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The Nehari manifold approach for $p(x)$-Laplacian problem with Neumann boundary condition

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Abstract

In this paper, we consider the system

$$\begin{cases} -\Delta_{p(x)}u + |u|^{p(x)-2}u = \lambda a(x)|u|^{r_1(x)-2}u + \frac{\alpha(x)}{\alpha(x)+\beta(x)}c(x)|u|^{\alpha(x)-2}u|v|^{\beta(x)} & in\ \Omega \\ -\Delta_{q(x)}v + |v|^{q(x)-2}v = \mu b(x)|v|^{r_2(x)-2}v + \frac{\beta(x)}{\alpha(x)+\beta(x)}c(x)|v|^{\beta(x)-2}v|u|^{\alpha(x)} & in\ \Omega \\ \frac{\partial u}{\partial \gamma} = \frac{\partial v}{\partial \gamma} = 0 & on\ \partial\Omega \end{cases}$$

where $\Omega \subset R^N$ is a bounded domain with smooth boundary and $\lambda, \mu > 0,\ \gamma$ is the outer unit normal to $\partial\Omega$. Under suitable assumptions, we prove the existence of positive solutions by using the Nehari manifold and some variational techniques.

Keywords: Nonstandard growth condition; $p(x)$-Laplacian problems; Nehari manifold; variable exponent Sobolev space.
AMS Subject Classification: 35J60, 35B30, 35B40

1. Introduction

In this paper, we prove the existence of positive solutions for the following system

$$\begin{cases} -\Delta_{p(x)}u + |u|^{p(x)-2}u = \lambda a(x)|u|^{r_1(x)-2}u + \frac{\alpha(x)}{\alpha(x)+\beta(x)}c(x)|u|^{\alpha(x)-2}u|v|^{\beta(x)} & in\ \Omega \\ -\Delta_{q(x)}v + |v|^{q(x)-2}v = \mu b(x)|v|^{r_2(x)-2}v + \frac{\beta(x)}{\alpha(x)+\beta(x)}c(x)|v|^{\beta(x)-2}v|u|^{\alpha(x)} & in\ \Omega \\ \frac{\partial u}{\partial \gamma} = \frac{\partial v}{\partial \gamma} = 0 & on\ \partial\Omega \end{cases} \quad (1)$$

where $\Omega \subset R^N$ is a bounded domain, $-\Delta_{p(x)}u = -\text{div}(|\nabla u|^{p(x)-2}\nabla u)$ is called $p(x)$-Laplacian, $\lambda, \mu > 0,\ \gamma$ is the outer unit normal to $\partial\Omega$, the functions $p, q, r_1, r_2, a, b, c, \alpha, \beta \in C(\bar{\Omega})$.

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In this paper, for any $\upsilon : \Omega \subset R^N \to R$, we denote

$$\upsilon^+ = \text{ess} \sup_{x\in\Omega} \upsilon(x), \qquad \upsilon^- = \text{ess} \inf_{x\in\Omega} \upsilon(x).$$

Through the paper, we always assume that

(H_0) $\alpha(x), \beta(x) > 1,\ 2 < \alpha(x) + \beta(x) < p(x) < r_1(x) < p^*(x)(p^*(x) = \frac{Np(x)}{N-p(x)}$ if $N > p(x), p^*(x) = \infty$ if $N \leq p(x))$ and

$$2 < \alpha^- + \beta^- \leq \alpha^+ + \beta^+ < p^- \leq p^+ < r_1^- \leq r_2^+$$

(H_1) $2 < \alpha(x) + \beta(x) < q(x) < r_2(x) < q^*(x)(q^*(x) = \frac{Nq(x)}{N-q(x)}$ if $N > q(x), q^*(x) = \infty$ if $N \leq q(x))$ and

$$2 < \alpha^- + \beta^- \leq \alpha^+ + \beta^+ < q^- \leq q^+ < r_2^- \leq r_2^+.$$

(H_2) $\min\{r_1^-, r_2^-\} > \max\{p^+, q^+\}$.

(H_3) $a(x), b(x), c(x) \geq 0, a(x) \in L^{k_1(x)}(\Omega), b(x) \in L^{k_2(x)}(\Omega), c(x) \in L^{k_3(x)}(\Omega), k_i \in C(\bar{\Omega})$ $(i = 1, 2, 3)$ where

$$\frac{1}{k_1(x)} + \frac{1}{p^*(x)/r_1(x)} = 1, \qquad \frac{1}{k_2(x)} + \frac{1}{q^*(x)/r_2(x)} = 1, \quad \frac{1}{k_3(x)} + \frac{1}{p^*(x)/\alpha(x)} + \frac{1}{q^*(x)/\beta(x)} = 1.$$

The study of differential equations and variational problems with nonstandard $p(x)$-growth conditions has been a new and interesting topic. Such problems arise from the study of electrorheological fluids, image processing, and the theory of nonlinear elasticity (see [1, 2, 12-15, 18, 19, 21]). When $p(x) \equiv p$ (a constant), $p(x)$-Laplacian is the usual p-Laplacian. There have been a large number of papers on the existence of solutions for p-Laplace equations.(see [3, 7]) However, the $p(x)$-Laplace operator possesses more complicated nonlinearity than p-Laplace operator, due to the fact that $-\Delta_{p(x)}$ is not homogeneous. This fact implies some difficulties; for example, we can not use the Lagrange Multiplier Theorem in many problems involving this operator.

In recent years, several authors use the Nehari manifold and fibering maps to solve semilinear and quasilinear problems (see [3-7, 14, 19]). Wu in [18] for the case $p = 2, r(x) = r, \alpha(x) = \alpha, \beta(x) = \beta$ and $1 < r < 2 < \alpha + \beta < 2^*$, proved that, there exists $C_0 > 0$ such that if the parameter λ, μ satisfy $0 < |\lambda|^{\frac{2}{2-q}} + |\mu|^{\frac{2}{2-q}} < C_0$, then problem (1) has at least two solutions (u_0^+, v_0^+) and (u_0^-, v_0^-) such that $u_0^\pm \geq 0, v_0^\pm \geq 0$ in Ω and $u_0^\pm \neq 0, v_0^\pm \neq 0$. By the fibering method, Drabek and Pohozaev [7], Bozhkov and Mitidieri [5] studied respectively the existence of multiple solutions to the following p-Laplacian single equation:

$$\begin{cases} -\Delta u(x) = \lambda a(x)|u(x)|^{p-2}u(x) + c(x)|u|^{\alpha-1}u(x) & x \in \ \Omega \\ u(x) = 0 & x \in \ \partial\Omega \end{cases}$$

and system

$$\begin{cases} -\Delta_p u = \lambda a(x)|u|^{p-2}u + (\alpha+1)c(x)|u|^{\alpha-1}u|v|^{\beta+1} & x \in \ \Omega \\ -\Delta_q v = \mu b(x)|v|^{q-2}v + (\beta+1)c(x)|v|^{\beta-1}v|u|^{\alpha+1} & x \ \in \ \Omega \\ u = v = 0 & x \in \ \partial\Omega \end{cases}$$

