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Oscillation of nonlinear impulsive differential equations with piecewise constant arguments

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Abstract Existence and uniqueness of the solutions of a class of first order nonlinear impulsive differential equation with piecewise constant arguments is studied. Moreover, sufficient conditions for the oscillation of the solutions are obtained.

Keywords: impulsive differential equation, piecewise constant argument, oscillation.

AMS Subject Classification: 34K11, 34K45.

1. Introduction

In this paper, we consider an impulsive differential equation with piecewise constant arguments of the form

$$x'(t) + a(t)\, x(t) + x([t-1])\, f(x[t]) = 0,\ t \neq n, \tag{1}$$

$$\Delta x(n) = d_n x(n),\ n \in \mathbb{N} = \{0, 1, 2, \ldots\}, \tag{2}$$

with the initial conditions

$$x(-1) = x_{-1}, \qquad x(0) = x_0, \tag{3}$$

where $a : [0, \infty) \to \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$ are continuous functions, $d_n : \mathbb{N} \to \mathbb{R} - \{1\}$, $\Delta x(n) = x(n^+) - x(n^-)$, $x(n^+) = \lim_{t \to n^+} x(t)$, $x(n^-) = \lim_{t \to n^-} x(t)$, $[.]$ denotes the greatest integer function, and x_{-1}, x_0 are given real numbers.
Since 1980's differential equations with piecewise constant arguments have attracted great deal of attention of researchers in mathematical and some of the others fields in science. Piecewise constant systems exist in a widely expanded areas such as biomedicine, chemistry, mechanical engineering, physics, etc. These kind of equations such as Eq.(1) are similar in structure to those found in certain sequential-continuous models of disease dynamics [1]. In 1994, Dai and Sing [2] studied the oscillatory motion of spring-mass systems with subject to piecewise constant forces of the form $f(x[t])$ or $f([t])$. Later, they improved an analytical and numerical method for solving linear and nonlinear vibration problems and they showed that a function $f([N(t)]/N)$ is a good approximation to the given continuous function $f(t)$ if N is sufficiently large [3].

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This method was also used to find the numerical solutions of a non-linear Froude pendulum and the oscillatory behavior of the pendulum [4].
In 1984, Cooke and Wiener [5] studied oscillatory and periodic solutions of a linear differential equation with piecewise constant argument and they note that such equations are comprehensively related to impulsive and difference equations. After this work, oscillatory and periodic solutions of linear differential equations with piecewise constant arguments have been dealt with by many authors [6, 7, 8] and the references cited therein. But, as we know, nonlinear differential equations with piecewise constant arguments have been studied in a few papers [9, 10, 11].
On the other hand, in 1994, the case of studying discontinuous solutions of differential equations with piecewise continuous arguments has been proposed as an open problem by Wiener [12]. Due to this open problem, the following linear impulsive differential equations have been studied [13, 14]:

$$\left\{\begin{array}{l} x'(t)+a(t)x(t)+b(t)x([t-1])=0,\ t\neq n, \\ x(n^{+})-x(n^{-})=d_{n}x(n),\ n\in\mathbb{N}=\{0,1,2,...\}, \end{array}\right. \tag{4}$$

and

$$\left\{\begin{array}{l} x'(t)+a(t)x(t)+b(t)x([t])+c(t)x([t+1])=f(t),\ t\neq n, \\ \Delta x\,(n)=d_{n}x(n),\ n\in\mathbb{N}=\{0,1,2,...\}. \end{array}\right.$$

Now, our aim is to consider the Wiener's open problem for the nonlinear problem (1)-(3). In this respect, we first prove existence and uniqueness of the solutions of Eq. (1)-(3) and we also obtain sufficient conditions for the existence of oscillatory solutions. Finally, we give some examples to illustrate our results.

