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Existence results for a coupled system of nonlinear fractional differential equations with boundary value problems on an unbounded domain

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Abstract. This paper deals with the existence results for solutions of coupled system of non-linear fractional differential equations with boundary value problems on an unbounded domain. Also, we give an illustrative example in order to indicate the validity of our assumptions.
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1 Introduction

Recently, fractional differential calculus has attracted a lot of attention by many researchers of different fields, such as: physics, chemistry, biology, economics, control theory and biophysics, etc. [11, 15, 16]. In particular, study of coupled systems involving fractional differential equations is also important in several problems.

Many authors have investigated sufficient conditions for the existence of solutions for the following coupled systems of nonlinear fractional differential equations with different boundary conditions on finite domain.

$$\begin{cases} D^{\alpha}u(t) = f(t, v(t)), \\ D^{\beta}v(t) = g(t, u(t)), \end{cases}$$

and more generally,

$$\begin{cases} D^{\alpha}u(t) = f(t, v(t), D^{\mu}v(t)), \\ D^{\beta}v(t) = g(t, u(t), D^{\nu}u(t)), \end{cases}$$

see for example [1, 2, 4, 7, 8, 9, 10, 17, 21, 22, 23]. However, to the best of our knowledge few papers consider the existence of solutions of fractional differential equations on the half-line. Arara et al. [3] studied the existence of bounded solutions for differential equations involving the Caputo fractional derivative on the unbounded domain given by

$$\begin{cases} cD^{\alpha}u(t) = f(t, u(t)), \quad t \in [0, \infty), \\ u(0) = u_0, \\ u \text{ is bounded on } [0, \infty), \end{cases} \tag{1}$$

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where $\alpha \in (1,2]$, cD^α is the Caputo fractional derivative of order $\alpha, u_0 \in \mathbb{R}$, and $f : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ is continuous.

Zhao and Ge [24] considered the following boundary value problem for fractional differential equations

$$\begin{cases} D^\alpha u(t) + f(t, u(t)) = 0, \qquad t \in (0,\infty), \\ u(0) = 0, \\ \lim_{t\to\infty} D^{\alpha-1}u(t) = \beta u(\xi), \end{cases} \tag{2}$$

where $\alpha \in (1,2), 0 < \xi < \infty, \beta \geq 0$, f is a given function and D^α is the Riemann–Liouville fractional derivative.

Su, Zhang [19] considered the following boundary value problem

$$\begin{cases} D^\alpha u(t) = f(t, u(t), D^{\alpha-1}u(t)),\ t \in [0,\infty), \\ u(0) = 0, \\ D^{\alpha-1}u(\infty) = u_0,\ u_0 \in \mathbb{R}, \end{cases} \tag{3}$$

where $\alpha \in (1,2]$, $f \in C([0,\infty) \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $D^\alpha, D^{\alpha-1}$ are the Riemann–Liouville fractional derivatives.

In [13], Liang and Zhang investigated the existence of three positive solutions for the following m-point fractional boundary value problem

$$\begin{cases} D^\alpha u(t) + a(t)f(u(t)) = 0, \quad t \in (0,\infty), \\ u(0) = u'(0) = 0, \\ D^{\alpha-1}u(\infty) = \sum_{i=1}^{m-2} \beta_i u(\xi_i), \end{cases} \tag{4}$$

where $\alpha \in (2,3), 0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < \infty, \beta_i \geq 0$ such that $0 < \sum_{i=1}^{m-2} \beta_i \xi_i^{\alpha-1} < \Gamma(\alpha)$ and D^α is the Riemann–Liouville fractional derivative.

Wang et al. [20] by using Schauder's fixed point theorem investigated the existence and uniqueness of solutions for the following coupled system of nonlinear fractional differential equations on an unbounded domain

$$\begin{cases} D^p u(t) + f(t, v(t)) = 0, \quad 2 < p < 3, \quad t \in J := [0,\infty), \\ D^q v(t) + g(t, u(t)) = 0, \quad 2 < q < 3, \quad t \in J := [0,\infty), \\ u(0) = u'(0) = 0, \quad D^{p-1}u(\infty) = \sum_{i=1}^{m-2} \beta_i u(\xi_i), \\ v(0) = v'(0) = 0, \quad D^{q-1}v(\infty) = \sum_{i=1}^{m-2} \gamma_i v(\xi_i), \end{cases} \tag{5}$$

where $f, g \in C(J \times \mathbb{R}, \mathbb{R})$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < \infty$, D^p and D^q denote Riemann–Liouville fractional derivatives of order p and q, respectively, as well as $\beta_i > 0$, $\gamma_i > 0$ are such that $0 < \sum_{i=1}^{m-2} \beta_i \xi_i^{p-1} < \Gamma(p)$ and $0 < \sum_{i=1}^{m-2} \gamma_i \xi_i^{q-1} < \Gamma(q)$.

Our aim in this paper is to generalize the above works on an infinite interval and more general boundary conditions, so we discuss the existence of the solutions of a coupled system of nonlinear fractional differential equations on an unbounded domain

$$\begin{cases} D^\alpha u(t) = f(t, v(t), D^{\beta-1}v(t)), \quad t \in J := [0,\infty), \\ D^\beta v(t) = g(t, u(t), D^{\alpha-1}u(t)), \quad t \in J := [0,\infty), \\ u(0) = 0, \\ v(0) = 0, \\ D^{\alpha-1}u(\infty) = u_0 + \sum_{i=1}^{m-2} a_i u(\xi_i) + \sum_{i=1}^{m-2} b_i D^{\alpha-1}u(\xi_i), \\ D^{\beta-1}v(\infty) = v_0 + \sum_{i=1}^{n-2} c_i v(\eta_i) + \sum_{i=1}^{n-2} d_i D^{\beta-1}v(\eta_i), \end{cases} \tag{6}$$

where $1 < \alpha, \beta \leq 2$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < \infty$, $0 < \eta_1 < \eta_2 < \cdots < \eta_{n-2} < \infty$, $a_i, b_i, c_i, d_i \geq 0$, $u_0, v_0 \geq 0$ are real numbers and $f, g \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and D is the Riemann–Liouville fractional derivative.

This paper is organized as follows: in Section 2, some facts and results about fractional calculus are given, while inspired by [19] we prove the main result and some corollaries in Section 3, and we conclude this paper by considering an example in Section 4.

