Electronic Journal of Qualitative Theory of Differential Equations
2013, No. **68**, 1-11; **http://www.math.u-szeged.hu/ejqtde/**

EXISTENCE OF A MOVING ATTRACTOR FOR SEMI-LINEAR PARABOLIC EQUATIONS

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Abstract. We analyze the existence of a moving attractor for a semi-linear parabolic equation. The analysis is carried out by using a coordinate transformation to obtain the equation in moving coordinates and the contractions of the solutions to traveling wave solutions. We show that a moving attractor exists if the linearized equation on the moving frame follows a stability condition. Application of moving attractor in long term error analysis is discussed with numerical examples.

Keywords: parabolic equation; long-time error estimation; contraction; moving attractor.

AMS(2000) Subject Classification: 65M15, 65L20

1. Introduction

In this paper, we consider the semi-linear parabolic initial boundary value problem of the form

$$
\begin{cases} u_t - u_{xx} = f(u), & x \in [a,b], \quad t > 0, \\ u(\cdot,0) = u_0, & x \in [a,b], \quad t > 0, \\ u_x(a) = u_x(b) = 0, & t > 0, \end{cases} \tag{1}
$$

where f is a twice continuously differentiable function.

Traditional methods for the error estimation of a numerical solution of (1) require consistency and the numerical stability of the scheme employed. In practice it is very difficult and tedious to obtain numerical stability of a scheme while solving a non-linear equation or a system. In a recent study by Sun et al. [8, 9], a new approach for error estimation is introduced which replaces the numerical stability by the numerical smoothing property of the scheme. This approach is a powerful tool in long time error analysis since it can be implemented without scheme complications.

The main features of this approach are to use error propagation of the exact solution and numerical smoothing while keeping the standard consistency requirement of the scheme. Numerical smoothing of the scheme depends on the numerical solution and can be monitored by using a stability smoothing indicator regardless of the scheme employed. Existence of a smoothing indicator for semi-linear parabolic equations was proved by Sun and Fillipova [9] when the stability smoothing indicator is bounded in each local time step. For more details regarding numerical smoothing and the stability smoothing indicator the reader is referred to [9] and references therein.

Since the exact solution of the problem (1) is unknown, or difficult to compute, it is not possible to obtain the error between the exact solution and the numerical solution. Therefore, the exact error propagation is obtained by computing the error between the numerical solution and a moving attractor [8]; a compact subset of space which attracts all

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the trajectories of the equation (1). We can use any contraction properties of the dynamical system to the moving attractor to obtain exact error estimates. The goal of this paper is to show existence of a moving attractor for a semi-linear parabolic problem given in (1).

To this end we consider *traveling wave* solutions of (1) in the form $u(x,t) = \phi(x - vt)$ where v is a positive constant. We use change of coordinates to obtain the equation in moving coordinates. A moving attractor to the original problem (1) is constructed by considering the linearized equation in moving coordinates assuming a certain property of the linearized equation.

Propagation of waves, described by nonlinear parabolic equations, were first considered in the paper by Lau [6]. These mathematical investigations arose in connection with a model for the propagation of dominant genes, a topic also considered by Fisher [3]. In systems with more than one stationary homogeneous solution, a typical solution is given by a traveling wave front. These solutions move with constant speed without changing their shape. Wave solutions of above type arise in numerous problems of physical interest; such as propagation of nerve impulses, propagation of favorable genes, shock waves, and propagation of flames (see [7] and references therein).

The outline of this paper is as follows. In Section 2, we introduce some preliminary results for traveling wave solutions. Section 3 is devoted to prove the contraction properties of traveling wave solutions. In Section 4, we define the moving attractor for the semi-linear equation and show that a moving attractor exists if the initial condition and linearized equation satisfy certain conditions. We also provide an application of moving attractor in long time error analysis. Finally in Section 5, some computational results are given.

