

# Observability of string vibrations

András Szijártó<sup>1</sup> and Jenő Hegedűs

Bolyai Institute, University of Szeged, Aradi vértanúk tere 1, Szeged, 6720, Hungary  
szijarto@math.u-szeged.hu, hegedusj@math.u-szeged.hu

**Abstract.** Transversal vibrations  $u = u(x, t)$  of a string of length  $l$  under three essential boundary conditions are studied, where  $u$  is governed by the Klein–Gordon equation:

$$u_{tt}(x, t) = a^2 u_{xx}(x, t) - cu(x, t), \quad (x, t) \in [0, l] \times \mathbb{R}; \quad 0 < a, c \in \mathbb{R}.$$

Sufficient conditions are obtained that guarantee the unique solvability of a general observation problem with the given state functions  $f, g \in D^s(0, l)$ ,  $s \in \mathbb{R}$  at two distinct instants of time  $-\infty < t_1 < t_2 < \infty$ :

$$\begin{aligned} A_1 u|_{t=t_1} + B_1 u_t|_{t=t_1} &= f, & |A_1| + |B_1| > 0, & A_1 B_1 \geq 0, \\ A_2 u|_{t=t_2} + B_2 u_t|_{t=t_2} &= g, & |A_2| + |B_2| > 0, & A_2 B_2 \leq 0. \end{aligned}$$

Here  $s$  is arbitrary, the space  $D^s(0, l)$  (see [2] and [13]) is some subspace of the Sobolev space  $H^s(0, l)$ . The essential condition of the solvability is that  $(t_2 - t_1)a/l$  is a rational number.

In fact, this result is a consequence of a general observability result related to the vibration  $u = u(x, t)$  governed by the equation

$$u_{tt} = (p(x)u_x)_x - q(x)u, \quad (x, t) \in [0, l] \times \mathbb{R}, \quad 0 < p, q \in C^\infty([0, l]),$$

subject to some initial data and linear boundary conditions (see in Proposition 1 below). This time the main restrictions are some Diophantine conditions and asymptotic properties of the eigenfrequencies  $\omega_n$  as  $n \rightarrow \infty$ . Some other results without these restrictions are also presented.

**Keywords:** observation problems, string vibrations, generalized solutions, method of Fourier expansions.

**2010 AMS Subject Classifications:** primary 35Q93, 81Q05; secondary 35L05, 35R30, 42A20.

## 1. INTRODUCTION

In control theory which is closely related to the subject of this paper, a number of monographs and articles dealt with the accessibility of a given complete final state

$$u|_{t=T} = f, \quad u_t|_{t=T} = g$$

---

<sup>1</sup>Corresponding author

of oscillations (in particular, string oscillations) in the time interval  $0 \leq t \leq T < \infty$ ; see, e.g., in [1–10].

In observation problems for the oscillations  $u = u(x, t)$  one looks for the initial functions

$$u|_{t=0} = \varphi, \quad u_t|_{t=0} = \psi$$

only from the partial states, e.g.,

$$u|_{t=t_1} = f, \quad u|_{t=t_2} = g, \quad -\infty < t_1 < t_2 < \infty,$$

so the aim is the same (roughly speaking) as in the control theory: to find  $\varphi$  and  $\psi$  occurring above. Although, only the short communication [12] dealt with observability of the string oscillations  $u$  on  $[0, l] \times [0, T]$  governed by  $u_{tt} = a^2 u_{xx}$ ; and another paper [13] where  $u$  is described by the Klein–Gordon equation

$$(1) \quad u_{tt}(x, t) = a^2 u_{xx}(x, t) - cu(x, t), \quad (x, t) \in \Omega := [0, l] \times \mathbb{R}; \quad 0 < a, c \in \mathbb{R},$$

with the initial conditions

$$(2) \quad u|_{t=0} = \varphi(x), \quad u_t|_{t=0} = \psi(x),$$

and boundary conditions

$$(3) \quad u|_{x=0} = u|_{x=l} = 0,$$

with  $u, \varphi, \psi$  from the corresponding generalized function spaces  $D^s$ ,  $s \in \mathbb{R}$ . In [12] these functions  $\varphi$  and  $\psi$  were constructed only for small  $t_1$  and  $t_2$ :  $0 \leq t_1 < t_2 \leq 2l/a$  under the additional assumption that the initial functions  $\varphi$  and  $\psi$  in question are known on some subinterval  $[h_1, h_2] \subset [0, l]$ . In [13] we dealt with the case of arbitrary  $t_1, t_2 \in \mathbb{R}$ ,  $-\infty < t_1 < t_2 < \infty$ , under the restriction that  $(t_2 - t_1)a/l$  is a rational number, and we considered the following observation conditions:

$$(a) \quad u|_{t=t_1} = f, \quad u|_{t=t_2} = g,$$

$$(b) \quad u_t|_{t=t_1} = f, \quad u|_{t=t_2} = g,$$

$$(c) \quad u|_{t=t_1} = f, \quad u_t|_{t=t_2} = g,$$

$$(d) \quad u_t|_{t=t_1} = f, \quad u_t|_{t=t_2} = g.$$

Now, our goal is to prove similar results related to the problem (1)–(2) with the following general observation conditions:

$$(e) \quad \begin{aligned} A_1 u|_{t=t_1} + B_1 u_t|_{t=t_1} &= f, & |A_1| + |B_1| &> 0, \\ A_2 u|_{t=t_2} + B_2 u_t|_{t=t_2} &= g, & |A_2| + |B_2| &> 0, \end{aligned}$$

under the following boundary conditions: either

$$(4) \quad u(0, t) = u_x(l, t) = 0;$$

or

$$(5) \quad u_x(0, t) = u_x(l, t) = 0;$$

or the Sturm–Liouville boundary conditions

$$(6) \quad \begin{aligned} u(0, t) \cos \alpha + u_x(0, t) \sin \alpha &= 0, & \cot \alpha < 0, \\ u(l, t) \cos \beta + u_x(l, t) \sin \beta &= 0, & \cot \beta > 0. \end{aligned}$$