2. Existence of solutions

Definition 1. It is said that a function $x:\mathbb{R}^{+}\cup\{-1\}\to\mathbb{R}$ is a solution of Eq. (1)-(2) if it satisfies the following conditions:
(i) $x(t)$ is continuous on $\mathbb{R}^{+}$ with the possible exception of the points $[t]\in[0,\infty)$,
(ii) $x(t)$ is right continuous and has left-hand limit at the points $[t]\in[0,\infty)$,
(iii) $x(t)$ is differentiable and satisfies (1) for any $t\in\mathbb{R}^{+}$, with the possible exception of the points $[t]\in[0,\infty)$ where one-sided derivatives exist,
(iv) $x(n)$ satisfies (2) for $n\in\mathbb{N}$.

Theorem 1. The initial value problem (1)-(3) has a unique solution $x(t)$ on $[0,\infty)\cup\{-1\}$. Moreover, for $n\leq t<n+1,\ n\in\mathbb{N},\ x$ has the form

$$\begin{aligned} x\,(t) = &\exp\left(-\int_{n}^{t}a\,(s)\,ds\right) \\ &\times\left(y(n)-y(n-1)f(y(n))\int_{n}^{t}\exp\left(\int_{n}^{u}a\,(s)\,ds\right)du\right), \end{aligned} \tag{5}$$

where $y(n) = x(n)$ and the sequence $\{y(n)\}_{n\geq -1}$ is the unique solution of the difference equation

$$\begin{aligned} y(n+1) = & \frac{1}{1-d_{n+1}} \exp\left(-\int_n^{n+1} a(s)\,ds\right) \\ & \times \left[y(n) - y(n-1)f(y(n)) \int_n^{n+1} \exp\left(\int_n^u a(s)\,ds\right) du\right], \; n \geq 0 \end{aligned} \tag{6}$$

with the initial conditions

$$y(-1) = x_{-1}, y(0) = x_0. \tag{7}$$

Proof. Let $x_n(t) \equiv x(t)$ be a solution of (1)-(2) on $n \leq t < n+1$. Eq. (1)-(2) is rewritten in the form

$$x'(t) + a(t)\,x(t) = -x(n-1)\,f(x(n)), \; n \leq t < n+1. \tag{8}$$

From (8), for $n \leq t < n+1$ we obtain

$$\begin{aligned} x_n(t) = & \exp\left(-\int_n^t a(s)\,ds\right) \\ & \times \left[x(n) - x(n-1)\,f(x(n)) \int_n^t \exp\left(\int_n^u a(s)\,ds\right) du\right]. \end{aligned} \tag{9}$$

On the other hand, if $x_{n-1}(t)$ is a solution of Eq.(1)-(2) on $n-1 \leq t < n$, then we get

$$\begin{aligned} x_{n-1}(t) &= \exp\left(-\int_{n-1}^t a(s)\,ds\right) \\ & \times \left[x(n-1) - x(n-2)\,f(x(n-1)) \int_{n-1}^t \exp\left(\int_{n-1}^u a(s)\,ds\right) du\right]. \end{aligned} \tag{10}$$

Using the impulse conditions (2), from (9) and (10), we obtain the difference equation

$$\begin{aligned} x(n+1) &= \frac{1}{1-d_{n+1}} \exp\left(-\int_n^{n+1} a(s)\,ds\right) \\ & \times \left[x(n) - x(n-1)f(x(n)) \int_n^{n+1} \exp\left(\int_n^u a(s)\,ds\right) du\right]. \end{aligned}$$

Considering the initial conditions (7), the solution of Equation (6) can be obtained uniquely. Thus, the unique solution of (1)-(3) is obtained as (5).

Theorem 2. The problem (1)-(3) has a unique backward continuation on $(-\infty, 0]$ given by (5)-(6) for $n \in \mathbb{Z}^- \cup \{0\}$.

