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Existence of solutions for some nonlinear elliptic unilateral problems with measure data

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Abstract

In this paper, we prove the existence of an entropy solution to unilateral problems associated to the equations of the type:

$$Au + H(x, u, \nabla u) - div\phi(u) = \mu \in L^1(\Omega) + W^{-1,p'(x)}(\Omega),$$

where A is a Leray-Lions operator acting from $W_0^{1,p(x)}(\Omega)$ into its dual $W^{-1,p(x)}(\Omega)$, the nonlinear term $H(x, s, \xi)$ satisfies some growth and the sign conditions and $\phi(u) \in C^0(R, R^N)$.

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1 Introduction

This paper is devoted to the study of the following nonlinear problem:

$$\begin{cases} Au + H(x, u, \nabla u) - div(\phi(u)) = f - div(F) & \text{in} \quad \Omega \\ u = 0 \qquad \text{on} \quad \partial\Omega. \end{cases} \tag{1.1}$$

In Problem (1.1) the framework is the following: Ω is a bounded open subset of $I\!R^N$, $N \geq 2$, and $p : \overline{\Omega} \to I\!R^+$ is a continuous function. The operator $Au \equiv -div(a(x, u, \nabla u))$ is a Leray-Lions operator defined on $W_0^{1,p(x)}(\Omega)$ (this space will be described in Section 2). The function ϕ is assumed to be continuous on $I\!R$ with values in $I\!R^N$ and the nonlinear term $H(x, s, \xi)$ satisfies some growth and the sign conditions. The data f and F respectively belong to $L^1(\Omega)$ and $(L^{p'(x)}(\Omega))^N$.

The study of problems with variable exponent is a new and interesting topic which raises many mathematical difficulties. One of our motivations for studying (1.1) comes from applications to electro-rheological fluids (we refer to [13] for more details) as an important class of non-Newtonian fluids (sometimes referred to as smart fluids). Other important applications are related to image processing (see [8]) and elasticity (see [16]).

Under our assumptions, problem (1.1) does not admit, in general, a weak solution since the term $\phi(u)$ may not belong to $(L^1_{loc}(\Omega))^N$ because the function ϕ is just assumed to be continuous on $I\!R$.

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In order to overcome this difficulty we use in this paper the framework of an entropy solution (see Definition 3.1). This notion was introduced by Bénilan et al. [1] for the study of nonlinear elliptic problems in case of a constant exponent $p(.) \equiv p$.

The first objective of our paper is to study the problem (1.1) in the generalized Lebesgue-Sobolev spaces with some general second member μ which lies in $L^1(\Omega) + W^{-1,p'(x)}(\Omega)$.

The second objective is to treat the unilateral problems, more precisely, we prove an existence result for solutions of the following obstacle problem:

$$\begin{cases} u \text{ is a measurable function such that } u \geq \psi \quad \text{a.e. in } \Omega,\ T_k(u) \in W_0^{1,p(x)}(\Omega) \text{ and } \forall k > 0 \\ \displaystyle\int_\Omega a(x,u,\nabla u)\nabla T_k(u-v)dx + \int_\Omega H(x,u,\nabla u)T_k(u-v)dx + \int_\Omega \phi(u)\nabla T_k(u-v)dx \\ \qquad \leq \displaystyle\int_\Omega fT_k(u-v)dx + \int_\Omega F\nabla T_k(u-v)dx \\ \forall v \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega) \text{ such that } v \geq \psi \text{ a.e. in } \Omega. \end{cases}$$

where ψ is a measurable function (see assumptions (3.6) and (3.7)), and for any non-negative real number k we denote by $T_k(r) = \min(k, \max(r, -k))$ the truncation function at height k.

The plan of the paper is as follows. In Section 2, we give some preliminaries and the definition of generalized Lebesgue-Sobolev spaces. In Section 3, we make precise all the assumptions and give some technical results and we establish the existence of the entropy solution to the problem (1.1). In Section 4 (Appendix), we give the proof of Lemma 3.5.

2 Mathematical preliminaries

In what follows, we recall some definitions and basic properties of Lebesgue and Sobolev spaces with variable exponents. For each open bounded subset Ω of $\mathbb{R}^N$ $(N \geq 2)$, we denote

$$C^+(\overline{\Omega}) = \Big\{p :\ \overline{\Omega} \longrightarrow \mathbb{R}^+ \text{ continuous function, such that } \quad 1 < p_- \leq p_+ < \infty\Big\},$$

where $p_- = \inf_{x\in\overline{\Omega}} p(x)$ and $p_+ = \sup_{x\in\overline{\Omega}} p(x)$. For $p \in C^+(\overline{\Omega})$, we define the variable exponent Lebesgue space by: $L^{p(x)}(\Omega) = \Big\{u : \Omega \longrightarrow \mathbb{R} \text{ is a measurable function such that } \displaystyle\int_\Omega |u(x)|^{p(x)}\,dx < \infty\Big\}$, the space $L^{p(x)}(\Omega)$ under the norm: $\|u\|_{p(x)} = \inf\Big\{\lambda > 0 \quad / \quad \displaystyle\int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} \leq 1\Big\}$ is a separable and reflexive Banach space, and its dual space is isomorphic to $L^{p'(x)}(\Omega)$ where $\dfrac{1}{p(x)} + \dfrac{1}{p'(x)} = 1$.

Proposition 2.1 *(see [9]).* ***(i)*** *For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have*

$$\left|\int_\Omega u\,v\,dx\right| \leq \Big(\frac{1}{p_-} + \frac{1}{p'_-}\Big)\|u\|_{p(x)}\,\|v\|_{p'(x)}\,.$$

(ii) *For all $p_1,\ p_2 \in C^+(\overline{\Omega})$ such that $p_1(x) \leq p_2(x)$ for any $x \in \overline{\Omega}$, then $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$ and the imbedding is continuous.*

Proposition 2.2 *(see [9]). If we denote $\rho(u) = \displaystyle\int_\Omega |u|^{p(x)}\,dx \quad \forall u \in L^{p(x)}(\Omega)$, then the following assertions hold:*

(i) $\|u\|_{p(x)} < 1 \quad (resp. = 1,\ > 1) \quad \Leftrightarrow \quad \rho(u) < 1 \quad (resp. = 1,\ > 1)$

(ii) $\|u\|_{p(x)} > 1 \quad \Rightarrow \quad \|u\|_{p(x)}^{p_-} \leq \rho(u) \leq \|u\|_{p(x)}^{p_+}$ *and* $\|u\|_{p(x)} < 1 \quad \Rightarrow \quad \|u\|_{p(x)}^{p_+} \leq \rho(u) \leq \|u\|_{p(x)}^{p_-}$

(iii) $\|u\|_{p(x)} \to 0 \quad \Leftrightarrow \quad \rho(u) \to 0$ *and* $\|u\|_{p(x)} \to \infty \quad \Leftrightarrow \quad \rho(u) \to \infty$.

We define also the variable exponent Sobolev space

$$W^{1,p(x)}(\Omega) = \Big\{ \ u \in L^{p(x)}(\Omega) \ \text{ and } \ |\nabla u| \in L^{p(x)}(\Omega) \Big\}$$

normed by $\|u\|_{1,p(x)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)}$ and denote $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ and $p^*(x) = \frac{N\,p(x)}{N-p(x)}$ for $p(x) < N$.

Proposition 2.3 *(see [9]).* ***(i)*** *Assuming $1 < p_- \leq p_+ < \infty$, the spaces $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces.*
(ii) *If $q \in C^+(\bar\Omega)$ and $q(x) < p^*(x)$ almost everywhere in Ω, then there is a continuous and compact embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$.*
(iii) *There is a constant $C > 0$ such that $\|u\|_{p(x)} \leq C\,\|\nabla u\|_{p(x)} \quad \forall u \in W_0^{1,p(x)}(\Omega)$.*

Remark 2.1 *By (iii) of Proposition 2.3, we know that $\|\nabla u\|_{p(x)}$ and $\|u\|_{1,p(x)}$ are equivalent norms on $W_0^{1,p(x)}(\Omega)$.*

Lemma 2.1 *(see [7]). Let $g \in L^{r(x)}(\Omega)$ and $g_n \in L^{r(x)}(\Omega)$ with $\|g_n\|_{r(x)} \leq C$ for $1 < r(x) < \infty$. If $g_n(x) \to g(x)$ a.e. on Ω, then $g_n \rightharpoonup g$ weakly in $L^{r(x)}(\Omega)$.*

3 Main general results

3.1 Basic assumptions and some lemmas

Throughout the paper, we assume that the following assumptions hold true:
The function $a : \Omega \times I\!R \times I\!R^N \to I\!R^N$ is a Carathéodory function satisfying the following conditions:

$$|a(x,s,\xi)| \leq \beta(k(x) + |s|^{p(x)-1} + |\xi|^{p(x)-1}) \tag{3.1}$$

for every $s \in I\!R$, $\xi \in I\!R^N$ and for almost every $x \in \Omega$, where $k(x)$ is a positive function in $L^{p'(x)}(\Omega)$ and β is a positive constants.

$$[a(x,s,\xi) - a(x,s,\eta)](\xi - \eta) > 0 \tag{3.2}$$

for almost every $x \in \Omega$ and for every $s \in I\!R$, ξ, $\eta \in I\!R^N$, with $\xi \neq \eta$.