Thus, we deal with observation problems related to the mixed problems (1), (2), (4); (1), (2), (5) and (1), (2), (6) respectively for the specific properties for all three mixed problems. We emphasize some common ideas that may be useful even for the study of some other problems (see Propositions 1, 2 and Lemma 1 below), e.g., for the class of the following mixed problems:

$$(7) \quad u_{tt} = (p(x)u_x)_x - q(x)u \equiv Lu, \quad (x, t) \in \Omega := [0, l] \times \mathbb{R}, \quad 0 < p, q \in C^\infty([0, l]),$$

$$(8) \quad u|_{t=0} = \varphi(x), \quad u_t|_{t=0} = \psi(x),$$

$$(9) \quad \mathcal{U}_i[u] \equiv \mathcal{U}_i(u|_{x=0}, u|_{x=l}, u_x|_{x=0}, u_x|_{x=l}) = 0, \quad i = 1, 2,$$

where  $\mathcal{U}_1, \mathcal{U}_2$  are independent, self-adjoint (see, e. g., [14]) linear expressions (they may be different from (4), (5), (6)), where  $u, \varphi, \psi$  are from the generalized function spaces  $D^s$ , and the conditions (e) and (9) are understood in the sense of generalized functions. It can be supposed without loss of the generality, that the ordinary homogeneous BVP for

$$\begin{aligned} y &= y(x), \quad L \sim (p(x)y')' - q(x)y \\ Ly &= 0, \quad \mathcal{U}_i[y] = 0, \quad i = 1, 2, \end{aligned}$$

has only the trivial solution. The ordinary differential operator  $L$  is formally self-adjoint, therefore the operator  $\mathcal{L} \sim (L, \mathcal{U}_1, \mathcal{U}_2)$  is also self-adjoint.

We use the definition of the spaces  $D^s(S)$ ,  $s \in \mathbb{R}$  given in [2] and [13]. Let the system  $\{X_n(x)\}_{n=0}^\infty$  be a complete orthonormal basis in  $L_2(S)$ . Given arbitrary real number  $s$ , we consider on the linear span  $D$  of the functions  $X_n(x)$ ,  $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ ,  $x \in \bar{S}$ , the following Euclidean norm:

$$\left\| \sum_{n=0}^{\infty} c_n X_n(x) \right\|_s := \left( \sum_{n=0}^{\infty} n^{2s} |c_n|^2 \right)^{\frac{1}{2}}.$$

Completing  $D$  with respect to this norm, we obtain a Hilbert space denoted by  $D^s$ . We use the notation  $S = (0, l)$  associated with string vibrations.

We will need that for every  $s \in \mathbb{R}$  and for every  $(\varphi, \psi) \in D^{s+1}(0, l) \times D^s(0, l)$ , the mixed problem (7), (8), (9) possesses the following good properties:

$$(10) \quad \exists! \text{ solution } u \text{ and } u \in C(D^{s+1}, \mathbb{R}) \cap C^1(D^s, \mathbb{R}) \cap C^2(D^{s-1}, \mathbb{R}),$$

moreover,  $u$  can be written as

$$(11) \quad u(x, t) = \sum_{n=0}^{\infty} [\alpha_n \cos(\omega_n t) + \beta_n \sin(\omega_n t)] X_n(x), \quad (x, t) \in \bar{\Omega}, \text{ where}$$

$$LX_n = -\omega_n^2 X_n, \quad \mathcal{U}_i X_n = 0, \quad i = 1, 2.$$

**Remark 1.** The set of problems satisfying our restrictions on  $L, \mathcal{U}_1, \mathcal{U}_2$  and (10), (11) is not empty. It contains, e.g. the problem considered in [13], the self-adjoint problems in Sections 3, 4 and the problem under the Sturm–Liouville boundary condition in Section 5. The self-adjointness of  $\mathcal{L} \sim (L, \mathcal{U}_1, \mathcal{U}_2)$  is also valid in the case where

$$\mathcal{U}_1 : a_{11}y'(0) + a_{12}y'(l) = 0$$

$$\mathcal{U}_2 : a_{21}y(0) + a_{22}y(l) = 0$$

if and only if  $a_{12}a_{22}p(0) = a_{11}a_{21}p(l)$ .

## 2. MAIN RESULTS

We will need the following lemma for certain estimates below.

**Lemma 1.** *Let the sequence  $r_n$  be such that  $0 < M/n < |r_n| \rightarrow 0$  with a positive constant  $M$  and let  $x_0 > 0$  be a rational number. Then for any fixed  $d \in \mathbb{R}$ , there exists a number  $N$  such that*

$$|\sin(n\pi x_0 + d + r_n)| > \frac{M}{2n}, \quad \forall n > N.$$

*Proof.* Since  $x_0$  is rational, it can be written as  $x_0 = p/q$ ;  $p, q \in \mathbb{N}$ , and then  $\sin(n\pi x_0 + d)$  can assume at most  $q$  different values. Let  $\mathbb{N}_1$  and  $\mathbb{N}_2$  be two disjoint sets of numbers,  $\mathbb{N}_1 \cup \mathbb{N}_2 = \mathbb{N}$  such that  $\sin(n\pi x_0 + d) = 0$  if  $n \in \mathbb{N}_1$  and  $\sin(n\pi x_0 + d) \neq 0$  if  $n \in \mathbb{N}_2$ . We will prove Lemma 1 according to these two cases.

First, if  $n \in \mathbb{N}_1$ , then we have

$$|\sin(n\pi x_0 + d + r_n)| = |\sin(r_n)| > \frac{1}{2}|r_n| > \frac{M}{2n}$$

for all large  $n \in \mathbb{N}_1$  since  $r_n \rightarrow 0$  as  $n \rightarrow \infty$ .

Second, if  $n \in \mathbb{N}_2$ , then this means that there is a constant  $d_1 > 0$  such that  $|\sin(n\pi x_0 + d)| > d_1$  for all  $n \in \mathbb{N}_2$ . Due to the uniform continuity of the sine function, it follows that

$$|\sin(n\pi x_0 + d + r_n)| > \frac{d_1}{2} > \frac{M}{2n}$$

for all large  $n \in \mathbb{N}_2$ .

