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Homoclinic solutions for second order Hamiltonian systems with general potentials near the origin*

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Abstract

In this paper, we study the existence of infinitely many homoclinic solutions for a class of second order Hamiltonian systems with general potentials near the origin. Recent results from the literature are generalized and significantly improved.
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1 Introduction and main results

Consider the following second order Hamiltonian system

$$\ddot{u} - L(t)u + W_u(t,u) = 0, \quad \forall\, t \in \mathbb{R}, \tag{HS}$$

where $u = (u_1, \ldots, u_N) \in \mathbb{R}^N$, $L \in C\left(\mathbb{R}, \mathbb{R}^{N^2}\right)$ is a symmetric matrix-valued function, and $W_u(t,u)$ denotes the gradient of $W(t,u)$ with respect to u. Here, as usual, we say that a solution u of (HS) is homoclinic (to 0) if $u \in C^2\left(\mathbb{R}, \mathbb{R}^N\right)$, $u(t) \not\equiv 0$, and $u(t) \to 0$ as $|t| \to \infty$.

With the aid of variational methods, the existence and multiplicity of homoclinic solutions for (HS) have been extensively investigated in the literature over the past several decades (see, e.g., [1–27] and the references therein). Many early papers (see, e.g., [1–3,

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6, 8–10, 15–17]) treated the periodic (including autonomous) case where $L(t)$ and $W(t,u)$ are either independent of t or periodic in t. Compared to the periodic case, the problem is quite different in nature for the nonperiodic case due to the lack of compactness of the Sobolev embedding. After the work of Rabinowitz and Tanaka [17], there are many papers (see, e.g., [4, 5, 7, 9, 11–14, 18–27]) concerning the nonperiodic case. For this case, the function L plays an important role. Actually, most of these mentioned papers assumed that L is either coercive or uniformly positively definite. Besides, we also note that all these papers required $W(t,u)$ to satisfy some kind of growth conditions at infinity with respect to u, such as superquadratic, asymptotically quadratic or subquadratic growth.

In the recent paper [28], the authors obtained infinitely many homoclinic solutions for (HS) without any conditions assumed on $W(t,u)$ for $|u|$ large. To be precise, $W(t,u)$ in that paper is only locally defined near the origin with respect to u, but L is required to satisfy a very strong coercivity condition. Motivated by [28], in the present paper, we will study the existence of infinitely many homoclinic solutions for (HS) in the case where L is unnecessarily coercive, and $W(t,u)$ is still only locally defined near the origin with respect to u. More precisely, we make the following assumptions:

(L_0) $l_0 := \inf_{t\in\mathbb{R}} \left[\min_{|u|=1,\ u\in\mathbb{R}^N} L(t)u\cdot u \right] > 0.$

(W_1) $W \in C^1(\mathbb{R}\times B_\delta(0),\mathbb{R})$ is even in u and $W(t,0)\equiv 0$, where $B_\delta(0)$ is the open ball in $\mathbb{R}^N$ centered at 0 with radius δ.

(W_2) There exist constants $\nu \in (1,2)$, $\mu_1 \in [1,2]$, $\mu_2 \in [1, 2/(2-\nu)]$ and nonnegative functions $\xi_i \in L^{\mu_i}(\mathbb{R},\mathbb{R})$ $(i = 1,2)$ such that

$$|W_u(t,u)| \le \xi_1(t) + \xi_2(t)|u|^{\nu-1}, \quad \forall\,(t,u)\in\mathbb{R}\times B_\delta(0).$$

(W_3) There exist a constant $\varrho > 0$, a closed interval $I_0 \subset \mathbb{R}$ and two sequences of positive numbers $\delta_n \to 0$, $M_n \to \infty$ as $n\to\infty$ such that

$$W(t,u) \ge -\varrho|u|^2, \quad \forall\, t\in I_0 \text{ and } |u| < \delta$$

and

$$W(t,u)/\delta_n^2 \ge M_n, \quad \forall\, t\in I_0,\ n\in\mathbb{N} \text{ and } |u| = \delta_n.$$

Our main result reads as follows.

