Regularity in Orlicz spaces for nondivergence elliptic operators with potentials satisfying a reverse Hölder condition*

Kelei Zhang[†]

Department of Applied Mathematics, Northwestern Polytechnical University,

Xi'an, Shaanxi, 710129, PR China

Abstract: The purpose of this paper is to obtain the global regularity in Orlicz spaces for nondivergence elliptic operators with potentials satisfying a reverse Hölder condition.

Keywords: nondivergence elliptic operator; regularity, Orlicz space; potential; reverse Hölder condition.

Mathematics Subject Classification (2010): 35J15, 46E30.

1 Introduction

In this paper we consider the following nondivergence elliptic operator

$$Lu \equiv Au + Vu \equiv -\sum_{i,j=1}^{n} a_{ij}(x)u_{x_ix_j} + Vu,$$
 (1.1)

where $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n (n \geq 3)$, and establish the regularity in Orlicz spaces for (1.1). It will be assumed that the following assumptions on the coefficients of the operator A and the potential V are satisfied

 (H_1) $a_{ij} \in L^{\infty}(\mathbb{R}^n)$ and $a_{ij} = a_{ji}$ for all i, j = 1, 2, ..., n, and there exists a positive constant Λ such that

$$|\Lambda^{-1}|\xi|^2 \le \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2$$

^{*}This work was supported by the National Natural Science Foundation of China (Grant Nos. 11271299, 11001221) and Natural Science Foundation Research Project of Shaanxi Province (Grant No. 2012JM1014).

[†]E-mail address: eaststonezhang@126.com

for any $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^n$;

 (H_2) $a_{ij}(x) \in VMO(\mathbb{R}^n)$, which means that for $i, j = 1, 2, \ldots, n$,

$$\eta_{ij}(r) = \sup_{\rho \le r} \sup_{x \in \mathbb{R}^n} \left(|B_{\rho}(x)|^{-1} \int_{B_{\rho}(x)} |a_{ij}(y) - a_{ij}^B| \, dy \right) \to 0, \ r \to 0^+,$$

where $a_{ij}^{B} = |B_{\rho}(x)|^{-1} \int_{B_{\rho}(x)} a_{ij}(y) dy;$

 (H_3) $V \in B_q$ for $n/2 \le q < \infty$, which means that $V \in L^q_{loc}(\mathbb{R}^n), V \ge 0$, and there exists a positive constant c_1 such that the reverse Hölder inequality

$$\left(|B|^{-1}\int_{B}V(x)^{q}dx\right)^{1/q} \le c_{1}\left(|B|^{-1}\int_{B}V(x)dx\right)$$

holds for every ball B in \mathbb{R}^n .

Note that when we say $V \in B_{\infty}$, it means

$$\sup_{B} V(x) \le c_1 \left(|B|^{-1} \int_{B} V(x) dx \right).$$

In fact, if $V \in B_{\infty}$, then it implies that $V \in B_q$ for $1 < q < \infty$.

Regularity theory for elliptic operators with potentials satisfying a reverse Hölder condition has been studied by many authors (see [4], [9]–[12], [14], [15]). When A is the Laplace operator and $V \in B_q$ $(n/2 \le q < \infty)$, Shen [10] derived L^p boundedness for 1 and showed that the range of <math>p is optimal. If A is the Laplace operator and $V \in B_{\infty}$, an extension of L^p estimates to the global Orlicz estimates was given by Yao [14] with modifying the iteration-covering method introduced by Acerbi and Mingione [1]. For $a_{ij} \in C^1(\mathbb{R}^n)$ and $V \in B_{\infty}$, regularity theory in Orlicz spaces for the operators $\sum_{i,j=1}^n \partial_{x_i}(a_{ij}\partial_{x_j}) + V$ was proved by Yao [15]. Recently, under the assumptions (H_1) – (H_3) , the global $L^p(\mathbb{R}^n)$ estimates for L in (1.1) has been deduced by Bramanti et al [4].

In this paper we will establish global estimates in Orlicz spaces for L which extends results in [4] to the case of the general Orlicz spaces. Our approach is based on an iteration-covering lemma (Lemma 3.1), the technique of "S. Agmon's idea" (see [3], p. 124) and an approximation procedure.

The definitions of Yong functions ϕ , Orlicz spaces $L^{\phi}(\mathbb{R}^n)$, Orlicz–Sobolev spaces $W^2L^{\phi}(\mathbb{R}^n)$, $W_V^2L^{\phi}(\mathbb{R}^n)$, and their properties will be described in Section 2.

We now state the main result of this paper.

