leq s \leq t < b, \end{aligned}$$

$j \in \{1, \dots, m\}$. From (3.8), (3.9), the last inequalities and (3.4), it is clear that all the assumptions of Theorem 1.1, with $l_{kj} = \frac{\kappa_{kj}}{2m-j} + \varepsilon$, $\gamma_{k0} = \gamma_{kj} = 1/2$, ($k = 0, 1, j = 1, \dots, m$) and $\max\{b - b_0, a_0 - a\} \leq \delta_1$, are fulfilled, and thus the corollary is valid. \square

Proof of Theorem 1.2. From Theorem 1.1 by conditions (1.18)-(1.21) it is obvious that problem (1.1), (1.2) has the Fredholm property. Thus, to prove Theorem 1.2, it will suffice to show that the homogeneous problem (1.1₀), (1.2) has only the trivial solution in the space $\tilde{C}^{2m, m+1}([a, b])$. Suppose that $u \in \tilde{C}^{2m, m+1}([a, b])$ is a nonzero solution of problem (1.1₀), (1.2) and $\tilde{u} = u'$. Then, in view of the condition $\varphi(b) - \varphi(a) \neq 0$, it is clear that $u \not\equiv \text{Const}$, and it follows from Remark 2.1 that the function \tilde{u} is a nonzero solution of problem (2.32), (2.33) such that

$$\rho = \int_a^b |\tilde{u}^{(m)}(s)|^2 ds < +\infty. \quad (3.10)$$

Multiplying both sides of (1.1₀) by $(-1)^m \tilde{u}(t)$ and integrating by parts from s to t , we obtain

$$\begin{aligned} w_{2m}(t) - w_{2m}(s) + \int_s^t |\tilde{u}^{(m)}(\xi)|^2 d\xi &= (-1)^m \sum_{j=1}^m \int_s^t p_j(\xi) \tilde{u}^{(j-1)}(\tau_j(\xi)) \tilde{u}(\xi) d\xi + \\ &+ (-1)^m \int_s^t p_0(s) \tilde{u}(s) \int_a^b G(s, \xi) \tilde{u}(\xi) d\xi ds, \end{aligned} \quad (3.11)$$

with $w_{2m}(t) = \sum_{j=1}^m (-1)^{m+j-1} \tilde{u}^{(2m-j)}(t) \tilde{u}(t)$, where, due Lemma 2.5, it is obvious that

$$\liminf_{s \rightarrow a} |w_{2m}(s)| = 0, \quad \liminf_{t \rightarrow b} |w_{2m}(t)| = 0. \quad (3.12)$$

According to (1.20), (1.21) and (3.10), all the conditions of Lemmas 2.3 and 2.4 with $\bar{p}_j(t) = (-1)^m p_j(t)$, $a_0 = b_0 = t^*$, $t_0 = a$, $t_1 = b$ and $\mu_j(t_0, t_1, t) = \tau_j(t)$ hold. Consequently, due to the equalities $\rho_0^{1/2}(\tau^*)\rho_0^{1/2}(t^*) \leq \rho$, $\rho_0^{1/2}(b)\rho_0^{1/2}(t^*) \leq \rho$, $\rho_1^{1/2}(\tau_*)\rho_1^{1/2}(t^*) \leq \rho$, $\rho_1^{1/2}(a)\rho_1^{1/2}(t^*) \leq \rho$, we have

$$\begin{aligned} (-1)^m \int_s^t p_0(s) \tilde{u}(s) \int_a^b G(s, \xi) \tilde{u}(\xi) d\xi ds &\leq \\ &\leq \bar{l}_{00} \beta_0(t^* - a, \gamma_{00}) \rho + \bar{l}_{10} \beta_0(b - t^*, \gamma_{10}) \rho + \\ &+ \alpha_0(\bar{l}_{00}, a_0 - a, \gamma_{00}) \rho_0^{1/2}(b) \rho_0^{1/2}(s) + \alpha_0(\bar{l}_{10}, b - b_0, \gamma_{10}) \rho_0^{1/2}(a) \rho_1^{1/2}(t) \end{aligned} \quad (3.13)$$