In [6] Brown and Zhang used the relationship between the Nehari manifold and fibering maps to show how existence and nonexistence results for positive solutions of the equation are linked to properties of the Nehari manifold. In [3] Afrouzi and Rasouli for the case $p(x) = p, r(x) = r, \alpha(x) = \alpha, \beta(x) = \beta$ discussed the existence and multiplicity results of nontrivial nonnegative solutions for the system. In [14] Mashiyev, Ogras, Yucedag and Avci studied the multiplicity of positive solutions for the following elliptic equation

$$\begin{cases} -\Delta_{p(x)}u = \lambda a(x)|u|^{q(x)-2}u + b(x)|u|^{h(x)-2}u & in\ \Omega \\ u(x) = 0 & on\ \partial\Omega \end{cases}$$

where $\Omega \subset R^N$ is a bounded domain with smooth boundary in R^N, $p, q, h \in C^1(\bar{\Omega})$ such that $1 < q(x) < p(x) < h(x) < p^*(x)(p^*(x) = \frac{Np(x)}{N-p(x)}$ if $N > p(x), p^*(x) = \infty$ if $N \leq p(x)), 1 < p^- := \text{ess inf}_{x\in\Omega}\, p(x) \leq \text{ess sup}_{x\in\Omega}\, p(x) < \infty, 1 < q^- \leq q^+ < p^- \leq p^+ < h^- \leq h^+, \lambda > 0 \in R$ and $a, b \in C(\bar{\Omega})$ are non-negative weight functions with compact supports in Ω.

In this paper, we have generalized the articles of Afrouzi-Rasouli [3] and Mashiyev, Ogras, Yucedag and Avci [14], to the $p(x)$-Laplacian by using the Nehari manifold under the similar conditions. We shall discuss the multiplicity of positive solutions for the problem (1) and prove the existence of at least two positive solutions.

This paper is divided into three parts. In the second part we introduce some basic properties of the variable exponent Sobolev spaces $W^{1,p(x)}(\Omega)$, where $\Omega \subset R^N$ is an open domain, section 3 gives main results and proofs.

2. Preliminary knowledge

In order to deal with $p(x)$-Laplacian problem, we need some theories on spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and properties of $p(x)$-Laplacian which we will use later (see [6]). If $\Omega \subset R^N$ is an open bounded domain, write

$$L_+^\infty(\Omega) = \{p \in L^\infty(\Omega) : \text{ess} \inf_{x\in\Omega} p(x) \geq 1\},$$

$$S(\Omega) = \{u \mid u \text{ is a measurable real-valued function on } \Omega\}$$

For any $p \in L_+^\infty(\Omega)$, we denote the variable exponent Lebesgue space by

$$L^{p(x)}(\Omega) = \{u \in S(\Omega) \mid \textstyle\int_\Omega |u|^{p(x)}dx < \infty\}.$$

We can introduce the norm on $L^{p(x)}(\Omega)$ by

$$|u|_{p(x)} = \inf\{\lambda > 0 \mid \textstyle\int_\Omega |\frac{u(x)}{\lambda}|^{p(x)}dx \leq 1\},$$

and $(L^{p(x)}(\Omega)\ ,\ |.|_{p(x)})$ becomes a Banach space, we call it variable exponent Lebesgue space.

Proposition 2.1 (see [10]). The space $(L^{p(x)}(\Omega)\ ,\ |.|_{p(x)})$ is a separable, reflexive and uniformly convex Banach space, and its conjugate space is $L^{p'(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have

$| \int_\Omega uv dx| \le (\frac{1}{p^-} + \frac{1}{p'^-})|u|_{p(x)}|v|_{p'(x)}$.

Proposition 2.2 (see [9])**.** If $\frac{1}{p(x)} + \frac{1}{p'(x)} + \frac{1}{p''(x)} = 1$, then for any $u \in L^{p(x)}(\Omega), v \in L^{p'(x)}(\Omega)$ and $w \in L^{p''(x)}(\Omega)$,

$$| \int_\Omega uvw dx| \le (\frac{1}{p^-} + \frac{1}{p'^-} + \frac{1}{p''^-})|u|_{p(x)}|v|_{p'(x)}|w|_{p''(x)} \le 3|u|_{p(x)}|v|_{p'(x)}|w|_{p''(x)}.$$

Proposition 2.3 (see [10])**.** Set

$$\rho(u) = \int_\Omega |u|^{p(x)} dx, \qquad \forall u \in L^{p(x)}(\Omega),$$

then

(i) $|u|_{p(x)} < 1 (= 1; > 1) \Leftrightarrow \rho(u) < 1(= 1; > 1)$;

(ii) $|u|_{p(x)} > 1 \Rightarrow |u|^{p^-}_{p(x)} \le \rho(u) \le |u|^{p^+}_{p(x)}$; $|u|_{p(x)} < 1 \Rightarrow |u|^{p^-}_{p(x)} \ge \rho(u) \ge |u|^{p^+}_{p(x)}$;

(iii) $|u|_{p(x)} \to 0 \Leftrightarrow \rho(u) \to 0$; $|u|_{p(x)} \to \infty \Leftrightarrow \rho(u) \to \infty$.

Proposition 2.4 (see [10])**.** If $u, u_n \in L^{p(x)}(\Omega), n = 1, 2, ...$, then the following statements are equivalent to each other:

(1) $\lim_{n\to\infty} |u_n - u|_{p(x)} = 0$;

(2) $\lim_{n\to\infty} \rho(u_n - u) = 0$;

(3) $u_n \to u$ in measure in Ω and $\lim_{n\to\infty} \rho(u_n) = \rho(u)$.

The space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) \mid |\nabla u| \in L^{p(x)}(\Omega)\},$$

and it can be equipped with the norm

$$\|u\|_{p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}, \quad \forall u \in W^{1,p(x)}(\Omega).$$

We denote by $W^{1,p(x)}_0(\Omega)$ the closure of $C^\infty_0(\Omega)$ in $W^{1,p(x)}(\Omega)$; then the Poincaré inequality

$$|u|_{p(x)} \le c|\nabla u|_{p(x)}$$

holds true. In this paper we will use the equivalent norm on $W^{1,p(x)}(\Omega)$:

$$\|u\|_{p(x)} = \inf\{\lambda > 0 : \int_\Omega \frac{|\nabla(u)|^{p(x)} + |u|^{p(x)}}{\lambda^{p(x)}} dx \le 1\}.$$

Proposition 2.5 (see [10])**.** The space $W^{1,p(x)}(\Omega)$ and $W^{1,p(x)}_0(\Omega)$ are separable and reflexive Banach spaces.