$$a(x,s,\xi)\xi \geq \alpha|\xi|^{p(x)} \tag{3.3}$$

for almost every $x \in \Omega$ and for every $s \in I\!R$, $\xi \in I\!R^N$, where α is a positive constant such that $\alpha \geq \|g\|_\infty$.
Let $H(x,s,\xi) : \Omega \times I\!R \times I\!R^N \to I\!R^N$ be a Carathéodory function such that for a.e $x \in \Omega$ and for all $s \in I\!R$, $\xi \in I\!R^N$ the sign and the growth conditions:

$$H(x,s,\xi)s \geq 0. \tag{3.4}$$

$$|H(x,s,\xi)| \leq \gamma(x) + g(s)|\xi|^{p(x)}, \tag{3.5}$$

are satisfied, where $g : I\!R \to I\!R^+$ is continuous increasing positive function that belongs to $L^\infty(I\!R)$ while $\gamma(x)$ belongs to $L^1(\Omega)$.
Let ψ be a measurable function such that for the convex set $K_\psi = \Big\{u \in W_0^{1,p(x)}(\Omega) \ \mid \ u \geq \psi \text{ a.e. in } \Omega\Big\}$

$$K_\psi \cap L^\infty(\Omega) \neq \emptyset. \tag{3.6}$$

holds. Finally, we suppose that

$$\phi \in C^0(I\!R, I\!R^N), \tag{3.7}$$

$$f \in L^1(\Omega), \tag{3.8}$$

$$F \in (L^{p'(x)}(\Omega))^N. \tag{3.9}$$

Let $p \in C^+(\overline{\Omega})$ be such that there is a vector $l \in I\!R^N \backslash \{0\}$ such that for any $x \in \Omega$,

$$h(t) = p(x + tl) \quad \text{is monotone for} \quad t \in I_x = \{t \ \mid \ x + tl \in \Omega\}. \tag{3.10}$$

Lemma 3.1 *(see [7]). Assume that* (3.1) − (3.3) *hold, and let* $(u_n)_n$ *be a sequence in* $W_0^{1,p(x)}(\Omega)$ *such that* $u_n \rightharpoonup u$ *weakly in* $W_0^{1,p(x)}(\Omega)$ *and*

$$\int_\Omega (a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u))\,\nabla(u_n-u)dx \longrightarrow 0, \tag{3.11}$$

then $u_n \longrightarrow u$ *strongly in* $W_0^{1,p(x)}(\Omega)$.

Lemma 3.2 *Assume that (3.10) holds, then there is a constant* $C>0$ *such that*

$$\rho(u) \le C\rho(\nabla u) \qquad \forall\, u \in W_0^{1,p(x)}(\Omega)\backslash\{0\}. \tag{3.12}$$

Proof. Let

$$\lambda_* = \inf_{u\in W_0^{1,p(x)}(\Omega)-\{0\}} \frac{\int_\Omega |\nabla u|^{p(x)}dx}{\int_\Omega |u|^{p(x)}dx}.$$

By Theorem 3.3 (see [10]), we have $\lambda_*>0$, which implies that

$$0<\lambda_* \le \frac{\int_\Omega |\nabla u|^{p(x)}dx}{\int_\Omega |u|^{p(x)}dx} \qquad \forall u \in W_0^{1,p(x)}(\Omega)\backslash\{0\},$$

consequently there is a constant $C>0$ such that $\rho(u)\le C\rho(\nabla u)$ for all $u\in W_0^{1,p(x)}(\Omega)\backslash\{0\}$. ■

Remark 3.1 *The inequality (3.12) holds true if we assume: there exists a function* $\xi \ge 0$ *such that* $\nabla p \nabla \xi \ge 0, \quad |\nabla\xi| \ne 0$ *in* $\overline{\Omega}$ *(see [6]).*

Lemma 3.3 *Let* $F: I\!R \to I\!R$ *be a Lipschitz uniform function with* $F(0)=0$ *and* $p\in C_+(\bar\Omega)$. *If* $u\in W_0^{1,p(x)}(\Omega)$, *then* $F(u)\in W_0^{1,p(x)}(\Omega)$, *moreover, if* D *the set of discontinuity points of* F' *is finite, then*

$$\frac{\partial(F\circ u)}{\partial x_i} = \begin{cases} F'(u)\frac{\partial u}{\partial x_i} & a.e \;\; in \;\; \{x\in\Omega \;/\; u(x)\notin D\} \\ 0 & a.e \;\; in \;\; \{x\in\Omega \;/\; u(x)\in D\}. \end{cases}$$

Proof. Taking at first the case of $F\in C^1(I\!R)$ and $F'\in L^\infty(I\!R)$. Let $u\in W_0^{1,p(x)}(\Omega)$, and since $\overline{C_0^\infty(\Omega)}^{W^{1,p(x)}(\Omega)} = W_0^{1,p(x)}(\Omega)$, then: $\exists\, u_n \in C_0^\infty(\Omega)$ such that $u_n \longrightarrow u$ in $W_0^{1,p(x)}(\Omega)$, we have $u_n\to u$ a.e. in Ω and $\nabla u_n \to \nabla u$ a.e. in Ω, then $F(u_n)\to F(u)$ a.e. in Ω. On the other hand, we have: $|F(u_n)| = |F(u_n)-F(0)| \le \|F'\|_\infty |u_n|$, then

$$|F(u_n)|^{p(x)} \le (\|F'\|_\infty+1)^{p_+}|u_n|^{p(x)} \quad \text{and} \quad \left|\frac{\partial F(u_n)}{\partial x_i}\right|^{p(x)} = \left|F'(u_n)\frac{\partial u_n}{\partial x_i}\right|^{p(x)} \le M\left|\frac{\partial u_n}{\partial x_i}\right|^{p(x)},$$

where $M=(\|F'\|_\infty+1)^{p_+}$. We conclude that $F(u_n)$ is bounded in $W_0^{1,p(x)}(\Omega)$ and we obtain: $F(u_n)$ converges to ν weakly in $W_0^{1,p(x)}(\Omega)$. Then $F(u_n)$ converges to ν strongly in $L^{q(x)}(\Omega)$ with $1<q(x)<p^*(x)$ and $p^*(x)=\frac{N\cdot p(x)}{N-p(x)}$, since $F(u_n)\to\nu$ a.e. in Ω, we obtain: $\nu = F(u)\in W_0^{1,p(x)}(\Omega)$.

Let $F: I\!R\to I\!R$ a Lipschitz uniform function, then $F_n = F*\varphi_n \to F$ uniformly on each compact set, where φ_n is a regularizing sequence, we conclude that $F_n\in C^1(I\!R)$ and $F_n'\in L^\infty(I\!R)$, from the first part, we have $F_n(u)\in W_0^{1,p(x)}(\Omega)$ and $F_n(u)\to F(u)$ a.e. in Ω. Since $(F_n(u))_n$ is bounded in $W_0^{1,p(x)}(\Omega)$, then $F_n(u)\rightharpoonup \overline{\nu}$ weakly in $W_0^{1,p(x)}(\Omega)$, we obtain $\overline{\nu} = F(u)\in W_0^{1,p(x)}(\Omega)$. ■

Lemma 3.4 *Let* Ω *be a bounded open subset of* $I\!R^N$ $(N\ge 1)$. *If* $u\in (W_0^{1,p(x)}(\Omega))^N$ *then*

$$\int_\Omega div(u)dx = 0.$$

Proof. Fix a vector $u = (u^1, \dots, u^N) \in (W_0^{1,p(x)}(\Omega))^N$. We have $W_0^{1,p(x)}(\Omega) = \overline{C_0^\infty(\Omega)}^{W^{1,p(x)}(\Omega)}$ and thus each term u^i can be approximated by a suitable sequence $u_k^i \in D(\Omega)$ such that, u_k^i converges to u^i strongly in $W_0^{1,p(x)}(\Omega)$. Moreover, due to the fact that $u_k^i \in C_0^\infty(\Omega)$, then the Green formula gives

$$\int_\Omega \frac{\partial u_k^i}{\partial x_i} dx = \int_{\partial\Omega} u_k^i \vec{n} ds = 0. \tag{3.13}$$

On the other hand, $\frac{\partial u_k^i}{\partial x_i} \to \frac{\partial u^i}{\partial x_i}$ strongly in $L^{p(x)}(\Omega)$. Thus $\frac{\partial u_k^i}{\partial x_i} \to \frac{\partial u^i}{\partial x_i}$ strongly in $L^1(\Omega)$, which gives in view of (3.13): $\int_\Omega div(u) dx = 0$. ■

3.2 Existence of an entropy solution

In this section, we study the existence of an entropy solution of problem (1.1). We now give the definition of an entropy solution

Definition 3.1 *A measurable function u is an entropy solution to problem (1.1) if for every $k \geq 0$:*

$$(P)\begin{cases} u \geq \psi \text{ a.e. in } \Omega,\ T_k(u) \in W_0^{1,p(x)}(\Omega), \\ \int_\Omega a(x,u,\nabla u)\nabla T_k(u-v)dx + \int_\Omega H(x,u,\nabla u)T_k(u-v)dx + \int_\Omega \phi(u)\nabla T_k(u-v)dx \\ \qquad \leq \int_\Omega f T_k(u-v)dx + \int_\Omega F\nabla T_k(u-v)dx \\ \text{for every function } v \in K_\psi \cap L^\infty(\Omega). \end{cases}$$

Our main result is

Theorem 3.1 *Under assumptions* (3.1)–(3.10)*, there exists at least an entropy solution of problem (1.1).*

Proof of Theorem 3.1. The proof is divided into 4 steps.

Step 1: The approximate problem

In this step, we introduce a family of approximate problems and prove the existence of solutions to such problems.