2 Preliminaries

In this section, we present some definitions and results which will be needed later.

Definition 2.1. [11] *The Riemann–Liouville fractional integral of order $\alpha > 0$ of a function $f : (0, \infty) \to \mathbb{R}$ is defined by*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \quad t > 0,$$

provided that the right-hand side is pointwise defined.

Definition 2.2. [11] *The Riemann–Liouville fractional derivative of order $\alpha > 0$ of a continuous function $f : (0, \infty) \to \mathbb{R}$ is defined by*

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} f(s) ds \quad t > 0,$$

where $n = [\alpha] + 1$, provided that the right-hand side is pointwise defined. In particular, for $\alpha = n$, $D^n f(t) = f^{(n)}(t)$.

Remark 1. *The following properties are well known:*

$D^\alpha I^\alpha f(t) = f(t),\ \alpha > 0,\ f(t) \in L^1(0, \infty),$
$D^\beta I^\alpha f(t) = I^{\alpha-\beta} f(t),\ \alpha > \beta > 0,\ f(t) \in L^1(0, \infty).$

The following two lemmas can be found in [5, 11].

Lemma 2.1. *Let $\alpha > 0$ and $u \in C(0,1) \cap L^1(0,1)$. Then the fractional differential equation $D^\alpha u(t) = 0$ has a unique solution*

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}, \quad c_i \in \mathbb{R},\ i = 1, \ldots, n,$$

where $n = [\alpha] + 1$ if $\alpha \notin \mathbb{N}$ and $n = \alpha$ if $\alpha \in \mathbb{N}$.

Lemma 2.2. *Assume that $u \in C(0,1) \cap L^1(0,1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0,1) \cap L^1(0,1)$. Then*

$$I^\alpha D^\alpha u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

for some $c_i \in \mathbb{R}, i = 1, \ldots, n$ and $n = [\alpha] + 1$ if $\alpha \notin \mathbb{N}$ and $n = \alpha$ if $\alpha \in \mathbb{N}$.

For the forthcoming analysis, we define the spaces

$$X = \left\{ u(t) \middle| u(t), D^{\alpha-1} u(t) \in C(J, \mathbb{R}),\ \sup_{t \in J} \frac{|u(t)|}{1 + t^{\alpha-1}} < \infty,\ \sup_{t \in J} |D^{\alpha-1} u(t)| < \infty \right\},$$

with the norm

$$||u||_X = \max\left\{\sup_{t\in J}\frac{|u(t)|}{1+t^{\alpha-1}},\ \sup_{t\in J}|D^{\alpha-1}u(t)|\right\},$$

and

$$Y = \left\{v(t)\middle|v(t), D^{\beta-1}v(t)\in C(J,\mathbb{R}),\ \sup_{t\in J}\frac{|v(t)|}{1+t^{\beta-1}}<\infty,\ \sup_{t\in J}|D^{\beta-1}v(t)|<\infty\right\},$$

with the norm

$$||v||_Y = \max\left\{\sup_{t\in J}\frac{|v(t)|}{1+t^{\beta-1}},\ \sup_{t\in J}|D^{\beta-1}v(t)|\right\}.$$

By Lemma 2.2 of [19], $(X,||\cdot||_X)$, $(Y,||\cdot||_Y)$ are Banach spaces. For $(u,v)\in X\times Y$, let $||(u,v)||_{X\times Y} = \max\{||u||_X, ||v||_Y\}$, then $(X\times Y, ||\cdot||_{X\times Y})$ is a Banach space. The Arzelà–Ascoli theorem fails to work in the Banach space X, Y due to the fact that the infinite interval $[0,\infty)$ is noncompact. The following compactness criterion will help us to resolve this problem.

Lemma 2.3. [19] *Let $Z\subseteq X(Y)$ be a bounded set. Then Z is relatively compact in $X(Y)$ if the following conditions hold.*

(i) For any $u(t)\in Z$, $\frac{u(t)}{1+t^{\alpha-1}}$ and $D^{\alpha-1}u(t)$ are equicontinuous on any compact interval of J.

(ii) Given $\epsilon>0$, there exists a constant $T=T(\epsilon)>0$ such that

$$\left|\frac{u(t_1)}{1+t_1^{\alpha-1}}-\frac{u(t_2)}{1+t_2^{\alpha-1}}\right|<\epsilon,$$

and

$$\left|D^{\alpha-1}u(t_1)-D^{\alpha-1}u(t_2)\right|<\epsilon,$$

for any $t_1,t_2\geq T$ and $u(t)\in Z$.

3 Main result

In this section, we investigate sufficient conditions for the existence and uniqueness solutions for the boundary value problem (6). Before we state our main result, for the convenience, we introduce the following notations:

$$\begin{aligned}
\Delta_\alpha &= \Gamma(\alpha)-\sum_{i=1}^{m-2}a_i\xi_i^{\alpha-1}-\Gamma(\alpha)\sum_{i=1}^{m-2}b_i,\\
\chi_\beta^1 &= \int_0^\infty\Big((1+s^{\beta-1})a(s)+b(s)\Big)ds,\\
\chi_\phi^2 &= \int_0^\infty\phi(s)ds,\\
\chi_{\alpha,\beta}^3 &= \int_0^{\xi_i}(\xi_i-s)^{\alpha-1}\Big((1+s^{\beta-1})a(s)+b(s)\Big)ds,\\
\chi_\alpha^4 &= \int_0^{\xi_i}(\xi_i-s)^{\alpha-1}\phi(s)ds.
\end{aligned}$$

By replacing a_i, b_i, ξ_i, ϕ, $a(s)$, $b(s)$ with c_i, d_i, η_i, ψ, $c(s)$, $d(s)$ respectively, and α with β we can introduce Δ_β, χ_α^1, χ_ψ^2, $\chi_{\beta,\alpha}^3$ and χ_β^4.

Now, we state sufficient conditions which allow us to establish the existence results for the system (6).