3. Oscillatory solutions

Definition 2. A function $x(t)$ defined on $[0,\infty)$ is called oscillatory if there exist two real valued sequences $\{t_n\}_{n\geq 0}$, $\{t'_n\}_{n\geq 0} \subset [0,\infty)$ such that $t_n \to +\infty$, $t'_n \to +\infty$ as $n \to +\infty$ and $x(t_n) \leq 0 \leq x(t'_n)$ for $n \geq N$ where N is sufficiently large. Otherwise, the solution is called nonoscillatory.

Remark 1. According to Definition 2, a piecewise continuous function $x : [0,\infty) \to \mathbb{R}$ can be oscillatory even if $x(t) \neq 0$ for all $t \in [0,\infty)$.

Definition 3. A solution $\{y_n\}_{n\geq -1}$ of Eq.(6) is said to be oscillatory if the sequence $\{y_n\}_{n\geq -1}$ is neither eventually positive nor eventually negative. Otherwise, the solution is called non-oscillatory.

Theorem 3. Let $x(t)$ be the unique solution of the problem (1)-(3) on $[0,\infty)$. If the solution $y(n)$, $n \geq -1$, of Eq. (6) with the initial conditions (7) is oscillatory, then the solution $x(t)$ is also oscillatory.

Proof. Since $x(t) = y(n)$ for $t = n$, the proof is clear.

Remark 2. We note that even if the solution $y(n)$, $n \geq -1$, of the Eq. (6) with the initial conditions (7) is nonoscillatory, the solution $x(t)$ of (1)-(3) might be oscillatory.

In the following theorem give a necessary and sufficient condition for the existence of nonoscillatory solution $x(t)$, when the solution of difference equation (6)-(7) is nonoscillatory.

Theorem 4. Let $\{y_n\}_{n\geq -1}$ be a nonoscillatory solution of Eq. (6) with the initial conditions (7). Then the solution $x(t)$ of the problem (1)-(3) is nonoscillatory iff there exist a $N \in \mathbb{N}$ such that

$$\frac{y(n)}{y(n-1)} > f(y(n)) \int_n^t \exp\left(\int_n^u a(s)\, ds \right) du, \quad n \leq t < n+1,\ n > N. \tag{11}$$

Proof. Without loss of generality we may assume that $y(n) = x(n) > 0$, $y(n-1) = x(n-1) > 0$ for $n > N$. If $x(t)$ is nonoscillatory, then $x(t) > 0$, $t > T \geq N$. So condition (11) is obtained from (5) easily.
Now, let us assume that (11) is true. We should show that $x(t)$ is nonoscillatory. For contradiction, let $x(t)$ be oscillatory. Therefore there exist sequences $\{t_k\}_{k\geq 0}$, $\{t'_k\}_{k\geq 0}$ such that $t_k \to +\infty$, $t'_k \to +\infty$ as $k \to +\infty$ and

$x(t_k) \le 0 \le x(t'_k)$. Let $n_k = [t_k]$. It is clear that $n_k \to +\infty$ as $k \to +\infty$. So, from (5) we get

$$x(t_k) = \exp\left(-\int_{n_k}^{t_k} a(s)\,ds\right) \times \left(y(n_k) - y(n_k-1)f(y(n_k))\int_{n_k}^{t_k} \exp\left(\int_{n_k}^{u} a(s)\,ds\right) du\right).$$

Since $y(n_k) > 0,\ y(n_k-1) > 0$ and $x(t_k) \le 0$ we obtain

$$\frac{y(n_k)}{y(n_k-1)} \le f(y(n_k))\int_{n_k}^{t_k} \exp\left(\int_{n_k}^{u} a(s)\,ds\right) du, \quad n_k \le t_k < n_k+1$$

which is a contradiction to (11).
If $y(n) = x(n) < 0,\ y(n-1) = x(n-1) < 0$ for $n > N$, then the proof is done by similar method.