Proposition 2.6 (see [9])**.** If we define $I(u) = \int_\Omega (|\nabla u(x)|^{p(x)} + |u(x)|^{p(x)})dx$, then for $u, u_k \in W^{1,p(x)}(\Omega)$:

(1) $\|u\|_{p(x)} < 1(= 1; > 1) \Leftrightarrow I(u) < 1(= 1; > 1)$;

(2) If $\|u\|_{p(x)} > 1$, then $\|u\|_{p(x)}^{p^-} \leq I(u) \leq \|u\|_{p(x)}^{p^+}$;

(3) $\|u\|_{p(x)} < 1$, then $\|u\|_{p(x)}^{p^+} \leq I(u) \leq \|u\|_{p(x)}^{p^-}$;

(4) $\|u_k\|_{p(x)} \to 0(\to \infty) \Leftrightarrow I(u_k) \to 0(\to \infty)$.

Proposition 2.7 (see [8]). If $p : \Omega \to R$ is Lipschitz continuous, and $p^+ < N$, then for $q(x) \in L_+^\infty(\Omega)$ with $p(x) \leq q(x) \leq p^*(x)$, there is a continuous embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$.

Proposition 2.8 (see [8]). If $s(x) \in C(\bar{\Omega})$ and $1 < s(x) < p^*(x)$ for all $x \in \bar{\Omega}$ then the embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{s(x)}(\Omega)$ is compact.

Proposition 2.9 (see [9]). If $|u|^{q(x)} \in L^{s(x)/q(x)}(\Omega)$, where $s(x), q(x) \in L_+^\infty(\Omega), q(x) \leq s(x)$, then $u \in L^{s(x)}(\Omega)$ and there is a number $\bar{q} \in [q^-, q^+]$ such that $\left||u|^{q(x)}\right|_{s(x)/q(x)} = (|u|_{s(x)})^{\bar{q}}$.

In what follows, W will denote the Cartesian product of two Sobolev spaces $W^{1,p(x)}(\Omega)$ and $W^{1,q(x)}(\Omega)$, i.e., $W = W^{1,p(x)}(\Omega) \times W^{1,q(x)}(\Omega)$. Let us choose on W the norm $\|.\|$ defined by

$$\|(u,v)\| = \max\{\|u\|_p, \|v\|_q\},$$

where $\|.\|_p$ is the norm of $W^{1,p(x)}(\Omega)$ and $\|.\|_q$ is the norm of $W^{1,q(x)}(\Omega)$.

3. Main results and proofs

Definition 3.1. We say that $(u, v) \in W$ is a weak solution of problem (1) if for all $(\xi, \eta) \in W$ we have

$$\begin{cases} \int_\Omega |\nabla u|^{p(x)-2}\nabla u \,.\, \nabla \xi dx + \int_\Omega |u|^{p(x)-2}u \; \xi dx = \\ \qquad\qquad \lambda \int_\Omega a(x)|u|^{r_1(x)-2}u \; \xi dx + \int_\Omega \frac{\alpha(x)}{\alpha(x)+\beta(x)} c(x)|u|^{\alpha(x)-2}u|v|^{\beta(x)} \; \xi dx, \\ \int_\Omega |\nabla v|^{q(x)-2}\nabla v \,.\, \nabla \eta dx + \int_\Omega |v|^{q(x)-2}v \; \eta dx = \\ \qquad\qquad \mu \int_\Omega b(x)|v|^{r_2(x)-2}v \; \eta dx + \int_\Omega \frac{\beta(x)}{\alpha(x)+\beta(x)} c(x)|v|^{\beta(x)-2}v|u|^{\alpha(x)} \; \eta dx. \end{cases}$$

It is clear that problem (1) has a variational structure. The energy functional corresponding to problem (1) is defined as $J_{\lambda,\mu} : W \to R$,

$$J_{\lambda,\mu}(u,v) = \int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})dx + \int_\Omega \frac{1}{q(x)}(|\nabla v|^{q(x)} + |v|^{q(x)})dx$$
$$-\lambda \int_\Omega \frac{1}{r_1(x)} a(x)|u|^{r_1(x)}dx - \mu \int_\Omega \frac{1}{r_2(x)} b(x)|v|^{r_2(x)}dx - \int_\Omega \frac{1}{\alpha(x)+\beta(x)} c(x)|u|^{\alpha(x)}|v|^{\beta(x)}dx.$$

Let

$$P(u,v) = \int_\Omega(|\nabla u|^{p(x)} + |u|^{p(x)})dx + \int_\Omega(|\nabla v|^{q(x)} + |v|^{q(x)})dx,$$
$$Q(u,v) = \lambda \int_\Omega a(x)|u|^{r_1(x)}dx + \mu \int_\Omega b(x)|v|^{r_2(x)}dx,$$

$R(u,v)=\int_\Omega c(x)|u|^{\alpha(x)}|v|^{\beta(x)}dx.$

It is well known that the weak solution of the problem (1) are the critical points of the energy functional $J_{\lambda,\mu}$. Let I be the energy functional associated with an elliptic problem on a Banach space X. If I is bounded below and I has a minimizer on X, then this minimizer is a critical point of I. So it is a solution of the corresponding elliptic problem. However, the energy functional $J_{\lambda,\mu}$ is not bounded below on the whole space W, but is bounded on an appropriate subset, and a minimizer on this set (if it exists) gives rise to a solution to (1). A good candidate for an appropriate subset of X is the Nehari manifold.

Then we introduce the following notation: for any functional $f : W \to R$ we denote by $f'(u,v)(h_1,h_2)$ the Gateaux derivative of f at $(u,v)\in W$ in the direction of $(h_1,h_2)\in W$, and

$$f^{(1)}(u,v)h_1=f'(u+\epsilon h_1,v)|_{\epsilon=0},\qquad f^{(2)}(u,v)h_2=f'(u,v+\delta h_2)|_{\delta=0}.$$

Consider the Nehari minimization problem for $\lambda,\mu>0$,

$$\alpha_0(\lambda,\mu)=\inf\{J_{\lambda,\mu}(u,v):(u,v)\in M_{\lambda,\mu}\},$$

where $M_{\lambda,\mu}=\{(u,v)\in W\backslash\{(0,0)\}:\langle J'_{\lambda,\mu}(u,v),(u,v)\rangle=\langle J^{(1)}_{\lambda,\mu}(u,v)u, J^{(2)}_{\lambda,\mu}(u,v)v\rangle\ =0\}$. It is clear that all critical points of $J_{\lambda,\mu}$ must lie on $M_{\lambda,\mu}$ which is known as the Nehari manifold and local minimizers on $M_{\lambda,\mu}$ are usually critical points of $J_{\lambda,\mu}$.