Theorem 1.1. *Suppose that* (L_0) *and* (W_1)–(W_3) *are satisfied. Then* (HS) *possesses a sequence of homoclinic solutions* $\{u_k\}$ *such that* $\max_{t\in\mathbb{R}}|u_k(t)| \to 0$ *as* $k \to \infty$.

Remark 1.2. Compared to Theorem 1.1 in [28], the matrix-valued function L in our Theorem 1.1 is not required to satisfy the coercivity condition (L_1) or the technical condition (L_2) of Theorem 1.1 in [28]. In addition, our Theorem 1.1 also essentially improves some related results in the existing literature. It is easy to see that the conditions of our Theorem 1.1 are weaker than those of Theorem 1.2 in [12, 18, 19]. Indeed, there is a function W which satisfies conditions (W_1)–(W_3) but does not satisfy the corresponding conditions of Theorem 1.2 in [12, 18, 19]. For example, let

$$W(t,u) = \begin{cases} te^{-t^2}|u|^\alpha \sin^2(\frac{1}{|u|^\epsilon}), & 0 < |u| < 1, \\ 0, & u = 0, \end{cases}$$

where $\epsilon > 0$ small enough and $\alpha \in (1+\epsilon, 2)$. Then it is easy to check that W satisfies conditions (W_1)–(W_3) with $\nu = \alpha-\epsilon$, $\xi_1(t) \equiv 0$, $\xi_2(t) = (\alpha+\epsilon)|t|e^{-t^2}$ and $\delta_n = (\frac{2}{(2n+1)\pi})^{1/\epsilon}$ for all $n \in \mathbb{N}$.

2 Variational setting and proof of the main result

Consider the space $E := \{u \in H^1(\mathbb{R},\mathbb{R}^N) \mid \int_{\mathbb{R}} L(t)u\cdot u dt < \infty\}$ equipped with the following inner product

$$(u,v) = \int_{\mathbb{R}} (\dot{u}\cdot\dot{v} + L(t)u\cdot v)dt.$$

Then E is a Hilbert space and we denote by $\|\cdot\|$ the associated norm. Moreover, we write E^* for the topological dual of E with norm $\|\cdot\|_{E^*}$, and $\langle\cdot,\cdot\rangle : E^*\times E \to \mathbb{R}$ for the dual pairing. Evidently, E is continuously embedded into $H^1\left(\mathbb{R},\mathbb{R}^N\right)$. Hence E is continuously embedded into $L^p \equiv L^p\left(\mathbb{R},\mathbb{R}^N\right)$ for all $p \in [2,\infty]$ and compactly embedded into $L^p_{\mathrm{loc}} \equiv L^p_{\mathrm{loc}}\left(\mathbb{R},\mathbb{R}^N\right)$ for all $p \in [1,\infty]$. Consequently, there exists $\tau_p > 0$ such that

$$\|u\|_p \le \tau_p\|u\|, \quad \forall\, u \in E, \tag{2.1}$$

where $\|\cdot\|_p$ denotes the usual norm in L^p for $p \in [2,\infty]$.

In order to prove our main result via the critical point theory, we need to modify $W(t,u)$ for u outside a neighborhood of the origin to get $\widetilde{W}(t,u)$ as follows.

Choose a constant $b \in (0, \delta/2)$ and define a cut-off function $\chi \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ such that $\chi(t) \equiv 1$ for $0 \le t \le b$, $\chi(t) \equiv 0$ for $t \ge 2b$, and $-2/b \le \chi'(t) < 0$ for $b < t < 2b$. Let

$$\widetilde{W}(t,u) = \chi(|u|)W(t,u), \quad \forall\,(t,u) \in \mathbb{R} \times \mathbb{R}^N. \tag{2.2}$$

Combining (W_1), (W_2) and the definition of χ, we have

$$\left|\widetilde{W}(t,u)\right| \le \xi_1(t)|u| + \xi_2(t)|u|^{\nu}, \quad \forall\,(t,u) \in \mathbb{R} \times \mathbb{R}^N \tag{2.3}$$

and

$$\left|\widetilde{W}_u(t,u)\right| \le c_1\left(\xi_1(t) + \xi_2(t)|u|^{\nu-1}\right), \quad \forall\,(t,u) \in \mathbb{R} \times \mathbb{R}^N \tag{2.4}$$

for some $c_1 > 0$.