Theorem 3.2 *Let $(f_n)_n$ be a sequence in $W^{-1,p'(x)}(\Omega) \cap L^1(\Omega)$ such that $f_n \longrightarrow f$ in $L^1(\Omega)$, and $\|f_n\|_1 \leq \|f\|_1$, and we consider the approximate problem:*

$$(P_n)\begin{cases} u_n \in K_\psi \\ \langle Au_n, u_n - v\rangle + \int_\Omega H_n(x,u_n,\nabla u_n)\nabla T_k(u_n - v)dx + \int_\Omega \phi(T_n(u_n))\nabla(u_n - v)dx \\ \leq \int_\Omega f_n(u_n - v)dx + \int_\Omega F\nabla(u_n - v)dx \quad \forall v \in K_\psi \cap L^\infty(\Omega), \end{cases}$$

where $\phi_n(s) = \phi(T_n(s))$, $Au_n = -div(a(x,u_n,\nabla u_n))$ and $H_n(x,s,\xi) = \dfrac{H(x,s,\xi)}{1 + \frac{1}{n}|H(x,s,\xi)|}$. Note that $H_n(x,s,\xi)s \geq 0$, $|H_n(x,s,\xi)| \leq |H(x,s,\xi)|$ and $|H_n(x,s,\xi)| \leq n$.
Assume that (3.1)–(3.10) *hold true, then there exists at least one weak solution u_n for the approximate problem (P_n).*

Proof. Indeed, we define the operator $G_n = -\text{div}\,(\phi_n) : W_0^{1,p(x)}(\Omega) \longrightarrow W^{-1,p'(x)}(\Omega)$, such that

$$\langle G_n(u), v\rangle = -\langle \text{div}\,\phi_n(u), v\rangle = \int_\Omega \phi_n(u)\nabla v\, dx \qquad \forall u,\ v \in W_0^{1,p(x)}(\Omega).$$

Using the Hölder inequality, we deduce

$$\begin{aligned}
\int_\Omega \phi_n(u)\nabla v\,dx &\le \left(\frac{1}{p_-}+\frac{1}{p'_-}\right)\|\phi_n(u)\|_{p'(x)}\|\nabla v\|_{p(x)}\\
&\le \left(\frac{1}{p_-}+\frac{1}{p'_-}\right)\left(\int_\Omega |\phi(T_n(u))|^{p'(x)}\,dx\right)^{\gamma_0}\|v\|_{1,p(x)}\\
&\le \left(\frac{1}{p_-}+\frac{1}{p'_-}\right)\left(meas(\Omega)(\sup_{|s|\le n}|\phi(s)|+1)^{p_+}\right)^{\gamma_0}\|v\|_{1,p(x)}\\
&\le C_0\|v\|_{1,p(x)}
\end{aligned}\tag{3.14}$$

where

$$\gamma_0=\begin{cases}\frac{1}{p'_-} & \text{if} \quad \|\phi_n(u)\|_{p'(x)}>1\\ \frac{1}{p'_+} & \text{if} \quad \|\phi_n(u)\|_{p'(x)}\le 1\end{cases}$$

and C_0 is a constant which depends only on ϕ, n and p.
We define the operator R_n : $W_0^{1,p(x)}(\Omega)\to W^{-1,p'(x)}(\Omega)$, by

$$\langle R_n(u),v\rangle=\int_\Omega H_n(x,u,\nabla u)v dx \qquad \forall v\in W_0^{1,p(x)}(\Omega),$$

using the Hölder inequality, we have for all $u,\ v\in W_0^{1,p(x)}(\Omega)$

$$\begin{aligned}
\int_\Omega H_n(x,u,\nabla u)v dx &\le \left(\frac{1}{p_-}+\frac{1}{p'_-}\right)\|H_n(x,u,\nabla u)\|_{p'(x)}\|\nabla v\|_{p(x)}\\
&\le \left(\frac{1}{p_-}+\frac{1}{p'_-}\right)\cdot\left(\int_\Omega |H_n(x,u,\nabla u)|^{p'(x)}\,dx+1\right)^{\frac{1}{p'_-}}\|v\|_{1,p(x)}\\
&\le \left(\frac{1}{p_-}+\frac{1}{p'_-}\right)\cdot\left(n^{p'_+}meas(\Omega)+1\right)^{\frac{1}{p'_-}}\|v\|_{1,p(x)}\\
&\le C_1\|v\|_{1,p(x)}.
\end{aligned}\tag{3.15}$$

Lemma 3.5 *The operator $B_n=A+R_n+G_n$ is pseudo-monotone from $W_0^{1,p(x)}(\Omega)$ into $W^{-1,p'(x)}(\Omega)$. Moreover, B_n is coercive in the following sense: there exists $v_0\in K_\psi$ such that:*

$$\frac{\langle B_n v, v-v_0\rangle}{||v||_{1,p(x)}}\to+\infty \text{ if } ||v||_{1,p(x)}\to\infty \text{ and } v\in K_\psi.$$

Proof. See the appendix. ∎

In view of Lemma 3.5, there exists at least one solution $u_n\in W_0^{1,p(x)}(\Omega)$ of the problem (P_n), (see [12]).

Step 2: A priori estimate

In this step, we establish a uniform estimate on u_n with respect to n.

Proposition 3.1 *Assume that (3.1)–(3.10) hold true. Let u_n be a solution of the approximate problem (P_n), then for all $k\ge 0$, there exists a constant $c(k)$ (which does not depend on n) such that*

$$\int_\Omega |\nabla T_k(u_n)|^{p(x)}dx\le c(k).\tag{3.16}$$

Proof. Let $v_0 \in K_\psi \cap L^\infty(\Omega),\ k \geq ||v_0||_\infty$ and $h > 0$, so as $v = T_h(u_n - T_k(u_n - v_0)) \in K_\psi \cap L^\infty(\Omega)$. Taking v as a test function in (P_n) and letting $h \to +\infty$, we obtain, for n large enough ($n \geq k+||v_0||_\infty$):

$$\begin{aligned}\int_\Omega a(x, u_n, \nabla u_n)\nabla T_k(u_n - v_0)dx + \int_\Omega H_n(x, u_n, \nabla u_n)T_k(u_n - v_0)dx + \int_\Omega \phi(u_n)\nabla T_k(u_n - v_0)dx \\ \leq \int_\Omega f_n T_k(u_n - v_0)dx + \int_\Omega F\nabla T_k(u_n - v_0)dx,\end{aligned}$$

which implies that

$$\begin{aligned}\int_\Omega a(x, u_n, \nabla u_n)\nabla T_k(u_n - v_0)dx \leq & \int_{\{|u_n - v_0| < k\}} H_n(x, u_n, \nabla u_n)v_0\, dx \\ & + \int_{\{|u_n - v_0| < k\}} \phi(T_{k+||v_0||}(u_n))|\nabla u_n|dx \\ & + \int_{\{|u_n - v_0| < k\}} \phi(T_{k+||v_0||}(u_n))|\nabla v_0|dx \\ & + k||f||_{L^1} + \int_{\{|u_n - v_0| < k\}} F|\nabla u_n|dx \\ & + \int_{\{|u_n - v_0| < k\}} F|\nabla v_0|dx.\end{aligned}$$

Thus,

$$\begin{aligned}\int_{\{|u_n - v_0| < k\}} a(x, u_n, \nabla u_n)\nabla u_n dx \leq & \int_{\{|u_n - v_0| < k\}} |a(x, u_n, \nabla u_n)||\nabla v_0|dx \\ & + \|v_0\|_\infty \int_{\{|u_n - v_0| < k\}} \gamma(x) + g(u_n)|\nabla u_n|^{p(x)}\, dx \\ & + \int_{\{|u_n - v_0| < k\}} |\phi(T_{k+||v_0||}(u_n))||\nabla u_n|dx \\ & + \int_{\{|u_n - v_0| < k\}} |\phi(T_{k+||v_0||}(u_n))||\nabla v_0|dx \\ & + k||f||_{L^1} + \int_{\{|u_n - v_0| < k\}} |F||\nabla u_n|dx \\ & + \int_{\{|u_n - v_0| < k\}} |F||\nabla v_0|dx.\end{aligned}$$

Since $\phi \in C^0(I\!R, I\!R^N),\ F \in (L^{p'(x)}(\Omega))^N$ and using Young's inequality, we obtain

$$\begin{aligned}\alpha \int_{\{|u_n - v_0| < k\}} |\nabla u_n|^{p(x)}dx \leq & c_0 \int_{\{|u_n - v_0| < k\}} |a(x, u_n, \nabla u_n)|^{p'(x)}dx \\ & + c_1 \int_{\{|u_n - v_0| < k\}} |\nabla u_n|^{p(x)}dx \\ & + \frac{\alpha}{3} \int_{\{|u_n - v_0| < k\}} |\nabla u_n|^{p(x)}dx + c(k).\end{aligned}$$

From (3.1) and (3.3), we deduce

$$\begin{aligned}\alpha \int_{\{|u_n - v_0| < k\}} |\nabla u_n|^{p(x)}dx \leq & \frac{\alpha}{6} \int_{\{|u_n - v_0| < k\}} (|u_n|^{p(x)} + |\nabla u_n|^{p(x)})dx \\ & + c_1 \int_{\{|u_n - v_0| < k\}} |\nabla u_n|^{p(x)}dx \\ & + \frac{\alpha}{3} \int_{\{|u_n - v_0| < k\}} |\nabla u_n|^{p(x)}dx + c(k),\end{aligned}$$

hence,

$$\left(\frac{\alpha}{2}-c_1\right)\int_{\{|u_n-v_0|<k\}}|\nabla u_n|^{p(x)}dx \leq c(k),$$

where $c(k)$ is a constant which depends of k. Since $\{|u_n|\leq k\}\subset\{|u_n-v_0|\leq k+||v_0||_\infty\}$, we deduce that $\displaystyle\int_\Omega |\nabla T_k(u_n)|^{p(x)}dx \leq c(k)$. ■

Step 3: Strong convergence of truncations

In this step, we prove the strong convergence of truncations.