Theorem 5. Suppose that $1 - d_n > 0$ for $n \in \mathbb{N}$ and there exist a $M > 0$ such that $f(x) \ge M$ for $x \in (-\infty, \infty)$ and

$$\lim_{n\to\infty} \sup\ (1-d_n)\int_{n}^{n+1} \exp\left(\int_{n-1}^{u} a(s)\,ds\right) du > \frac{1}{M}. \tag{12}$$

Then, all solutions of Eq. (6) are oscillatory.

Proof. We prove that the existence of eventually positive (or negative) solutions leads to a contradiction. Let $y(n)$ be a solution of Eq. (6). Assume that $y(n) > 0,\ y(n-1) > 0,\ y(n-2) > 0$ for $n > N$, where N is sufficiently large. From (6)

$$(1-d_n)y(n)\exp\left(\int_{n-1}^{n} a(s)\,ds\right) = y(n-1) - y(n-2)f(y(n-1))\int_{n-1}^{n} \exp\left(\int_{n-1}^{u} a(s)\,ds\right) du.$$

Since $y(n-2) > 0$ and $f(y(n-1)) > 0$, we have

$$(1-d_n)y(n)\exp\left(\int_{n-1}^{n} a(s)\,ds\right) < y(n-1). \tag{13}$$

By using inequality (13) and Eq. (6), we obtain

$$y(n)\left[1-(1-d_n)f(y(n))\int\limits_{n}^{n+1}\exp\left(\int\limits_{n-1}^{u}a\left(s\right)ds\right)du\right]$$

$$>y(n)-y(n-1)f(y(n))\int\limits_{n}^{n+1}\exp\left(\int\limits_{n}^{u}a\left(s\right)ds\right)du$$

$$=(1-d_{n+1})y(n+1)\exp\left(\int\limits_{n}^{n+1}a\left(s\right)ds\right). \tag{14}$$

Since $y(n)>0,\ y(n+1)>0,\ 1-d_{n+1}>0$ and $f(x)\geq M$, from (14), we get

$$\frac{1}{M}\geq\lim_{n\to\infty}\sup\ (1-d_n)\int\limits_{n}^{n+1}\exp\left(\int\limits_{n-1}^{u}a\left(s\right)ds\right)du,$$

which is a contradiction to (12). The proof is the same in case of existence of an eventually negative solution.

Corollary 1. Under the hypotheses of Theorem 5, all solutions of (1)-(2) are oscillatory.

Remark 3. If $f(x)=b,\ b>0$ is a constant function, then we have a linear equation in the form

$$\begin{cases} x'(t)+a(t)x(t)+bx([t-1])=0,\ t\neq n, \\ x(n^+)-x(n^-)=d_nx(n),\ n\in\mathbb{N}=\{0,1,2,...\}, \end{cases} \tag{15}$$

which is a special case of (4). In this case, condition (12) reduces to the following condition

$$\lim_{n\to\infty}\sup\ (1-d_n)b\int\limits_{n}^{n+1}\exp\left(\int\limits_{n-1}^{u}a\left(s\right)ds\right)du>1,$$

which is stated in [13] for $b(t)\equiv b>0$.

Now, consider following nonimpulsive equation

$$x'\left(t\right)+a\left(t\right)x\left(t\right)+x\left(\left[t-1\right]\right)f(x\left[t\right])=0, \tag{16}$$

where $a:[0,\infty)\to\mathbb{R},\ f:\mathbb{R}\to\mathbb{R}$ are continuous functions.

Corollary 2. Assume that there exists a constant $M>0$ such that $f(x)\geq M$. If

$$\lim_{n\to\infty}\sup\ \int\limits_{n}^{n+1}\exp\left(\int\limits_{n-1}^{u}a\left(s\right)ds\right)du>\frac{1}{M},$$

then all solutions of Eq. (16) are oscillatory.