Combining the two cases we get the statement of Lemma 1. □

**Proposition 1.** *Let*

$$(12) \quad f \in D^{s+2}, \quad g \in D^{s+2}, \quad s \in \mathbb{R},$$

and suppose that there are constants  $0 < A \in \mathbb{Q}$ ,  $B \in \mathbb{R}$ ,  $0 < M_1 \in \mathbb{R}$  and a sequence  $C_n \in \mathbb{R} \setminus \{0\}$  such that

$$(13) \quad \begin{aligned} &\omega_n(t_2 - t_1) + \gamma_n - \delta_n = n\pi A + B + C_n, \\ &0 < \frac{M_1}{n} < |C_n|, \quad \forall n \in \mathbb{N} \quad \text{and} \quad C_n \rightarrow 0 \text{ as } n \rightarrow \infty, \end{aligned}$$

and

$$(14) \quad \sin(\omega_n(t_2 - t_1) + \gamma_n - \delta_n) \neq 0, \quad n \in \mathbb{N}_0.$$

Here the angles  $\gamma_n, \delta_n \in [0, 2\pi)$  are uniquely determined by the following relationships:

$$\begin{aligned} \sin \gamma_n &= \frac{A_1}{\sqrt{A_1^2 + B_1^2 \omega_n^2}}, & \cos \gamma_n &= \frac{B_1 \omega_n}{\sqrt{A_1^2 + B_1^2 \omega_n^2}}, \\ \sin \delta_n &= \frac{A_2}{\sqrt{A_2^2 + B_2^2 \omega_n^2}}, & \cos \delta_n &= \frac{B_2 \omega_n}{\sqrt{A_2^2 + B_2^2 \omega_n^2}}. \end{aligned}$$

If the solution of the mixed problem (7), (8), (9) satisfies (10), (11) and conditions (12), (13), (14) hold, then the observation problem posed for (7), (9) under the observation condition (e) has a unique solution for  $(\varphi, \psi) \in D^{s+1} \times D^s$ .

**Remark 2.** According to condition (13), we can use Lemma 1 for the estimation of

$$\sin(\omega_n(t_2 - t_1) + \gamma_n - \delta_n) = \sin(n\pi A + B + C_n),$$

which means that  $\sin(\omega_n(t_2 - t_1) + \gamma_n - \delta_n) \neq 0$  automatically holds for all large  $n$ , say when  $n$  is greater than a threshold number  $N$ .

*Proof of Proposition 1.* Clearly, the initial functions can be uniquely expanded into Fourier series with respect to the system  $\{X_n\}$  with – yet unknown – coefficients  $\alpha_n, \omega_n\beta_n; n \in \mathbb{N}_0$ :

$$(15) \quad \varphi(x) = u(x, 0) = \sum_{n=0}^{\infty} \alpha_n X_n(x),$$

$$(16) \quad \psi(x) = u_t(x, 0) = \sum_{n=0}^{\infty} \omega_n \beta_n X_n(x).$$

Since the solution  $u$  of the problem (7), (8), (9) has the representation (11) with some coefficients  $\alpha_n, \beta_n; n \in \mathbb{N}_0$ , the observation problem can be reduced to the problem of finding the appropriate choices of  $\alpha_n$  and  $\beta_n$  such that (e) is satisfied. For this reason, we substitute  $t_1$  and  $t_2$  into (11), and use both conditions in (e). As a result, we get the following necessary conditions for  $\alpha_n$  and  $\beta_n$ :

$$(17) \quad f(x) = A_1 u(x, t_1) + B_1 u_t(x, t_1) = \\ = \sum_{n=0}^{\infty} [\alpha_n (A_1 \cos(\omega_n t_1) - B_1 \omega_n \sin(\omega_n t_1)) + \beta_n (A_1 \sin(\omega_n t_1) + B_1 \omega_n \cos(\omega_n t_1))] X_n(x),$$

$$(18) \quad g(x) = A_2 u(x, t_2) + B_2 u_t(x, t_2) = \\ = \sum_{n=0}^{\infty} [\alpha_n (A_2 \cos(\omega_n t_2) - B_2 \omega_n \sin(\omega_n t_2)) + \beta_n (A_2 \sin(\omega_n t_2) + B_2 \omega_n \cos(\omega_n t_2))] X_n(x).$$

Since  $f \in D^{s+2}$  and  $g \in D^{s+2}$ , the coefficients of the Fourier expansions (with respect to the system  $\{X_n\}$ ) of the functions  $f(x), g(x)$  are unambiguously determined, and comparing these Fourier series with (17) and (18), we get the following conditions for  $\alpha_n$  and  $\beta_n, n \in \mathbb{N}_0$ :

$$(19) \quad \alpha_n (A_1 \cos(\omega_n t_1) - B_1 \omega_n \sin(\omega_n t_1)) + \beta_n (A_1 \sin(\omega_n t_1) + B_1 \omega_n \cos(\omega_n t_1)) = f_n, \\ \alpha_n (A_2 \cos(\omega_n t_2) - B_2 \omega_n \sin(\omega_n t_2)) + \beta_n (A_2 \sin(\omega_n t_2) + B_2 \omega_n \cos(\omega_n t_2)) = g_n.$$

where

$$f_n = \int_0^l f(x) X_n(x) dx \quad \text{and} \quad g_n = \int_0^l g(x) X_n(x) dx.$$

By substituting  $\gamma_n$  and  $\delta_n$  into (19) and using trigonometric identities, we get that

$$-\alpha_n \sin(\omega_n t_1 - \gamma_n) + \beta_n \cos(\omega_n t_1 - \gamma_n) = \frac{f_n}{\sqrt{A_1^2 + B_1^2 \omega_n^2}}, \\ -\alpha_n \sin(\omega_n t_2 - \delta_n) + \beta_n \cos(\omega_n t_2 - \delta_n) = \frac{g_n}{\sqrt{A_2^2 + B_2^2 \omega_n^2}}.$$