Proposition 3.2 *Let u_n be a solution of the problem (P_n), then there exists a measurable function u such that*

$$T_k(u_n)\to T_k(u) \text{ strongly in } W_0^{1,p(x)}(\Omega).$$

In order to prove Proposition 3.2, we will use the following lemma:

Lemma 3.6 *Assume that* (3.1)–(3.10) *hold true. Let u_n be a solution of the approximate problem (P_n). Then*

$$\int_\Omega |\nabla T_k(u_n-T_h(u_n))|^{p(x)}dx \leq k\,c \tag{3.17}$$

for all $k>h>||v_0||_\infty$, where c is a constant independent of k and $v_0\in K_\psi\cap L^\infty(\Omega)$.

Proof. Let $l\geq ||v_0||_\infty$. It is easy to see that $v=T_l(u_n-T_k(u_n-T_h(u_n)))\in K_\psi\cap L^\infty(\Omega)$. By using v as test function in (P_n) and letting $l\to\infty$, we obtain

$$\begin{aligned}\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n-T_h(u_n))dx+\int_\Omega H_n(x,u_n,\nabla u_n)T_k(u_n-T_h(u_n))dx\\ +\int_\Omega \phi(T_h(u_n))\nabla T_k(u_n-T_h(u_n))dx\\ \leq \int_\Omega f_nT_k(u_n-T_h(u_n))dx+\int_\Omega F\nabla T_k(u_n-T_h(u_n))dx.\end{aligned} \tag{3.18}$$

Let us define

$$\chi_{hk}(t)=\begin{cases}1 & \text{if}\quad h<|t|<h+k\\ 0 & \text{otherwise.}\end{cases}$$

We consider $\theta(t)=\phi(t)\chi_{hk}(t)$ and $\tilde\theta(t)=\displaystyle\int_0^t\theta(s)ds$. Then by Lemma 3.4, we obtain

$$\int_\Omega \phi(u_n)\nabla T_k(u_n-T_h(u_n))dx=\int_\Omega \phi(u_n)\chi_{hk}(u_n)\nabla u_ndx=\int_\Omega \theta(u_n)\nabla u_ndx=\int_\Omega div(\tilde\theta(u_n))dx=0.$$

Then, the second term of the left side of the inequality (3.18) vanishes for n large enough, which implies that

$$\begin{aligned}\int_\Omega a(x,\nabla u_n)\nabla T_k(u_n-T_h(u_n))dx+\int_\Omega H_n(x,u_n,\nabla u_n)u_n\chi_{hk}(u_n)dx\\ \leq k||f||_{L^1(\Omega)}+\int_\Omega F\nabla T_k(u_n-T_h(u_n))dx.\end{aligned}$$

By using Young's inequality, we can deduce that

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n-T_h(u_n))dx\leq k||f||_{L^1(\Omega)}+c_1+\frac{\alpha}{2}\int_\Omega |\nabla T_k(u_n-T_h(u_n))|^{p(x)}dx.$$

Since $\nabla T_k(u_n - T_h(u_n)) = \nabla u_n \chi_{hk}$ a.e. in Ω, then

$$\int_\Omega a(x, u_n, \nabla T_k(u_n - T_h(u_n)))\nabla T_k(u_n - T_h(u_n))dx \le kc_2 + \frac{\alpha}{2}\int_\Omega |\nabla T_k(u_n - T_h(u_n))|^{p(x)}dx.$$

Finally, from (3.3), we deduce (3.17) of Lemma 3.6. ■

Proof of Proposition 3.2. We will show firstly that $(u_n)_n$ is a Cauchy sequence in measure. Let $v_0 \in K_\psi \cap L^\infty(\Omega)$ and $k > 2h > 2||v_0||_\infty$ large enough, we have

$$k\ meas(\{|u_n - T_h(u_n)| > k\}) \le \int_{\{|u_n - T_h(u_n)|>k\}} |T_k(u_n - T_h(u_n))|\,dx.$$

Using (3.17) and applying Hölder's inequality and Poincaré's inequality, we obtain that

$$\begin{aligned} k\ meas(\{|u_n - T_h(u_n)| > k\}) &\le \int_\Omega |T_k(u_n - T_h(u_n))|\ dx \\ &\le \left(\frac{1}{p_-} + \frac{1}{p'_-}\right) \|1\|_{p'(x)} \|T_k(u_n - T_h(u_n))\|_{p(x)} \\ &\le \left(\frac{1}{p_-} + \frac{1}{p'_-}\right)\ (meas(\Omega)+1)^{\frac{1}{p'_-}}\ \|T_k(u_n - T_h(u_n))\|_{p(x)} \\ &\le C_4 k^{\frac{1}{\gamma}}, \end{aligned} \tag{3.19}$$

where

$$\gamma = \begin{cases} \frac{1}{p^-} & \text{if } \ \|\nabla T_k(u_n - T_h(u_n))\|_{p(x)} > 1 \\ \frac{1}{p^+} & \text{if } \ \|\nabla T_k(u_n - T_h(u_n))\|_{p(x)} \le 1. \end{cases} \tag{3.20}$$

Finally, for $k > 2h > 2||v_0||_\infty$, we have

$$meas\{|u_n| > k\} \le meas\{|u_n - T_h(u_n)| > k - h\} \le \frac{c}{(k-h)^{1-\frac{1}{\gamma}}}. \tag{3.21}$$

Passing to the limit as k goes to infinity, we deduce

$$meas(\{|u_n| > k\}) \longrightarrow 0, \tag{3.22}$$

then, for every $\varepsilon > 0$, there exists k_0 such that

$$meas\{|u_n| > k\} \le \frac{\varepsilon}{3} \quad \text{and} \quad meas\{|u_m| > k\} \le \frac{\varepsilon}{3} \qquad \forall k \ge k_0. \tag{3.23}$$

For every $\delta > 0$, we have

$$meas\ \{|u_n - u_m| > \delta\} \le meas\ \{|u_n| > k\} + meas\ \{|u_m| > k\} + meas\ \{|T_k(u_n) - T_k(u_m)| > \delta\}.$$

By (3.16), the sequence $(T_k(u_n))_n$ is bounded in $W_0^{1,p(x)}(\Omega)$, then there exists a subsequence $(T_k(u_n))_n$ such that $T_k(u_n)$ converges to η_k weakly in $W_0^{1,p(x)}(\Omega)$ as $n \to \infty$, and by the compact imbedding, we have $T_k(u_n)$ converges to η_k strongly in $L^{p(x)}(\Omega)$ a.e. in Ω. Thus, we can assume that $(T_k(u_n))_n$ is a Cauchy sequence in measure in Ω, then there exists an n_0 which depends on δ and ε such that

$$meas\{|T_k(u_n) - T_k(u_m)| > \delta\} \le \frac{\varepsilon}{3} \qquad \forall\ m,\ n \ge n_0 \text{ and } k \ge k_0. \tag{3.24}$$

By combining (3.23) and (3.24), we obtain

$$\forall\ \delta > 0,\ \exists\ \varepsilon > 0: \quad meas\{|u_n - u_m| > \delta\} \le \varepsilon \qquad \forall n,\ m \ge n_0(k_0, \delta).$$

Then $(u_n)_n$ is a Cauchy sequence in measure in Ω, thus, there exists a subsequence still denoted by u_n which converges almost everywhere to some measurable function u, then u_n converges to u a.e. in Ω, by Lemma 2.1, we obtain

$$\begin{cases} T_k(u_n) \rightharpoonup T_k(u) & \text{weakly in } W_0^{1,p(x)}(\Omega) \\ T_k(u_n) \to T_k(u) & \text{strongly in } L^{p(x)}(\Omega) \text{ and a.e. in } \Omega. \end{cases} \tag{3.25}$$

Now, we choose $v = T_l\Big(u_n - \exp(G(u_n))h_m(u_n - v_0)(T_k(u_n) - T_k(u))\Big)$ as test function in (P_n), where $G(s) = \int_0^s \frac{g(t)}{\alpha} dt$ and

$$h_m(s) = \begin{cases} 1 & \text{if } |s| \leq m \\ 0 & \text{if } |s| \geq m+1 \\ m+1-|s| & \text{if } m \leq |s| \leq m+1. \end{cases} \tag{3.26}$$

For every $n > m+1$, and by letting $l \to \infty$, we obtain that

$$\begin{aligned} &\int_\Omega a(x, u_n, \nabla u_n)\nabla \exp(G(u_n))h_m(u_n - v_0)(T_k(u_n) - T_k(u))dx \\ &+ \int_\Omega H_n(x, u_n, \nabla u_n)\exp(G(u_n))h_m(u_n - v_0)(T_k(u_n) - T_k(u))dx \\ &+ \int_\Omega \phi(u_n)\nabla \exp(G(u_n))h_m(u_n - v_0)(T_k(u_n) - T_k(u))dx \\ \leq &\int_\Omega f_n \exp(G(u_n))h_m(u_n - v_0)(T_k(u_n) - T_k(u))dx \\ &+ \int_\Omega F\nabla \exp(G(u_n))h_m(u_n - v_0)(T_k(u_n) - T_k(u))dx, \end{aligned}$$

which implies that

$$\begin{aligned} &\int_\Omega a(x, u_n, \nabla u_n)\nabla(T_k(u_n) - T_k(u))h_m(u_n - v_0)\exp(G(u_n))dx \\ &+ \int_\Omega a(x, u_n, \nabla u_n)\nabla u_n \frac{g(u_n)}{\alpha}(T_k(u_n) - T_k(u))h_m(u_n - v_0)\exp(G(u_n))dx \\ &+ \int_\Omega a(x, u_n, \nabla u_n)\nabla(u_n - v_0)h'_m(u_n - v_0)(T_k(u_n) - T_k(u))\exp(G(u_n))dx \\ &+ \int_\Omega \phi(u_n)\nabla(u_n - v_0)h'_m(u_n - v_0)(T_k(u_n) - T_k(u))\exp(G(u_n))dx \\ &+ \int_\Omega \phi(u_n)\nabla(T_k(u_n) - T_k(u))h_m(u_n - v_0)\exp(G(u_n))dx \\ &+ \int_\Omega \phi(u_n)\nabla u_n \frac{g(u_n)}{\alpha}(T_k(u_n) - T_k(u))h_m(u_n - v_0)\exp(G(u_n))dx \\ \leq &\int_\Omega (f_n + \gamma(x))h_m(u_n - v_0)(T_k(u_n) - T_k(u))\exp(G(u_n))dx \\ &+ \int_\Omega g(u_n)|\nabla u_n|^{p(x)}h_m(u_n - v_0)(T_k(u_n) - T_k(u))\exp(G(u_n))dx \\ &+ \int_\Omega F\nabla(u_n - v_0)h'_m(u_n - v_0)(T_k(u_n) - T_k(u))\exp(G(u_n))dx \\ &+ \int_\Omega F\nabla u_n \frac{g(u_n)}{\alpha}(T_k(u_n) - T_k(u))h_m(u_n - v_0)\exp(G(u_n))dx \\ &+ \int_\Omega F\nabla(T_k(u_n) - T_k(u))h_m(u_n - v_0)\exp(G(u_n))dx. \end{aligned}$$