Now we introduce the following modified Hamiltonian system

$$\ddot{u} - L(t)u + \widetilde{W}_u(t,u) = 0, \quad \forall\,t \in \mathbb{R} \tag{$\widetilde{\mathrm{HS}}$}$$

and define the variational functional Φ associated with $(\widetilde{\mathrm{HS}})$ by

$$\begin{aligned}\Phi(u) &= \frac{1}{2}\int_{\mathbb{R}}(|\dot{u}|^2 + L(t)u\cdot u)dt - \Psi(u)\\ &= \frac{1}{2}\|u\|^2 - \Psi(u), \quad \text{where } \Psi(u) = \int_{\mathbb{R}}\widetilde{W}(t,u)dt. \end{aligned}\tag{2.5}$$

Proposition 2.1. *Let* (L_0), (W_1) *and* (W_2) *be satisfied. Then* $\Psi \in C^1(E,\mathbb{R})$ *and* $\Psi' : E \to E^*$ *is compact, and hence* $\Phi \in C^1(E,\mathbb{R})$. *Moreover,*

$$\langle \Psi'(u), v\rangle = \int_{\mathbb{R}}\widetilde{W}_u(t,u)\cdot v dt, \tag{2.6}$$

$$\begin{aligned}\langle \Phi'(u), v\rangle &= (u,v) - \langle \Psi'(u), v\rangle\\ &= (u,v) - \int_{\mathbb{R}}\widetilde{W}_u(t,u)\cdot v dt\end{aligned}\tag{2.7}$$

for all $u, v \in E$, *and nontrivial critical points of* Φ *on* E *are homoclinic solutions of* $(\widetilde{\mathrm{HS}})$.

Proof. First, we show that Φ and Ψ are both well defined. For notational simplicity, we set

$$\mu_1^* := \frac{\mu_1}{\mu_1 - 1}, \quad \mu_2^* := \frac{\nu\mu_2}{\mu_2 - 1} \quad (\mu_i^* = \infty \text{ if } \mu_i = 1,\ i = 1,2),$$

and always use these notations in the sequel. Since $\mu_1 \in [1,2]$, $\mu_2 \in [1, 2/(2-\nu)]$ in (W_2), it is easy to see that $\mu_i^* \in [2,\infty]$ for $i=1,2$. For any $u \in E$, by (2.1), (2.3) and the Hölder inequality, we have

$$\begin{aligned}\int_{\mathbb{R}} \big|\widetilde{W}(t,u)\big| dt &\leq \int_{\mathbb{R}} \xi_1(t)|u|dt + \int_{\mathbb{R}} \xi_2(t)|u|^{\nu}dt \\ &\leq |\xi_1|_{\mu_1}\|u\|_{\mu_1^*} + |\xi_2|_{\mu_2}\|u\|_{\mu_2^*}^{\nu} \\ &\leq \tau_{\mu_1^*}|\xi_1|_{\mu_1}\|u\| + \tau_{\mu_2^*}^{\nu}|\xi_2|_{\mu_2}\|u\|^{\nu} < \infty, \end{aligned} \tag{2.8}$$

where $|\cdot|_{\mu_i}$ denotes the usual norm of in $L^{\mu_i}(\mathbb{R},\mathbb{R})$ and $\tau_{\mu_i^*}$ is the constant given in (2.1) for $i=1,2$. This together with (2.5) implies that Φ and Ψ are both well defined.