This linear system can be uniquely solved for the unknown coefficients  $\alpha_n$  and  $\beta_n$ , due to the fact that  $\sin(\omega_n(t_2 - t_1) + \gamma_n - \delta_n) \neq 0$ ,  $n \in \mathbb{N}_0$ :

$$(20) \quad \begin{aligned} \alpha_n &= \frac{\sqrt{A_2^2 + B_2^2 \omega_n^2} \cos(\omega_n t_2 - \delta_n) f_n - \sqrt{A_1^2 + B_1^2 \omega_n^2} \cos(\omega_n t_1 - \gamma_n) g_n}{\sqrt{A_1^2 + B_1^2 \omega_n^2} \sqrt{A_2^2 + B_2^2 \omega_n^2} \sin(\omega_n(t_2 - t_1) + \gamma_n - \delta_n)}, \\ \beta_n &= \frac{\sqrt{A_2^2 + B_2^2 \omega_n^2} \sin(\omega_n t_2 - \delta_n) f_n - \sqrt{A_1^2 + B_1^2 \omega_n^2} \sin(\omega_n t_1 - \gamma_n) g_n}{\sqrt{A_1^2 + B_1^2 \omega_n^2} \sqrt{A_2^2 + B_2^2 \omega_n^2} \sin(\omega_n(t_2 - t_1) + \gamma_n - \delta_n)}. \end{aligned}$$

Thus, the unknown initial functions  $\varphi$  and  $\psi$  are uniquely determined in the form of (15) and (16). It remains to show that  $\varphi$ ,  $\psi$  are from the classes  $D^{s+1}$ ,  $D^s$ , respectively, i.e., to show that the following inequality holds:

$$(21) \quad \max\{\|\varphi\|_{s+1}, \|\psi\|_s\} = \max\left\{\sum_{n=0}^{\infty} n^{2s+2} |\alpha_n|^2, \sum_{n=0}^{\infty} n^{2s} |\omega_n \beta_n|^2\right\} < \infty.$$

We note that condition (13) involves that  $|\omega_n| = O(n)$ , so by using Lemma 1 and (20), we get that there is a constant  $M$  such that

$$\begin{aligned} |\alpha_n| &\leq \left| M \frac{n}{1 + B_1 n} f_n \right| + \left| M \frac{n}{1 + B_2 n} g_n \right|, \quad \forall n > N, \\ |\beta_n| &\leq \left| M \frac{n}{1 + B_1 n} f_n \right| + \left| M \frac{n}{1 + B_2 n} g_n \right|, \quad \forall n > N, \end{aligned}$$

which means that

$$(22) \quad \max\{|\alpha_n|, |\beta_n|\} < c_1 n \max\{|f_n|, |g_n|\}, \quad n \in \mathbb{N}_0,$$

for a suitable constant  $c_1$ . Obviously, the inequality (22) (and thus condition (12)) can be improved when  $B_1 \neq 0$  or  $B_2 \neq 0$  (see Remark 4 below).

By using (22), we get that

$$\begin{aligned} \max\left\{\sum_{n=0}^{\infty} n^{2s+2} |\alpha_n|^2, \sum_{n=0}^{\infty} n^{2s} |\omega_n \beta_n|^2\right\} &\leq \sum_{n=0}^{\infty} C^2 n^{2s+2} \max\{|\alpha_n|^2, |\beta_n|^2\} < \\ &< c_1^2 C^2 \sum_{n=0}^{\infty} n^{2s+4} \max\{|f_n|^2, |g_n|^2\}, \end{aligned}$$

where

$$\sum_{n=0}^{\infty} n^{2s+4} \max\{|f_n|^2, |g_n|^2\} \leq \|f\|_{s+2}^2 + \|g\|_{s+2}^2 < \infty$$

and  $(f, g) \in D^{s+2} \times D^{s+2}$  according to the definition of the  $s+2$  norm.  $\square$

**Remark 3.** The above examinations and estimates show that the operator  $\mathcal{A}$ , which assigns to the partial states  $(f, g)$  at  $t_1$  and  $t_2$  the couple  $(\varphi, \psi)$ , is a continuous (bounded) operator from  $D^{s+2} \times D^{s+2}$  into  $D^{s+1} \times D^s$ .

On the other hand, we get from the system of equations (19) that

$$\alpha_n \sin(\omega_n(t_2 - t_1) + \gamma_n - \delta_n) = \frac{f_n \cos(\omega_n t_2 - \delta_n)}{v_n} - \frac{g_n \cos(\omega_n t_1 - \gamma_n)}{w_n},$$

$$\beta_n \sin(\omega_n(t_2 - t_1) + \gamma_n - \delta_n) = \frac{f_n \sin(\omega_n t_2 - \delta_n)}{v_n} - \frac{g_n \sin(\omega_n t_1 - \gamma_n)}{w_n},$$

where

$$v_n = \sqrt{A_1^2 + B_1^2 \omega_n^2}, \quad w_n = \sqrt{A_2^2 + B_2^2 \omega_n^2}$$

and obviously

$$v_n = |A_1| \text{ if } B_1 = 0, \quad w_n = |A_2| \text{ if } B_2 = 0,$$

$$v_n = O(n) \text{ if } B_1 \neq 0, \quad w_n = O(n) \text{ if } B_2 \neq 0, \quad n \rightarrow \infty.$$

Based on the behaviour of the Fourier coefficients  $\alpha_n, \beta_n$  as  $n \rightarrow \infty$  we get that

$$\sum_{n=0}^{\infty} (n^{2s+4} |\alpha_n|^2 + n^{2s+2} |\omega_n \beta_n|^2) \sin^2(\omega_n(t_2 - t_1) + \gamma_n - \delta_n) \leq$$

$$\leq C \sum_{n=0}^{\infty} (n^{2s+4} |f_n|^2 + n^{2s+4} |g_n|^2) \leq C (\|f\|_{s+2}^2 + \|g\|_{s+2}^2),$$

which indicates a slightly stronger smoothness for the couple  $(\varphi, \psi) = \mathcal{A}(f, g)$ .

This suggests to prove a slightly sharper statement than the one in Proposition 1. To this effect, let us introduce the subspaces  $D_0^s \subset D^s$  that contain the functions  $f \in D^s$  whose Fourier coefficients  $f_n$  for  $n \in \mathbb{N}_1$  (the set  $\mathbb{N}_1$  depends on the problem, see its definition in Lemma 1) have the following property:

$$\sum_{n \in \mathbb{N}_1} |f_n|^2 |n|^{2s+2} < \infty.$$

Certainly this involves that  $D^{s+1} \subset D_0^s$ .