Theorem 6. Assume that

$$f(x) \geq M > 0, \tag{17}$$

$$1 - d_n \geq K > 0, \quad n = 0, 1, 2, ..., \tag{18}$$

and

$$\frac{1}{4KM} < \lim_{n\to\infty} \inf \exp \int_n^{n+1} a(s)ds \lim_{n\to\infty} \inf \int_n^{n+1} \exp(\int_n^u a(s)ds)du < \infty. \tag{19}$$

Then, all solutions of Eq. (6) are oscillatory.

Proof. Let $y(n)$ be a solution of Eq. (6). Assume that $y(n) > 0,\ y(n-1) > 0$ for $n > N$, where N is sufficiently large. From Eq. (6), we have

$$\begin{aligned}(1 - d_{n+1})y(n+1) \exp \int_n^{n+1} a(s)ds =& y(n) - y(n-1)f(y(n)) \\ & \times \int_n^{n+1} \exp\left(\int_n^u a\,(s)\,ds\right) du.\end{aligned} \tag{20}$$

Let $w_n = \frac{y(n)}{y(n-1)}$. Since $w_n > 0$, we consider two cases:

Case 1. Let $\lim\limits_{n\to\infty} \inf w_n = \infty$. Then from (20), we have

$$1 \geq (1 - d_{n+1})w_{n+1} \exp \int_n^{n+1} a(s)ds + \frac{M}{w_n} \int_n^{n+1} \exp\left(\int_n^u a\,(s)\,ds\right) du. \tag{21}$$

Taking the inferior limit on both sides of inequality (21), we get

$$\begin{aligned}1 \quad \geq & \quad \lim_{n\to\infty} \inf(1 - d_{n+1}) \lim_{n\to\infty} \inf w_{n+1} \lim_{n\to\infty} \inf \exp \int_n^{n+1} a(s)ds \\ & +M \lim_{n\to\infty} \inf \frac{1}{w_n} \lim_{n\to\infty} \inf \int_n^{n+1} \exp\left(\int_n^u a\,(s)\,ds\right) du,\end{aligned}$$

which is a contradiction to the $\lim\limits_{n\to\infty} \inf w_n = \infty$. So, we consider the second case;

Case 2. Let $0 \leq \lim\limits_{n\to\infty} \inf w_n < \infty$. Dividing Eq. (20) by $y(n-1)$, we have

$$\begin{aligned}\frac{y(n)}{y(n-1)} \quad = & \quad (1 - d_{n+1})\frac{y(n+1)}{y(n-1)} \exp\left(\int_n^{n+1} a\,(s)\,ds\right) \\ & +f(y(n)) \int_n^{n+1} \exp\left(\int_n^u a\,(s)\,ds\right) du,\end{aligned}$$

which yields

$$
\begin{aligned}
w_n \geq & (1-d_{n+1}) w_n w_{n+1} \exp\left(\int_n^{n+1} a(s)\,ds\right) \\
& + M \int_n^{n+1} \exp\left(\int_n^{u} a(s)\,ds\right) du.
\end{aligned} \tag{22}
$$

Let $\lim_{n\to\infty} \inf w_n = W$, $\lim_{n\to\infty} \inf \exp \int_n^{n+1} a(s)ds = A$,
$\lim_{n\to\infty} \inf \int_n^{n+1} \exp(\int_n^u a(s)ds)du = B$. Taking the inferior limit on both sides of inequality (22), we have

$$W \geq \lim_{n\to\infty} \inf(1-d_{n+1})W^2 A + MB \tag{23}$$

Now, from (18), there are two subcases:

(i) If $\lim_{n\to\infty} \inf(1-d_{n+1}) = \infty$, then we obtain a contradiction from (23).

(ii) If $\lim_{n\to\infty} \inf(1-d_{n+1}) < \infty$, then from (23) we have

$$AKW^2 - W + MB \leq 0,$$

or

$$KA\left[\left(W - \frac{1}{2KA}\right)^2 + \frac{4MBKA-1}{4K^2A^2}\right] \leq 0,$$

which contradicts to (19). So Eq. (6) cannot have an eventually positive solution. Similarly, existence of an eventually negative solution leads us a contradiction. Thus all solutions of (6) are oscillatory.