Theorem 1.1 Let ϕ be a Young function and satisfy the global $\Delta_2 \cap \nabla_2$ condition. Assume that the operator L satisfies the assumptions (H_1) , (H_2) and (H_3) for $q \geq \max\{n/2, \alpha_1\}$, $f \in L^{\phi}(\mathbb{R}^n)$. If $u \in W_V^2 L^{\phi}(\mathbb{R}^n)$ satisfies

$$Lu - \mu u = f, \quad x \in \mathbb{R}^n, \tag{1.2}$$

then there exists a constant C > 0 such that for any $\mu \gg 1$ large enough, we have

$$\mu^{\alpha_2} \int_{\mathbb{R}^n} \phi(|u|) dx + \mu^{\alpha_2/2} \int_{\mathbb{R}^n} \phi(|Du|) dx + \int_{\mathbb{R}^n} \phi(|Vu|) dx + \int_{\mathbb{R}^n} \phi(|D^2u|) dx$$

$$\leq C \int_{\mathbb{R}^n} \phi(|f|) dx, \tag{1.3}$$

where the constants α_1 and α_2 appear in Orlicz spaces, see (2.4), C depends only on n, q, Λ , c_1 , α_1 , α_2 and the VMO moduli of the leading coefficients a_{ij} .

The proof of Theorem 1.1 is based on the following result.

Theorem 1.2 Under the same assumptions on ϕ , a_{ij} , V, q, f as in Theorem 1.1, let $u \in C_0^{\infty}(\mathbb{R}^n)$ satisfy Lu = f in \mathbb{R}^n . Then there exists a constant C > 0 such that

$$\int_{\mathbb{R}^n} \phi\left(\left|D^2 u\right|\right) dx + \int_{\mathbb{R}^n} \phi\left(\left|V u\right|\right) dx \le C \left\{ \int_{\mathbb{R}^n} \phi\left(\left|f\right|\right) dx + \int_{\mathbb{R}^n} \phi\left(\left|u\right|\right) dx \right\}, \quad (1.4)$$

where C depends only on n, q, Λ , c_1 , a, K and the VMO moduli of a_{ij} .

Note that Theorem 1.2 and Definition 2.9 easily imply the following result by using the monotonicity, convexity of ϕ , (2.2) and Remark 2.7.

Corollary 1.3 Under the same assumptions on ϕ , a_{ij} , V, q, f as in Theorem 1.1, let $u \in W_V^2 L^{\phi}(\mathbb{R}^n)$ satisfy Lu = f in \mathbb{R}^n . Then there exists a constant C > 0 such that

$$\int_{\mathbb{R}^n} \phi\left(\left|D^2 u\right|\right) dx + \int_{\mathbb{R}^n} \phi\left(\left|V u\right|\right) dx \le C \left\{ \int_{\mathbb{R}^n} \phi\left(\left|f\right|\right) dx + \int_{\mathbb{R}^n} \phi\left(\left|u\right|\right) dx \right\},$$

where C depends only on n, q, Λ , c_1 , a, K and the VMO moduli of a_{ij} .

Remark 1.4 When we take $\phi(t) = t^p$, $t \ge 0$ for $1 , then (1.4) is reduced to the classical <math>L^p$ estimates (see [4, Theorem 1]).

This paper will be organized as follows. In Section 2 some basic facts about Orlicz spaces and Orlicz–Sobolev spaces are recalled. In Section 3 we prove Theorem 1.2 by describing an iteration-covering lemma (Lemma 3.1) and using the

results in [4]. Section 4 is devoted to the proof of Theorem 1.1. We first assume $u \in C_0^{\infty}(B_{R_0/2})$ satisfying (1.2) and prove that (1.3) is valid by using Theorem 1.2 and "S. Agmon's idea"(see [3], p. 124); then we show that the assumption $u \in C_0^{\infty}(B_{R_0/2})$ can be removed by an approximation procedure and a covering lemma in [5].

Dependence of constants. Throughout this paper, the letter C denotes a positive constant which may vary from line to line.

2 Preliminaries

We collect here some facts about Orlicz spaces and Orlicz–Sobolev spaces which will be needed in the following. For more properties, we refer the readers to [2] and [8].

We use the following notation:

$$\Phi = \{\phi : [0, +\infty) \to [0, +\infty) \mid \phi \text{ is increasing and convex}\}.$$

Definition 2.1 A function $\phi \in \Phi$ is said to be a Young function if

$$\phi(0) = 0$$
, $\lim_{t \to +\infty} \phi(t) = +\infty$, $\lim_{t \to 0^+} \frac{\phi(t)}{t} = \lim_{t \to +\infty} \frac{t}{\phi(t)} = 0$. (2.1)

Definition 2.2 A Young function ϕ is said to satisfy the global Δ_2 condition denoted by $\phi \in \Delta_2$, if there exists a positive constant K such that for any t > 0,

$$\phi(2t) \le K\phi(t). \tag{2.2}$$

Definition 2.3 A Young function ϕ is said to satisfy the global ∇_2 condition denoted by $\phi \in \nabla_2$, if there exists a positive constant a > 1 such that for any t > 0,

$$\phi(at) \ge 2a\phi(t). \tag{2.3}$$

The following result was obtained in [7].