**Proposition 2.** *If the conditions of Proposition 1 hold and  $f, g \in D_0^{s+1}$  (instead of  $D^{s+2}$ ), then the observation problem posed for (7), (8), (9) under condition (e) has a unique solution for  $(\varphi, \psi) \in D^{s+1} \times D^s$ .*

*In other terms, the operator  $\mathcal{A}: \mathcal{A}(f, g) := (\varphi, \psi)$  maps  $D_0^{s+1} \times D_0^{s+1}$  into  $D^{s+1} \times D^s$  as a continuous (bounded) operator.*

*Proof of Proposition 2.* We can do the same steps as in the proof of Proposition 1. It remains only to show that (21) also holds for  $(f, g) \in D_0^{s+1} \times D_0^{s+1}$ . Indeed, we have

$$\max \left\{ \sum_{n=0}^{\infty} n^{2s+2} |\alpha_n|^2, \sum_{n=0}^{\infty} n^{2s} |\omega_n \beta_n|^2 \right\} \leq \sum_{n=0}^{\infty} C^2 n^{2s+2} \max\{|\alpha_n|^2, |\beta_n|^2\} =$$



$$= \sum_{n \in \mathbb{N}_1} C^2 n^{2s+2} \max\{|\alpha_n|^2, |\beta_n|^2\} + \sum_{n \in \mathbb{N}_2} C^2 n^{2s+2} \max\{|\alpha_n|^2, |\beta_n|^2\}.$$

Using inequality (22) for the first sum, we get

$$\sum_{n \in \mathbb{N}_1} C^2 n^{2s+2} \max\{|\alpha_n|^2, |\beta_n|^2\} \leq c_1^2 C^2 \sum_{n \in \mathbb{N}_1} n^{2s+4} \max\{|f_n|^2, |g_n|^2\},$$

which is finite by the definition of the spaces  $D_0^{s+1}$ .

For  $n \in \mathbb{N}_2$  we can improve inequality (22). Namely, for these values of  $n$  (if  $n$  is large enough) we have  $\sin(\omega_n(t_2 - t_1) + \gamma_n - \delta_n) > d_1/2$  as it was shown in the proof of Lemma 1, whence it follows that

$$\max\{|\alpha_n|, |\beta_n|\} < c_2 \max\{|f_n|, |g_n|\} \quad \forall n \in \mathbb{N}_2$$

with a suitable constant  $c_2$ . So, we get, that

$$\sum_{n \in \mathbb{N}_2} C^2 n^{2s+2} \max\{|\alpha_n|^2, |\beta_n|^2\} \leq c_2^2 C^2 \sum_{n \in \mathbb{N}_2} n^{2s+2} \max\{|f_n|^2, |g_n|^2\} < \infty,$$

due to the fact that  $D_0^{s+1} \subset D^{s+1}$ . □

**Remark 4.** As we mentioned before, the inequality (22) can be improved when  $B_1 \neq 0$  or  $B_2 \neq 0$ . More precisely, if  $B_1 \neq 0$  then  $f \in D^{s+1}$ ; while if  $B_2 \neq 0$  then  $g \in D^{s+1}$  is sufficient in condition (12).

In the special case when one of  $A_1, B_1$  equals zero and one of  $A_2, B_2$  equals zero, the observation condition (e) simplifies to one of the condition (a), (b), (c), (d) (after suitably dividing with  $A_1, A_2, B_1$ , or  $B_2$ ). In this case, we have  $\gamma_n \equiv 0$  or  $\gamma_n \equiv \pi/2$ ;  $\delta_n \equiv 0$  or  $\delta_n \equiv \pi/2$ , and conditions (13) and (14) are simplified accordingly. These terms correspond to our previous results in [13].

**Remark 5.** Condition (13) is suitable for the usage of Lemma 1 to have the estimation

$$\frac{1}{\sin(\omega_n(t_2 - t_1) + \gamma_n - \delta_n)} < \frac{n}{c_3}, \quad \forall n > N.$$

However condition (13) of Proposition 1 is not necessary, it can be replaced, e.g., with the assumption that  $f_n = g_n = 0$  for all  $n \in \mathcal{K}(\epsilon)$ , where for the arbitrary small fixed  $\epsilon > 0$ ,  $\mathcal{K}(\epsilon) \subset \mathbb{N}$  is the collection of all  $n$  such that

$$-\epsilon < \omega_n(t_2 - t_1) + \gamma_n - \delta_n < \epsilon \pmod{\pi}.$$

Now, we will show three applications of Proposition 1 in the following Sections 3–5.

### 3. THE VIBRATING STRING WITH FIXED LEFT AND FREE RIGHT ENDS

Let  $\Omega = \{(x, t) : 0 < x < l, t \in \mathbb{R}\}$ . Consider the problem of the vibrating  $[0, l]$  string with a fixed left end and a free right end under the assumption that there is an elastic withdrawing force proportional to the transversal deflection  $u(x, t)$  of the point  $x$  of the string at the instant denoted by  $t$ . This phenomenon is described by the Klein–Gordon equation as follows:

$$(1) \quad u_{tt}(x, t) = a^2 u_{xx}(x, t) - cu(x, t), \quad (x, t) \in \bar{\Omega}, \quad 0 < a, c \in \mathbb{R},$$

with the initial conditions

$$(2) \quad u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x), \quad 0 \leq x \leq l,$$

and the homogeneous boundary conditions

$$(4) \quad u(0, t) = 0, \quad u_x(l, t) = 0, \quad t \in \mathbb{R}.$$

Some of the results of [2] (see Section 1.1–1.3) and [10] say that for arbitrary  $s \in \mathbb{R}$  with  $(\varphi, \psi) \in D^{s+1} \times D^s$ , the solution of the mixed problem (1), (2), (4) satisfies (10), and (11); and using  $q(x) \equiv c > 0$  we also have

$$X_n = \sqrt{\frac{2}{l}} \sin\left(\frac{(n + \frac{1}{2})\pi}{l}x\right), \quad n \in \mathbb{N}_0,$$

and

$$u(x, t) = \sum_{n=0}^{\infty} [\alpha_n \cos(\omega_n t) + \beta_n \sin(\omega_n t)] \sqrt{\frac{2}{l}} \sin\left(\frac{(n + \frac{1}{2})\pi}{l}x\right), \quad (x, t) \in \bar{\Omega}.$$