for $a < s < t^* < t < b$ and

$$\begin{aligned} (-1)^m \int_s^t p_j(\xi) \tilde{u}^{(j-1)}(\tau_j(\xi)) \tilde{u}(\xi) d\xi &\leq \\ &\leq \bar{l}_{0j} \beta_j(t^* - a, \gamma_{0j}) \rho + l_{0j} \frac{(2m - j) 2^{2m-j+1}}{(2m - 1)!! (2m - 2j + 1)!!} \rho_0(t^*) + \\ &+ \bar{l}_{1j} \beta_j(b - t^*, \gamma_{1j}) \rho + l_{1j} \frac{(2m - j) 2^{2m-j+1}}{(2m - 1)!! (2m - 2j + 1)!!} \rho_1(t^*) + \\ &+ \alpha_j(l_{0j}, \bar{l}_{0j}, a_0 - a, \gamma_{0j}) \rho_0^{1/2}(\tau^*) \rho_0^{1/2}(s) + \alpha_j(l_{1j}, \bar{l}_{1j}, b - b_0, \gamma_{1j}) \rho_1^{1/2}(\tau_*) \rho_1^{1/2}(t) \end{aligned} \quad (3.14)$$

for $a < s < t^* < t < b$. On the other hand, due to conditions (1.18) and (1.19), the number $\nu \in]0, 1[$ can be chosen such that inequalities

$$\begin{aligned} B_0 &\equiv \bar{l}_{00} \beta_0(t^* - a, \gamma_{00}) + \\ &+ \sum_{j=1}^m \left(l_{0j} \frac{(2m - j) 2^{2m-j+1}}{(2m - 1)!! (2m - 2j + 1)!!} + \bar{l}_{0j} \beta_j(t^* - a, \gamma_{0j}) \right) < \frac{1 - \nu}{2}, \\ B_1 &\equiv \bar{l}_{10} \beta_0(b - t^*, \gamma_{10}) + \\ &+ \sum_{j=1}^m \left(l_{1j} \frac{(2m - j) 2^{2m-j+1}}{(2m - 1)!! (2m - 2j + 1)!!} + \bar{l}_{1j} \beta_j(b - t^*, \gamma_{1j}) \right) < \frac{1 - \nu}{2}, \end{aligned} \quad (3.15)$$

are satisfied. Thus if we pass to limit with $s \rightarrow s$, $t \rightarrow b$, in (3.11), according to (3.12)-(3.15), and the fact that $\rho_0(a) = \rho_1(b) = 0$, we get the inequality $\rho \leq (1 - \nu)\rho$, and

consequently, $\rho = 0$. Hence, by

$$|\tilde{u}(t)| = \frac{1}{(k-1)!} \left| \int_a^t (t-s)^{m-1} \tilde{u}^{(m)}(s) ds \right| \leq (t-a)^{m-1/2} \rho \quad \text{for } a < t < b,$$

we have the contradiction with the fact that $\tilde{u}(t) \equiv 0$. Therefore, our assumption is wrong and, thus, problem (1.1), (1.2) has only the trivial solution in the space $\tilde{C}^{2m, m+1}([a, b])$. \square

Proof of Remark 1.1. Let u be a solution of problem (1.1), (1.2). Then, by Remark 2.1, the function \tilde{u} , where $u(t) = \int_a^b G(t, s) \tilde{u}(s) ds$, is a solution of problem (2.32), (2.33) and, in view of Theorem 1.1, the inclusion $u \in \tilde{C}^{2m, m+1}([a, b])$ holds, i.e.

$$\rho \equiv \int_a^b |u^{(m+1)}(s)|^2 ds \rho = \int_a^b |\tilde{u}^{(m)}(s)|^2 ds < +\infty. \quad (3.16)$$

Furthermore, if t_{0k}, t_{1k} are defined by equalities (3.1), it is clear from the proof of Theorem 1.1 that for any $k \in N$ problem (2.34), (2.35) has a unique solution $\tilde{u}_k \in \tilde{C}^{2m, m-1}([a, b])$ such that (3.2) and (3.3) hold.

Multiplying equation (2.34) by $(-1)^m \tilde{u}_k$ and then integrating by parts from t_{0k} to t_{1k} , we obtain

$$\begin{aligned} w_{2m, k}(t) - w_{2m, k}(s) + \int_s^t |\tilde{u}_k^{(m)}(\xi)|^2 d\xi &= (-1)^m \int_s^t q(s) \tilde{u}_k(s) ds + \\ &+ (-1)^m \sum_{j=1}^m \int_s^t p_j(\xi) \tilde{u}_k^{(j-1)}(\tau_j(\xi)) \tilde{u}_k(\xi) d\xi + \\ &+ (-1)^m \int_s^t p_0(s) \tilde{u}_k(s) \int_a^b G(s, \xi) \chi_{t_{0k} t_{1k}}(\tilde{u}_k)(\xi) d\xi ds, \end{aligned} \quad (3.17)$$

for $a < s \leq t < b$, with $w_{2m, k}(t) = \sum_{j=1}^m (-1)^{m+j-1} \tilde{u}_k^{(2m-j)}(t) \tilde{u}_k(t)$, where, due to (3.3), we have