Corollary 3. Under the hypotheses of Theorem 6, all solutions of (1)-(2) are oscillatory.

Corollary 4. Assume that $f(x) \geq M > 0$, and

$$\frac{1}{4M} < \lim_{n\to\infty} \inf \exp \int_n^{n+1} a(s)ds \lim_{n\to\infty} \inf \int_n^{n+1} \exp(\int_n^u a(s)ds)du < \infty.$$

then all solutions of (16) are oscillatory.

Remark 4. If $f(x) = b$, $b > 0$ is a constant, and $d_n \equiv 0$ for all $n \in \mathbb{N}$, then Eq.(1)-(2) reduces to the linear nonimpulsive equation

$$x'(t) + a(t)x(t) + bx([t-1]) = 0, \tag{24}$$

which is the same as Eq.(1) with $b(t) \equiv b$ in [7]. In this case, conditions (12) and (19), respectively, correspond to conditions (2) and (8) in [7].

Equation (24) is also special case of Eq.(1.1) in [9]. In this case, condition (19) reduces to condition (2.3) in [9] with $b(t) \equiv b$. Moreover, if $a(t) \equiv a$ (constant), then condition (19) reduces to the condition

$$b > \frac{ae^{-a}}{4(e^a - 1)}$$

which is known as the best possible for the oscillation [7, 9].

Remark 5. In the case of $f(x) \equiv b,\ a(t) \equiv a,\ d_n \equiv d,\ n \in \mathbb{N},\ a, b, d$ are constants, Theorem 9 in [13] can be applied to Eq. (1)-(2) to obtain existence of periodic solutions.

Consider the following equation.

$$\begin{cases} x'(t) + ax(t) + bx([t-1]) = 0,\ t \neq n, \\ x(n^+) - x(n^-) = dx(n),\ n \in \mathbb{N} = \{0, 1, 2, ...\}. \end{cases} \tag{25}$$

Corollary 5. Let $1 - d > K > 0$. A necessary and sufficient condition for every oscillatory solution of Eq.(25) to be periodic with period k is

$$\frac{ae^a(1-d)}{e^a - 1} = b \text{ and } a = -\ln\left(2(1-d)\cos\frac{2\pi m}{k}\right), \tag{26}$$

where m and k are relatively prime and $m = 1, 2, ..., [(k-1)/4]$.

4. Examples

In this section, we give some examples to illustrate our results.
Example 1. Let us consider the following differential equation

$$x'(t) + x(t) + (x^2[t] + 1)x([t-1]) = 0,\ t \neq n, \tag{27}$$

$$\Delta x(n) = \frac{e-1}{e}x(n),\ n \in \mathbb{N}, \tag{28}$$

which is a special case of (1)-(2) with $a(t) = 1,\ f(x) = x^2 + 1,\ d_n = \frac{e-1}{e},\ n \in \mathbb{N}$. It is easily checked that the Eq. (27)-(28) satisfies all hypotheses of Theorem 6. Thus every solution of equation (27)-(28) is oscillatory. The solution $x_n(t)$ of Eq. (27)-(28) with the initial conditions $x(-1) = 0,\ x(0) = 0.001$ for $n = 0, 1, 2, 3, 4$ is demonstrated in Figure 1.
Example 2. Consider the equation

$$x'(t) + x(t) + x([t-1]) = 0,\ t \neq n, \tag{29}$$

$$\Delta x(n) = \frac{1}{2}x(n),\ n \in \mathbb{N}, \tag{30}$$

that is a special case of Eq. (1)-(2) with $a(t) = 1,\ f(x) = 1$ and $d_n = \frac{1}{2},\ n \in \mathbb{N}$. Since all hypotheses of Theorem 5 are satisfied, every solution of Eq.(29)-(30) is oscillatory. Indeed, the solution $x(t)$ of Eq.(29)-(30) is in the form

$$x_n(t) = e^{-t+n}[y(n) - y(n-1)\left(e^{t-n} - 1\right)],\ n \leq t < n+1, \tag{31}$$