Thus $(u,v)\in M_{\lambda,\mu}$ if and only if

$$\begin{aligned}I_{\lambda,\mu}(u,v):=\langle J'_{\lambda,\mu}(u,v),(u,v)\rangle&=\int_\Omega(|\nabla u|^{p(x)}+|u|^{p(x)})dx+\int_\Omega(|\nabla v|^{q(x)}+|v|^{q(x)})dx\\&-\lambda\int_\Omega a(x)|u|^{r_1(x)}dx-\mu\int_\Omega b(x)|v|^{r_2(x)}dx\\&-\int_\Omega c(x)|u|^{\alpha(x)}|v|^{\beta(x)}dx=0.\end{aligned}\tag{2}$$

Then for $(u,v)\in M_{\lambda,\mu}$, we have

$$\begin{aligned}\langle I'_{\lambda,\mu}(u,v),(u,v)\rangle&=\int_\Omega p(x)(|\nabla u|^{p(x)}+|u|^{p(x)})dx+\int_\Omega q(x)(|\nabla v|^{q(x)}+|v|^{q(x)})dx\\&-\lambda\int_\Omega r_1(x)a(x)|u|^{r_1(x)}dx-\mu\int_\Omega r_2(x)b(x)|v|^{r_2(x)}dx\\&-\int_\Omega(\alpha(x)+\beta(x))c(x)|u|^{\alpha(x)}|v|^{\beta(x)}dx.\end{aligned}$$

Now, we split $M_{\lambda,\mu}$ into three parts:

$$M^+_{\lambda,\mu}=\{(u,v)\in M_{\lambda,\mu}:\langle I'_{\lambda,\mu}(u,v),(u,v)\rangle>0\},$$
$$M^0_{\lambda,\mu}=\{(u,v)\in M_{\lambda,\mu}:\langle I'_{\lambda,\mu}(u,v),(u,v)\rangle=0\},$$
$$M^-_{\lambda,\mu}=\{(u,v)\in M_{\lambda,\mu}:\langle I'_{\lambda,\mu}(u,v),(u,v)\rangle<0\}.$$

Theorem 3.1. Suppose that (u_0,v_0) is a local maximum or minimum for $J_{\lambda,\mu}$ on $M_{\lambda,\mu}$. If $(u_0,v_0)\notin M^0_{\lambda,\mu}$, then (u_0,v_0) is a critical point of $J_{\lambda,\mu}$.

Proof. The proof of Theorem 3.1 can be obtained directly from the following lemmas.

Lemma 3.2. There exists $\delta > 0$ such that for $0 < \lambda + \mu < \delta$, we have $M^0_{\lambda,\mu} = \emptyset$

Proof. Suppose otherwise, then for

$\delta = \frac{(\min\{p^-,q^-\}-\alpha^+-\beta^+)}{(\max\{r_1^+,r_2^+\}-\alpha^+-\beta^+)C_3}\left[\frac{(\min\{r_1^-,r_2^-\}-\max\{p^+,q^+\})}{C_4(\min\{r_1^-,r_2^-\}-\alpha^--\beta^-)}\right]^{\frac{\max\{r_1^+,r_2^+\}-\min\{p^-,q^-\}}{\min\{p^-,q^-\}-\alpha^+-\beta^+}}$, where C_3, C_4 are positive constants and will specified later, there exists (λ,μ) with $0 < \lambda+\mu < \delta$ such that $M^0_{\lambda,\mu} \neq \emptyset$. Then for $(u,v) \in M^0_{\lambda,\mu}$ we have

$$\begin{aligned}
0 = \langle I'_{\lambda,\mu}(u,v),(u,v)\rangle &= \textstyle\int_\Omega p(x)(|\nabla u|^{p(x)} + |u|^{p(x)})dx + \int_\Omega q(x)(|\nabla v|^{q(x)} + |v|^{q(x)})dx \\
&-\lambda \textstyle\int_\Omega a(x)r_1(x)|u|^{r_1(x)}dx - \mu\int_\Omega b(x)r_2(x)|v|^{r_2(x)}dx \\
&- \textstyle\int_\Omega(\alpha(x)+\beta(x))c(x)|u|^{\alpha(x)}|v|^{\beta(x)}dx \\
&\geq \min\{p^-,q^-\}\left[\textstyle\int_\Omega(|\nabla u|^{p(x)} + |u|^{p(x)})dx + \int_\Omega(|\nabla v|^{q(x)} + |v|^{q(x)})dx\right] \\
&- \max\{r_1^+,r_2^+\}[\lambda\textstyle\int_\Omega a(x)|u|^{r_1(x)}dx + \mu\int_\Omega b(x)|v|^{r_2(x)}dx] \\
&-(\alpha^+ + \beta^+)\textstyle\int_\Omega c(x)|u|^{\alpha(x)}|v|^{\beta(x)}dx \\
&= (\min\{p^-,q^-\}-\alpha^+-\beta^+)P(u,v) + (\alpha^++\beta^+-\max\{r_1^+,r_2^+\})Q(u,v), \qquad (3)
\end{aligned}$$

and

$$\begin{aligned}
0 = \langle I'_{\lambda,\mu}(u,v),(u,v)\rangle &= \textstyle\int_\Omega p(x)(|\nabla u|^{p(x)} + |u|^{p(x)})dx + \int_\Omega q(x)(|\nabla v|^{q(x)} + |v|^{q(x)})dx \\
&-\lambda \textstyle\int_\Omega r_1(x)a(x)|u|^{r_1(x)}dx - \mu\int_\Omega r_2(x)b(x)|v|^{r_2(x)}dx \\
&- \textstyle\int_\Omega(\alpha(x)+\beta(x))b(x)|u|^{\alpha(x)}|v|^{\beta(x)}dx \\
&\leq \max\{p^+,q^+\}\left[\textstyle\int_\Omega(|\nabla u|^{p(x)} + |u|^{p(x)})dx + \int_\Omega(|\nabla v|^{q(x)} + |v|^{q(x)})dx\right] \\
&- \min\{r_1^-,r_2^-\}[\lambda\textstyle\int_\Omega a(x)|u|^{r_1(x)}dx + \mu\int_\Omega b(x)|v|^{r_2(x)}dx] \\
&-(\alpha^- + \beta^-)\textstyle\int_\Omega b(x)|u|^{\alpha(x)}|v|^{\beta(x)}dx \\
&= (\max\{p^+,q^+\}-\min\{r_1^-,r_2^-\})P(u,v) + (\min\{r_1^-,r_2^-\}-\alpha^--\beta^-)R(u,v). \qquad (4)
\end{aligned}$$