Here

$$\omega_n = \sqrt{\left(\frac{(n + \frac{1}{2})\pi a}{l}\right)^2 + c}, \quad n \in \mathbb{N}_0,$$

thus we have

$$\begin{aligned} \omega_n &= \frac{(n + \frac{1}{2})\pi a}{l} + \left(\omega_n - \frac{(n + \frac{1}{2})\pi a}{l}\right) = \frac{(n + \frac{1}{2})\pi a}{l} + \frac{\omega_n^2 - \left(\frac{(n + \frac{1}{2})\pi a}{l}\right)^2}{\omega_n + \frac{(n + \frac{1}{2})\pi a}{l}} = \\ &= n \frac{\pi a}{l} + \frac{\pi a}{2l} + \frac{c}{\omega_n + \frac{(n + \frac{1}{2})\pi a}{l}}. \end{aligned}$$

Let us consider the following observation conditions:

$$(e) \quad \begin{aligned} A_1 u|_{t=t_1} + B_1 u_t|_{t=t_1} &= f, & |A_1| + |B_1| &> 0, \\ A_2 u|_{t=t_2} + B_2 u_t|_{t=t_2} &= g, & |A_2| + |B_2| &> 0, \end{aligned}$$

and let us suppose that  $A_1B_1 \geq 0$  and  $A_2B_2 \leq 0$ . Without loss of the generality, we can take  $A_1, B_1, B_2 \geq 0$  and  $A_2 \leq 0$ . Hence it follows that  $\sin \gamma_n \rightarrow 0^+$ ,  $\gamma_n \rightarrow 0^+$ ,  $\sin \delta_n \rightarrow 0^-$ ,  $\delta_n \rightarrow 0^-$  as  $n \rightarrow \infty$ . Therefore, there exists  $N \in \mathbb{N}$ , such that  $\forall n > N$ ,

$$0 \leq \frac{2}{\pi} \gamma_n \leq \sin \gamma_n \leq \gamma_n, \quad \delta_n \leq \sin \delta_n \leq \frac{2}{\pi} \delta_n \leq 0.$$

Due to the facts that

$$\sin \gamma_n = \frac{A_1}{\sqrt{A_1^2 + B_1^2 \omega_n^2}}, \quad \sin \delta_n = \frac{A_2}{\sqrt{A_2^2 + B_2^2 \omega_n^2}}, \quad \omega_n = O(n),$$

there is a constant  $M \in \mathbb{R}^+ \cup \{0\}$  such that

$$0 \leq \frac{M}{n} \leq \gamma_n \rightarrow 0, \quad 0 \leq \frac{M}{n} \leq -\delta_n \rightarrow 0.$$

So, the condition (13) is fulfilled with

$$A = (t_2 - t_1) \frac{a}{l}, \quad B = (t_2 - t_1) \frac{\pi a}{2l}, \quad C_n = (t_2 - t_1) \frac{c}{\omega_n + \frac{(n+\frac{1}{2})\pi a}{l}} + \gamma_n - \delta_n,$$

provided  $(t_2 - t_1) \frac{a}{l} \in \mathbb{Q}$ .

Consequently, we can apply Proposition 1 to the vibrating string with fixed left end and free right end as follows.

**Theorem 1.** *Let*

$$f \in D^{s+2}, \quad g \in D^{s+2}, \quad s \in \mathbb{R},$$

*and suppose that*

$$\sin((t_2 - t_1)\omega_n + \gamma_n - \delta_n) \neq 0, \quad n \in \mathbb{N}_0.$$

*Then the observation problem posed for (1), (2), (4) under the observation conditions*

$$\begin{aligned} A_1 u|_{t=t_1} + B_1 u_t|_{t=t_1} &= f, & |A_1| + |B_1| > 0, & A_1 B_1 \geq 0, \\ A_2 u|_{t=t_2} + B_2 u_t|_{t=t_2} &= g, & |A_2| + |B_2| > 0, & A_2 B_2 \leq 0, \end{aligned}$$

*has a unique solution for  $(\varphi, \psi) \in D^{s+1} \times D^s$  provided the elapsed time between  $t_1$  and  $t_2$  is a rational multiple of  $l/a$ .*

#### 4. THE VIBRATING STRING WITH FREE ENDS

Let  $\Omega = \{(x, t) : 0 < x < l, t \in \mathbb{R}\}$ . Consider the problem of the vibrating  $[0, l]$  string with free ends when there is an elastic withdrawing force proportional to the transversal deflection  $u(x, t)$  of the point  $x$  of the string at the instant denoted by  $t$ . This phenomenon is described by the Klein–Gordon equation as follows:

$$(1) \quad u_{tt}(x, t) = a^2 u_{xx}(x, t) - cu(x, t), \quad (x, t) \in \overline{\Omega}, \quad 0 < a, c \in \mathbb{R},$$

with the initial conditions

$$(2) \quad u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x), \quad 0 \leq x \leq l,$$

and the homogeneous boundary conditions of the second kind

$$(5) \quad u_x(0, t) = 0, \quad u_x(l, t) = 0, \quad t \in \mathbb{R}.$$

From the same results of [2] and [10] as in Section 3 it follows that for arbitrary  $s \in \mathbb{R}$  with  $(\varphi, \psi) \in D^{s+1} \times D^s$  the solution of the mixed problem (1), (2), (5) satisfies (10), (11), and due to the fact that  $q(x) \equiv c > 0$ , we have

$$X_0 = \sqrt{\frac{1}{l}} \text{ and } X_n = \sqrt{\frac{2}{l}} \cos\left(\frac{n\pi}{l}x\right), \quad n \in \mathbb{N},$$

and

$$u(x, t) = \sum_{n=0}^{\infty} [\alpha_n \cos(\omega_n t) + \beta_n \sin(\omega_n t)] X_n(x), \quad (x, t) \in \bar{\Omega}.$$

Here

$$\omega_n = \sqrt{\left(\frac{n\pi a}{l}\right)^2 + c}, \quad n \in \mathbb{N}_0,$$

so

$$\omega_n = \frac{n\pi a}{l} + \left(\omega_n - \frac{n\pi a}{l}\right) = n\frac{\pi a}{l} + \frac{\omega_n^2 - \left(\frac{n\pi a}{l}\right)^2}{\omega_n + \frac{n\pi a}{l}} = n\frac{\pi a}{l} + \frac{c}{\omega_n + \frac{n\pi a}{l}}.$$

If  $A_1 B_1 \geq 0$  and  $A_2 B_2 \leq 0$ , then the same arguments for  $\gamma_n, \delta_n$  as in Section 3 give that condition (13) is fulfilled with

$$A = (t_2 - t_1)\frac{a}{l}, \quad B = 0, \quad C_n = (t_2 - t_1)\frac{c}{\omega_n + \frac{n\pi a}{l}} + \gamma_n - \delta_n,$$

provided  $(t_2 - t_1)\frac{a}{l} \in \mathbb{Q}$ .