Lemma 2.4 If $\phi \in \Delta_2 \cap \nabla_2$, then for any t > 0 and $0 < \theta_2 \le 1 \le \theta_1 < \infty$,

$$\phi(\theta_1 t) \le K \theta_1^{\alpha_1} \phi(t) \text{ and } \phi(\theta_2 t) \le 2a \theta_2^{\alpha_2} \phi(t),$$
 (2.4)

where $\alpha_1 = \log_2 K$, $\alpha_2 = \log_a 2 + 1$ and $\alpha_1 \ge \alpha_2$.

Definition 2.5 (Orlicz spaces) Given a Young function ϕ , we define the Orlicz class $K^{\phi}(\mathbb{R}^n)$ which consists of all the measurable functions $g: \mathbb{R}^n \to \mathbb{R}$ satisfying

$$\int_{\mathbb{R}^n} \phi\left(|g|\right) dx < \infty$$

and the Orlicz space $L^{\phi}(\mathbb{R}^n)$ which is the linear hull of $K^{\phi}(\mathbb{R}^n)$.

In the Orlicz spaces $L^{\phi}(\mathbb{R}^n)$, we use the following Luxembourg norm

$$||u||_{L^{\phi}(\mathbb{R}^n)} = \inf \left\{ k > 0 : \int_{\mathbb{R}^n} \phi(|u|/k) \, dx \leqslant 1 \right\}.$$
 (2.5)

The space $L^{\phi}(\mathbb{R}^n)$ equipped with the Luxembourg norm $\|\cdot\|_{L^{\phi}(\mathbb{R}^n)}$ is a Banach space. In general, $K^{\phi} \subset L^{\phi}$. Moreover, if ϕ satisfies the global Δ_2 condition, then $K^{\phi} = L^{\phi}$ and C_0^{∞} is dense in L^{ϕ} (see [2], pp. 266–274).

Definition 2.6 (Convergence in mean) A sequence $\{u_k\}$ of functions in $L^{\phi}(\mathbb{R}^n)$ is said to converge in mean to $u \in L^{\phi}(\mathbb{R}^n)$ if

$$\lim_{k \to \infty} \int_{\mathbb{R}^n} \phi(|u_k(x) - u(x)|) dx = 0.$$

Remark 2.7 (see [2], p. 270)

- (i) The norm convergence in $L^{\phi}(\mathbb{R}^n)$ implies the mean convergence.
- (ii) If $\phi \in \Delta_2$, then the mean convergence implies the norm convergence.

Definition 2.8 (Orlicz–Sobolev spaces) The Orlicz–Sobolev space $W^2L^{\phi}(\mathbb{R}^n)$ is the set of all functions u which satisfy $|D^{\alpha}u(x)| \in L^{\phi}(\mathbb{R}^n)$ for $0 \le |\alpha| \le 2$. The norm is defined by

$$||u||_{W^{2}L^{\phi}(\mathbb{R}^{n})} = ||u||_{L^{\phi}(\mathbb{R}^{n})} + ||Du||_{L^{\phi}(\mathbb{R}^{n})} + ||D^{2}u||_{L^{\phi}(\mathbb{R}^{n})},$$

where
$$Du(x) = \{u_{x_i}\}_{i=1}^n$$
, $D^2u(x) = \{u_{x_ix_j}\}_{i,j=1}^n$, $\|Du\|_{L^{\phi}(\mathbb{R}^n)} = \sum_{i=1}^n \|u_{x_i}\|_{L^{\phi}(\mathbb{R}^n)}$, $\|D^2u\|_{L^{\phi}(\mathbb{R}^n)} = \sum_{i,j=1}^n \|u_{x_ix_j}\|_{L^{\phi}(\mathbb{R}^n)}$.

The following definition is analogous to the definition of the space $W_V^{2,p}(\mathbb{R}^n)$ introduced by Bramanti, Brandolini, Harboure and Viviani in [4].

Definition 2.9 The space $W_V^2L^{\phi}(\mathbb{R}^n)$ is the closure of $C_0^{\infty}(\mathbb{R}^n)$ in the norm

$$||u||_{W_V^2 L^{\phi}(\mathbb{R}^n)} = ||u||_{W^2 L^{\phi}(\mathbb{R}^n)} + ||Vu||_{L^{\phi}(\mathbb{R}^n)}.$$

Remark 2.10 (see e.g. [13]) If $g \in L^{\phi}(\mathbb{R}^n)$, then $\int_{\mathbb{R}^n} \phi(|g|) dx$ can be easily rewritten in an integral form

$$\int_{\mathbb{R}^n} \phi(|g|) dx = \int_0^\infty |\{x \in \mathbb{R}^n : |g| > t\}| d[\phi(t)]. \tag{2.6}$$

As usual, we denote by $B_R(x)$ the open ball in \mathbb{R}^n of radius R centered at x and $B_R = B_R(0)$.