2. Preliminaries

Traveling wave solutions of equation (1) are of the form $u(x,t) = \phi(y)$ with $y = x - vt$, where v is the speed of the traveling wave with $v \neq 0$. When $u(x,t) = \phi(y)$, the semi-linear equation (1) takes the form

$$-v\frac{\partial \phi}{\partial y} - \frac{\partial^2 \phi}{\partial y^2} = f(\phi). \tag{2}$$

The existence and uniqueness of the solution to (2) can be obtained by standard phase plane arguments. We consider perturbations ϕ of (2) to find a moving attractor to solutions of (1).

If $\phi(y)$ is a solution to (2), under the change of coordinates $y = x - vt$ we have $U(y,t) = u(y + vt, t)$, where u is a solution to (1) and $U(y,t) = \phi(y)$. Thus on the moving frame we obtain

$$\frac{\partial U}{\partial t} - v\frac{\partial U}{\partial y} - \frac{\partial^2 U}{\partial y^2} = f(U). \tag{3}$$

Linearizing (3) about ϕ in $\tilde{U}$ leads to the equation

$$\frac{\partial \tilde{U}}{\partial t}(y,t) - v\frac{\partial \tilde{U}}{\partial y}(y,t) - \frac{\partial^2 \tilde{U}}{\partial y^2}(y,t) = \frac{\partial f}{\partial \phi}(\phi(y))\tilde{U}(y,t), \tag{4}$$

where $\tilde{U}(y,t) = \frac{d\phi}{dy}(y),\ t \geq 0$.

Solutions of equations similar to (3) and (4) can be obtained in infinite domain by considering the fundamental solution of the equation in moving coordinates [1]. Here we use semigroup solution operator to obtain the solutions of (1) and (4) in finite domain.

If the semigroup operator $E(t)$ is the solution of the homogenous problem

$$u_t - u_{xx} = 0, \qquad u(0) = u_0,$$

by Duhamel's principle it follows that solutions of (1) satisfy

$$u(x,t) = E(t)u_0 + \int_0^t E(t-r)f(u(x,r))dr, \quad t > 0,$$

where $(E(t)\psi)(x) = [\exp(-x^2/4t)/\sqrt{4\pi t}]\psi(x)$ for some bounded and piecewise function ψ. Since (3) related to (1) by change of coordinates, solution $U(y,t)$ with initial value U_0 satisfies

$$U(y,t) = E(t)U_0 + \int_0^t E(t-r)f(U(y+vt-vr,r))dr, \quad t > 0. \tag{5}$$

In an identical fashion, solution $\tilde{U}$ corresponds to (4) satisfies

$$\tilde{U}(y,t) = E(t)\tilde{U}_0 + \int_0^t E(t-r)\frac{\partial f}{\partial \phi}(\phi(y+vt-vr))\tilde{U}(y+vt-vr,r)dr, \quad t > 0. \tag{6}$$

The following lemma is due to Evans [1] and was proved by using the fundamental solution of the linear and non-linear systems. Here we prove the lemma by using Duhamel's principle. The lemma gives a relation between the solutions $\tilde{U}(y,t)$ and $U(t)$, of linear and nonlinear equations in the L^2-norm $\|\cdot\|$. Let

$$\rho(t) = \|U(\cdot,t) - \phi(t) - \tilde{U}(\cdot,t)\|. \tag{7}$$

Lemma 2.1. *If $\tilde{U}$ of (6) is bounded by M for all $t \geq 0$, then*

$$\rho(t) \leq C_l \rho(0) e^{C_l L t} + \frac{M^2 Q}{L}(e^{C_l L t} - 1),$$

where L and Q are upper bounds for $\left|\frac{\partial f(\phi)}{\partial \phi}\right|$ and $\left|\frac{\partial^2 f(\phi)}{\partial \phi^2}\right|$ respectively.