Next, we prove $\Psi \in C^1(E,\mathbb{R})$ and $\Psi' : E \to E^*$ is compact. For any given $u \in E$, define an associated linear operator $\mathcal{J}(u) : E \to \mathbb{R}$ by

$$\langle \mathcal{J}(u), v\rangle = \int_{\mathbb{R}} \widetilde{W}_u(t,u)\cdot v dt, \quad \forall\, v \in E.$$

By (2.1), (2.4) and the Hölder inequality, there holds

$$\begin{aligned}|\langle \mathcal{J}(u), v\rangle| &\leq \int_{\mathbb{R}} \big|\widetilde{W}_u(t,u)\big||v|dt \\ &\leq c_1\left(\int_{\mathbb{R}} \xi_1(t)|v|dt + \int_{\mathbb{R}} \xi_2(t)|u|^{\nu-1}|v|dt\right) \\ &\leq c_1\left(|\xi_1|_{\mu_1}\|v\|_{\mu_1^*} + |\xi_2|_{\mu_2}\|u\|_{\mu_2^*}^{\nu-1}\|v\|_{\mu_2^*}\right) \\ &\leq c_1(\tau_{\mu_1^*}|\xi_1|_{\mu_1} + \tau_{\mu_2^*}^{\nu}|\xi_2|_{\mu_2}\|u\|^{\nu-1})\|v\|, \quad \forall\, v \in E,\end{aligned}$$

where c_1 is the constant given in (2.4). This implies that $\mathcal{J}(u)$ is well defined and bounded. By (2.4), for any $\eta \in [0,1]$, there holds

$$\big|\widetilde{W}_u(t,u+\eta v)\cdot v\big| \leq c_1\left[\xi_1(t)|v| + 2\xi_2(t)\left(|u|^{\nu-1}|v| + |v|^{\nu}\right)\right], \quad \forall\, t \in \mathbb{R} \text{ and } u,v \in \mathbb{R}^N.$$

Therefore, for any $u, v \in E$, by the Mean Value Theorem and Lebesgue's Dominated Convergence Theorem, we have

$$\begin{aligned}\lim_{s\to 0}\frac{\Psi(u+sv)-\Psi(u)}{s} &= \lim_{s\to 0}\int_{\mathbb{R}} \widetilde{W}_u\left(t, u+\theta(t)sv\right)\cdot v dt \\ &= \int_{\mathbb{R}} \widetilde{W}_u(t,u)\cdot v dt \\ &= \langle \mathcal{J}(u), v\rangle,\end{aligned} \tag{2.9}$$

where $\theta(t) \in [0,1]$ depends on u, v, s. This implies that Ψ is Gâteaux differentiable on E and the Gâteaux derivative of Ψ at $u \in E$ is $\mathcal{J}(u)$. Let $u_n \rightharpoonup u$ in E as $n \to \infty$, then $\{u_n\}$ is bounded in E and

$$u_n \to u \text{ in } L^\infty_{\text{loc}} \quad \text{as } n \to \infty. \tag{2.10}$$

Consequently, there exists a constant $D_0 > 0$ such that

$$\|u_n\|^{\nu-1} + \|u\|^{\nu-1} \le D_0, \quad \forall\, n \in \mathbb{N}. \tag{2.11}$$

For any $\epsilon > 0$, by (W_2), there exists $T_\epsilon > 0$ such that

$$\left(\int_{|t|>T_\epsilon} \xi_1(t)^{\mu_1} dt\right)^{1/\mu_1} < \frac{\epsilon}{8c_1\tau_{\mu_1^*}} \tag{2.12}$$

and

$$\left(\int_{|t|>T_\epsilon} \xi_2(t)^{\mu_2} dt\right)^{1/\mu_2} < \frac{\epsilon}{4c_1 D_0 \tau^\nu_{\mu_2^*}}. \tag{2.13}$$