$$\liminf_{k \rightarrow +\infty} |w_{2m, k}(t)| = |w_{2m}(t)|, \quad \liminf_{k \rightarrow +\infty} |w_{2m, k}(t)| = |w_{2m}(t)|, \quad (3.18)$$

and, therefore, it is obvious from Lemma 2.5 that equalities (3.12) hold. Furthermore, due to conditions (1.18) and (1.19), the number $\nu \in]0, 1[$ can be chosen so that inequalities (3.15) hold, and then

$$0 < \nu < 1 - 2 \max\{B_0, B_1\}. \quad (3.19)$$

It is obvious that the maximum of ν depend only on the numbers $l_{kj}, \bar{l}_{k0}, \bar{l}_{kj}, \gamma_{k0}, \gamma_{kj}$ ($k = 0, 1; j = 1, \dots, m$), and a, b, t^* . If we now put $c = (a + b)/2$, then, by using Lemmas 2.1, 2.2, conditions (2.35), and the Young inequality, we get

$$\begin{aligned} & \left| \int_{t_{0k}}^{t_{1k}} q(\psi) \tilde{u}_k(\psi) d\psi \right| \leq \left| \int_{t_{0k}}^c q(\psi) \tilde{u}_k(\psi) d\psi \right| + \left| \int_c^{t_{1k}} q(\psi) \tilde{u}_k(\psi) d\psi \right| = \\ & = \left| \int_{t_{0k}}^c \tilde{u}'_k(\psi) \left(\int_{\psi}^c q(\xi) d\xi \right) d\psi \right| + \left| \int_c^{t_{1k}} \tilde{u}'_k(\psi) \left(\int_c^{\psi} q(\xi) d\xi \right) d\psi \right| \leq \\ & \leq \left(\int_{t_{0k}}^c \frac{\tilde{u}'_k{}^2(\psi)}{(\psi - a)^{2m-2}} d\psi \right)^{1/2} \times \left(\int_{t_{0k}}^c (\psi - a)^{2m-2} \left(\int_{\psi}^c q(\xi) d\xi \right)^2 d\psi \right)^{1/2} + \\ & + \left(\int_c^{t_{1k}} \frac{\tilde{u}'_k{}^2(\psi)}{(b - \psi)^{2m-2}} d\psi \right)^{1/2} \times \left(\int_c^{t_{1k}} (b - \psi)^{2m-2} \left(\int_c^{\psi} q(\xi) d\xi \right)^2 d\psi \right)^{1/2} \leq \\ & \leq \frac{2^m}{(2m - 3)!!} \|q\|_{\tilde{L}_{2m-2, 2m-2}^2} \left(\int_a^b |\tilde{u}_k^{(m)}(s)|^2 ds \right)^{1/2} \leq \\ & \leq \frac{\nu}{2} \int_a^b |\tilde{u}_k^{(m)}(s)|^2 ds + \frac{1}{2\nu} \left(\frac{2^m}{(2m - 1)!!} \right)^2 \|q\|_{\tilde{L}_{2m-2, 2m-2}^2}^2. \end{aligned} \quad (3.20)$$

Using Lemmas 2.3 and 2.4 and conditions (1.20), (1.21), we get the inequalities (3.13) and (3.14) with $s = t_{0k}$, $t = t_{1k}$.

Now if we pass to the limit as $k \rightarrow +\infty$ in (3.17), according to (3.3), (3.12), (3.13), (3.14), (3.18), (3.20), and equalities $\rho_0(a) = \rho_1(a) = 0$ we get

$$\rho \leq (1 - \nu)\rho + \frac{\nu}{2}\rho + \frac{1}{2\nu} \left(\frac{2^m}{(2m - 1)!!} \right)^2 \|q\|_{\tilde{L}_{2m-2, 2m-2}^2}^2. \quad (3.21)$$

From (3.19) and (3.21) immediately follows that

$$\|u^{(m)}\|_{L^2} \leq r \|q\|_{\tilde{L}_{2m-2, 2m-2}^2}, \quad (3.22)$$

with

$$r = \frac{2^m}{(1 - 2 \max\{B_0, B_1\})(2m - 1)!!},$$

where it is clear from definition of the numbers B_0, B_1 that r depend only on the numbers $l_{kj}, \bar{l}_{k0}, \bar{l}_{kj}, \gamma_{k0}, \gamma_{kj}$ ($k = 0, 1; j = 0, \dots, m$), and a, b, t^* . By virtue of (3.16), the last inequality implies estimate (1.22). \square

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