Proof. Recall that U has the representation (5)

$$U(y,t) = E(t)U_0 + \int_0^t E(t-r)f(U(y+vt-vr,r))dr.$$

Similarly ϕ has the representation

$$\phi(y) = E(t)\phi(0) + \int_0^t E(t-r)f(\phi(y+vt-vr))dr$$

and $\tilde{U}$ has the representation (6)

$$\tilde{U}(y,t) = E(t)\tilde{U}_0 + \int_0^t E(t-r)\frac{\partial f}{\partial \phi}(\phi(y+vt-vr))\tilde{U}(y+vt-vr,r)dr.$$

Using the standard fact [10]

$$\|E(t)v\| \leq C_l \|v\|, \quad t > 0 \tag{8}$$

we have

$$\|U(y,t) - \phi(y) - \tilde{U}(y,t)\| \leq C_l \|U(\cdot,0) - \phi(0) - \tilde{U}(\cdot,0)\| + C_l \int_0^t H(r)dr \tag{9}$$

for all x, where

$$H(r) = \left\| f(U(y+vt-vr,r)) - f(\phi(y+vt-vr)) - \frac{\partial f}{\partial \phi}(\phi(y+vt-vr))\tilde{U}(y+vt-vr,r) \right\|.$$

Now letting $V = U - \phi$, by the mean value theorem and Taylor's expansion we have

$$\begin{aligned} \left\| f(\phi+V) - f(\phi) - \frac{\partial f}{\partial \phi}\tilde{U} \right\| &= \left\| f(\phi+V) - f(\phi+\tilde{U}) + f(\phi+\tilde{U}) - f(\phi) - \frac{\partial f}{\partial \phi}\tilde{U} \right\| \\ &\leq L\|V - \tilde{U}\| + Q\|\tilde{U}\|^2. \end{aligned} \tag{10}$$

If $\|\tilde{U}\|$ is bounded by M for $t > 0$, then (9) and (10) give

$$\rho(t) \leq C_l\rho(0) + C_l\int_0^t (L\rho(r) + QM^2)dr. \tag{11}$$

Then by Gronwall's inequality, we obtain

$$\rho(t) \leq C_l\rho(0)e^{C_lLt} + (M^2Q/L)(e^{C_lLt} - 1).$$

□

3. Contraction of solutions

In this section we prove that if the linearized equation (4) has a certain stability property, and if the initial condition is a perturbation of traveling wave, then the solution U of (3) is contractive to a traveling wave. We define contraction as a local property on each time step. U is said to be contractive to a traveling wave if it is closer to traveling wave at the end of the time step than it is at the beginning of the step. Also, if the initial values are of the form $\|U(y,0) - \phi(y)\| \leq \varepsilon$ for sufficiently small $\varepsilon > 0$, in general one cannot expect that $U \to \phi$ as $t \to \infty$. Instead we are looking for $U \to \phi_\gamma$ as $t \to \infty$ for some γ where $\phi_\gamma = \phi(x+\gamma)$ [7]. Moreover, it is clear that if ϕ is a solution of (2) then ϕ_γ is a solution of (2) as well. Therefore we define the contraction property of solutions as follows:

Definition 3.1. *Let U be the solution of equation (3). Then U contracts to a displaced traveling wave ϕ_δ at time $t+\tau, \tau > 0$, if there exists a constant $\theta_\tau \in (0,1)$ such that*

$$\|U(t+\tau) - \phi_\delta(t+\tau)\| \leq \theta_\tau\|U(t) - \phi_\gamma(t)\|,$$

for some $\delta, \gamma > 0$.

Furthermore, we say that equation (4) is stable at $\frac{d\phi}{dy}$, for given $\|\tilde{U}(0)\| < \varepsilon$, if there exists $\vartheta_\tau \in (0,1)$ such that

$$\left\| \tilde{U}(\tau) - \delta\left(\frac{d\phi}{dy}(\tau)\right) \right\| \leq \vartheta_\tau\|\tilde{U}(0)\|, \tag{12}$$

where $\delta < C_\delta\|\tilde{U}(0)\|$, $C_\delta, \varepsilon > 0$. Sattinger [7] and Evans [2] have proved similar stability property in L^∞ norm for parabolic and nerve axon equations respectively. In this paper we assume stability property (12) and leave the proof as a future research problem.