By Propositions 2.1, 2.2 , 2.7, 2.9 we have

$$\begin{aligned}
Q(u,v) &= \lambda\textstyle\int_\Omega a(x)|u|^{r_1(x)}dx + \mu\int_\Omega b(x)|v|^{r_2(x)})dx \\
&\leq 2\lambda|a(x)|_{r_1(x)}\left||u|^{r_1(x)}\right|_{\frac{p^*(x)}{r_1(x)}} + 2\mu|b(x)|_{k_2(x)}\left||v|^{r_2(x)}\right|_{\frac{q^*(x)}{r_2(x)}} \\
&\leq 2\lambda|a(x)|_{k_1(x)}(|u|_{p^*(x)})^{\bar{r_1}} + 2\mu|b(x)|_{k_2(x)}(|v|_{q^*(x)})^{\bar{r_2}} \\
&\leq 2\lambda c_1\|u\|_{p(x)}^{\bar{r_1}} + 2\mu c_2\|v\|_{q(x)}^{\bar{r_2}} \\
&\leq \lambda C_1\|(u,v)\|^{r_1^+} + \mu C_2\|(u,v)\|^{r_2^+} \\
&\leq (\lambda+\mu)C_3\|(u,v)\|^{\max\{r_1^+,r_2^+\}} \qquad (5)
\end{aligned}$$

and

$$R(u,v) = \int_\Omega c(x)|u|^{\alpha(x)}|v|^{\beta(x)}dx$$
$$\leq 3|c(x)|_{k_3(x)} \left||u|^{\alpha(x)}\right|_{\frac{p^*(x)}{\alpha(x)}} \left||v|^{\beta(x)}\right|_{\frac{q^*(x)}{\beta(x)}}$$
$$\leq 3|c(x)|_{k_3(x)}(|u|_{p^*(x)})^{\bar{\alpha}}(|v|_{q^*(x)})^{\bar{\beta}}$$
$$\leq c_4\|u\|^{\bar{\alpha}}_{p(x)}\|v\|^{\bar{\beta}}_{q(x)}$$
$$\leq C_4\|(u,v)\|^{\alpha^+ + \beta^+}. \tag{6}$$

By using (5), (6) in (3) and (4) we get

$$\|(u,v)\| \geq \left[\frac{(\min\{p^-,q^-\}-\alpha^+-\beta^+)}{(\lambda+\mu)C_3(\max\{r_1^+,r_2^+\}-\alpha^+-\beta^+)}\right]^{\frac{1}{\max\{r_1^+,r_2^+\}-\min\{p^-,q^-\}}} \tag{7}$$

and

$$\|(u,v)\| \leq \left[\frac{C_4(\min\{r_1^-,r_2^-\}-\alpha^--\beta^-)}{(\min\{r_1^-,r_2^-\}-\max\{p^+,q^+\})}\right]^{\frac{1}{\min\{p^-,q^-\}-\alpha^+-\beta^+}}. \tag{8}$$

This implies $\lambda + \mu \geq \delta$ which is a contradiction. Thus we can conclude that there exists $\delta > 0$ such that for $0 < \lambda + \mu < \delta$, we have $M^0_{\lambda,\mu} = \emptyset$.

Lemma 3.3. The energy functional $J_{\lambda,\mu}$ is coercive and bounded below on $M_{\lambda,\mu}$.

Proof. If $(u,v) \in M_{\lambda,\mu}$ and $\|(u,v)\| > 1$. Without loss of generality, we may assume $\|u\|_{p(x)}, \|v\|_{q(x)} > 1$, we have

$$J_{\lambda,\mu}(u,v) = \int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})dx + \int_\Omega \frac{1}{q(x)}(|\nabla v|^{q(x)} + |v|^{q(x)})dx$$
$$-\lambda \int_\Omega \frac{1}{r_1(x)}a(x)|u|^{r_1(x)} - \mu \int_\Omega \frac{1}{r_2(x)}b(x)|v|^{r_2(x)}dx$$
$$-\int_\Omega \frac{1}{\alpha(x)+\beta(x)}c(x)|u|^{\alpha(x)}|v|^{\beta(x)}dx$$
$$\geq \frac{1}{p^+}\int_\Omega(|\nabla u|^{p(x)} + |u|^{p(x)})dx + \frac{1}{q^+}\int_\Omega(|\nabla v|^{q(x)} + |v|^{q(x)})dx$$
$$-\frac{\lambda}{r_1^-}\int_\Omega a(x)|u|^{r_1(x)}dx - \frac{\mu}{r_2^-}\int_\Omega b(x)|v|^{r_2(x)}dx$$
$$-\frac{1}{\alpha^-+\beta^-}\int_\Omega c(x)|u|^{\alpha(x)}|v|^{\beta(x)}dx$$
$$\geq \left(\frac{1}{\max\{p^+,q^+\}} - \frac{1}{\min\{r_1^-,r_2^-\}}\right)\left[\int_\Omega(|\nabla u|^{p(x)} + |u|^{p(x)})dx + \int_\Omega(|\nabla v|^{q(x)} + |v|^{q(x)})dx\right]$$
$$-\left(\frac{1}{\alpha^-+\beta^-} - \frac{1}{\min\{r_1^-,r_2^-\}}\right)\int_\Omega c(x)|u|^{\alpha(x)}|v|^{\beta(x)}dx$$
$$\geq \left(\frac{1}{\max\{p^+,q^+\}} - \frac{1}{\min\{r_1^-,r_2^-\}}\right)\|(u,v)\|^{\min\{p^-,q^-\}}$$
$$-\left(\frac{1}{\alpha^-+\beta^-} - \frac{1}{\min\{r_1^-,r_2^-\}}\right)C_4\|(u,v)\|^{\alpha^++\beta^+}.$$

Since $p^-, q^- > (\alpha^+ + \beta^+)$ so, $J_{\lambda,\mu}(u,v) \to \infty$ as $\|(u,v)\| \to \infty$. This implies $J_{\lambda,\mu}(u,v)$ is coercive and bounded below on $M_{\lambda,\mu}$.

By Lemma 3.1, for $0 < \lambda + \mu < \delta$, we can write $M_{\lambda,\mu} = M^+_{\lambda,\mu} \cup M^-_{\lambda,\mu}$ and define

$$\alpha_0^+(\lambda, \mu) = \inf_{(u,v)\in M^+_{\lambda,\mu}} J_{\lambda,\mu}(u, v) \text{ and } \alpha_0^-(\lambda, \mu) = \inf_{(u,v)\in M^-_{\lambda,\mu}} J_{\lambda,\mu}(u, v)$$

Lemma 3.4. If $0 < \lambda + \mu < \delta$, then for all $(u, v) \in M^+_{\lambda,\mu}$, $J_{\lambda,\mu}(u, v) < 0$.