Consequently, we can apply Proposition 1 to the vibrating string with free ends as follows.

**Theorem 2.** *Let*

$$f \in D^{s+2}, \quad g \in D^{s+2}, \quad s \in \mathbb{R},$$

*and suppose that*

$$\sin((t_2 - t_1)\omega_n + \gamma_n - \delta_n) \neq 0, \quad n \in \mathbb{N}_0.$$

*Then the observation problem posed for (1), (2), (5) under observation conditions*

$$\begin{aligned} A_1 u|_{t=t_1} + B_1 u_t|_{t=t_1} &= f, & |A_1| + |B_1| > 0, \quad A_1 B_1 \geq 0, \\ A_2 u|_{t=t_2} + B_2 u_t|_{t=t_2} &= g, & |A_2| + |B_2| > 0, \quad A_2 B_2 \leq 0, \end{aligned}$$

*has a unique solution for  $(\varphi, \psi) \in D^{s+1} \times D^s$  provided the elapsed time between  $t_1$  and  $t_2$  is a rational multiple of  $l/a$ .*

5. THE VIBRATING STRING WITH BOUNDARY CONDITION  
OF THE THIRD KIND  
(STURM-LIOUVILLE BOUNDARY CONDITIONS)

Let  $\Omega = \{(x, t) : 0 < x < l, t \in \mathbb{R}\}$ . Consider the problem of the vibrating  $[0, l]$  string when there exists a varying withdrawing force proportional to the transversal deflection  $u(x, t)$  of the point  $x$  of the string at the instant denoted by  $t$ . This phenomenon is described by the Klein-Gordon equation as follows:

$$(1') \quad u_{tt}(x, t) = a^2 u_{xx}(x, t) - c(x)u(x, t), \quad (x, t) \in \bar{\Omega}, \quad 0 < a \in \mathbb{R}, \quad 0 < c(x) \in C^2[0, l],$$

$$(2) \quad u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x), \quad 0 \leq x \leq l,$$

and the boundary conditions

$$(6) \quad \begin{aligned} u(0, t) \cos \alpha + u_x(0, t) \sin \alpha &= 0, & \cot \alpha < 0, \\ u(l, t) \cos \beta + u_x(l, t) \sin \beta &= 0, & \cot \beta > 0. \end{aligned}$$

The conditions for the sign of  $\cot \alpha$  and  $\cot \beta$  in (6) are sufficient to ensure the conservation of energy, hence guaranteeing the uniqueness of the solution of the mixed problem (1'), (2), (6) in the classical case. Moreover, this condition also ensures that the (later introduced) constant  $\gamma$  is strictly positive.

It follows again from [2] and [10] that for arbitrary  $s \in \mathbb{R}$  with  $(\varphi, \psi) \in D^{s+1} \times D^s$ , the solution of the mixed problem (1'), (2), (6) satisfies (10) and (11). The orthonormal basis  $\{X_n\}$ ,  $n \in \mathbb{N}_0$ , in (11) can be derived from [11] as follows:

$$X_n(x) = \sqrt{\frac{2}{\pi}} \left[ \cos\left(\frac{n\pi}{l}x\right) + \frac{\beta(x)}{n} \sin\left(\frac{n\pi}{l}x\right) \right] + O\left(\frac{1}{n^2}\right),$$

$$\beta(x) = -\gamma \frac{\pi}{l} x - \frac{l}{\pi} \cot \alpha + \frac{1}{2a^2} \int_0^x \frac{l}{\pi} c(\tau) d\tau,$$

$$u(x, t) = \sum_{n=0}^{\infty} [\alpha_n \cos(\omega_n t) + \beta_n \sin(\omega_n t)] X_n(x), \quad (x, t) \in \bar{\Omega}.$$

Here we have

$$\omega_n = \frac{a\pi}{l} \left( n + \frac{\gamma}{n} + O\left(\frac{1}{n^3}\right) \right), \quad \gamma = \frac{l}{\pi^2} \left( -\cot \alpha + \cot \beta + \frac{1}{2a^2} \int_0^l c(\tau) d\tau \right),$$

where  $\omega_n$  can be written in the form of:

$$\omega_n = n \frac{\pi a}{l} + \left( \frac{\gamma a \pi}{n l} + O\left(\frac{1}{n^3}\right) \right).$$

If  $A_1B_1 \geq 0$  and  $A_2B_2 \leq 0$ , with the same arguments for  $\gamma_n, \delta_n$  as in Section 3, we get that condition (13) is fulfilled with

$$A = (t_2 - t_1)\frac{a}{l}, \quad B = 0, \quad C_n = (t_2 - t_1)\frac{\gamma a \pi}{nl} + \gamma_n - \delta_n + O\left(\frac{1}{n^3}\right),$$

provided  $(t_2 - t_1)\frac{a}{l} \in \mathbb{Q}$ .

Consequently, we can apply Proposition 1 to the vibrating string with fixing (6) as follows.