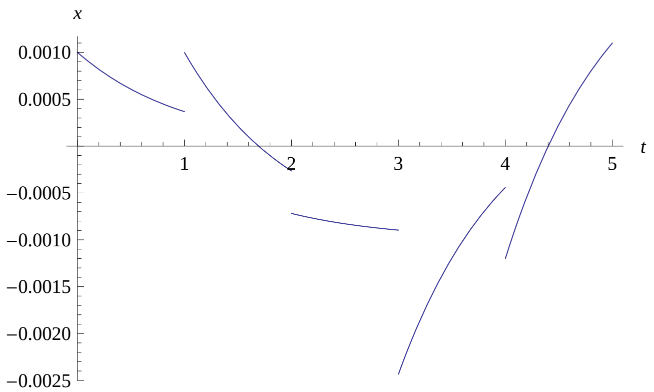

Figure 1: Oscillatory solutions of Eq. (27)-(28) with the initial conditions $x(-1) = 0,\ x\,(0) = 0.001$

where $y(n)$ is the solution of the following linear difference equation

$$y(n+2) - 2e^{-1}y(n+1) + (2 - 2e^{-1})y(n) = 0, \tag{32}$$

which has the complex characteristic roots

$$\lambda_{1,2} = \frac{1}{e}[1 \pm i\sqrt{-1 - 2e + 2e^2}].$$

So, Eq. (32) has only oscillatory solutions. Hence from Corollary 1, Eq. (29)-(30) has only oscillatory solutions too. The solution $x_n(t),\ n = 0, 1, ...11,$ of (29)-(30) with the initial conditions $x(-1) = \sqrt{2e^2 - 2e - 1}/(2 - 2e),\ x(0) = 0$ is given in Figure 2.

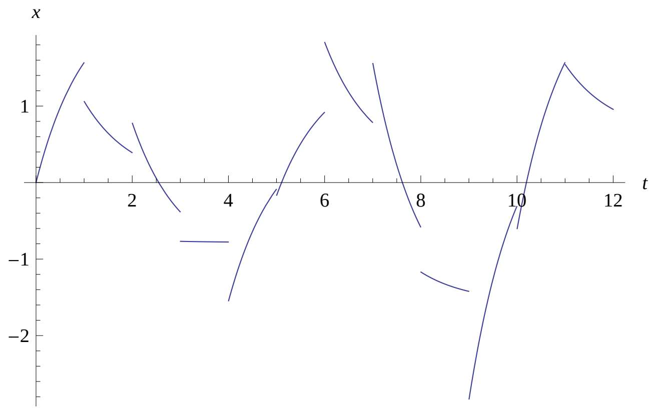

Figure 2: Oscillatory solutions of Eq. (29)-(30) with the initial conditions $x(-1) = \sqrt{2e^2 - 2e - 1}/(2 - 2e),\ x(0) = 0$

Example 3. Finally we consider the equation

$$x'(t) + (\ln 2)x(t) + \frac{\ln 4}{\sqrt{5}-1}x([t-1]) = 0,\ t \neq n, \tag{33}$$

$$\Delta x(n) = \frac{\sqrt{5}-2}{\sqrt{5}-1}x(n),\ n \in \mathbb{N}. \tag{34}$$

Since $a(t) = \ln 2$, $f(x) = \frac{\ln 4}{\sqrt{5}-1}$ and $d_n = \frac{\sqrt{5}-2}{\sqrt{5}-1}$, $n \in \mathbb{N}$, verify the hypotheses of Theorem 5, all solutions of Eq. (33)-(34) are oscillatory. On the other hand, Since Eq. (33)-(34) satisfies the hypotheses of Corollary 5, all solutions of (33)-(34) are periodic with period 5. This fact can be seen in Figure 3.