In view of (3.3) we have

$$\begin{aligned}
&\int_\Omega a(x,u_n,\nabla u_n)\nabla(T_k(u_n)-T_k(u))h_m(u_n-v_0)\exp(G(u_n))dx\\
&+\int_\Omega a(x,u_n,\nabla u_n)\nabla(u_n-v_0)h_m'(u_n-v_0)(T_k(u_n)-T_k(u))\exp(G(u_n))dx\\
&+\int_\Omega \phi(u_n)\nabla(u_n-v_0)h_m'(u_n-v_0)(T_k(u_n)-T_k(u))\exp(G(u_n))dx\\
&+\int_\Omega \phi(u_n)\nabla(T_k(u_n)-T_k(u))h_m(u_n-v_0)\exp(G(u_n))dx\\
&+\int_\Omega \phi(u_n)\nabla u_n\frac{g(u_n)}{\alpha}(T_k(u_n)-T_k(u))h_m(u_n-v_0)\exp(G(u_n))dx\\
\le&\int_\Omega (f_n+\gamma(x))h_m(u_n-v_0)(T_k(u_n)-T_k(u))\exp(G(u_n))dx\\
&+\int_\Omega F\nabla(u_n-v_0)h_m'(u_n-v_0)(T_k(u_n)-T_k(u))\exp(G(u_n))dx\\
&+\int_\Omega F\nabla u_n\frac{g(u_n)}{\alpha}(T_k(u_n)-T_k(u))h_m(u_n-v_0)\exp(G(u_n))dx\\
&+\int_\Omega F\nabla(T_k(u_n)-T_k(u))h_m(u_n-v_0)\exp(G(u_n))dx.
\end{aligned}$$

The pointwise convergence of u_n to u, the bounded character of h_m and T_k make it possible to conclude that $h_m(u_n-v_0)(T_k(u_n)-T_k(u))$ converges to 0 in $L^\infty(\Omega)$ weakly-$*$, as $n\to\infty$, remark that $\exp(G(u_n))\le\exp\left(\frac{\|g\|_{L^1(I\!R)}}{\alpha}\right)$ then

$$\int_\Omega (f_n+\gamma(x))h_m(u_n-v_0)(T_k(u_n)-T_k(u))\exp(G(u_n))dx=\epsilon(n), \tag{3.27}$$

where $\epsilon(n)$ tends to 0 as n tends to $+\infty$. Moreover, by using Lebesgue's theorem, we get $\phi(u_n)h_m(u_n-v_0)$ converges to $\phi(u)h_m(u-v_0)$ strongly in $L^{p'(x)}(\Omega)$, and since $\nabla T_k(u_n)$ converges to $\nabla T_k(u)$ weakly in $L^{p(x)}(\Omega)$, we can deduce that

$$\int_\Omega \phi(u_n)\nabla(T_k(u_n)-T_k(u))h_m(u_n-v_0)\exp(G(u_n))dx=\epsilon(n). \tag{3.28}$$

Similarly we have

$$\int_\Omega F\nabla(T_k(u_n)-T_k(u))h_m(u_n-v_0)\exp(G(u_n))dx=\epsilon(n). \tag{3.29}$$

On the other hand, remark that

$$\begin{aligned}
&\left|\int_\Omega F\nabla(u_n-v_0)h_m'(u_n-v_0)(T_k(u_n)-T_k(u))dx\right|\\
=&\left|\int_\Omega F\nabla(u_n-v_0)(T_k(u_n)-T_k(u))\chi_{\{m<|u_n-v_0|<m+1\}}dx\right|\\
\le&\int_\Omega |F\nabla(T_M(u_n)-v_0)(T_k(u_n)-T_k(u))|\,dx
\end{aligned}$$

with $M=m+1+||v_0||_\infty$. By Lebesgue's dominated convergence theorem, we deduce that $F(T_k(u_n)-T_k(u))$ converges to 0 strongly in $L^{p'(x)}(\Omega)$, and since $\nabla(T_M(u_n)-v_0)$ converges to $\nabla(T_M(u)-v_0)$ weakly in $(L^{p(x)}(\Omega))^N$, we obtain

$$\left|\int_\Omega F\nabla(u_n-v_0)h_m'(u_n-v_0)(T_k(u_n)-T_k(u))\exp(G(u_n))dx\right|=\epsilon(n). \tag{3.30}$$

Similarly, we can write

$$\int_\Omega \phi(u_n)\nabla(u_n - v_0)h'_m(u_n - v_0)(T_k(u_n) - T_k(u))\exp(G(u_n))dx = \epsilon(n). \tag{3.31}$$

Moreover, by using Lebesgue's theorem, we have

$$F\frac{g(u_n)}{\alpha}h_m(u_n - v_0)(T_k(u_n) - T_k(u)) \to 0 \quad \text{in} \quad L^{p'(x)}(\Omega),$$

and since $\nabla u_n \to \nabla u$ weakly in $(L^{p(x)}(\Omega))^N$, we have

$$\int_\Omega F\nabla u_n \frac{g(u_n)}{\alpha}(T_k(u_n) - T_k(u))h_m(u_n - v_0)\exp(G(u_n))dx = \epsilon(n). \tag{3.32}$$

Similarly, we can write

$$\int_\Omega \phi(u_n)\nabla u_n \frac{g(u_n)}{\alpha}(T_k(u_n) - T_k(u))h_m(u_n - v_0)\exp(G(u_n))dx = \epsilon(n). \tag{3.33}$$

We claim that

$$\int_\Omega a(x, u_n, \nabla u_n)\nabla(u_n - v_0)h'_m(u_n - v_0)(T_k(u_n) - T_k(u))dx = \epsilon(n). \tag{3.34}$$

Indeed, we have

$$\begin{aligned}
&\left|\int_\Omega a(x, u_n, \nabla u_n)\nabla(u_n - v_0)h'_m(u_n - v_0)(T_k(u_n) - T_k(u))dx\right| \\
\le& 2k \int_{\{m\le|u_n - v_0|\le m+1\}} \Big|a(x, u_n, \nabla u_n)\nabla(u_n - v_0)\Big|dx \\
\le& 2k\left(\int_{\{l\le|u_n|\le l+s\}} a(x, u_n, \nabla u_n)\nabla u_n dx + \int_{\{l\le|u_n|\le l+s\}} |a(x, \nabla u_n)||\nabla v_0|dx\right)
\end{aligned}$$

where $l = m - ||v_0||_\infty$ and $s = 2||v_0||_\infty + 1$. Now we choose $v = u_n - T_s(u_n - T_l(u_n))$ as test function in (P_n), we get

$$\begin{aligned}
&\int_{\{l\le|u_n|\le l+s\}} a(x, u_n, \nabla u_n)\nabla u_n dx + \int_{\{l\le|u_n|\le l+s\}} H_n(x, u_n, \nabla u_n)u_n dx + \int_\Omega div(\tilde{\theta}(u_n))dx \\
\le& \int_\Omega f_n T_s(u_n - T_l(u_n))dx + \int_\Omega F\nabla T_s(u_n - T_l(u_n))dx,
\end{aligned}$$

where $\tilde{\theta_s}(t) = \displaystyle\int_0^t \theta_s(z)dz$ and $\theta_s(z) = \phi(z)\chi_{sl}(z)$ with

$$\chi_{sl} = \begin{cases} 1 & l \le t \le l + s \\ 0 & \text{otherwise.} \end{cases}$$

Since $\tilde{\theta}(u_n) \in (W_0^{1,p(x)}(\Omega))^N$ and by Lemma 3.4, we get

$$\int_{\{l\le|u_n|\le l+s\}} a(x, u_n, \nabla u_n)\nabla u_n dx \le s\int_{\{|u_n|>l\}} |f_n|dx + \int_{\{l\le|u_n|\le l+s\}} F\nabla u_n dx. \tag{3.35}$$

Firstly, we show that

$$\int_{\{l\le|u_n|\le l+s\}} F\nabla u_n dx = \epsilon(n, l).$$

Indeed, by (3.35) and Young's inequality, we get

$$\int_{\{l\leq|u_n|\leq l+s\}} a(x,u_n,\nabla u_n)\nabla u_n dx \leq s\int_{\{|u_n|>l\}} |f_n|dx + c\int_{\{|u_n|>l\}} |F|^{p'(x)}dx$$
$$+\frac{\alpha}{2}\int_{\{l\leq|u_n|\leq l+s\}} |\nabla u_n|^{p(x)}dx.$$

By (3.3), we obtain

$$\frac{\alpha}{2}\int_{\{l\leq|u_n|\leq l+s\}} |\nabla u_n|^{p(x)}dx \leq s\int_{\{|u_n|>l\}} |f_n|dx + c\int_{\{|u_n|>l\}} |F|^{p'(x)}dx,$$

which implies that

$$\int_\Omega |\nabla T_s(u_n - T_l(u_n))|^{p(x)}dx \leq \frac{2s}{\alpha}\int_{\{|u_n|>l\}} |f_n|dx + \frac{2c}{\alpha}\int_{\{|u_n|>l\}} |F|^{p'(x)}dx.$$