By (2.4), (2.11)–(2.13) and the Hölder inequality, we have

$$\begin{aligned}
&\int_{|t|>T_\epsilon} \left|\widetilde{W}_u(t,u_n) - \widetilde{W}_u(t,u)\right||v|dt \\
&\le \int_{|t|>T_\epsilon} c_1 \left[2\xi_1(t) + \xi_2(t)\left(|u_n|^{\nu-1} + |u|^{\nu-1}\right)\right]|v|dt \\
&\le 2c_1 \int_{|t|>T_\epsilon} \xi_1(t)|v|dt + c_1 \int_{|t|>T_\epsilon} \xi_2(t)\left(|u_n|^{\nu-1} + |u|^{\nu-1}\right)|v|dt \\
&\le 2c_1 \left(\int_{|t|>T_\epsilon} \xi_1(t)^{\mu_1} dt\right)^{1/\mu_1} \|v\|_{\mu_1^*} \\
&\quad + c_1 \left(\int_{|t|>T_\epsilon} \xi_2(t)^{\mu_2} dt\right)^{1/\mu_2} \left(\|u_n\|^{\nu-1}_{\mu_2^*} + \|u\|^{\nu-1}_{\mu_2^*}\right)\|v\|_{\mu_2^*} \\
&\le 2c_1\tau_{\mu_1^*} \left(\int_{|t|>T_\epsilon} \xi_1(t)^{\mu_1} dt\right)^{1/\mu_1} + c_1\tau^\nu_{\mu_2^*} \left(\int_{|t|>T_\epsilon} \xi_2(t)^{\mu_2} dt\right)^{1/\mu_2} \left(\|u_n\|^{\nu-1} + \|u\|^{\nu-1}\right) \\
&< \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}, \quad \forall\, n \in \mathbb{N} \text{ and } \|v\| = 1.
\end{aligned} \tag{2.14}$$

For the T_ϵ given above, by (2.1), (2.10) and the continuity of $\widetilde{W}$, there exists $N_\epsilon \in \mathbb{N}$ such that

$$
\begin{aligned}
&\int_{-T_\epsilon}^{T_\epsilon} \left|\widetilde{W}_u(t,u_n) - \widetilde{W}_u(t,u)\right||v|dt \\
&\le \tau_\infty \int_{-T_\epsilon}^{T_\epsilon} \left|\widetilde{W}_u(t,u_n) - \widetilde{W}_u(t,u)\right|dt \\
&< \frac{\epsilon}{2}, \quad \forall n \ge N_\epsilon \text{ and } \|v\| = 1,
\end{aligned}
\tag{2.15}
$$

where τ_∞ is the constant given in (2.1). Now for any $\epsilon > 0$, combining (2.14) and (2.15), we have

$$
\begin{aligned}
\|\mathcal{J}(u_n) - \mathcal{J}(u)\|_{E^*} &= \sup_{\|v\|=1} |\langle \mathcal{J}(u_n) - \mathcal{J}(u), v\rangle| \\
&= \sup_{\|v\|=1} \left|\int_{\mathbb{R}} \left(\widetilde{W}_u(t,u_n) - \widetilde{W}_u(t,u)\right) \cdot v dt\right| \\
&\le \sup_{\|v\|=1} \int_{-T_\epsilon}^{T_\epsilon} \left|\widetilde{W}_u(t,u_n) - \widetilde{W}_u(t,u)\right||v|dt \\
&\quad + \sup_{\|v\|=1} \int_{|t|>T_\epsilon} \left|\widetilde{W}_u(t,u_n) - \widetilde{W}_u(t,u)\right||v|dt \\
&\le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \quad \forall n \ge N_\epsilon.
\end{aligned}
$$

This means that $\mathcal{J}$ is completely continuous. Thus $\Psi \in C^1(E, \mathbb{R})$ and (2.6) holds with $\Psi' = \mathcal{J}$. Consequently, Ψ' is completely continuous. This together with the reflexivity of Hilbert space E implies that Ψ' is compact. In addition, due to the form of Φ in (2.5), we know that $\Phi \in C^1(E, \mathbb{R})$ and (2.7) also holds.