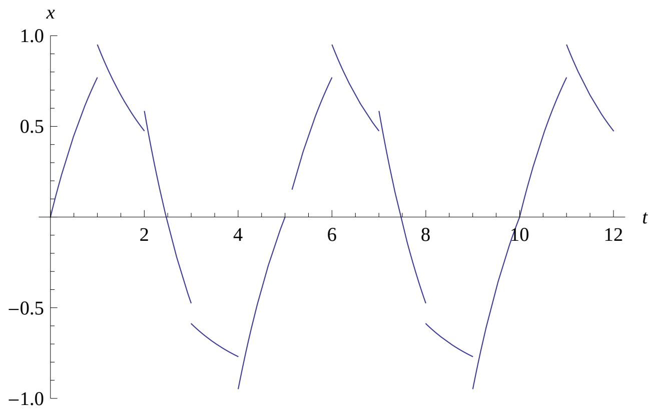

Figure 3: Oscillatory solutions of Eq. (33)-(34) with the initial conditions $x(-1) = \sqrt{10+2\sqrt{5}}/4$, $x(0) = 0$.

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References

[1] S. Busenberg and K.L. Cooke. Models of vertically transmitted diseases with sequential-continuous dynamics. In: Nonlinear Phenomena in Mathematical Sciences, V. Lakshmikantham (Ed.), Academic Press, (1982) 179-187.

[2] L. Dai and M. C. Singh, On oscillatory motion of spring mass systems subject to piecewise constant forces. J. Sound Vib., 173 (1994) 217-233.

[3] L. Dai and M. C. Singh, An analytical and numerical method for solving linear and nonlinear vibration problems. Int. J. Solids Struct. 34 (1997) 2709-2731.

[4] L. Dai and M. C. Singh, Periodic, quasiperiodic and chaotic behavior of a driven Froude pendulum. Int. J. Nonlinear Mech. 33 (1998) 947-965.

[5] K. L. Cooke and J. Wiener, Retarded differential equations with piecewise constant delays. J. Math. Anal. Appl., 99 (1984) 265-297.

[6] A. R. Aftabizadeh and J. Wiener, Oscillatory properties of first order linear functional differential equations. Applicable Anal., 20 (1985) 165-187.

[7] A. R. Aftabizadeh, J. Wiener and J.-M. Xu, Oscillatory and periodic solutions of delay differential equations with piecewise constant argument. Proc. of American Math. Soc., 99 (1987) 673-679.

[8] J. Wiener and A. R. Aftabizadeh, Differential equations alternately of retarded and advanced type. J. Math. Anal. Appl. 129 (1988) 243-255.

[9] J. H. Shen and I. P. Stavroulakis, Oscillatory and nonoscillatory delay equations with piecewise constant argument. J. Math. Anal. Appl. 248 (2000) 385-401.

[10] L. A. V. Carvalho and K. L. Cooke, a nonlinear equation with piecewise continuous argument. Differential and Integral Equations 1 (1988) 359-367.

[11] A. Cabada and J. B. Ferreiro, First order differential equations with piecewise constant arguments and nonlinear boundary value conditions. J. Math. Anal. Appl. 380 (2011) 124-136.

[12] J. Wiener, Generalized Solutions of Functional Differential Equations. World Scientific, Singapore, 1994.

[13] F. Karakoc, H. Bereketoglu and G. Seyhan, Oscillatory and periodic solutions of impulsive differential equations with piesewise constant argument. Acta Appl. Math. 110 (2010) 499-510.

[14] H. Bereketoglu, G. Seyhan and A. Ogun, Advanced impulsive differential equations with piecewise constant arguments. Math. Model. Anal. 15 (2010) 175–187.

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