In the following theorem we show that solution $U(y,t)$ of (3) is contractive to a traveling wave ϕ_δ for any given t under small perturbation of a solution $\phi(y)$ of (2) given that $\tilde{U}$ satisfies the linear stability property (12).

Theorem 3.2. *If there exists a sufficiently small $\varepsilon > 0$ such that $\|U(y,0) - \phi(y)\| \le \varepsilon$ and if $\tilde{U}$ of linearized equation (4) satisfies property (12), then for any $t > 0$ there are constants δ, γ and τ such that*

$$\|U(t+\tau) - \phi_\delta(t+\tau)\| \le \theta_\tau \|U(t) - \phi_\gamma(t)\|.$$

Proof. Let N and R be the upper bounds of $\|d\phi/dy\|$ and $\|d^2\phi/dy^2\|$ respectively. Constants Q, L and M be defined as in the previous Lemma. Since contraction is a local property we prove there is a contractive interval $[t, t+\tau]$ with time step size τ for a given t. We start with dividing the interval $[0, t+\tau]$ into subintervals with interval length τ_i for $i = 1, 2, \ldots$, and let $s_k := \sum_{i=1}^{k} \tau_i$ with $s_0 = 0$ for some k. We choose τ_i such a way that $t = s_{k-1}$ and $\vartheta_{\tau_i} < 1$ for any given $t > 0$.

Now, for any $\tau_i > 0$, choose ε such that

$$\varepsilon \le \min\left(\frac{L\Gamma_{\tau_i}}{Qe^{C_l L\tau_i}(\vartheta_{\tau_i} + C_\delta N)^2}, \frac{\Gamma_{\tau_i}}{RC_\delta^2}\right),$$

where ε is defined as $\|U(0) - \phi(0)\| < \varepsilon$ and $\Gamma_\tau = \beta - \vartheta_\tau$ with $\beta < (1+\vartheta_\tau)/2$. With this type of relation between τ_i and ε, we can determine a better value for the constant ε. This is the reasoning behind the partitioning for subintervals.

Therefore, for the interval $[t, t+\tau_k]$ it is sufficient to prove that

$$\|U(s_k) - \phi_\delta(s_k)\| \le \theta_{\tau_k}\|U(s_{k-1}) - \phi_\delta(s_{k-1})\|.$$

We prove this using mathematical induction.

First consider the case when $k = 1$. That is, when $t = 0$. For the interval $[0, s_1]$, if $\tilde{U}(s_1)$ is a solution of (4) at s_1 with initial value $\tilde{U}(0) = U(0) - \phi(0)$, equation (12) implies that,

$$\left\|\tilde{U}(s_1) - \delta\frac{d\phi}{dy}(s_1)\right\| \le \vartheta_{\tau_1}\|U(0) - \phi(0)\|, \tag{13}$$

with $|\delta| \le C_\delta\|U(0) - \phi(0)\|$. Also note that $\rho(0) = 0$ since the initial condition of the linearized equation is $\tilde{U}(0) = U(0) - \phi(0)$. Therefore Lemma 2.1 implies that

$$\rho(s_1) \le \frac{M^2 Q}{L}(e^{C_l L s_1} - 1). \tag{14}$$

From (13) we have that

$$\begin{aligned} \|\tilde{U}(s_1)\| &\le \left\|\tilde{U}(s_1) - \delta\frac{d\phi}{dy}(s_1)\right\| + \left\|\delta\frac{d\phi}{dy}(s_1)\right\| \\ &\le \vartheta_{\tau_1}\|U(0) - \phi(0)\| + |\delta|N \\ &\le \vartheta_{\tau_1}\|U(0) - \phi(0)\|\left(1 + \frac{NC_\delta}{\vartheta_{\tau_1}}\right) \\ &= M. \end{aligned} \tag{15}$$