Proof. Let $(u, v) \in M^+_{\lambda,\mu}(\Omega)$. We have

$$\max\{p^+, q^+\} \left(\int_\Omega(|\nabla u|^{p(x)} + |u|^{p(x)})dx + \int_\Omega(|\nabla v|^{q(x)} + |v|^{q(x)})dx\right)$$
$$- \min\{r_1^-, r_2^-\} \left(\lambda \int_\Omega a(x)|u|^{r_1(x)}dx + \mu \int_\Omega b(x)|v|^{r_2(x)}dx\right)$$
$$-(\alpha^- + \beta^-) \int_\Omega c(x)|u|^{\alpha(x)}|v|^{\beta(x)}dx > 0. \quad (9)$$

By definition of $J_{\lambda,\mu}(u, v)$ we can write

$$J_{\lambda,\mu}(u, v) \leq \left(\frac{1}{\min\{p^-,q^-\}} - \frac{1}{\alpha^++\beta^+}\right) \left[(\int_\Omega(|\nabla u|^{p(x)} + |u|^{p(x)})dx + \int_\Omega(|\nabla v|^{q(x)} + |v|^{q(x)})dx\right)$$
$$+ \left(\frac{1}{\alpha^++\beta^+} - \frac{1}{\max\{r_1^+,r_2^+\}}\right) \left(\lambda \int_\Omega a(x)|u|^{r_1(x)}dx + \mu \int_\Omega b(x)|v|^{r_2(x)}dx\right). \quad (10)$$

Now, if we multiply (2) by $-(\alpha^- + \beta^-)$ and add with (9), we get

$$Q(u, v) \leq \frac{\max\{p^+,q^+\}-\alpha^--\beta^-}{\min\{r_1^-,r_2^-\}-\alpha^--\beta^-} P(u, v), \quad (11)$$

and applying (11) in (10), it follows

$$J_{\lambda,\mu}(u, v) \leq \left[\frac{\alpha^++\beta^+-\min\{p^-,q^-\}}{\min\{p^-,q^-\}(\alpha^++\beta^+)} + \frac{\max\{p^+,q^+\}-\alpha^--\beta^-}{\max\{r_1^+,r_2^+\}(\alpha^++\beta^+)}\right] P(u, v)$$
$$\leq -\left[\frac{(\min\{p^-,q^-\}-\alpha^+-\beta^+)(\max\{r_1^+,r_2^+\}-\min\{p^-,q^-\})}{\max\{r_1^+,r_2^+\}(\alpha^++\beta^+)\min\{p^-,q^-\}}\right] P(u, v) < 0.$$

Thus $\alpha_0^+(\lambda, \mu) = \inf_{(u,v)\in M^+_{\lambda,\mu}} J_{\lambda,\mu}(u, v) < 0$.

Lemma 3.5. If $0 < \lambda + \mu < \delta$, there exists a minimizer of $J_{\lambda,\mu}(u, v)$ on $M^+_{\lambda,\mu}$.

Proof. Since $J_{\lambda,\mu}$ is bounded below on $M_{\lambda,\mu}$ and so on $M^+_{\lambda,\mu}$, then, there exists a minimizing sequence $\{(u_n^+, v_n^+)\} \subseteq M^+_{\lambda,\mu}$ such that

$$\lim_{n\to\infty} J_{\lambda,\mu}(u_n^+, v_n^+) = \inf_{(u,v)\in M^+_{\lambda,\mu}} J_{\lambda,\mu}(u, v) = \alpha_0^+(\lambda, \mu) < 0$$

Since $J_{\lambda,\mu}$ is coercive, $\{(u_n^+, v_n^+)\}$ is bounded below in W. Thus, we may assume that, without loss of generality, $(u_n^+, v_n^+) \rightharpoonup (u_0^+, v_0^+)$ in W. Hence $u_n^+ \rightharpoonup u_0^+$ in $W^{1,p(x)}(\Omega)$, $v_n^+ \rightharpoonup v_0^+$ in $W^{1,q(x)}(\Omega)$ and by the compact embeddings we have

$$u_n^+ \to u_0^+ \text{ in } L^{r_1(x)}(\Omega) \text{ and in } L^{\alpha(x)+\beta(x)}(\Omega),$$
$$v_n^+ \to v_0^+ \text{ in } L^{r_2(x)}(\Omega) \text{ and in } L^{\alpha(x)+\beta(x)}(\Omega).$$

This implies

$$Q(u_n^+, v_n^+) \to Q(u_0^+, v_0^+) \quad \text{as } n \to \infty,$$
$$R(u_n^+, v_n^+) \to R(u_0^+, v_0^+) \quad \text{as } n \to \infty.$$

Now, we shall prove $u_n^+ \to u_0^+$ in $W^{1,p(x)}(\Omega)$, $v_n^+ \to v_0^+$ in $W^{1,q(x)}(\Omega)$. Suppose otherwise, then either

$$\|u_0^+\|_p < \liminf_{n\to\infty} \|u_n^+\|_p \quad \text{or} \quad \|v_0^+\|_q < \liminf_{n\to\infty} \|v_n^+\|_q.$$

Using the fact that $\langle J'_{\lambda,\mu}(u_n^+, v_n^+), (u_n^+, v_n^+)\rangle = 0$ and (5) we can write the followings

$$J_{\lambda,\mu}(u_n^+, v_n^+) > \left(\tfrac{1}{\max\{p^+,q^+\}} - \tfrac{1}{\min\{r_1^-,r_2^-\}}\right) P(u_n, v_n) - \left(\tfrac{1}{\alpha^-+\beta^-} - \tfrac{1}{\min\{r_1^-,r_2^-\}}\right) R(u_n, v_n),$$

$$\begin{aligned}\lim_{n\to\infty} J_{\lambda,\mu}(u_n^+, v_n^+) > &\left(\tfrac{1}{\max\{p^+,q^+\}} - \tfrac{1}{\min\{r_1^-,r_2^-\}}\right) \lim_{n\to\infty} P(u_n, v_n)\\ &- \left(\tfrac{1}{\alpha^-+\beta^-} - \tfrac{1}{\min\{r_1^-,r_2^-\}}\right) \lim_{n\to\infty} R(u_n, v_n),\end{aligned}$$

$$\begin{aligned}\alpha_0^+(\lambda,\mu) &= \inf_{(u,v)\in M^+_{\lambda,\mu}} J_{\lambda,\mu}(u,v)\\ &> \left(\tfrac{1}{\max\{p^+,q^+\}} - \tfrac{1}{\min\{r_1^-,r_2^-\}}\right) \|(u_0^+, v_0^+)\|^{\min\{p^-,q^-\}}\\ &- \left(\tfrac{1}{\alpha^-+\beta^-} - \tfrac{1}{\min\{r_1^-,r_2^-\}}\right) \|(u_0^+, v_0^+)\|^{\alpha^++\beta^+},\end{aligned}$$

since $\min\{p^-, q^-\} > \alpha^+ + \beta^+$, for $\|(u_0^+, v_0^+)\| > 1$, we have

$$\alpha_0^+(\lambda,\mu) = \inf_{(u,v)\in M^+_{\lambda,\mu}} J_{\lambda,\mu}(u,v) > 0.$$

So that is a contradiction. Hence

$$u_n^+ \to u_0^+ \text{ in } W^{1,p(x)}(\Omega),$$
$$v_n^+ \to v_0^+ \text{ in } W^{1,q(x)}(\Omega).$$

This implies

$$J_{\lambda,\mu}(u_n^+, v_n^+) \to J_{\lambda,\mu}(u_0^+, v_0^+) = \inf_{u,v\in M^+_{\lambda,\mu}} J_{\lambda,\mu}(u,v) \quad \text{as } n\to\infty.$$

Thus, (u_0^+, v_0^+) is a minimizer for $J_{\lambda,\mu}$ on $M^+_{\lambda,\mu}$.