3 Proof of Theorem 1.2

Before the proof of Theorem 1.2, some notions and two useful lemmas are given. Let us introduce the notation

$$p = \frac{1 + \alpha_2}{2} > 1.$$

For $u \in C_0^{\infty}(\mathbb{R}^n)$ satisfying Lu = f, set

$$\lambda_0^p = \int_{\mathbb{R}^n} |Vu|^p dx + \varepsilon^{-p} \left(\int_{\mathbb{R}^n} |f|^p dx + \int_{\mathbb{R}^n} |u|^p dx \right),$$

where $\varepsilon \in (0,1)$ is a small enough constant to be determined later. Let

$$u_{\lambda} = \frac{u}{\lambda_0 \lambda}$$
 and $f_{\lambda} = \frac{f}{\lambda_0 \lambda}$, for any $\lambda > 0$.

Then u_{λ} satisfies $Lu_{\lambda} = f_{\lambda}$. For any ball B in \mathbb{R}^n , we use the notations

$$J_{\lambda}[B] = \frac{1}{|B|} \int_{B} |Vu_{\lambda}|^{p} dx + \frac{1}{\varepsilon^{p} |B|} \left(\int_{B} |f_{\lambda}|^{p} dx + \int_{B} |u_{\lambda}|^{p} dx \right)$$

and

$$E_{\lambda}(1) = \{x \in \mathbb{R}^n : |Vu_{\lambda}| > 1\}.$$

The following lemma is just an analogous version of the result given in [15, Lemma 2.2]. Here the selection of λ_0 and the condition of V are different from [15].

Lemma 3.1 (Iteration-covering lemma) For any $\lambda > 0$, there exists a family of disjoint balls $\{B_{\rho_{x_i}}(x_i)\}$ with $x_i \in E_{\lambda}(1)$ and $\rho_{x_i} = \rho(x_i, \lambda) > 0$ such that

$$J_{\lambda}[B_{\rho_{x_i}}(x_i)] = 1, \ J_{\lambda}[B_{\rho}(x_i)] < 1 \ \text{for any } \rho > \rho_{x_i},$$
 (3.1)

and

$$E_{\lambda}(1) \subset \bigcup_{i>1} B_{5\rho_{x_i}}(x_i) \bigcup F, \tag{3.2}$$

where F is a zero measure set. Moreover,

$$\left| B_{\rho_{x_{i}}}(x_{i}) \right| \leq \frac{3^{p-1}}{3^{p-1} - 1} \left\{ \int_{\left\{ x \in B_{\rho_{x_{i}}}(x_{i}) : |Vu_{\lambda}| > \frac{1}{3} \right\}} |Vu_{\lambda}|^{p} dx \right. \\
\left. + \varepsilon^{-p} \int_{\left\{ x \in B_{\rho_{x_{i}}}(x_{i}) : |f_{\lambda}| > \frac{\varepsilon}{3} \right\}} |f_{\lambda}|^{p} dx + \varepsilon^{-p} \int_{\left\{ x \in B_{\rho_{x_{i}}}(x_{i}) : |u_{\lambda}| > \frac{\varepsilon}{3} \right\}} |u_{\lambda}|^{p} dx \right\}.$$
(3.3)

We omit the proof of Lemma 3.1 because it is actually similar to that of [15, Lemma 2.2].

In analogy with [4, Theorem 13], the following lemma holds by using [4, Theorem 2, Theorem 3], and standard techniques involving cutoff functions and the interpolation inequality (see e.g. [6]).

Lemma 3.2 Under the assumptions (H_1) – (H_3) , for any $\gamma \in (1, q]$, there exists a positive constant C such that for any x_i , ρ_{x_i} as in Lemma 3.1 and $u \in C_0^{\infty}(\mathbb{R}^n)$,

$$\int_{B_{5\rho_{x_{i}}}(x_{i})} |Vu|^{\gamma} dx \le C \left\{ \int_{B_{10\rho_{x_{i}}}(x_{i})} |Lu|^{\gamma} dx + \int_{B_{10\rho_{x_{i}}}(x_{i})} |u|^{\gamma} dx \right\},\,$$

where C depends only on $n, \gamma, q, c_1, \Lambda$ and the VMO moduli of a_{ij} .

Proof of Theorem 1.2. In order to prove (1.4), the first step is to check the following estimate

$$\int_{\mathbb{R}^n} \phi(|Vu|) dx \le C \left(\int_{\mathbb{R}^n} \phi(|f|) dx + \int_{\mathbb{R}^n} \phi(|u|) dx \right). \tag{3.4}$$