We use the $L^1(\Omega)$ strong convergence of f_n and since $F \in L^{p'(x)}(\Omega)$, we have by using Lebesgue's theorem, as first n and then l tends to infinity

$$\lim_{l\to+\infty}\lim_{n\to+\infty}\int_\Omega |\nabla T_s(u_n - T_l(u_n))|^{p(x)}dx = 0,$$

which implies by Hölder's inequality that

$$\lim_{l\to+\infty}\lim_{n\to+\infty}\int_\Omega F\nabla T_s(u_n - T_l(u_n))dx = 0.$$

So that

$$\int_{\{l\leq|u_n|\leq l+s\}} F\nabla u_n dx = \epsilon(n,l). \tag{3.36}$$

Finally by (3.35) and (3.36) we deduce

$$\int_{\{l\leq|u_n|\leq l+s\}} a(x,u_n,\nabla u_n)\nabla u_n dx = \epsilon(n,l). \tag{3.37}$$

On the other hand

$$\begin{aligned}
&\int_{\{l\leq|u_n|\leq l+s\}} |a(x,u_n,\nabla u_n)||\nabla v_0|dx\\
\leq &c\left(\int_\Omega |a(x,\nabla T_s(u_n - T_l(u_n)))|^{p'(x)}dx\right)^\gamma \|\nabla v_0\chi_{\{|u_n|>l\}}\|_{p(x)}\\
\leq &c\left(\int_\Omega |k(x) + |\nabla T_s(u_n - T_l(u_n))|^{p(x)} + |T_s(u_n - T_l(u_n))|^{p(x)}dx\right)^\gamma \|\nabla v_0\chi_{\{|u_n|>l\}}\|_{p(x)},
\end{aligned} \tag{3.38}$$

where

$$\gamma = \begin{cases} \frac{1}{p'^-} & \text{if} \quad ||a(x,\nabla T_s(u_n - T_l(u_n)))||_{p'(x)} \geq 1 \\ \frac{1}{p'^+} & \text{if} \quad ||a(x,\nabla T_s(u_n - T_l(u_n)))||_{p'(x)} < 1. \end{cases}$$

Furthermore, by Lemma 3.6 we have

$$\int_\Omega |\nabla T_s(u_n - T_l(u_n))|^{p(x)}dx \leq c(s), \tag{3.39}$$

and

$$\int_\Omega |T_s(u_n - T_l(u_n))|^{p(x)}dx \leq c'(s), \tag{3.40}$$

where $c(s)$ and $c'(s)$ are two constants independent of l. By (3.38), (3.39) and (3.40), we obtain

$$\int_{\{l\leq|u_n|\leq l+s\}} |a(x,u_n,\nabla u_n)||\nabla v_0|dx = \epsilon(n,l). \tag{3.41}$$

Finally, from (3.37) and (3.41) follows the estimate (3.34) combining (3.27), (3.28), (3.29), (3.30), (3.31), (3.34) and $l = m - ||v_0||_\infty$, we get

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla(T_k(u_n) - T_k(u))h_m(u_n - v_0)dx \leq \epsilon(n,m). \tag{3.42}$$

Splitting the first integral on the left hand side of (3.42) where $|u_n| \leq k$ and $|u_n| > k$, we can write

$$\begin{aligned}
&\int_\Omega a(x,u_n,\nabla u_n)\nabla(T_k(u_n) - T_k(u))h_m(u_n - v_0)dx \\
&= \int_{\{|u_n|\leq k\}} a(x,T_k(u_n),\nabla T_k(u_n))\nabla(T_k(u_n) - T_k(u))h_m(u_n - v_0)dx \\
&\quad - \int_{\{|u_n|>k\}} a(x,u_n,\nabla u_n)\nabla T_k(u)h_m(u_n - v_0)dx \\
&\geq \int_{\{|u_n|\leq k\}} a(x,T_k(u_n),\nabla T_k(u_n))\nabla(T_k(u_n) - T_k(u))h_m(u_n - v_0)dx \\
&\quad - \int_\Omega |a(x,T_M(u_n),\nabla T_M(u_n))||\nabla T_k(u)|\chi_{\{|u_n|>k\}}dx,
\end{aligned}$$

where $M = m + ||v_0||_\infty + 1$. Since $a(x,T_M(u_n),\nabla T_M(u_n))$ is bounded in $(L^{p'(x)}(\Omega))^N$, we have for a subsequence $a(x,T_M(u_n),\nabla T_M(u_n)) \rightharpoonup l_m$ weakly in $(L^\infty(\Omega))^N$ as $n \to +\infty$. Since $\left|\frac{\partial T_k(u_n)}{\partial x_i}\right| \chi_{\{|u_n|>k\}}$ converges to $\left|\frac{\partial T_k(u)}{\partial x_i}\right| \chi_{\{|u|>k\}} = 0$ strongly in $L^{p(x)}(\Omega)$, we get

$$\int_\Omega |a(x,T_M(u_n),\nabla T_M(u_n))||\nabla T_k(u)|\chi_{\{|u_n|>k\}}dx = \epsilon(n). \tag{3.43}$$

From (3.42) and (3.43), we have

$$\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla(T_k(u_n) - T_k(u))h_m(u_n - v_0)dx \leq \epsilon(n,m). \tag{3.44}$$

It is easy to see that

$$\begin{aligned}
&\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla(T_k(u_n) - T_k(u))h_m(u_n - v_0)dx \\
&= \int_\Omega \Big(a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))\Big)\nabla(T_k(u_n) - T_k(u))h_m(u_n - v_0)dx \\
&\quad + \int_\Omega a(x,T_k(u_n),\nabla T_k(u))\nabla(T_k(u_n) - T_k(u))h_m(u_n - v_0)dx \\
&= \int_\Omega \Big(a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))\Big)\nabla(T_k(u_n) - T_k(u))h_m(u_n - v_0)dx \\
&\quad + \int_\Omega a(x,T_k(u_n),\nabla T_k(u))\nabla T_k(u_n)h_m(u_n - v_0)dx \\
&\quad - \int_\Omega a(x,T_k(u_n),\nabla T_k(u))\nabla T_k(u)h_m(u_n - v_0)dx.
\end{aligned} \tag{3.45}$$

By using the continuity of the Nemytskii operator, we have that $a(x,T_k(u_n),\nabla T_k(u))h_m(u_n - v_0)$ converges to $a(x,T_k(u),\nabla T_k(u))h_m(u - v_0)$ strongly in $(L^{p'(x)}(\Omega)^N$ while $\frac{\partial T_k(u_n)}{\partial x_i}$ converges to $\frac{\partial T_k(u)}{\partial x_i}$

weakly in $L^{p(x)}(\Omega)$, the second and the third term of the right hand side of (3.45) tend respectively to $\int_\Omega a(x,T_k(u),\nabla T_k(u))\nabla T_k(u)h_m(u-v_0)dx$ and $-\int_\Omega a(x,T_k(u),\nabla T_k(u))\nabla T_k(u)h_m(u-v_0)dx$. So that (3.44) and (3.45) yield

$$\int_\Omega \Big(a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u))\Big)\nabla(T_k(u_n)-T_k(u))h_m(u_n-v_0)dx \le \epsilon(n,m) \quad (3.46)$$

which implies that

$$\begin{aligned}
&\int_\Omega \Big(a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u))\Big)\nabla(T_k(u_n)-T_k(u))dx\\
=&\int_\Omega \Big(a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u))\Big)\nabla(T_k(u_n)-T_k(u))h_m(u_n-v_0)dx\\
&+\int_\Omega \Big(a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u))\Big)\nabla(T_k(u_n)-T_k(u))(1-h_m(u_n-v_0))\,dx.
\end{aligned} \quad (3.47)$$

Since $1-h_m(u_n-v_0)=0$ in $\{x\in\Omega : |u_n-v_0|<m\}$ and since $\{x\in\Omega : |u_n|<k\}\subset\{x\in\Omega : |u_n-v_0|<m\}$ for m large enough, we deduce from (3.47)

$$\begin{aligned}
&\int_\Omega \Big(a(x,T_k(u_n),\nabla T_k(u_n))-a(x,\nabla T_k(u))\Big)\nabla(T_k(u_n)-T_k(u))dx\\
=&\int_\Omega \Big(a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u))\Big)\nabla(T_k(u_n)-T_k(u))h_m(u_n-v_0)dx\\
&-\int_{\{|u_n|>k\}} a(x,T_k(u_n),\nabla T_k(u))\nabla T_k(u)dx.
\end{aligned}$$

It is easy to see that, the last term of the last inequality tends to zero as $n\to+\infty$, which implies that

$$\begin{aligned}
&\int_\Omega \Big(a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u))\Big)\nabla(T_k(u_n)-T_k(u))dx\\
=&\int_\Omega \Big(a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u))\Big)\nabla(T_k(u_n)-T_k(u))h_m(u_n-v_0)dx\\
&+\ \epsilon(n).
\end{aligned} \quad (3.48)$$

Combining (3.46) and (3.48), we obtain

$$\int_\Omega \Big(a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u))\Big)\nabla(T_k(u_n)-T_k(u))dx\le\epsilon(n,m).$$

By passing to the lim-sup over n and letting m tend to infinity, we obtain

$$\limsup_{m\to\infty}\limsup_{n\to\infty}\int_\Omega \Big(a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u))\Big)\nabla(T_k(u_n)-T_k(u))dx=0,$$

thus implies by Lemma 3.1

$$T_k(u_n)\to T_k(u) \text{ strongly in } W_0^{1,p(x)}(\Omega). \quad (3.49)$$

■

Step 4: Passing to the limit in (P_n)