(H_1) There exist nonnegative functions $a(t), b(t), \phi(t) \in C[0,\infty)$ such that

$$|f(t,x,y)| \le a(t)|x| + b(t)|y| + \phi(t), \quad \Delta_\alpha > 0, \chi^2_\phi < \infty.$$

(H_2) There exist nonnegative functions $c(t), d(t), \psi(t) \in C[0,\infty)$ such that

$$|g(t,x,y)| \le c(t)|x| + d(t)|y| + \psi(t), \quad \Delta_\beta > 0, \chi^2_\psi < \infty.$$

(H_3) $$\frac{\chi^1_\beta + \frac{\sum_{i=1}^{m-2} a_i \chi^3_{\alpha,\beta}}{\Gamma(\alpha)} + \sum_{i=1}^{m-2} b_i \chi^1_\beta + \frac{\Delta_\alpha \chi^1_\beta}{\Gamma(\alpha)}}{\Delta_\alpha} < 1,$$

(H_4) $$\frac{\chi^1_\alpha + \frac{\sum_{i=1}^{n-2} c_i \chi^3_{\beta,\alpha}}{\Gamma(\beta)} + \sum_{i=1}^{n-2} d_i \chi^1_\alpha + \frac{\Delta_\beta \chi^1_\alpha}{\Gamma(\beta)}}{\Delta_\beta} < 1.$$

Lemma 3.1. *Let $h \in C[0,\infty)$, then the boundary value problem*

$$\begin{cases} (D^\alpha u)(t) = h(t), \quad 0 < t < \infty, \quad 1 < \alpha \le 2, \\ u(0) = 0, \\ D^{\alpha-1}u(\infty) = u_0 + \sum_{i=1}^{m-2} a_i u(\xi_i) + \sum_{i=1}^{m-2} b_i D^{\alpha-1} u(\xi_i), \end{cases} \tag{7}$$

has a unique solution

$$\begin{aligned} u(t) &= \frac{t^{\alpha-1}}{\Delta_\alpha}\Bigg(u_0 - \int_0^\infty h(s)ds + \frac{\sum_{i=1}^{m-2} a_i}{\Gamma(\alpha)} \int_0^{\xi_i} (\xi_i - s)^{\alpha-1} h(s)ds \\ &\quad + \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} h(s)ds \Bigg) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds, \end{aligned} \tag{8}$$

where $\Delta_\alpha \neq 0$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < \infty$ and $a_i, b_i, u_0 \ge 0$ for $i = \{1, \ldots, m-2\}$.

Proof. We apply Lemma (2.2) to convert the boundary value problem (7) into the integral equation

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + I^\alpha h(t).$$

Since $u(0) = 0$, so $c_2 = 0$ and

$$D^{\alpha-1}u(t) = c_1\Gamma(\alpha) + I^1 h(t).$$

Now using the second boundary condition we obtain c_1. Since

$$\begin{aligned} c_1\Gamma(\alpha) + \int_0^\infty h(t) &= u_0 + \sum_{i=1}^{m-2} a_i \Big(c_1 \xi_i^{\alpha-1} + I^\alpha h(\xi_i) \Big) \\ &\quad + \sum_{i=1}^{m-2} b_i \Big(c_1 \Gamma(\alpha) + I^1 h(\xi_i) \Big), \end{aligned}$$

then

$$\begin{aligned} c_1 &= \frac{1}{\Delta_\alpha}\Bigg(u_0 - \int_0^\infty h(s)ds + \frac{\sum_{i=1}^{m-2} a_i}{\Gamma(\alpha)} \int_0^{\xi_i} (\xi_i - s)^{\alpha-1} h(s)ds \\ &\quad + \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} h(s)ds \Bigg). \end{aligned}$$

Therefore

$$
\begin{aligned}
u(t) &= \frac{t^{\alpha-1}}{\Delta_\alpha}\Bigg(u_0 - \int_0^\infty h(s)ds + \frac{\sum_{i=1}^{m-2} a_i}{\Gamma(\alpha)} \int_0^{\xi_i} (\xi_i - s)^{\alpha-1} h(s)ds \\
&\quad + \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} h(s)ds\Bigg) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds,
\end{aligned}
$$

and

$$
\begin{aligned}
D^{\alpha-1}u(t) &= \frac{\Gamma(\alpha)}{\Delta_\alpha}\Bigg(u_0 - \int_0^\infty h(s)ds + \frac{\sum_{i=1}^{m-2} a_i}{\Gamma(\alpha)} \int_0^{\xi_i} (\xi_i - s)^{\alpha-1} h(s)ds \\
&\quad + \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} h(s)ds\Bigg) + \int_0^t h(s)ds,
\end{aligned}
$$

and the proof is completed. □

Define the operator $T : X \times Y \to X \times Y$ by

$$
T(u, v)(t) = (Av(t), Bu(t)),
$$

where

$$
\begin{aligned}
Av(t) &= \frac{t^{\alpha-1}}{\Delta_\alpha}\Bigg(u_0 - \int_0^\infty f(s, v(s), D^{\beta-1}v(s))ds \\
&\quad + \frac{\sum_{i=1}^{m-2} a_i}{\Gamma(\alpha)} \int_0^{\xi_i} (\xi_i - s)^{\alpha-1} f(s, v(s), D^{\beta-1}v(s))ds \\
&\quad + \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} f(s, v(s), D^{\beta-1}v(s))ds\Bigg) \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, v(s), D^{\beta-1}v(s))ds,
\end{aligned}
$$

and

$$
\begin{aligned}
Bu(t) &= \frac{t^{\beta-1}}{\Delta_\beta}\Bigg(v_0 - \int_0^\infty g(s, u(s), D^{\alpha-1}u(s))ds \\
&\quad + \frac{\sum_{i=1}^{n-2} c_i}{\Gamma(\beta)} \int_0^{\eta_i} (\eta_i - s)^{\beta-1} g(s, u(s), D^{\alpha-1}u(s))ds \\
&\quad + \sum_{i=1}^{n-2} d_i \int_0^{\eta_i} g(s, u(s), D^{\alpha-1}u(s))ds\Bigg) \\
&\quad + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g(s, u(s), D^{\alpha-1}u(s))ds.
\end{aligned}
$$

Note that by conditions (H_1)–(H_4),

$$
\int_0^\infty \Big|f(s, v(s), D^{\beta-1}v(s))\Big| ds \le ||v||_Y \int_0^\infty \Big[(1 + s^{\beta-1})a(s) + b(s)\Big] ds + \int_0^\infty \phi(s)ds < \infty.
$$

$$
\int_0^\infty \Big|g(s, u(s), D^{\alpha-1}u(s))\Big| ds \le ||u||_X \int_0^\infty \Big[(1 + s^{\alpha-1})c(s) + d(s)\Big] ds + \int_0^\infty \psi(s)ds < \infty.
$$