Since $u \in C_0^{\infty}(\mathbb{R}^n)$, then there exists some constant $R_0 > 0$ such that u is compactly supported in B_{R_0} . It follows from $q \ge \max\{n/2, \alpha_1\}$ and (2.4) that

$$\int_{\mathbb{R}^{n}} \phi(|Vu|) dx = \int_{\{x \in \mathbb{R}^{n}: |Vu| \ge 1\}} \phi(|Vu|) dx + \int_{\{x \in \mathbb{R}^{n}: |Vu| \le 1\}} \phi(|Vu|) dx
\leq K\phi(1) \int_{\mathbb{R}^{n}} |Vu|^{\alpha_{1}} dx + 2a\phi(1) \int_{\mathbb{R}^{n}} |Vu|^{\alpha_{2}} dx
\leq C \left(\sup_{B_{R_{0}}} |u|^{\alpha_{1}} + \sup_{B_{R_{0}}} |u|^{\alpha_{2}} \right) \left(\int_{B_{R_{0}}} |V|^{\alpha_{1}} dx + \int_{B_{R_{0}}} |V|^{\alpha_{2}} dx \right)
< \infty,$$

that is $|Vu| \in L^{\phi}(\mathbb{R}^n)$. Hence by (2.6), it yields

$$\int_{\mathbb{R}^n} \phi(|Vu|) dx = \int_0^\infty |\{x \in \mathbb{R}^n : |Vu| > \lambda_0 \lambda\}| d[\phi(\lambda_0 \lambda)].$$

Due to (3.2),

$$|\{x \in \mathbb{R}^n : |Vu| > \lambda_0 \lambda\}| \le \sum_{i=1}^{\infty} |\{x \in B_{5\rho_{x_i}}(x_i) : |Vu_{\lambda}| > 1\}|.$$

Thus the key is to estimate $|\{x \in B_{5\rho_{x_i}}(x_i) : |Vu_{\lambda}| > 1\}|$. Applying Lemma 3.2, (3.1) and (3.3) we deduce

$$\begin{aligned}
& \left| \left\{ x \in B_{5\rho_{x_{i}}}(x_{i}) : |Vu_{\lambda}| > 1 \right\} \right| \\
& \leq \int_{B_{5\rho_{x_{i}}}(x_{i})} |Vu_{\lambda}|^{p} dx \\
& \leq C \left\{ \int_{B_{10\rho_{x_{i}}}(x_{i})} |f_{\lambda}|^{p} dx + \int_{B_{10\rho_{x_{i}}}(x_{i})} |u_{\lambda}|^{p} dx \right\} \\
& \leq \varepsilon^{p} C(p,n) \left| B_{\rho_{x_{i}}}(x_{i}) \right| \\
& \leq C(p,n) \left\{ \varepsilon^{p} \int_{\left\{ x \in B_{\rho_{x_{i}}}(x_{i}) : |Vu_{\lambda}| > \frac{1}{3} \right\}} |Vu_{\lambda}|^{p} dx + \int_{\left\{ x \in B_{\rho_{x_{i}}}(x_{i}) : |f_{\lambda}| > \frac{\varepsilon}{3} \right\}} |f_{\lambda}|^{p} dx \\
& + \int_{\left\{ x \in B_{\rho_{x_{i}}}(x_{i}) : |u_{\lambda}| > \frac{\varepsilon}{3} \right\}} |u_{\lambda}|^{p} dx \right\}.
\end{aligned}$$

Set $\tilde{\lambda} = \lambda_0 \lambda$ and observe that

$$\int_{\mathbb{R}^{n}} \phi(|Vu|) dx = \int_{0}^{\infty} \left| \left\{ x \in \mathbb{R}^{n} : |Vu| > \tilde{\lambda} \right\} \right| d[\phi(\tilde{\lambda})]$$

$$\leq C(p, n) \varepsilon^{p} \int_{0}^{\infty} \tilde{\lambda}^{-p} \left\{ \int_{\left\{ x \in \mathbb{R}^{n} : |Vu| > \tilde{\lambda}/3 \right\}} |Vu|^{p} dx \right\} d[\phi(\tilde{\lambda})]$$

$$+ C(p, n) \int_{0}^{\infty} \tilde{\lambda}^{-p} \left\{ \int_{\left\{ x \in \mathbb{R}^{n} : |u| > \varepsilon \tilde{\lambda}/3 \right\}} |f|^{p} dx \right\} d[\phi(\tilde{\lambda})]$$

$$+ C(p, n) \int_{0}^{\infty} \tilde{\lambda}^{-p} \left\{ \int_{\left\{ x \in \mathbb{R}^{n} : |u| > \varepsilon \tilde{\lambda}/3 \right\}} |u|^{p} dx \right\} d[\phi(\tilde{\lambda})]$$

$$=: C(p, n) (\varepsilon^{p} I_{1} + I_{2} + I_{3}).$$

By Fubini's theorem, integration by parts and (2.4), it implies that

$$I_{1} = \int_{\mathbb{R}^{n}} |Vu|^{p} \left\{ \int_{0}^{3|Vu|} \frac{d\phi(\tilde{\lambda})}{\tilde{\lambda}^{p}} \right\} dx$$

$$= \frac{1}{3^{p}} \int_{\mathbb{R}^{n}} \phi(3|Vu|) dx + p \int_{\mathbb{R}^{n}} |Vu|^{p} \left\{ \int_{0}^{3|Vu|} \frac{\phi(\tilde{\lambda})}{\tilde{\lambda}^{p+1}} d\tilde{\lambda} \right\} dx$$