Finally, a standard argument shows that nontrivial critical points of Φ on E are homoclinic solutions of $(\widetilde{\text{HS}})$. The proof is completed. □

We will use the following variant symmetric mountain pass lemma due to Kajikiya [29] to prove that $(\widetilde{\text{HS}})$ possesses a sequence of homoclinic solutions. Before stating this theorem, we first recall the notion of genus.

Let E be a Banach space and A a subset of E. A is said to be symmetric if $u \in A$ implies $-u \in A$. Denote by Γ the family of all closed symmetric subset of E which does not contain 0. For any $A \in \Gamma$, define the genus $\gamma(A)$ of A by the smallest integer k such that there exists an odd continuous mapping from A to $\mathbb{R}^k \setminus \{0\}$. If there does not exist such a k, define $\gamma(A) = \infty$. Moreover, set $\gamma(\phi) = 0$. For each $k \in \mathbb{N}$, let $\Gamma_k = \{A \in \Gamma \mid \gamma(A) \geq k\}$.

Theorem 2.2 ([29, Theorem 1]). *Let E be an infinite dimensional Banach space and $\Phi \in C^1(E, \mathbb{R})$ an even functional with $\Phi(0) = 0$. Suppose that Φ satisfies*

(Φ_1) *Φ is bounded from below and satisfies* (PS) *condition.*

(Φ_2) *For each $k \in \mathbb{N}$, there exists an $A_k \in \Gamma_k$ such that $\sup_{u \in A_k} \Phi(u) < 0$.*

Then either (i) or (ii) below holds.

(i) *There exists a critical point sequence $\{u_k\}$ such that $\Phi(u_k) < 0$ and $\lim_{k\to\infty} u_k = 0$.*

(ii) *There exist two critical point sequences $\{u_k\}$ and $\{v_k\}$ such that $\Phi(u_k) = 0$, $u_k \neq 0$, $\lim_{k\to\infty} u_k = 0$, $\Phi(v_k) < 0$, $\lim_{k\to\infty} \Phi(v_k) = 0$, and $\{v_k\}$ converges to a non-zero limit.*

In order to apply Theorem 2.2, we will show in the following lemmas that the functional Φ defined in (2.5) satisfies conditions (Φ_1) and (Φ_2) in Theorem 2.2.

Lemma 2.3. *Let* (L_0), (W_1) *and* (W_2) *be satisfied. Then Φ is bounded from below and satisfies* (PS) *condition.*

Proof. We first prove that Φ is bounded from below. By (2.5) and (2.8), there holds

$$\begin{aligned} \Phi(u) &\geq \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} \big|\widetilde{W}(t,u)\big| dt \\ &\geq \frac{1}{2}\|u\|^2 - \tau_{\mu_1^*}|\xi_1|_{\mu_1}\|u\| - \tau_{\mu_2^*}^{\nu}|\xi_2|_{\mu_2}\|u\|^{\nu}, \quad \forall\, u \in E. \end{aligned} \tag{2.16}$$

Since $\nu < 2$, it follows that Φ is bounded from below.

Next, we show that Φ satisfies (PS) condition. Let $\{u_n\}_{n\in\mathbb{N}} \subset E$ be a (PS)-sequence, i.e.,

$$|\Phi(u_n)| \leq D_1, \text{ and } \Phi'(u_n) \to 0 \quad \text{as } n \to \infty \tag{2.17}$$

for some $D_1 > 0$. By (2.16) and (2.17), we have

$$D_1 \geq \frac{1}{2}\|u_n\|^2 - \tau_{\mu_1^*}|\xi_1|_{\mu_1}\|u_n\| - \tau_{\mu_2^*}^{\nu}|\xi_2|_{\mu_2}\|u_n\|^{\nu},$$

which implies that $\{u_n\}_{n\in\mathbb{N}} \subset E$ is bounded in E since $\nu < 2$. Thus there exists a subsequence $\{u_{n_k}\}_{k\in\mathbb{N}} \subset E$ such that

$$u_{n_k} \rightharpoonup u_0 \quad \text{as } k \to \infty \tag{2.18}$$

for some $u_0 \in E$. By virtue of the Riesz Representation Theorem, $\Phi' : E \to E^*$ and $\Psi' : E \to E^*$ can be viewed as $\Phi' : E \to E$ and $\Psi' : E \to E$ respectively. This together with (2.7) yields

$$u_{n_k} = \Phi'(u_{n_k}) + \Psi'(u_{n_k}), \quad \forall\, k \in \mathbb{N}. \tag{2.19}$$