Theorem 3.1. *Assume that the conditions (H1)–(H4) hold. Then the coupled system (6) has at least one solution.*

Proof. Take

$$R > \max\left[\frac{\frac{1}{\Delta_\alpha}\left(u_0+\chi_\phi^2+\frac{\sum_{i=1}^{m-2}a_i\chi_\alpha^4}{\Gamma(\alpha)}+\sum_{i=1}^{m-2}b_i\chi_\phi^2+\frac{\chi_\phi^2\Delta_\alpha}{\Gamma(\alpha)}\right)}{1-\frac{\chi_\beta^1+\frac{\sum_{i=1}^{m-2}a_i\chi_{\alpha,\beta}^3}{\Gamma(\alpha)}+\sum_{i=1}^{m-2}b_i\chi_\beta^1+\frac{\Delta_\alpha\chi_\beta^1}{\Gamma(\alpha)}}{\Delta_\alpha}},\right.$$
$$\left.\frac{\frac{1}{\Delta_\beta}\left(v_0+\chi_\psi^2+\frac{\sum_{i=1}^{n-2}c_i\chi_\beta^4}{\Gamma(\beta)}+\sum_{i=1}^{m-2}d_i\chi_\psi^2+\frac{\chi_\psi^2\Delta_\beta}{\Gamma(\beta)}\right)}{1-\frac{\chi_\alpha^1+\frac{\sum_{i=1}^{n-2}c_i\chi_{\beta,\alpha}^3}{\Gamma(\beta)}+\sum_{i=1}^{n-2}d_i\chi_\alpha^1+\frac{\Delta_\beta\chi_\alpha^1}{\Gamma(\beta)}}{\Delta_\beta}}\right]$$

and define a ball

$$B_R=\left\{(u,v)\in X\times Y\,\Big|\;||(u,v)||_{X\times Y}\le R\right\}.$$

First, we prove that $T: B_R\to B_R$. In view of

$$\begin{aligned}D^{\alpha-1}Av(t) &= \frac{\Gamma(\alpha)}{\Delta_\alpha}\Bigg(u_0-\int_0^\infty f(s,v(s),D^{\beta-1}v(s))ds\\ &\quad+\frac{\sum_{i=1}^{m-2}a_i}{\Gamma(\alpha)}\int_0^{\xi_i}(\xi_i-s)^{\alpha-1}f(s,v(s),D^{\beta-1}v(s))ds\\ &\quad+\sum_{i=1}^{m-2}b_i\int_0^{\xi_i}f(s,v(s),D^{\beta-1}v(s))ds\Bigg)+\int_0^t f(s,v(s),D^{\beta-1}v(s))ds,\end{aligned}$$

together with the definition of $Av(t)$ and continuity of f we have $Av(t)$, $D^{\alpha-1}Av(t)$ and similarly $Bu(t)$, $D^{\beta-1}Bu(t)$ are continuous on J.

For any $(u,v)\in B_R$, we have

$$\begin{aligned}\frac{|Av(t)|}{1+t^{\alpha-1}} &\le \frac{1}{\Delta_\alpha}.\frac{t^{\alpha-1}}{1+t^{\alpha-1}}\Bigg[u_0+\int_0^\infty\Big|f(s,v(s),D^{\beta-1}v(s))\Big|ds\\ &\quad+\frac{\sum_{i=1}^{m-2}a_i}{\Gamma(\alpha)}\int_0^{\xi_i}\Big|(\xi_i-s)^{\alpha-1}f(s,v(s),D^{\beta-1}v(s))\Big|ds\\ &\quad+\sum_{i=1}^{m-2}b_i\int_0^{\xi_i}\Big|f(s,v(s),D^{\beta-1}v(s))\Big|ds\Bigg]\\ &\quad+\frac{1}{\Gamma(\alpha)}\int_0^t\frac{(t-s)^{\alpha-1}}{1+t^{\alpha-1}}\Big|f(s,v(s),D^{\beta-1}v(s))\Big|ds\\ &\le \frac{1}{\Delta_\alpha}\Bigg[u_0+\int_0^\infty\Big|f(s,v(s),D^{\beta-1}v(s))\Big|ds\\ &\quad+\frac{\sum_{i=1}^{m-2}a_i}{\Gamma(\alpha)}\int_0^{\xi_i}\Big|(\xi_i-s)^{\alpha-1}f(s,v(s),D^{\beta-1}v(s))\Big|ds\\ &\quad+\sum_{i=1}^{m-2}b_i\int_0^{\xi_i}\Big|f(s,v(s),D^{\beta-1}v(s))\Big|ds\Bigg]+\frac{1}{\Gamma(\alpha)}\int_0^\infty\Big|f(s,v(s),D^{\beta-1}v(s))\Big|ds\\ &\le \frac{1}{\Delta_\alpha}\Bigg[u_0+\int_0^\infty\Big(a(s)|v(s)|+b(s)|D^{\beta-1}v(s)|+\phi(s)\Big)ds\end{aligned}$$