Then, from (14) and (15),

$$
\begin{aligned}
&\left\|\left(U-\phi-\tilde{U}\right)(s_1)\right\| \\
&\le M^2\frac{Q}{L}(e^{C_l L s_1}-1) \\
&=\Gamma_{\tau_1}\|U(0)-\phi(0)\|\left[\frac{1}{\Gamma_{\tau_1}}\|U(0)-\phi(0)\|(\vartheta_{\tau_1}+NC_\delta)^2\frac{Q}{L}(e^{C_l L s_1}-1)\right] \qquad (16)\\
&\le\Gamma_{\tau_1}\|U(0)-\phi(0)\|\left[\frac{1}{\Gamma_{\tau_1}}\varepsilon(\vartheta_{\tau_1}+NC_\delta)^2\frac{Q}{L}e^{C_l L s_1}\right] \\
&\le\Gamma_{\tau_1}\|U(0)-\phi(0)\|
\end{aligned}
$$

Also,

$$
\begin{aligned}
R\delta^2 &\le RC_\delta^2\|U(0)-\phi(0)\|^2 g \\
&\le \Gamma_{\tau_1}\|U(0)-\phi(0)\|\left(\frac{RC_\delta^2}{\Gamma_{\tau_1}}\|U(0)-\phi(0)\|\right) \qquad (17)\\
&\le \Gamma_{\tau_1}\|U(0)-\phi(0)\|\left(\frac{RC_\delta^2}{\Gamma_{\tau_1}}\varepsilon\right) \\
&\le \Gamma_{\tau_1}\|U(0)-\phi(0)\|
\end{aligned}
$$

Since R is an upper bound for $d^2\phi/dy^2$, using (13), (16), (17) and by Taylor expansion, we have that,

$$
\begin{aligned}
\|U(s_1)-\phi_\delta(s_1)\| &\le \left\|U-\phi-\tilde{U}\right\|+\left\|\tilde{U}-\delta\frac{d\phi}{dy}\right\|+\left\|\phi_\delta-\phi-\delta\frac{d\phi}{dy}\right\| \\
&\le \Gamma_{\tau_1}\|U(0)-\phi(0)\|+\vartheta_{\tau_1}\|U(0)-\phi(0)\|+R\delta^2 \\
&\le (2\beta-\vartheta_{\tau_1})\|U(0)-\phi(0)\| \\
&= \theta_{\tau_1}\|U(0)-\phi(0)\| \\
&= \theta_{\tau_1}\|U(0)-\phi_\gamma(0)\|, \quad \text{where } \gamma=0.
\end{aligned}
$$

Now assume that when $k=p$: $\|U(s_p)-\phi_\delta(s_p)\|\le\theta_{\tau_p}\|U(s_{p-1})-\phi_\gamma(s_{p-1})\|$ and $\|U(s_{p-1})-\phi_\gamma(s_{p-1})\|<\varepsilon$. That is, since $\|U(s_p)-\phi_\delta(s_p)\|<\varepsilon$, if $U(s_p)-\phi_\delta(s_p)$ is the initial value for $\tilde{U}$of (4) for the interval $[s_p,s_{p+1}]$, then from (12) we have that,

$$
\left\|\tilde{U}(s_{p+1})-\delta\frac{d\phi}{dy}(s_{p+1})\right\|\le\vartheta_{\tau_{p+1}}\|U(s_p)-\phi_\delta(s_p)\|. \qquad (18)
$$

And as in (15),

$$
\|\tilde{U}(s_{p+1})\|\le\vartheta_{\tau_{p+1}}\|U(s_p)-\phi_\delta(s_p)\|\left(1+\frac{NC_\delta}{\vartheta_{s_{p+1}}}\right)=M. \qquad (19)
$$

Then inequalities (16) and (17) become

$$
\begin{aligned}
\left\|\left(U-\phi-\tilde{U}\right)(s_{p+1})\right\| &\le \Gamma_{\tau_{p+1}}\|U(s_p)-\phi_\delta(s_p)\| \qquad (20)\\
R\delta^2 &\le \Gamma_{\tau_{p+1}}\|U(s_p)-\phi_\delta(s_p)\| \qquad (21)
\end{aligned}
$$

respectively. Thus from (18), (20) and (21) we obtain

$$
\|U(s_{p+1})-\phi_\delta(s_{p+1})\|\le\theta_{\tau_{p+1}}\|U(s_p)-\phi_\delta(s_p)\|, \quad \text{where } \gamma=\delta.
$$