In order to pass to the limit in approximate equation, we now show that

$$H_n(x, u_n, \nabla u_n) \to H(x, u, \nabla u) \text{ strongly in } L^1(\Omega).$$

In particular, it is enough to prove the equi-integrability of the sequence $(H_n(x, u_n, \nabla u_n))_n$. To this purpose, we take $T_{l+1}(u_n) - T_l(u_n)$ as test function in P_n we obtain

$$\int_{\{|u_n|>l+1\}} |H_n(x, u_n, \nabla u_n)|dx \leq \int_{\{|u_n|>l\}} |f_n|dx. \tag{3.50}$$

Let $\varepsilon > 0$ be fixed. Then there exists $l(\varepsilon) \geq 1$, such that

$$\int_{\{|u_n|>l(\varepsilon)\}} |H_n(x, u_n, \nabla u_n)|dx \leq \frac{\varepsilon}{2}. \tag{3.51}$$

For any measurable subset $E \subset \Omega$, we have

$$\int_E |H_n(x, u_n, \nabla u_n)|dx \leq \int_E (\gamma(x) + g(l(\varepsilon)|\nabla T_{l(\varepsilon)}(u_n)|^{p(x)})dx + \int_{\{|u_n|>l(\varepsilon)\}} |H_n(x, u_n, \nabla u_n)|dx.$$

In view of (3.49), there exists $\eta(\varepsilon) > 0$, such that

$$\int_E \gamma(x) + g(l(\varepsilon)|\nabla T_{l(\varepsilon)}(u_n)|^{p(x)}dx \leq \frac{\varepsilon}{2} \quad \text{for all } E \text{ such that } meas(E) < \eta(\varepsilon). \tag{3.52}$$

Finally, by combining (3.51) and (3.52) we have

$$\int_E |H_n(x, u_n, \nabla u_n)|dx \leq \varepsilon \quad \text{for all } E \text{ such that } meas(E) < \eta(\varepsilon),$$

then, we deduce that $(H_n(x, u_n, \nabla u_n))_n$ are uniformly equi-integrable in Ω.
Let $v \in K_\psi \cap L^\infty(\Omega)$, we take $T_l(u_n - T_k(u_n - v))$ as test function in (P_n) and letting l to ∞, we can write, for n large enough $(n > k + ||v||_\infty)$

$$\begin{aligned}&\int_\Omega a(x, u_n, \nabla u_n)\nabla T_k(u_n - v)dx + \int_\Omega H_n(x, u_n, \nabla u_n)T_k(u_n - v)dx \int_\Omega \phi(u_n)\nabla T_k(u_n - v)dx \\ &\leq \int_\Omega f_n T_k(u_n - v)dx + \int_\Omega F\nabla T_k(u_n - v)dx.\end{aligned}$$

We get

$$\begin{aligned}&\int_\Omega a(x, T_{k+||v||_\infty}(u_n), \nabla T_{k+||v||_\infty}(u_n))\nabla T_k(u_n - v)dx + \int_\Omega H_n(x, u_n, \nabla u_n)T_k(u_n - v)dx \\ &+ \int_\Omega \phi(T_{k+||v||_\infty}(u_n))\nabla T_k(u_n - v)dx \leq \int_\Omega f_n T_k(u_n - v)dx + \int_\Omega F\nabla T_k(u_n - v)dx.\end{aligned}$$

By Fatou's lemma and by the fact that $a(x, T_{k+||v||_\infty}(u_n), \nabla T_{k+||v||_\infty}(u_n))$ converges weakly in $(L^{p'(x)}(\Omega))^N$ to $a(x, T_{k+||v||_\infty}(u), \nabla T_{k+||v||_\infty}(u))$, it is easy to see that

$$\begin{aligned}&\int_\Omega a(x, T_{k+||v||_\infty}(u), \nabla T_{k+||v||_\infty}(u))\nabla T_k(u - v)dx \\ &\leq \liminf_{n\to\infty} \int_\Omega a(x, T_{k+||v||_\infty}(u_n), \nabla T_{k+||v||_\infty}(u_n))\nabla T_k(u_n - v)dx.\end{aligned} \tag{3.53}$$

On the other hand, since $F \in (L^{p'(x)}(\Omega))^N$, we deduce that the integral

$$\int_\Omega F\nabla T_k(u_n - v)dx \to \int_\Omega F\nabla T_k(u - v)dx \quad \text{as} \quad n \to \infty. \tag{3.54}$$

Moreover, using Lebesgue's dominated convergence theorem, we deduce that $\phi(T_{k+||v||_\infty}(u_n))$ converges to $\phi(T_{k+||v||_\infty}(u))$ strongly in $(L^{p'(x)}(\Omega))^N$ and $\nabla T_k(u_n - v)$ converges to $\nabla T_k(u-v)$ weakly in $(L^{p(x)}(\Omega))^N$ as $n \to +\infty$, so that

$$\int_\Omega \phi(T_{k+||v||_\infty}(u_n))\nabla T_k(u_n - v)dx \to \int_\Omega \phi(T_{k+||v||_\infty}(u))\nabla T_k(u-v)dx \quad \text{as } n \to \infty. \tag{3.55}$$

Similarly, we have

$$\int_\Omega f_n T_k(u_n - v)dx \to \int_\Omega f T_k(u-v)dx. \tag{3.56}$$

$$\int_\Omega H_n(x, u_n, \nabla u_n)T_k(u_n - v)dx \to \int_\Omega H(x, u, \nabla u)T_k(u-v)dx. \tag{3.57}$$

Using (3.53), (3.54), (3.55), (3.56) and (3.57), we can pass to the limit in (3.53) then we have

$$\begin{aligned}&\int_\Omega a(x,u,\nabla u)\nabla T_k(u-v)dx + \int_\Omega H(x,u,\nabla u)T_k(u-v)dx + \int_\Omega \phi(u)\nabla T_k(u-v)dx \\ &\le \int_\Omega f T_k(u-v)dx + \int_\Omega F\nabla T_k(u-v)dx.\end{aligned}$$

As a conclusion of Step 1, Step 2, Step 3 and Step 4, the proof of Theorem 3.1 is complete. ■

Remark 3.2 *Note that the condition (3.10) is used essentially to prove the coercivity of the operator B_n. We can prove the coercivity of the operator B_n if we replace the condition* (3.10) *by the condition*

$$p^+ - p^- < 1, \tag{3.58}$$

this is the objective of the following theorem:

Theorem 3.3 *Assume that (3.1)–(3.9) hold. In addition, let us assume that (3.58) also holds. Then, there exists at least one solution of the unilateral problem (P).*

Proof. Following the same steps of the argument of the proof of Theorem 3.1, it suffices to show the coercivity of the operator B_n. Let $v_0 \in K_\psi$, from Hölder's inequality and the growth condition we have

$$\begin{aligned}\langle Av, v_0\rangle &= \int_\Omega a(x,v,\nabla v)\nabla v_0 dx \\ &\le C(\frac{1}{p^-} + \frac{1}{p'^-})\left(\int_\Omega |a(x,v,\nabla v)|^{p'(x)}\, dx\right)^{\gamma'} ||v_0||_{W_0^{1,p(x)}(\Omega)} \\ &\le C(\frac{1}{p^-} + \frac{1}{p'^-})||v_0||_{W_0^{1,p(x)}(\Omega)}\left(\int_\Omega \beta(k(x)^{p'(x)} + |v|^{p(x)} + |\nabla v|^{p(x)})dx\right)^{\gamma'} \\ &\le C_0\left(C_1 + \rho(v) + \rho(\nabla v)\right)^{\gamma'} \\ &\le C_0\left(C_1 + C(\rho(\nabla v))^{\frac{p^+}{p^-}} + \rho(\nabla v)\right)^{\gamma'},\end{aligned}$$

where

$$\gamma' = \begin{cases} \frac{1}{p'^-} & \text{if} \quad \|a(x,v,\nabla v)\|_{L^{p'(x)}(\Omega)} > 1 \\ \frac{1}{p'^+} & \text{if} \quad \|a(x,v,\nabla v)\|_{L^{p'(x)}(\Omega)} \le 1. \end{cases} \tag{3.59}$$

From (3.3) we have

$$\frac{\langle Av, v\rangle}{||v||_{1,p(x)}} - \frac{\langle Av, v_0\rangle}{||v||_{1,p(x)}} \ge \frac{1}{||v||_{1,p(x)}}\left(\alpha\rho(\nabla v) - C_0(C_1 + C(\rho(\nabla v))^{\frac{p^+}{p^-}} + \rho(\nabla v)\right)^{\gamma'}. \tag{3.60}$$

Since $||v||_{1,p(x)} \to \infty$, we have $||a(x,v,\nabla v)||_{L^{p'(x)}(\Omega)} > 1$, then $\gamma' = \frac{1}{p'^-}$, and due to the fact that $p^+ - p^- < 1$, we have $\frac{p^+}{p'^- p^-} < 1$, then

$$\frac{\langle Av, v\rangle}{||v||_{1,p(x)}} - \frac{\langle Av, v_0\rangle}{||v||_{1,p(x)}} \to \infty \quad \text{as} \quad ||v||_{1,p(x)} \to \infty.$$

Since

$$\frac{< G_n v, v >}{||v||_{1,p(x)}}, \ \frac{< G_n v, v_0 >}{||v||_{1,p(x)}}, \ \frac{< R_n v, v >}{||v||_{1,p(x)}} \text{ and } \frac{< R_n v, v_0 >}{||v||_{1,p(x)}}$$

are bounded, we have:

$$\frac{< B_n v, v - v_0 >}{||v||_{1,p(x)}} \longrightarrow \infty \text{ as } ||v||_{1,p(x)} \to \infty.$$