Lemma 3.6. If $0 < \lambda + \mu < \delta$, then for all $(u,v) \in M^-_{\lambda,\mu}$, $J_{\lambda,\mu}(u,v) > 0$.

Proof. Let $(u,v) \in M^-_{\lambda,\mu}$. We have

$$\begin{aligned}&\min\{p^-,q^-\}[\textstyle\int_\Omega(|\nabla u|^{p(x)} + |u|^{p(x)})dx + \int_\Omega(|\nabla v|^{q(x)} + |v|^{q(x)})dx]\\ &\quad - \max\{r_1^+, r_2^+\}[\lambda \textstyle\int_\Omega a(x)|u|^{r_1(x)}dx + \int_\Omega b(x)|v|^{r_2(x)}dx]\\ &\quad -(\alpha^+ + \beta^+) \textstyle\int_\Omega c(x)|u|^{\alpha(x)}|v|^{\beta(x)}dx < 0. \qquad (12)\end{aligned}$$

By definition of $J_{\lambda,\mu}(u,v)$ and (2), we have

$$J_{\lambda,\mu}(u,v) > \left(\frac{1}{\max\{p^+,q^+\}} - \frac{1}{\min\{r_1^-,r_2^-\}}\right) P(u,v) - \left(\frac{1}{\alpha^-+\beta^-} - \frac{1}{\min\{r_1^-,r_2^-\}}\right) R(u,v). \qquad (13)$$

Now, if we multiply (2) by $-\max\{r_1^+, r_2^+\}$ and add with (12), we get

$$R(u,v) \leq \frac{(\max\{r_1^+,r_2^+\}-\min\{p^-,q^-\})}{\max\{r_1^+,r_2^+\}-\alpha^+-\beta^+} P(u,v), \qquad (14)$$

and applying (14) in (13), it follows

$$\begin{aligned} J_{\lambda,\mu}(u,v) &\geq \left(\frac{\min\{r_1^-,r_2^-\}-\max\{p^+,q^+\}}{\min\{r_1^-,r_2^-\}\max\{p^+,q^+\}}\right) P(u,v) + \left(\frac{\max\{r_1^+,r_2^+\}-\min\{p^-,q^-\}}{\min\{r_1^-,r_2^-\}(\alpha^-+\beta^-)}\right) P(u,v) \\ &\geq \frac{(\min\{r_1^-,r_2^-\}-\max\{p^+,q^+\})(\alpha^-+\beta^-+\max\{p^+,q^+\})}{\min\{r_1^-,r_2^-\}\max\{p^+,q^+\}(\alpha^-+\beta^-)} P(u,v) > 0. \end{aligned}$$

Theorem 3.7. If $0 < \lambda + \mu < \delta$, there exists a minimizer of $J_{\lambda,\mu}$ on $M^-_{\lambda,\mu}$.

Proof. Since $J_{\lambda,\mu}$ is bounded below on $M_{\lambda,\mu}$ and so on $M^-_{\lambda,\mu}$, then there exists a minimizing sequence$\{(u_n^-, v_n^-)\} \subseteq M^-_{\lambda,\mu}$ such that

$$\lim_{n\to\infty} J_{\lambda,\mu}(u_n^-, v_n^-) = \inf_{(u,v)\in M^-_{\lambda,\mu}} J_{\lambda,\mu}(u,v) = \alpha_0^-(\lambda,\mu).$$

Since $J_{\lambda,\mu}$ is coercive, $\{(u_n^-, v_n^-)\}$ is bounded below in W. Thus, we may assume that, without loss of generality, $(u_n^-, v_n^-) \rightharpoonup (u_0^-, v_0^-)$ in W. Hence $u_n^- \rightharpoonup u_0^-$ in $W^{1,p(x)}(\Omega)$, $v_n^- \rightharpoonup v_0^-$ in $W^{1,q(x)}(\Omega)$ and by the compact embeddings we have

$$\begin{aligned} &u_n^- \to u_0^- \text{ in } L^{r_1(x)}(\Omega) \text{ and in } L^{\alpha(x)+\beta(x)}(\Omega),\\ &v_n^- \to v_0^- \text{ in } L^{r_2(x)}(\Omega) \text{ and in } L^{\alpha(x)+\beta(x)}(\Omega). \end{aligned}$$

This implies

$$\begin{aligned} &Q(u_n^-, v_n^-) \to Q(u_0^-, v_0^-) \text{ as } n \to \infty,\\ &R(u_n^-, v_n^-) \to R(u_0^-, v_0^-) \text{ as } n \to \infty. \end{aligned}$$

Moreover, if $(u_0^-, v_0^-) \in M^-_{\lambda,\mu}$, then there is a constant $t > 0$ such that $(tu_0^-, tv_0^-) \in M^-_{\lambda,\mu}$ and $J_{\lambda,\mu}(u_0^-, v_0^-) \geq J_{\lambda,\mu}(tu_0^-, tv_0^-)$. Indeed, since

$$\begin{aligned} I'_{\lambda,\mu}(u,v) &= \textstyle\int_\Omega p(x)(|\nabla u|^{p(x)} + |u|^{p(x)})dx + \int_\Omega q(x)(|\nabla v|^{q(x)} + |v|^{q(x)})dx\\ &\quad -\lambda \textstyle\int_\Omega r_1(x)a(x)|u|^{r_1(x)}dx - \mu \int_\Omega r_2(x)b(x)|v|^{r_2(x)}dx\\ &\quad - \textstyle\int_\Omega(\alpha(x)+\beta(x))c(x)|u|^{\alpha(x)}|v|^{\beta(x)}dx, \end{aligned}$$