$$\leq \frac{1}{3^{p}} \int_{\mathbb{R}^{n}} \phi(3|Vu|) dx + \frac{2ap}{3^{p}(\alpha_{2} - p)} \int_{\mathbb{R}^{n}} \phi(3|Vu|) dx$$

$$\leq C(n, p, a, K) \int_{\mathbb{R}^{n}} \phi(|Vu|) dx.$$

Similarly,

$$I_2 \le C(n, p, a, K) \varepsilon^{p-\alpha_1} \int_{\mathbb{R}^n} \phi(|f|) dx$$

and

$$I_3 \le C(n, p, a, K) \varepsilon^{p-\alpha_1} \int_{\mathbb{R}^n} \phi(|u|) dx.$$

Therefore,

$$\int_{\mathbb{R}^n} \phi(|Vu|) dx \le C \left\{ \varepsilon^p \int_{\mathbb{R}^n} \phi(|Vu|) dx + \varepsilon^{p-\alpha_1} \int_{\mathbb{R}^n} \phi(|f|) dx + \varepsilon^{p-\alpha_1} \int_{\mathbb{R}^n} \phi(|u|) dx \right\}.$$

Choosing a suitable ε such that $C(n, p, a, K)\varepsilon^p < \frac{1}{2}$, (3.4) is obtained.

Next, taking into account [16, Theorem 2.8], the convexity of ϕ , (2.2) and (3.4), we have

$$\int_{\mathbb{R}^{n}} \phi\left(\left|D^{2}u\right|\right) dx \leq C \int_{\mathbb{R}^{n}} \phi\left(\left|f-Vu\right|\right) dx$$

$$\leq \frac{C}{2} \int_{\mathbb{R}^{n}} \phi\left(\left|2f\right|\right) dx + \frac{C}{2} \int_{\mathbb{R}^{n}} \phi\left(\left|2Vu\right|\right) dx$$

$$\leq \frac{KC}{2} \int_{\mathbb{R}^{n}} \phi\left(\left|f\right|\right) dx + \frac{KC}{2} \int_{\mathbb{R}^{n}} \phi\left(\left|Vu\right|\right) dx$$

$$\leq C \left\{ \int_{\mathbb{R}^{n}} \phi\left(\left|f\right|\right) dx + \int_{\mathbb{R}^{n}} \phi\left(\left|u\right|\right) dx \right\}. \tag{3.5}$$

Thus, (3.5) implies (1.4). The proof is finished. \square

4 Proof of Theorem 1.1

By the technique of "S. Agmon's idea" (see [3], p. 124) and Theorem 1.2, we first prove the following lemma.

Lemma 4.1 Under the same assumptions on ϕ , a_{ij} , V, q, f as in Theorem 1.1, let $u \in C_0^{\infty}(B_{R_0/2})$ satisfy the following equation

$$Lu - \mu u = f, \ x \in \mathbb{R}^n.$$

Then for any $\mu \gg 1$ large enough,

$$\mu^{\alpha_2} \int_{\mathbb{R}^n} \phi(|u|) dx + \mu^{\alpha_2/2} \int_{\mathbb{R}^n} \phi(|Du|) dx + \int_{\mathbb{R}^n} \phi(|Vu|) dx + \int_{\mathbb{R}^n} \phi(|D^2u|) dx$$

$$\leq C \int_{\mathbb{R}^n} \phi(|Lu - \mu u|) dx = C \int_{\mathbb{R}^n} \phi(|f|) dx, \tag{4.1}$$

where the constant C is independent of μ , and R_0 , α_2 are the constants in the proofs of Theorem 1.2 and (2.4), respectively.

Proof Let $\xi \in C_0^{\infty}(-R_0/2, R_0/2)$ be a cutoff function (not identically zero) and set

$$\tilde{u}(z) = \tilde{u}(x,t) = \xi(t)\cos(\sqrt{\mu}t)u(x) \tag{4.2}$$

and

$$\tilde{L}\tilde{u}(z) = L\tilde{u} + \tilde{u}_{tt},\tag{4.3}$$

where $\mu \geq 1$ will be chosen later, then $\tilde{u}(z) \in C_0^{\infty}(B_{R_0/2} \times (-R_0/2, R_0/2))$. It is easy to verify that the coefficients matrix

$$\left(\begin{array}{cc} (a_{ij})_{n\times n} & 0\\ 0 & 1 \end{array}\right)$$

of the operator \tilde{L} still satisfies the assumptions (H_1) and (H_2) . Furthermore, in view of (4.2) and (4.3) we find that

$$\tilde{L}\tilde{u}(z) = \tilde{f}(z), \tag{4.4}$$

where

$$\tilde{f}(z) = \xi(t)\cos(\sqrt{\mu}t)(Lu - \mu u) + (\xi''(t)\cos(\sqrt{\mu}t) - 2\sqrt{\mu}\xi'(t)\sin(\sqrt{\mu}t))u. \quad (4.5)$$