The proof of induction completes the proof of the theorem. ◻

4. Moving attractor

We recall that the concept of the moving attractor initially introduced in [8] and [9]. It is a compact subset of phase space that attracts all the trajectories of the dynamical system. As such, we can expect the set of trajectories that lie in the attractor to cover all the possible dynamical behaviors of the system.

We also need an invariant condition, which guarantees that the absorbing set does not decrease as $t \to \infty$. If $\mathcal{M}$ is a one-parameter family of sets in L^2, $\mathcal{M} = \left\{\Phi_t \subset L^2 | t > T\right\}$, we say that $\mathcal{M}$ is positively invariant under the dynamical system if for any $u(t) \in \Phi_t$ and $p > 0$, $u(t+p) \in \Phi_{t+p}$. Following is the definition of the moving attractor given in [9].

Definition 4.1. *A positively invariant one-parameter family of sets $\mathcal{M}$ in L^2 as defined above is called a moving attractor, if there exists a real number $\theta_\tau \in (0,1)$ depending on τ, and a one-parameter family of open sets $\mathcal{U} = \left\{U_t \subset L^2 | t > T\right\}$, positively invariant under the dynamical system, with $\Phi_t \subset U_t$ for all $t > T$, such that for any $u \in U_t$*

$$d(u(t+\tau), \Phi_{t+\tau}) \leq \theta_\tau d(u(t), \Phi_t),$$

where $d(u, \Phi) = \inf_{w \in \Phi} \|u - w\|$. $\mathcal{U}$ is called a basin of the moving attractor.

Following theorem proves the existence of a moving attractor for the semi-linear parabolic equation (1).

Theorem 4.2. *Let $\phi(x - vt)$ and $u(x,t)$ be wave profile and solution of equation (1) respectively. Assume that conditions given in Theorem 3.2 are satisfied. Define Φ_t and U_t:*

$$\Phi_t = \{\phi_h(t); \phi_h(t) = \phi(x - vt + h) | h \in \mathbb{R}\},$$

$$U_t = \{u(x,t),\ |u(x,0) = \phi(x) + \epsilon u_0(x)\}.$$

Then the family of sets

$$\mathcal{M} = \{\Phi_t | t > T_0\}$$

is a moving attractor for the equation (1). The family of sets $\mathcal{U} = \{U_t | t > T_0\}$ is the basin of the moving attractor.

Proof. If $\phi_h(t)$ is a solution of equation (1) then $\phi_h(t+\tau)$ is also a solution after time τ. In other words if $\phi_h(t) \in \Phi_t$ then $\phi_h(t+\tau) \in \Phi_{t+\tau}$. Therefore, the family of sets $\mathcal{M}$ is positively invariant. Clearly the family of sets

$$\mathcal{U} = \{U_t | t > T_0\}$$

is also positively invariant and $\Phi_t \subset U_t$.

Then, under the conditions defined in Theorem 3.2, there exist $\theta_\tau \in (0,1)$, $\delta, \gamma \in \mathbb{R}$ such that

$$\|u(t+\tau) - \phi_\delta(t+\tau)\| \leq \theta_\tau \|u(t) - \phi_\gamma(t)\|$$

for any $u(t) \in U_t$. Thus

$$d(u(t+\tau), \Phi_{t+\tau}) \leq \theta_\tau d(u(t), \Phi_t).$$

◻

Remark: For a system of parabolic equations of the form

$$\begin{aligned}
u_t - \Delta u &= f(u) \qquad x \in \Omega \quad t > 0, \\
u(\cdot, 0) &= u_0(x), \qquad x \in \Omega \quad t > 0, \\
\frac{\partial u}{\partial n} &= 0, \qquad t > 0,
\end{aligned}$$

where f and u are vector valued functions and n is normal to the surface, we can prove the existence of a moving attractor in a similar fashion. Wave solutions of the system are of the from $u(x,t) = \phi(x_1 - vt, x')$, where ϕ is function of m variables, $x = (x_1, \dots, x_m)$ and $x' = (x_2, \dots, x_m)$. The space $\Omega \subset \mathbb{R}^m$ is a cylinder and the system of coordinates is chosen so that axis x_1 is directed along the axis of the cylinder.