$$
\begin{aligned}
&+\frac{\sum_{i=1}^{m-2} a_i}{\Gamma(\alpha)}\int_0^{\xi_i}(\xi_i-s)^{\alpha-1}\Big(a(s)|v(s)|+b(s)|D^{\beta-1}v(s)|+\phi(s)\Big)ds\\
&+\sum_{i=1}^{m-2} b_i\int_0^{\xi_i}\Big(a(s)|v(s)|+b(s)|D^{\beta-1}v(s)|+\phi(s)\Big)ds\Bigg]\\
&+\frac{1}{\Gamma(\alpha)}\int_0^{\infty}\Big(a(s)|v(s)|+b(s)|D^{\beta-1}v(s)|+\phi(s)\Big)ds\\
\leq\ & \frac{1}{\Delta_\alpha}\Bigg[u_0+||v||_Y\int_0^{\infty}\Big((1+s^{\beta-1})a(s)+b(s)\Big)ds+\int_0^{\infty}\phi(s)ds\\
&+\frac{\sum_{i=1}^{m-2} a_i}{\Gamma(\alpha)}||v||_Y\int_0^{\xi_i}(\xi_i-s)^{\alpha-1}\Big((1+s^{\beta-1})a(s)+b(s)\Big)ds\\
&+\frac{\sum_{i=1}^{m-2} a_i}{\Gamma(\alpha)}\int_0^{\xi_i}(\xi_i-s)^{\alpha-1}\phi(s)ds\\
&+\sum_{i=1}^{m-2} b_i||v||_Y\int_0^{\xi_i}\Big((1+s^{\beta-1})a(s)+b(s)\Big)ds+\sum_{i=1}^{m-2} b_i\int_0^{\xi_i}\phi(s)ds\Bigg]\\
&+\frac{||v||_Y}{\Gamma(\alpha)}\int_0^{\infty}\Big((1+s^{\beta-1})a(s)+b(s)\Big)ds+\frac{1}{\Gamma(\alpha)}\int_0^{\infty}\phi(s)ds\\
\leq\ & \frac{1}{\Delta_\alpha}\Bigg[u_0+||v||_Y\Big(\chi^1_\beta+\frac{\sum_{i=1}^{m-2} a_i}{\Gamma(\alpha)}\chi^3_{\alpha,\beta}+\sum_{i=1}^{m-2} b_i\chi^1_\beta+\frac{\Delta_\alpha}{\Gamma(\alpha)}\chi^1_\beta\Big)\\
&+\Big(\chi^2_\phi+\frac{\sum_{i=1}^{m-2} a_i}{\Gamma(\alpha)}\chi^4_\alpha+\sum_{i=1}^{m-2} b_i\chi^2_\phi+\frac{\Delta_\alpha}{\Gamma(\alpha)}\chi^2_\phi\Big)\Bigg]\leq R.
\end{aligned}
$$

In a similar way, we can get

$$
\frac{|Bu(t)|}{1+t^{\beta-1}}\leq R.
$$

Now, we show that $\Big|D^{\alpha-1}Av(t)\Big|\leq R$, $\Big|D^{\beta-1}Bu(t)\Big|\leq R$. To do it note that,

$$
\begin{aligned}
\Big|D^{\alpha-1}Av(t)\Big| \leq\ & \frac{\Gamma(\alpha)}{\Delta_\alpha}\Bigg[u_0+\int_0^{\infty}\Big|f(s,v(s),D^{\beta-1}v(s))\Big|ds\\
&+\frac{\sum_{i=1}^{m-2} a_i}{\Gamma(\alpha)}\int_0^{\xi_i}\Big|(\xi_i-s)^{\alpha-1}f(s,v(s),D^{\beta-1}v(s))\Big|ds\\
&+\sum_{i=1}^{m-2} b_i\int_0^{\xi_i}\Big|f(s,v(s),D^{\beta-1}v(s))\Big|ds\Bigg]\\
&+\int_0^{t}\Big|f(s,v(s),D^{\beta-1}v(s))\Big|ds\\
\leq\ & \frac{\Gamma(\alpha)}{\Delta_\alpha}\Bigg[u_0+||v||_Y\Big(\chi^1_\beta+\frac{\sum_{i=1}^{m-2} a_i}{\Gamma(\alpha)}\chi^3_{\alpha,\beta}+\sum_{i=1}^{m-2} b_i\chi^1_\beta+\frac{\Delta_\alpha}{\Gamma(\alpha)}\chi^1_\beta\Big)\\
&+\Big(\chi^2_\phi+\frac{\sum_{i=1}^{m-2} a_i}{\Gamma(\alpha)}\chi^4_\alpha+\sum_{i=1}^{m-2} b_i\chi^2_\phi+\frac{\Delta_\alpha}{\Gamma(\alpha)}\chi^2_\phi\Big)\Bigg]\leq\Gamma(\alpha)R\leq R.
\end{aligned}
$$

Similarly, we can get

$$
\Big|D^{\beta-1}Bu(t)\Big|\leq R.
$$

Hence, $||T(u,v)||_{X\times Y} \leq R$, and this shows that $T: B_R \to B_R$.

Now, we show that $T: B_R \to B_R$ is a continuous operator. Let $(u_n, v_n), (u,v) \in B_R$, $n = 1, 2, \ldots$ and $||(u_n, v_n) - (u,v)||_{X\times Y} \to 0$ as $n \to \infty$. Then we have to show that $||T(u_n, v_n) - T(u,v)||_{X\times Y} \to 0$ as $n \to \infty$.

$$\begin{aligned}
\left|\frac{Av_n(t)}{1+t^{\alpha-1}} - \frac{Av(t)}{1+t^{\alpha-1}}\right| &\leq \frac{1}{\Delta_\alpha}\left[\int_0^\infty \left|f(s, v_n(s), D^{\beta-1}v_n(s)) - f(s, v(s), D^{\beta-1}v(s))\right| ds\right.\\
&\quad + \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \frac{(\xi_i - s)^{\alpha-1}}{\Gamma(\alpha)} \left|f(s, v_n(s), D^{\beta-1}v_n(s)) - f(s, v(s), D^{\beta-1}v(s))\right| ds\\
&\quad \left.+ \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \left|f(s, v_n(s), D^{\beta-1}v_n(s)) - f(s, v(s), D^{\beta-1}v(s))\right| ds\right]\\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{(t-s)^{\alpha-1}}{1+t^{\alpha-1}} \left|f(s, v_n(s), D^{\beta-1}v_n(s)) - f(s, v(s), D^{\beta-1}v(s))\right| ds\\
&\leq \left(\frac{1}{\Delta_\alpha} + \frac{1}{\Gamma(\alpha)}\right)\left(\chi_\beta^1(||v_n||_Y + ||v||_Y) + 2\chi_\phi^2\right)\\
&\quad + \frac{\sum_{i=1}^{m-2} a_i}{\Gamma(\alpha)\Delta_\alpha}\chi_{\alpha,\beta}^3(||v_n||_Y + ||v||_Y) + 2\frac{\sum_{i=1}^{m-2} a_i}{\Gamma(\alpha)\Delta_\alpha}\chi_\alpha^4\\
&\quad + \frac{\sum_{i=1}^{m-2} b_i}{\Delta_\alpha}\chi_\beta^1(||v_n||_Y + ||v||_Y) + 2\frac{\sum_{i=1}^{m-2} b_i}{\Delta_\alpha}\chi_\phi^2\\
&\leq 2R\left(\left(\frac{1}{\Delta_\alpha} + \frac{1}{\Gamma(\alpha)}\right)\chi_\beta^1 + \frac{\sum_{i=1}^{m-2} a_i}{\Gamma(\alpha)\Delta_\alpha}\chi_{\alpha,\beta}^3 + \frac{\sum_{i=1}^{m-2} b_i}{\Delta_\alpha}\chi_\beta^1\right)\\
&\quad + 2\left(\frac{1}{\Delta_\alpha} + \frac{1}{\Gamma(\alpha)}\right)\chi_\phi^2 + 2\frac{\sum_{i=1}^{m-2} a_i}{\Gamma(\alpha)\Delta_\alpha}\chi_\alpha^4 + \frac{\sum_{i=1}^{m-2} b_i}{\Delta_\alpha}\chi_\alpha^2 + \frac{2\sum_{i=1}^{m-2} b_i\chi_\phi^2}{\Delta(\alpha)}.
\end{aligned}$$