For the sake of convenience, we use the following notation

$$D_{zz}^{2}\tilde{u}(z) = \{D_{xx}^{2}\tilde{u}(z), \tilde{u}_{xt}(z), \tilde{u}_{tt}(z)\},\$$

where

$$D_{xx}^2 \tilde{u}(z) = \{\tilde{u}_{x_i x_j}\}_{i,j=1}^n \text{ and } \tilde{u}_{xt} = \{\tilde{u}_{x_i t}\}_{i=1}^n.$$

Applying Theorem 1.2 to (4.4),

$$\int_{\mathbb{R}^{n+1}} \phi\left(\left|D_{zz}^{2}\tilde{u}\right|\right) dx dt + \int_{\mathbb{R}^{n+1}} \phi\left(\left|V\tilde{u}\right|\right) dx dt
\leq C \left\{ \int_{\mathbb{R}^{n+1}} \phi\left(\left|\tilde{f}\right|\right) dx dt + \int_{\mathbb{R}^{n+1}} \phi\left(\left|\tilde{u}\right|\right) dx dt \right\}.$$
(4.6)

If $|\xi(t)\cos(\sqrt{\mu}t)| > 0$, by (2.4) we have

$$\begin{split} \phi\left(\left|D^2 u(x)\right|\right) &= \phi\left(\left|\left(\xi(t)\cos(\sqrt{\mu}t)\right)^{-1}\xi(t)\cos(\sqrt{\mu}t)D^2 u(x)\right|\right) \\ &\leq K|\xi(t)\cos(\sqrt{\mu}t)|^{-\alpha_1}\phi\left(\left|\xi(t)\cos(\sqrt{\mu}t)D^2 u(x)\right|\right). \end{split}$$

This and (4.2) yield

$$\int_{\mathbb{R}^{n}} \phi\left(\left|D^{2}u(x)\right|\right) dx$$

$$= \left(\int_{\mathbb{R}} K^{-1} |\xi(t) \cos(\sqrt{\mu}t)|^{\alpha_{1}} dt\right)^{-1}$$

$$\times \int_{\mathbb{R}^{n+1}} K^{-1} |\xi(t) \cos(\sqrt{\mu}t)|^{\alpha_{1}} \phi\left(\left|D^{2}u(x)\right|\right) dx dt$$

$$\leq C \int_{\mathbb{R}^{n+1}} K^{-1} |\xi(t) \cos(\sqrt{\mu}t)|^{\alpha_{1}} \phi\left(\left|D^{2}u(x)\right|\right) dx dt$$

$$= C \int_{\{(x,t)\in\mathbb{R}^{n+1}||\xi(t)\cos(\sqrt{\mu}t)|>0\}} K^{-1} |\xi(t)\cos(\sqrt{\mu}t)|^{\alpha_{1}} \phi\left(\left|D^{2}u(x)\right|\right) dx dt$$

$$\leq C \int_{\mathbb{R}^{n+1}} \phi\left(\left|D^{2}_{xx}\tilde{u}(z)\right|\right) dx dt$$

$$\leq C \int_{\mathbb{R}^{n+1}} \phi\left(\left|D^{2}_{zx}\tilde{u}(z)\right|\right) dx dt.$$

$$(4.7)$$

Similarly to (4.7) we get

$$\int_{\mathbb{R}^n} \phi(|Vu|) dx \le C \int_{\mathbb{R}^{n+1}} \phi(|V\tilde{u}(z)|) dx dt. \tag{4.8}$$

Using (2.4),

$$\phi\left(|Du(x)|\right) \le K|\xi(t)\sin(\sqrt{\mu}t)|^{-\alpha_1}\phi\left(|\xi(t)\sin(\sqrt{\mu}t)Du|\right).$$

Thus,

$$\int_{\mathbb{R}^n} \phi\left(|Du(x)|\right) dx \le C \int_{\mathbb{R}^{n+1}} \phi\left(|\xi(t)\sin(\sqrt{\mu}t)Du|\right) dx dt$$

$$\le C \sum_{i=1}^n \int_{\mathbb{R}^{n+1}} \phi\left(\mu^{-1/2} |\xi'(t)\cos(\sqrt{\mu}t)u_{x_i} - \tilde{u}_{x_it}|\right) dx dt$$

$$\le C \mu^{-\alpha_2/2} \left(\int_{\mathbb{R}^n} \phi\left(|Du|\right) dx + \int_{\mathbb{R}^{n+1}} \phi\left(|\tilde{u}_{xt}|\right) dx dt\right).$$

By choosing $\mu \gg 1$ large enough, we obtain the following

$$\mu^{\alpha_2/2} \int_{\mathbb{R}^n} \phi(|Du(x)|) dx \leq C \int_{\mathbb{R}^{n+1}} \phi(|\tilde{u}_{xt}(z)|) dx dt$$

$$\leq C \int_{\mathbb{R}^{n+1}} \phi(|D_{zz}^2 \tilde{u}(z)|) dx dt. \tag{4.9}$$