4.1. **Application on long time error.** In this subsection we provide an application of moving attractor in numerical error approximation.

Suppose we use a convergent numerical scheme. That is if u_N and u are numerical solution and exact solution respectively, then there exists $\epsilon_{\tau,h}$ such that $\|u_N(ns+T_0) - u(ns+T_0)\| \leq \epsilon_{\tau,h}$. The term $\epsilon_{\tau,h}$ depends on temporal discretization $\tau = s/k$ for some integer k, and spacial discretisation h such that $\epsilon_{\tau,h} \to 0$ as $\tau, h \to 0$. Then, according to Theorem 4.6 of [8] if there is a moving attractor and convergent numerical scheme, the long time error satisfies

$$d(u_N(ns+T_0), \Phi_{T_0+ns}) \leq \frac{\epsilon_{\tau,h}}{1+\theta_s} + \theta_s^n d(u_N(T_0), \Phi_{T_0}). \tag{22}$$

In other words by Theorem 4.2 there are constants γ and ϵ_2 such that $\|u_N - \phi_\gamma\| \leq \epsilon_2$. Here ϵ_2 is an upper bound for the right-hand side of the error estimate (22). The following theorem shows if there is a moving attractor with convergent and mass conservative numerical scheme then error of a numerical solution is uniformly bounded. A solution is mass conservative if $\int_a^b u dx$ is computed accurately.

Theorem 4.3. *If (a) the error on total mass is bounded by ϵ_1: $|\int_a^b u_N dx - \int_a^b \phi dx| \leq \epsilon_1$ (b) the solution is close to traveling wave: $\|u_N - \phi_\gamma\| \leq \epsilon_2$, then for $\epsilon = \max(\epsilon_1, \epsilon_2)$, $\|u_N - \phi\| \leq \epsilon + \sqrt{2C_{amp}\left(\epsilon(b-a)^{1/2} + \epsilon\right)}$, and thus the global error is uniformly bounded in time.*

Proof. First notice that there exists a constant C_{amp} such that $\|\phi_\gamma - \phi\|_{L^\infty[a,b]} \leq 2C_{amp}$. where C_{amp} is the amplitude of the wave. Thus

$$\begin{aligned}
\|\phi_\gamma - \phi\|^2 &\leq \|\phi_\gamma - \phi\|_{L^\infty[a,b]} \left| \int_a^b (\phi_\gamma - \phi) dx \right| \\
&\leq 2C_{amp} \left(\left| \int_a^b (\phi_\gamma - u_N) dx \right| + \left| \int_a^b (u_N - \phi) dx \right| \right) \\
&\leq 2C_{amp} \left(\epsilon_2 \sqrt{b-a} + \epsilon_1 \right).
\end{aligned}$$

Now

$$\begin{aligned}
\|u_N - \phi\| &\leq \|u_N - \phi_\gamma\| + \|\phi_\gamma - \phi\| \\
&\leq \epsilon_2 + \sqrt{2C_{amp}\left(\epsilon_2\sqrt{b-a} + \epsilon_1\right)}.
\end{aligned}$$

□

5. Numerical application

In this section, we give two examples to illustrate an application of moving attractor in numerical error approximation. To obtain numerical solutions of the examples we used finite element method for spatial discretization with piecewise linear elements and backward Euler method for temporal discretization. We considered the wave solution of both examples. In practice however, we are interested in approximation of the front of the wave. By wave front we mean solution $\phi(y)$ of (2) with $\lim_{y\to\pm\infty}\phi(y)=\omega_\pm$ as $y\to\pm\infty$. In general $\omega_+\neq\omega_-$. If $\omega_+=\omega_-$ wave front is called a pulse.