Also

$$\begin{aligned}
\left|D^{\alpha-1}Av_n(t) - D^{\alpha-1}Av(t)\right| &\leq 2R\left(\left(\frac{\Gamma(\alpha)}{\Delta_\alpha} + 1\right)\chi_\beta^1 + \frac{\sum_{i=1}^{m-2} a_i}{\Delta_\alpha}\chi_{\alpha,\beta}^3 + \frac{\sum_{i=1}^{m-2} b_i\Gamma(\alpha)}{\Delta_\alpha}\chi_\beta^1\right)\\
&\quad + 2\left(\frac{\Gamma(\alpha)}{\Delta_\alpha} + 1\right)\chi_\phi^2 + 2\frac{\sum_{i=1}^{m-2} a_i}{\Delta_\alpha}\chi_\alpha^4 + 2\frac{\sum_{i=1}^{m-2} b_i\Gamma(\alpha)}{\Delta_\alpha}\chi_\phi^2.
\end{aligned}$$

Similar process can be repeated for B and then Lebesgue's dominated convergence theorem asserts that T is continuous.

Now we show that T maps bounded sets of $X \times Y$ to relatively compact sets of $X \times Y$. It suffices to prove that both A and B map bounded sets to relatively compact sets.

Now, for a bounded subset V of Y and U of X, by Lemma (2.3), we show that AV, BU are relatively compact. Let $I \subseteq J$ be a compact interval, $t_1, t_2 \in I$ and $t_1 < t_2$; then for any $v(t) \in V$, we have

$$\begin{aligned}
\left|\frac{Av(t_2)}{1+t_2^{\alpha-1}} - \frac{Av(t_1)}{1+t_1^{\alpha-1}}\right| &\leq \frac{1}{\Delta_\alpha}\left|\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}} - \frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right|\left[u_0 + \int_0^\infty \left|f(s, v(s), D^{\beta-1}v(s))\right| ds\right.\\
&\quad + \sum_{i=1}^{m-1} a_i \int_0^{\xi_i} \left|\frac{(\xi_i - s)^{\alpha-1}}{\Gamma(\alpha)} f(s, v(s), D^{\beta-1}v(s))\right| ds\\
&\quad \left.+ \sum_{i=1}^{m-1} b_i \int_0^{\xi_i} \left|f(s, v(s), D^{\beta-1}v(s))\right| ds\right]
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\alpha)}\bigg|\int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}f(s,v(s),D^{\beta-1}v(s))ds\\
&-\int_0^{t_1}\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}f(s,v(s),D^{\beta-1}v(s))ds\\
&+\int_0^{t_1}\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}f(s,v(s),D^{\beta-1}v(s))ds\\
&-\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}f(s,v(s),D^{\beta-1}v(s))ds\bigg|\\
\leq\ &\frac{1}{\Delta_\alpha}\left|\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}-\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right|\bigg[u_0+\int_0^\infty\Big|f(s,v(s),D^{\beta-1}v(s))\Big|ds\\
&+\sum_{i=1}^{m-1}a_i\int_0^{\xi_i}\left|\frac{(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,v(s),D^{\beta-1}v(s))\right|ds\\
&+\sum_{i=1}^{m-1}b_i\int_0^{\xi_i}\Big|f(s,v(s),D^{\beta-1}v(s))\Big|ds\bigg]\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}\Big|f(s,v(s),D^{\beta-1}v(s))\Big|ds\\
&+\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left|\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}-\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}\right|\ \Big|f(s,v(s),D^{\beta-1}v(s))\Big|ds,
\end{aligned}
$$

and

$$\Big|D^{\alpha-1}Av(t_2)-D^{\alpha-1}Av(t_1)\Big|\leq\int_{t_1}^{t_2}\Big|f(s,v(s),D^{\beta-1}v(s))\Big|ds.$$

Also for $v(t)\in V$, $f(t,v(t),D^{\beta-1}v(t))$ is bounded on I. Then it is easy to see that $\frac{Av(t)}{1+t^{\alpha-1}}$ and $D^{\alpha-1}Av(t)$ are equicontinuous on I.

Next we show that for any $v(t)\in V$, functions $\frac{Av(t)}{1+t^{\alpha-1}}$ and $D^{\alpha-1}Av(t)$ satisfy the condition (ii) of Lemma (2.3). Based on condition (H_1)

$$\int_0^\infty\Big|f(t,v(t),D^{\beta-1}v(t))\Big|dt<||v||_Y\int_0^\infty\Big((1+t^{\beta-1})a(t)+b(t)\Big)dt+\int_0^\infty\phi(t)dt<\infty,$$

we know that for given $\epsilon>0$, there exists a constant $L>0$ such that

$$\int_L^\infty\Big|f(t,v(t),D^{\beta-1}v(t))\Big|dt<\epsilon.$$

On the other hand, since $\lim_{t\to\infty}\frac{t^{\alpha-1}}{1+t^{\alpha-1}}=1$, there exists a constant $T_1>0$ such that for any $t_1,t_2\geq T_1$,

$$\left|\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}-\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}\right|\leq\left|1-\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right|+\left|1-\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}\right|.$$