By Proposition 2.1, $\Psi' : E \to E$ is also compact. Combining this with (2.17) and (2.18), the right-hand side of (2.19) converges strongly in E and hence $u_{n_k} \to u_0$ in E as $k \to \infty$. Thus Φ satisfies (PS) condition. The proof is completed. □

Lemma 2.4. *Let* (L_0), (W_1) *and* (W_3) *be satisfied. Then for each* $k \in \mathbb{N}$, *there exists an* $A_k \in E$ *with genus* $\gamma(A_k) = k$ *such that* $\sup_{u\in A_k} \Phi(u) < 0$.

Proof. We follow the idea of dealing with elliptic problems in Kajikiya [29]. Let d_0 be the length of the closed interval I_0 in (W_3). For any fixed $k \in \mathbb{N}$, we divide I_0 equally into k closed sub-intervals and denote them by I_i with $1 \le i \le k$. Then the length of each I_i is $a \equiv d_0/k$. For each $1 \le i \le k$, let t_i be the center of I_i and J_i be the closed interval centered at t_i with length $a/2$. Choose a function $\varphi \in C_0^\infty(\mathbb{R}, \mathbb{R}^N)$ such that $|\varphi(t)| \equiv 1$ for $t \in [-a/4, a/4]$, $\varphi(t) \equiv 0$ for $t \in \mathbb{R} \setminus [-a/2, a/2]$, and $|\varphi(t)| \le 1$ for all $t \in \mathbb{R}$. Now for each $1 \le i \le k$, define $\varphi_i \in C_0^\infty(\mathbb{R}, \mathbb{R}^N)$ by

$$\varphi_i(t) = \varphi(t - t_i), \quad \forall\, t \in \mathbb{R}.$$

Then it is easy to see that

$$\mathrm{supp}\varphi_i \subset I_i \tag{2.20}$$

and

$$|\varphi_i(t)| = 1,\ \forall\, t \in J_i, \quad |\varphi_i(t)| \le 1,\ \forall\, t \in \mathbb{R} \tag{2.21}$$

for all $1 \le i \le k$. Set

$$V_k \equiv \left\{ (r_1, r_2, \ldots, r_k) \in \mathbb{R}^k \mid \max_{1\le i\le k} |r_i| = 1 \right\}$$

and

$$W_k \equiv \left\{ \sum_{i=1}^{k} r_i \varphi_i \mid (r_1, r_2, \ldots, r_k) \in V_k \right\}.$$

Evidently, V_k is homeomorphic to the unit sphere in $\mathbb{R}^k$ by an odd mapping. Thus $\gamma(V_k) = k$. If we define the mapping $\mathcal{F} : V_k \to W_k$ by

$$\mathcal{F}(r_1, r_2, \dots, r_k) = \sum_{i=1}^{k} r_i \varphi_i, \quad \forall\, (r_1, r_2, \dots, r_k) \in V_k,$$

then $\mathcal{F}$ is odd and homeomorphic. Therefore $\gamma(W_k) = \gamma(V_k) = k$. Moreover, it is evident that W_k is compact and hence there is a constant $C_k > 0$ such that

$$\|u\|^2 \leq C_k, \quad \forall\, u \in W_k. \tag{2.22}$$

For any $s \in (0, b)$ and $u = \sum_{i=1}^{k} r_i \varphi_i \in W_k$, by (2.5), (2.20) and (2.21), we have