**Theorem 3.** *Let*

$$f \in D^{s+2}, \quad g \in D^{s+2}, \quad s \in \mathbb{R},$$

*and suppose that*

$$\sin((t_2 - t_1)\omega_n + \gamma_n - \delta_n) \neq 0, \quad n \in \mathbb{N}_0.$$

*Then the observation problem posed for (1'), (2), (6) under the observation conditions*

$$\begin{aligned} A_1u|_{t=t_1} + B_1u_t|_{t=t_1} &= f, & |A_1| + |B_1| > 0, \quad A_1B_1 \geq 0, \\ A_2u|_{t=t_2} + B_2u_t|_{t=t_2} &= g, & |A_2| + |B_2| > 0, \quad A_2B_2 \leq 0, \end{aligned}$$

*has a unique solution for  $(\varphi, \psi) \in D^{s+1} \times D^s$  provided the elapsed time between  $t_1$  and  $t_2$  is a rational multiple of  $l/a$ .*

## 6. OBSERVABILITY RESULTS FOR ARBITRARY $t_1 < t_2$

Considering the observation problem posed in Proposition 1 without conditions (13), (14) and for randomly chosen  $t_1 < t_2$ , we have to investigate the necessity of the description of all possibilities of the observability. We shall give an alternative treatment, and for this reason we introduce the following notations: for every  $n \in \mathbb{N}_0$ , let  $d_n$  denote the distance of  $h_n = \omega_n(t_2 - t_1) + \gamma_n - \delta_n$  to the nearest zero of the sine function, i.e.,

$$d_n := \rho(h_n, \mathbb{N}_0\pi), \quad n \in \mathbb{N}_0, \quad \mathbb{N}_0\pi := \bigcup_{k=0}^{\infty} \{k\pi\}.$$

The promised alternative is the following:

1. If  $d_n > 0$  for all  $n \in \mathbb{N}_0$  then the observation problem is (formally) uniquely solvable, since the coefficients  $\alpha_n, \beta_n$  for the definition of  $\varphi, \psi$  can be uniquely found from the linear system (19) for every  $n \in \mathbb{N}_0$ . Although, for the estimates of  $\alpha_n, \beta_n$  (and for the belonging of  $\varphi, \psi$  to the corresponding spaces  $D^{s+1}, D^s$ , respectively), some

special tools are needed (e.g., we used Diophantine ones in Sections 1–5). A general result can be the following: The formal series:

$$\varphi(x) \sim \sum_{n=0}^{\infty} \alpha_n X_n(x), \quad \psi(x) \sim \sum_{n=0}^{\infty} \omega_n \beta_n X_n(x)$$

have the following properties:

$$(24) \quad \sum_{n=0}^{\infty} \alpha_n d_n X_n(x) \in D^{s+1}, \quad \sum_{n=0}^{\infty} \omega_n \beta_n d_n X_n(x) \in D^s.$$

Moreover, the relations in (24) also hold in the case if we replace  $d_n$  by  $\tilde{d}_n$ , where

$$\tilde{d}_n = \begin{cases} 1, & d_n \geq \delta > 0, \\ b_n, & d_n < \delta. \end{cases} \quad n \in \mathbb{N}_0,$$

with arbitrarily small fixed  $\delta$  and arbitrary  $|b_n| \leq d_n$ .

2. Let us denote by  $\mathbb{N}(0)$  the set of all  $n \in \mathbb{N}_0$  such that  $d_n = 0$ . Then for every  $n \in \mathbb{N}(0)$ , the linear system (19) can be solved for  $(\alpha_n, \beta_n)$  if and only if

$$\frac{f_n}{g_n} = \frac{A_1 \cos(\omega_n t_1) - B_1 \omega_n \sin(\omega_n t_1)}{A_2 \cos(\omega_n t_2) - B_2 \omega_n \sin(\omega_n t_2)} = \left( \frac{A_1 \sin(\omega_n t_1) + B_1 \omega_n \cos(\omega_n t_1)}{A_2 \sin(\omega_n t_2) + B_2 \omega_n \cos(\omega_n t_2)} \right), \quad n \in \mathbb{N}(0),$$

but the solution  $(\alpha_n, \beta_n)$  as well as  $(\varphi, \psi)$  are not unique, and the belonging of the constructed  $(\varphi, \psi)$  to  $D^{s+1} \times D^s$  is not guaranteed.

We emphasize, that all remarks of the present section are also valid for the observation problems posed for the case  $c = 0$  of the Klein–Gordon equation, i.e., for the standard vibrating string.

#### ACKNOWLEDGEMENTS

We would like to sincerely thank Professor Ferenc Mórnicz for his encouragement and constant help on a previous draft of the present paper.

#### REFERENCES

- [1] Lions, J. L., *Contrôle optimal de systèmes gouvernés par des équations aux dérivées partielles (Études mathématiques)*, Paris: Dunod, 1968.
- [2] Komornik, V., *Exact controllability and stabilization. The multiplier method.*, Research in Applied Mathematics 36. Chichester: Wiley. Paris: Masson, 1994.

- [3] Komornik, V., *Exact controllability in short time for the wave equation*, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 6, No. 2, pp. 153–164, 1989.
- [4] Joó, I., *On the vibration of a string.*, Studia Sci. Math. Hungar 22, No. 1–4, pp. 1–9, 1987.
- [5] Horváth, M., *Vibrating strings with free ends*, Acta Math. Hungar 51, No. 1–2, pp. 171–180, 1988.
- [6] Il'in, V. A., *Boundary control of the vibration process at two ends*, Dokl. Acad. Nauk 369, No. 5, pp. 592–596, 1999.
- [7] Moiseev, E. I., Kholomeeva, A. A., *Optimal boundary displacement control of string vibrations with nonlocal oddness condition of the first kind*, Differential Equations 46, No. 11, pp. 1624–1630, 2010.
- [8] Il'in, V. A., Moiseev, E. I., *Optimization of boundary controls of string vibrations*, Uspekhi Mat. Nauk 60, No. 6, pp. 89–114, 2005.
- [9] Lions, J. L., *Exact controllability, stabilization and perturbations for distributed systems*, SIAM Rev. 30, No. 1, pp. 1–68, 1988.
- [10] Lions, J. L., Magenes, E., *Problèmes aux limites non homogènes et applications I–III*, Dunod, Paris, 1968–1970.
- [11] Levitan, B. M., Sargsjan, I. S., *Introduction to spectral theory*, Nauka, Moscow, 1970. English translation: Translations of Mathematical Monographs, Vol. 39. American Mathematical Society, Providence, R.I., 1975.
- [12] Znamenskaya, L. N., *State observability of elastic string vibrations under the boundary conditions of the first kind*, Differential Equations 46, No. 5, pp. 748–752, 2010.
- [13] Szijártó, A., Hegedűs, J. *Observation problems posed for the Klein–Gordon equation*, E. J. Qualitative Theory of Differential Equations, No. 7, pp. 1–13, 2012.
- [14] Naimark, M. A., *Linear differential operators. Part I–II*, Frederick Ungar Publishing Co., New York.

(Received November 5, 2012)