Similarly, $\lim_{t\to\infty}\frac{(t-L)^{\alpha-1}}{1+t^{\alpha-1}}=1$ and thus there exists a constant $T_2>L>0$ such that for any $t_1,t_2\geq T_2$ and $0\leq s\leq L$

$$
\begin{aligned}
\left|\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}-\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}\right|&\leq\left(1-\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}\right)+\left(1-\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}\right)\\
&\leq\left(1-\frac{(t_1-L)^{\alpha-1}}{1+t_1^{\alpha-1}}\right)+\left(1-\frac{(t_2-L)^{\alpha-1}}{1+t_2^{\alpha-1}}\right)\leq\epsilon.
\end{aligned}
$$

Now choose $T > \max\{T_1, T_2\}$; then for $t_1, t_2 \geq T$, we can obtain

$$\begin{aligned}
\left|\frac{Av(t_2)}{1+t_2^{\alpha-1}} - \frac{Av(t_1)}{1+t_1^{\alpha-1}}\right| \leq\; & \frac{1}{\Delta_\alpha}\left|\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}} - \frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right|\Bigg(u_0 + \int_0^\infty \left|f(s, v(s), D^{\beta-1}v(s))\right|ds \\
& + \sum_{i=1}^{m-1} a_i \int_0^{\xi_i} \frac{(\xi_i - s)^{\alpha-1}}{\Gamma(\alpha)}\left|f(s, v(s), D^{\beta-1}v(s))\right|ds \\
& + \sum_{i=1}^{m-1} b_i \int_0^{\xi_i} \left|f(s, v(s), D^{\beta-1}v(s))\right|ds\Bigg) \\
& + \frac{1}{\Gamma(\alpha)}\left|\int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}} f(s, v(s), D^{\beta-1}v(s))ds\right. \\
& \left. - \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}} f(s, v(s), D^{\beta-1}v(s))ds\right| \\
\leq\; & \frac{1}{\Delta_\alpha}\left|\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}} - \frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right|\Bigg(u_0 + \int_0^\infty \left|f(s, v(s), D^{\beta-1}v(s))\right|ds \\
& + \sum_{i=1}^{m-1} a_i \int_0^{\xi_i} \frac{(\xi_i - s)^{\alpha-1}}{\Gamma(\alpha)}\left|f(s, v(s), D^{\beta-1}v(s))\right|ds \\
& + \sum_{i=1}^{m-1} b_i \int_0^{\xi_i} \left|f(s, v(s), D^{\beta-1}v(s))\right|ds\Bigg) \\
& + \frac{1}{\Gamma(\alpha)}\int_0^L \left|\frac{(t_2-s)^{\alpha-1}}{1+t_1^{\alpha-1}} - \frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}\right|\left|f(s, v(s), D^{\beta-1}v(s))\right|ds \\
& + \frac{1}{\Gamma(\alpha)}\int_L^{t_1} \frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}\left|f(s, v(s), D^{\beta-1}v(s))\right|ds \\
& + \frac{1}{\Gamma(\alpha)}\int_L^{t_2} \frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}\left|f(s, v(s), D^{\beta-1}v(s))\right|ds \\
\leq\; & \frac{1}{\Delta_\alpha}\left|\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}} - \frac{t_1^{\beta-1}}{1+t_1^{\alpha-1}}\right|\Bigg(u_0 + \int_0^\infty \left|f(s, v(s), D^{\beta-1}v(s))\right|ds \\
& + \sum_{i=1}^{m-1} a_i \int_0^{\xi_i} \frac{(\xi_i - s)^{\alpha-1}}{\Gamma(\alpha)}\left|f(s, v(s), D^{\beta-1}v(s))\right|ds \\
& + \sum_{i=1}^{m-1} b_i \int_0^{\xi_i} \left|f(s, v(s), D^{\beta-1}v(s))ds\right|\Bigg) \\
& + \frac{\max_{t\in[0,L], u\in V}\left|f(t, v(t), D^{\beta-1}v(t))\right|}{\Gamma(\alpha)}\int_0^L \left|\frac{(t_2-s)^{\alpha-1}}{1+t_1^{\alpha-1}} - \frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}\right| ds \\
& + \frac{1}{\Gamma(\alpha)}\int_L^\infty \left|f(t, v(t), D^{\beta-1}v(t))\right|dt + \frac{1}{\Gamma(\alpha)}\int_L^\infty \left|f(t, v(t), D^{\beta-1}v(t))\right|dt \\
\leq\; & \frac{1}{\Delta_\alpha}\left|\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}} - \frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right|\epsilon \\
& + \frac{\max_{t\in[0,L], u\in V}\left|f(t, v(t), D^{\beta-1}v(t))\right|}{\Gamma(\alpha)} L\epsilon + \frac{2}{\Gamma(\alpha)}\epsilon,
\end{aligned}$$

and

$$\left|D^{\alpha-1}Av(t_2)-D^{\alpha-1}Av(t_1)\right| \le \int_{t_1}^{t_2}\left|f(s,v(s),D^{\beta-1}v(s))\right|ds$$
$$\le \int_{L}^{\infty}\left|f(s,v(s),D^{\beta-1}v(s))\right|ds<\epsilon.$$

Similar process can be repeated for B, thus T is relatively compact. Therefore, by Schauder's fixed point theorem the boundary value problem (6) has at least one solution. □

Corollary 3.1. *Assume that conditions* $(H_1),(H_2)$ $(\Delta_\alpha=\Gamma(\alpha),\Delta_\beta=\Gamma(\beta))$ *and*

$$\chi^1_\beta<\frac{\Gamma(\alpha)}{2},\qquad \chi^1_\alpha<\frac{\Gamma(\beta)}{2},$$

holds. Then there exists at least one solution $(u(t),v(t))\in X\times Y$ *solving the following problem.*

$$\begin{cases} D^\alpha u(t)=f(t,v(t),D^{\beta-1}v(t)), & t\in[0,\infty),\\ D^\beta v(t)=g(t,u(t),D^{\alpha-1}u(t)), & t\in[0,\infty),\\ u(0)=0, D^{\beta-1}u(\infty)=u_0, \\ v(0)=0, D^{\alpha-1}v(\infty)=v_0, \end{cases} \tag{9}$$

where $\alpha,\beta\in(1,2]$, $f\in C([0,\infty)\times\mathbb{R}\times\mathbb{R},\mathbb{R})$.