Therefore Theorem 3.3 holds true. ■

4 Appendix

Proof of Lemma 3.5. Let $v_0 \in K_\psi$. From Hölder's inequality and using the growth condition, we obtain

$$\begin{aligned}
\langle Av, v_0\rangle &= \int_\Omega a(x,v,\nabla v)\nabla v_0 dx \\
&\le C(\frac{1}{p^-} + \frac{1}{p'^-})\Big(\int_\Omega |a(x,v,\nabla v)|^{p'(x)}\Big)^{\gamma'} ||v_0||_{W_0^{1,p(x)}(\Omega)} \\
&\le C(\frac{1}{p^-} + \frac{1}{p'^-})||v_0||_{W_0^{1,p(x)}(\Omega)}\Big(\int_\Omega \beta(k(x)^{p'(x)} + |v|^{p(x)} + |\nabla v|^{p(x)})dx\Big)^{\gamma'} \\
&\le C_0\Big(C_1 + \rho(v) + \rho(\nabla v)\Big)^{\gamma'} \\
&\le C_0\Big(C_1 + C\rho(\nabla v) + \rho(\nabla v)\Big)^{\gamma'},
\end{aligned}$$

where

$$\gamma' = \begin{cases} \frac{1}{p'^-} & \text{if } ||a(x,v,\nabla v)||_{L^{p'(x)}(\Omega)} \ge 1 \\ \frac{1}{p'^+} & \text{if } ||a(x,v,\nabla v)||_{L^{p'(x)}(\Omega)} \le 1. \end{cases} \tag{4.1}$$

From (3.3) we have

$$\frac{\langle Av, v\rangle}{||v||_{1,p(x)}} - \frac{\langle Av, v_0\rangle}{||v||_{1,p(x)}} \ge \frac{1}{||v||_{1,p(x)}}\Big(\alpha\rho(\nabla v) - C_0(C_1 + C\rho(\nabla v) + \rho(\nabla v))^{\gamma'}\Big). \tag{4.2}$$

Hence $\dfrac{\rho(\nabla v)}{||v||_{1,p(x)}} \to \infty$ as $||v||_{1,p(x)} \to \infty$, we have

$$\frac{\langle Av, v\rangle}{||v||_{1,p(x)}} - \frac{\langle Av, v_0\rangle}{||v||_{1,p(x)}} \longrightarrow \infty \text{ as } ||v||_{1,p(x)} \to \infty.$$

Since

$$\frac{\langle G_n v, v\rangle}{||v||_{1,p(x)}}, \ \frac{\langle G_n v, v_0\rangle}{||v||_{1,p(x)}}, \ \frac{\langle R_n v, v\rangle}{||v||_{1,p(x)}} \text{ and } \frac{\langle R_n v, v_0\rangle}{||v||_{1,p(x)}}$$

are bounded, we have

$$\frac{\langle B_n v, v - v_0\rangle}{||v||_{1,p(x)}} = \frac{\langle Av, v - v_0\rangle}{||v||_{1,p(x)}} + \frac{\langle G_n v, v\rangle}{||v||_{1,p(x)}} - \frac{\langle G_n v, v_0\rangle}{||v||_{1,p(x)}} \longrightarrow \infty \text{ as } ||v||_{1,p(x)} \to \infty.$$

It remains to show that B_n is pseudo-monotone. Let $(u_k)_k$ be a sequence in $W_0^{1,p(x)}(\Omega)$ such that

$$\begin{cases} u_k \rightharpoonup u \quad \text{weakly in } \ W_0^{1,p(x)}(\Omega) \\ B_n u_k \rightharpoonup \chi \text{weakly in } \ W^{-1,p'(x)}(\Omega) \\ \limsup_{k\to\infty} \langle B_n u_k, u_k\rangle \le \langle \chi, u\rangle. \end{cases} \tag{4.3}$$

We will prove that

$$\chi = B_n u \ \text{ and } \ \langle B_n u_k, u_k\rangle \longrightarrow \langle \chi, u\rangle \ \text{ as } \ k\to+\infty.$$

Firstly, since $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$, then $u_k \to u$ in $L^{p(x)}(\Omega)$ for a subsequence still denoted by $(u_k)_k$. We have $(u_k)_k$ is a bounded sequence in $W_0^{1,p(x)}(\Omega)$, then by the growth condition, $(a(x,u_k,\nabla u_k))_k$ is bounded in $(L^{p'(x)}(\Omega))^N$, therefore there exists a function $\varphi \in (L^{p'(x)}(\Omega))^N$ such that

$$a(x,u_k,\nabla u_k) \rightharpoonup \varphi \text{ weakly in } \ (L^{p'(x)}(\Omega))^N \ \text{ as } \ k\to\infty. \tag{4.4}$$

We have $\phi_n = \phi\circ T_n$ is a continuous function, and since $u_k \to u$ in $L^{p(x)}(\Omega)$ then

$$\phi_n(u_k) \to \phi_n(u) \text{ strongly in } \ (L^{p'(x)}(\Omega))^N \ \text{ as } \ k\to\infty. \tag{4.5}$$

Similarly, since $(H_n(x,u_k,\nabla u_k))_k$ is bounded in $L^{p'(x)}(\Omega)$, then there exists a function $\psi_n \in L^{p'(x)}(\Omega)$ such that

$$H_n(x,u_k,\nabla u_k) \rightharpoonup \psi_n \text{ weakly in } \ L^{p'(x)}(\Omega) \ \text{ as } \ k\to\infty. \tag{4.6}$$

It is clear that, for all $v \in W_0^{1,p(x)}(\Omega)$,

$$\begin{aligned} \langle \chi, v\rangle &= \lim_{k\to\infty} \langle B_n u_k, v\rangle \\ &= \lim_{k\to\infty}\int_\Omega a(x,u_k,\nabla u_k)\nabla v\,dx + \lim_{k\to\infty}\int_\Omega H_n(x,u_k,\nabla u_k)v\,dx - \lim_{k\to\infty}\int_\Omega \phi_n(u_k)\nabla v\,dx \\ &= \int_\Omega \varphi\nabla v\,dx + \int_\Omega \psi_n v\,dx - \int_\Omega \phi_n(u)\nabla v\,dx. \end{aligned} \tag{4.7}$$

On the one hand, by (4.5) we have

$$\int_\Omega \phi_n(u_k)\nabla u_k\,dx \longrightarrow \int_\Omega \phi_n(u)\nabla u\,dx \ \text{ as } \ k\to\infty, \tag{4.8}$$

and by (4.6) we have

$$\int_\Omega H_n(x,u_k,\nabla u_k)u_k\,dx \longrightarrow \int_\Omega \psi_n u\,dx \ \text{ as } \ k\to\infty, \tag{4.9}$$

by combining (4.3) and (4.7), we have

$$\begin{aligned} &\limsup_{k\to\infty}\langle B_n(u_k), u_k\rangle \\ &= \limsup_{k\to\infty}\left\{\int_\Omega a(x,u_k,\nabla u_k)\nabla u_k\,dx + \int_\Omega H_n(x,u_k,\nabla u_k)u_k\,dx - \int_\Omega \phi_n(u_k)\nabla u_k\,dx\right\} \\ &\le \int_\Omega \varphi\nabla u\,dx + \int_\Omega \psi_n u\,dx - \int_\Omega \phi_n(u)\nabla u dx. \end{aligned} \tag{4.10}$$

Therefore

$$\limsup_{k\to\infty}\int_\Omega a(x,u_k,\nabla u_k)\nabla u_k\,dx \le \int_\Omega \varphi\nabla u\,dx. \tag{4.11}$$

On the other hand, thanks to (3.3), we have

$$\int_\Omega (a(x,u_k,\nabla u_k) - a(x,u_k,\nabla u))(\nabla u_k - \nabla u)\,dx > 0, \tag{4.12}$$

then

$$\int_\Omega a(x,u_k,\nabla u_k)\nabla u_k\,dx \geq -\int_\Omega a(x,u_k,\nabla u)\nabla u\,dx + \int_\Omega a(x,u_k,\nabla u_k)\nabla u\,dx + \int_\Omega a(x,u_k,\nabla u)\nabla u_k\,dx,$$

and by (4.4), we get

$$\liminf_{k\to\infty}\int_\Omega a(x,u_k,\nabla u_k)\nabla u_k\,dx \geq \int_\Omega \varphi\nabla u\,dx.$$

This implies, by using (4.11), that

$$\lim_{k\to\infty}\int_\Omega a(x,u_k,\nabla u_k)\nabla u_k\,dx = \int_\Omega \varphi\nabla u\,dx. \tag{4.13}$$

By combining (4.7), (4.8) and (4.13), we obtain

$$\langle B_n u_k, u_k\rangle \longrightarrow \langle \chi, u\rangle \quad \text{as} \quad k\to+\infty.$$

On the other hand, by (4.13), and the fact that $a(x,u_k,\nabla u)$ converges to $a(x,u,\nabla u)$ strongly in $(L^{p'(x)}(\Omega))^N$, we can deduce that

$$\lim_{k\to+\infty}\int_\Omega (a(x,u_k,\nabla u_k)-a(x,u_k,\nabla u))(\nabla u_k-\nabla u)\,dx = 0,$$

and by Lemma 3.1, we obtain u_k converges to u strongly in $W_0^{1,p(x)}(\Omega)$ and a.e. in Ω, we deduce that $a(x,u_k,\nabla u_k)$ converges to $a(x,u,\nabla u)$ weakly in $(L^{p'(x)}(\Omega))^N$, $\phi_n(u_k)$ converges to $\phi_n(u)$ strongly in $(L^{p'(x)}(\Omega))^N$, and $H_n(x,u_k,\nabla u_k)$ converges to $H_n(x,u,\nabla u)$ strongly in $L^{p'(x)}(\Omega)$ then $\chi = B_n u$, which completes the proof of Lemma 3.5. ■

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