then,

$$\begin{aligned} I'_{\lambda,\mu}(tu_0^-, tv_0^-) &= \textstyle\int_\Omega p(x)(|\nabla tu_0^-|^{p(x)} + |tu_0^-|^{p(x)})dx + \int_\Omega q(x)(|\nabla tv_0^-|^{q(x)} + |tv_0^-|^{q(x)})dx\\ &\quad -\lambda \textstyle\int_\Omega r_1(x)a(x)|tu_0^-|^{r_1(x)}dx - \int_\Omega r_2(x)b(x)|tv_0^-|^{r_2(x)}dx\\ &\quad - \textstyle\int_\Omega(\alpha(x)+\beta(x))c(x)|tu_0^-|^{\alpha(x)}|tv_0^-|^{\beta(x)}dx\\ &\leq t^{\max\{p^+,q^+\}} \max\{p^+, q^+\} \left(\textstyle\int_\Omega(|\nabla u_0^-|^{p(x)} + |u_0^-|^{p(x)})dx + \int_\Omega(|\nabla v_0^-|^{q(x)} + |v_0^-|^{q(x)})dx\right) \end{aligned}$$

$$
\begin{aligned}
&-t^{r_1^-} r_1^- \lambda \int_\Omega a(x)|u_0^-|^{r_1(x)}dx - t^{r_2^-} r_2^- \mu \int_\Omega b(x)|v_0^-|^{r_2(x)})dx\\
&-t^{\alpha^-+\beta^-}(\alpha^- + \beta^-) \int_\Omega c(x)|u_0^-|^{\alpha(x)}|v_0^-|^{\beta(x)}dx\\
&\le \left(t^{\max\{p^+,q^+\}} \max\{p^+,q^+\} - t^{\min\{r_1^-,r_2^-\}} \min\{r_1^-,r_2^-\}\right) P(u_0^-,v_0^-)\\
&+ \left(t^{\min\{r_1^-,r_2^-\}} \min\{r_1^-,r_2^-\} - t^{\alpha^-+\beta^-}(\alpha^-+\beta^-)\right) R(u_0^-,v_0^-)\\
&\le 2\left(t^{\max\{p^+,q^+\}} \max\{p^+,q^+\} - t^{\min\{r_1^-,r_2^-\}} \min\{r_1^-,r_2^-\}\right) \|(u_0^-,v_0^-)\|^{\max\{p^+,q^+\}}\\
&+C_2\left(t^{\min\{r_1^-,r_2^-\}} \min\{r_1^-,r_2^-\} - t^{\alpha^-+\beta^-}(\alpha^-+\beta^-)\right) \|(u_0^-,v_0^-)\|^{\alpha^++\beta^+}.
\end{aligned}
$$

By $(H_0)-(H_2)$ it follows $I'_{\lambda,\mu}(tu_0^-,tv_0^-) < 0$. Hence by the definition of $M^-_{\lambda,\mu}$, $(tu_0^-,tv_0^-) \in M^-_{\lambda,\mu}$.

Now, we shall prove $u_n^- \to u_0^-$ in $W^{1,p(x)}(\Omega)$, $v_n^- \to v_0^-$ in $W^{1,q(x)}(\Omega)$. Suppose otherwise, then either

$$
\|u_0^-\|_p < \liminf_{n\to\infty} \|u_n^-\|_p \quad \text{or} \quad \|v_0^-\|_q < \liminf_{n\to\infty} \|v_n^-\|_q.
$$

We have

$$
\begin{aligned}
J_{\lambda,\mu}(tu_0^-,tv_0^-) &\le \tfrac{t^{\max\{p^+,q^+\}}}{\min\{p^-,q^-\}} P(u_0^-,v_0^-) - \tfrac{t^{\min\{r_1^-,r_2^-\}}}{\max\{r_1^+,r_2^+\}} Q(u_0^-,v_0^-) - \tfrac{t^{\alpha^-+\beta^-}}{\alpha^++\beta^+} R(u_0^-,v_0^-)\\
&< \lim_{n\to\infty}\left[\tfrac{t^{\max\{p^+,q^+\}}}{\min\{p^-,q^-\}} P(u_n^-,v_n^-) - \tfrac{t^{\min\{r_1^-,r_2^-\}}}{\max\{r_1^+,r_2^+\}} Q(u_n^-,v_n^-) - \tfrac{t^{\alpha^-+\beta^-}}{\alpha^++\beta^+} R(u_n^-,v_n^-)\right]\\
&\le \lim_{n\to\infty} J_{\lambda,\mu}(tu_n^-,tv_n^-) \le \lim_{n\to\infty} J_{\lambda,\mu}(u_n^-,v_n^-) = \inf_{(u,v)\in M^-_{\lambda,\mu}} J_{\lambda,\mu}(u,v) = \alpha_0^-(\lambda,\mu).
\end{aligned}
$$

This implies that $J_{\lambda,\mu}(tu_0^-,tv_0^-) < \inf_{(u,v)\in M^-_{\lambda,\mu}} J_{\lambda,\mu}(u,v) = \alpha_0^-(\lambda,\mu)$, which is a contradiction. Hence

$$
\begin{aligned}
&u_n^- \to u_0^- \text{ in } W^{1,p(x)}(\Omega),\\
&v_n^- \to v_0^- \text{ in } W^{1,q(x)}(\Omega).
\end{aligned}
$$

This implies

$$
J_{\lambda,\mu}(u_n^-,v_n^-) \to J_{\lambda,\mu}(u_0^-,v_0^-) = \inf_{(u,v)\in M^-_{\lambda,\mu}} J_{\lambda,\mu}(u,v) \quad \text{as } n\to\infty.
$$

Thus, (u_0^-,v_0^-) is a minimizer for $J_{\lambda,\mu}$ on $M^-_{\lambda,\mu}$.

Corollary 3.8. By Theorem 3.5 and 3.7 we conclude that there exist $(u_0^+,v_0^+) \in M^+_{\lambda,\mu}$ and $(u_0^-,v_0^-) \in M^-_{\lambda,\mu}$ such that

$$
J_{\lambda,\mu}(u_0^+,v_0^+) = \inf_{(u,v)\in M^+_{\lambda,\mu}} J_{\lambda,\mu}(u,v) \;\text{ and }\; J_{\lambda,\mu}(u_0^-,v_0^-) = \inf_{(u,v)\in M^-_{\lambda,\mu}} J_{\lambda,\mu}(u,v).
$$

Moreover, since $J_{\lambda,\mu}(u_0^\pm,v_0^\pm) = J_{\lambda,\mu}(|u_0^\pm|,|v_0^\pm|)$ and $(|u_0^\pm|,|v_0^\pm|) \in M^\pm_{\lambda,\mu}$, we may assume $(u_0^\pm,v_0^\pm) \ge 0$. By Theorem 3.1, $(u_0^\pm,v_0^\pm)$ are critical points of $J_{\lambda,\mu}$ on W and hence are weak solutions. Finally, by the Harnack inequality due to [19 , 21], we obtain that $(u_0^{\pm}, v_0^{\pm})$ are positive solutions of (1).

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