Corollary 3.2. *Assume that*

- *There exist nonnegative functions* $a(t),\phi(t)\in C[0,\infty)$ *such that*

$$|f(t,x,y)|\le a(t)|x|+\phi(t),\quad \Delta_\alpha=\Gamma(\alpha)-a\xi^{\alpha-1}>0,\ \chi^2_\phi<\infty.$$

- *There exist nonnegative functions* $c(t),\psi(t)\in C[0,\infty)$ *such that*

$$|g(t,x,y)|\le c(t)|x|+\psi(t),\quad \Delta_\beta=\Gamma(\beta)-c\eta^{\beta-1}>0,\ \chi^2_\psi<\infty.$$

- $\dfrac{\chi^1_\beta+\frac{a\chi^3_{\alpha,\beta}}{\Gamma(\alpha)}+\frac{\Delta_\alpha\chi^1_\beta}{\Gamma(\alpha)}}{\Delta_\alpha}<1,$
- $\dfrac{\chi^1_\alpha+\frac{c\chi^3_{\beta,\alpha}}{\Gamma(\beta)}+\frac{\Delta_\beta\chi^1_\alpha}{\Gamma(\beta)}}{\Delta_\beta}<1.$

Then

$$\begin{cases} D^\alpha u(t)=f(t,v(t),D^{\beta-1}v(t)), & t\in[0,\infty),\\ D^\beta v(t)=g(t,u(t),D^{\alpha-1}u(t)), & t\in[0,\infty),\\ u(0)=0, D^{\alpha-1}u(\infty)=au(\xi), \\ v(0)=0, D^{\beta-1}v(\infty)=cv(\eta), \end{cases} \tag{10}$$

has at least one solution, where $\alpha,\beta\in(1,2], 0<\xi,\eta<\infty, a,c\ge 0,$ f *is a continuous function.*

Corollary 3.3. *Assume that*

- *There exist nonnegative functions* $a(t), \phi(t) \in C[0,\infty)$ *such that*

$$|f(t,x,y)| \le a(t)|x| + \phi(t), \quad \Delta_\alpha = \Gamma(\alpha) - \sum_{i=1}^{m-2} a_i \xi_i^{\alpha-1} > 0, \ \chi_\phi^2 < \infty.$$

- *There exist nonnegative functions* $c(t), \psi(t) \in C[0,\infty)$ *such that*

$$|g(t,x,y)| \le c(t)|x| + \psi(t), \quad \Delta_\beta = \Gamma(\beta) - \sum_{i=1}^{n-2} c_i \eta_i^{\beta-1} > 0, \ \chi_\psi^2 < \infty.$$

- $\dfrac{\chi_\beta^1 + \frac{\sum_{i=1}^{m-2} a_i \chi_{\alpha,\beta}^3}{\Gamma(\alpha)} + \frac{\Delta_\alpha \chi_\beta^1}{\Gamma(\alpha)}}{\Delta_\alpha} < 1,$

- $\dfrac{\chi_\alpha^1 + \frac{\sum_{i=1}^{n-2} c_i \chi_{\beta,\alpha}^3}{\Gamma(\beta)} + \frac{\Delta_\beta \chi_\alpha^1}{\Gamma(\beta)}}{\Delta_\beta} < 1.$

Then

$$\begin{cases} D^\alpha u(t) = f(t, v(t), D^{\beta-1}v(t)), & t \in [0,\infty), \\ D^\beta v(t) = f(t, u(t), D^{\alpha-1}u(t)), & t \in [0,\infty), \\ u(0) = 0, \ v(0) = 0, \\ D^{\alpha-1}u(\infty) = \sum_{i=1}^{m-2} a_i u(\xi_i), \\ D^{\beta-1}v(\infty) = \sum_{i=1}^{n-2} c_i v(\eta_i), \end{cases} \tag{11}$$

has at least one solution, where $\alpha, \beta \in (1,2], 0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < \infty, 0 < \eta_1 < \eta_2 < \cdots < \eta_{n-2} < \infty, a_i, c_i \ge 0.$

4 An example

Consider the following boundary value problem on an unbounded domain.

$$\begin{cases} D^{1.5}u(t) = \frac{1}{10+20t^2} + \frac{|v(t)|}{(10+20t^2)(1+t^{0.2})} + \frac{\ln(1+|D^{0.2}v(t)|)}{10+20t^2}, & t \in J := [0,\infty), \\ D^{1.2}v(t) = \frac{|u(t)|}{(t+2)^2(1+t^{0.5})} + \frac{|\sin D^{0.5}u(t)|}{(t+8)^2}, \\ u(0) = 0, \quad D^{0.5}u(\infty) = \frac{1}{2} + \frac{1}{8}u(2) + \frac{1}{10}D^{0.5}u(2), \\ v(0) = 0, \quad D^{0.2}v(\infty) = \frac{1}{4} + \frac{1}{9}v(3) + \frac{1}{5}D^{0.2}v(3). \end{cases} \tag{12}$$

Here,

$$f(t,x,y) = \frac{1}{10+20t^2} + \frac{|x|}{(10+20t^2)(1+t^{0.2})} + \frac{\ln(1+|y|)}{10+20t^2}.$$

For

$$a(t) = \frac{1}{(1+t^{0.2})(10+20t^2)}, \quad b(t) = \frac{1}{10+20t^2}, \quad \phi(t) = \frac{1}{10+20t^2},$$

by easy calculation we have

$$\begin{aligned}
\Delta_\alpha &= 0.61 > 0,\\
\chi_\beta^1 &= \int_0^\infty \Big((1+s^{0.2})\frac{1}{(1+s^{0.2})(10+20s^2)} + \frac{1}{10+20s^2}\Big)ds = 0.2,\\
\chi_\phi^2 &= \int_0^\infty \frac{1}{10+20s^2}ds = 0.1,\\
\chi_{\alpha,\beta}^3 &= \int_0^2 (2-s)^{0.5}\Big((1+s^{0.2})\frac{1}{(1+s^{0.2})(10+20s^2)} + \frac{1}{10+20s^2}\Big)ds = 0.19,
\end{aligned}$$

and it is easy to verify that condition (H_3) holds.

Similarly, we can show that for the second equation condition (H_4) holds. Thus all the conditions of theorem (3.1) are satisfied and the problem (12) has at least one solution.

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