Consider Huxley's equation [5] on the following form

$$\begin{aligned} u_t &= u_{xx}+u(1-u)(u-0.25)\\ u(x,0) &= \begin{cases} 1, & \text{if } -100<x<-75\\ -0.04x-2, & \text{if } -75<x<-50\\ 0, & \text{if } -50<x<100. \end{cases} \end{aligned}$$

Figure 1 shows computed moving front of the Huxley's equation at time $t=20,50,100,$ $1000,2000,\ldots,10000$.

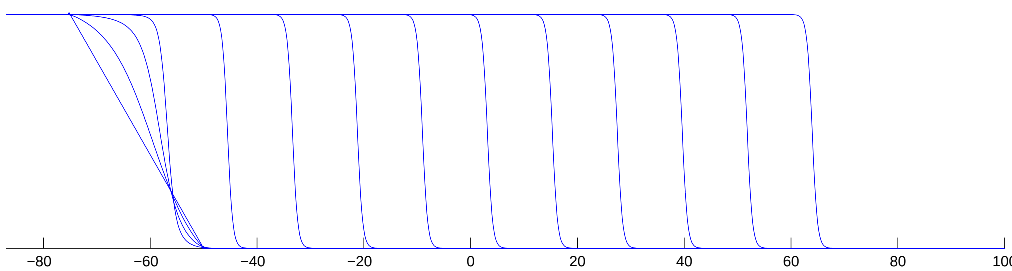

Figure 1. Traveling wave fronts of Huxley's equation.

For the second example we choose FitzHugh's nerve axon equation [4] given by

$$\begin{aligned} \frac{\partial u_1}{\partial t} &= \frac{\partial^2 u_1}{\partial x^2}+10(u_1-\frac{1}{3}u_1^3-u_2),\\ \frac{\partial u_1}{\partial t} &= 0.8(1.5+1.25u_1-u_2),\\ u_1(0)=-1.5, &\qquad u_2(0)=-\frac{3}{8}. \end{aligned}$$

Figure 2 shows the traveling wave front of a nerve impulse at time $t=6,7,8,9,\ldots,13,14$.

Both in Figure 1 and 2 the shape of the moving fronts is consistent. Therefore, we can conclude that the error of the numerical solution is not growing with time. Since the position of the front determines the total mass, we can also conclude that mass is conservative during the computation.

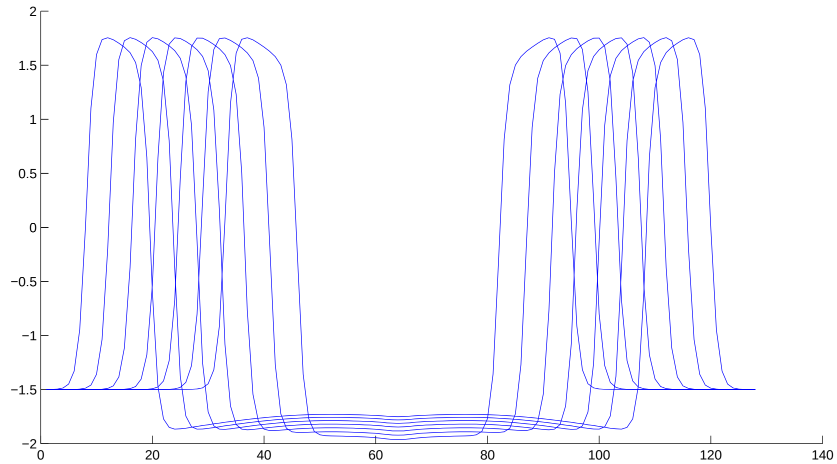

FIGURE 2. Traveling wave pulses of FitzHugh's nerve axon equation.

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(Received April 24, 2013)

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