$$\begin{aligned} \Phi(su) &= \frac{1}{2}\|su\|^2 - \int_{\mathbb{R}} \widetilde{W}(t, s\sum_{i=1}^{k} r_i\varphi_i)dt \\ &= \frac{s^2}{2}\|u\|^2 - \sum_{i=1}^{k}\int_{I_i} \widetilde{W}(t, sr_i\varphi_i)dt \\ &= \frac{s^2}{2}\|u\|^2 - \sum_{i=1}^{k}\int_{I_i} W(t, sr_i\varphi_i)dt, \end{aligned} \tag{2.23}$$

where the last equality holds by the definition of $\widetilde{W}$ in (2.2) and the fact that $|sr_i\varphi_i(t)| < b$ for all $1 \leq i \leq k$. Observing the definition of V_k, for every $u = \sum_{i=1}^{k} r_i\varphi_i \in W_k$, there exists some integer $1 \leq i_u \leq k$ such that $|r_{i_u}| = 1$. Then it follows that

$$\begin{aligned} &\sum_{i=1}^{k}\int_{I_i} W(t, sr_i\varphi_i)dt \\ &= \int_{J_{i_u}} W(t, sr_{i_u}\varphi_{i_u})dt + \int_{I_{i_u}\setminus J_{i_u}} W(t, sr_{i_u}\varphi_{i_u})dt \\ &\quad + \sum_{i\neq i_u}\int_{I_i} W(t, sr_i\varphi_i)dt. \end{aligned} \tag{2.24}$$

By (W_3), (2.21) and the the definition of V_k, there holds

$$\int_{I_{i_u}\setminus J_{i_u}} W(t, sr_{i_u}\varphi_{i_u})dt + \sum_{i\neq i_u}\int_{I_i} W(t, sr_i\varphi_i)dt \geq -\varrho d_0 s^2, \tag{2.25}$$

where d_0 is given at the beginning of the proof. For each $\delta_n \in (0, b)$, combining (W_3), (2.2) and (2.21)–(2.25), we have

$$\begin{aligned}\Phi(\delta_n u) &\leq \frac{C_k\delta_n^2}{2} + \varrho d_0\delta_n^2 - \int_{J_{i_u}} W(t, \delta_n r_{i_u}\varphi_{i_u})dt \\ &\leq \delta_n^2\left(\frac{C_k}{2} + \varrho d_0 - \frac{M_n d_0}{2k}\right).\end{aligned} \tag{2.26}$$

Here we use the fact that $|\delta_n r_{i_u}\varphi_{i_u}(t)| \equiv \delta_n$ for $t \in J_{i_u}$. Note that $\delta_n \to 0$ and $M_n \to \infty$ as $n \to \infty$ in (W_3). Then we can choose $n_0 \in \mathbb{N}$ large enough such that the right-hand side of (2.26) is negative. Define

$$A_k = \{\delta_{n_0}u \mid u \in W_k\}. \tag{2.27}$$

Then we have

$$\gamma(A_k) = \gamma(W_k) = k \quad \text{and} \quad \sup_{u\in A_k} \Phi(u) < 0.$$

The proof is completed. □

Now we are in the position to give the proof of our main result.

Proof of Theorem 1.1. Lemmas 2.3 and 2.4 show that the functional Φ defined in (2.5) satisfies conditions (Φ_1) and (Φ_2) in Theorem 2.2. Therefore, by Theorem 2.2, we get a sequence nontrivial critical points $\{u_k\}$ for Φ satisfying $\Phi(u_k) \leq 0$ for all $k \in \mathbb{N}$ and $u_k \to 0$ in E as $k \to \infty$. By virtue of Proposition 2.1, $\{u_k\}$ is a sequence of homoclinic solutions of $(\widetilde{\text{HS}})$. Since E is continuously embedded into L^∞, then it follows that $\max_{t\in\mathbb{R}} |u_k(t)| \to 0$ as $k \to \infty$. Hence, there exists $k_0 \in \mathbb{N}$ such that u_k is a homoclinic solution of (HS) for each $k \geq k_0$. This ends the proof. □

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