Since

$$-\mu \xi(t) \cos(\sqrt{\mu}t) u(x) = \tilde{u}_{tt}(z) - (\xi''(t) \cos(\sqrt{\mu}t) - 2\sqrt{\mu}\xi'(t) \sin(\sqrt{\mu}t)) u(x),$$

EJQTDE, 2013 No. 78, p. 11

we get

$$\mu^{\alpha_2} \int_{\mathbb{R}^n} \phi(|u(x)|) dx \leq C \int_{\mathbb{R}^{n+1}} \phi(|\tilde{u}_{tt}(z)|) dx dt$$

$$\leq C \int_{\mathbb{R}^{n+1}} \phi(|D_{zz}^2 \tilde{u}(z)|) dx dt. \tag{4.10}$$

Combining (4.5)–(4.10) and noting that

$$-\sqrt{\mu}\xi'(t)\sin(\sqrt{\mu}t)u(x) = \left(\left(\xi'(t)\cos(\sqrt{\mu}t)\right)_t - \xi''(t)\cos(\sqrt{\mu}t)\right)u(x),$$

we immediately find that

$$\mu^{\alpha_{2}} \int_{\mathbb{R}^{n}} \phi(|u|) dx + \mu^{\alpha_{2}/2} \int_{\mathbb{R}^{n}} \phi(|Du|) dx + \int_{\mathbb{R}^{n}} \phi(|Vu|) dx + \int_{\mathbb{R}^{n}} \phi(|D^{2}u|) dx$$

$$\leq C \left\{ \int_{\mathbb{R}^{n+1}} \phi(|D^{2}_{zz}\tilde{u}|) dx dt + \int_{\mathbb{R}^{n+1}} \phi(|V\tilde{u}|) dx dt \right\}$$

$$\leq C \left\{ \int_{\mathbb{R}^{n+1}} \phi(|\tilde{f}|) dx dt + \int_{\mathbb{R}^{n+1}} \phi(|\tilde{u}|) dx dt \right\}$$

$$\leq C \left(\int_{\mathbb{R}^{n}} \phi(|Lu - \mu u|) dx + \int_{\mathbb{R}^{n}} \phi(|u|) dx \right).$$

The desired estimate (4.1) follows by taking $\mu \gg 1$ large enough. The lemma is proved. \square

Furthermore, we shall show that the assumption $C_0^{\infty}(B_{R_0/2})$ can be removed. A covering lemma in a locally invariant quasimetric space was proved by Bramanti et al. in [5]. Since the Euclidean space \mathbb{R}^n is a special locally invariant quasimetric space, the covering lemma also holds in \mathbb{R}^n . For the convenience to readers, we describe it as follows.

Lemma 4.2 For given R_0 and any $\kappa > 1$, there exist $R_1 \in (0, R_0/2)$, a positive integer M and a sequence of points $\{x_i\}_{i=1}^{\infty} \subset \mathbb{R}^n$ such that

$$\mathbb{R}^n = \bigcup_{i=1}^{\infty} B_{R_1}(x_i);$$

$$\sum_{i=1}^{\infty} \chi_{B_{\kappa R_1}(x_i)}(y) \le M \quad \text{for any} \quad y \in \mathbb{R}^n,$$

where $\chi_{B_{\kappa R_1}(x_i)}(y)$ is the characteristic function of $B_{\kappa R_1}(x_i)$, that is, the function equal to 1 in $B_{\kappa R_1}(x_i)$ and 0 in $\mathbb{R}^n \backslash B_{\kappa R_1}(x_i)$.

Proof of Theorem 1.1. Let $\rho(x)$ be a cutoff function on $B_{R_0/2}$ relative to B_{R_1} , namely, $\rho(x) \in C_0^{\infty}(B_{R_0/2})$, $0 \le \rho(x) \le 1$ and $\rho(x) \equiv 1$ on B_{R_1} , where R_1 is as in Lemma 4.2. For any fixed $x_0 \in \mathbb{R}^n$, we set

$$u^{0}(x) = u(x)\rho(x - x_{0}) =: u(x)\rho^{0}(x)$$
(4.11)

and observe that

$$Lu^{0}(x) - \mu u^{0}(x) = f\rho^{0} - 2a_{ij}u_{x_{i}}\rho_{x_{i}}^{0} - a_{ij}u\rho_{x_{i}x_{i}}^{0} =: f^{0}.$$

By Definition 2.9, there exists a sequence $\{u_k\}$ of functions in $C_0^{\infty}(\mathbb{R}^n)$ such that

$$||u_k - u||_{W^2L^{\phi}(\mathbb{R}^n)} + ||Vu_k - Vu||_{L^{\phi}(\mathbb{R}^n)} \to 0, \text{ as } k \to \infty.$$
 (4.12)

It follows from Remark 2.7 that

$$\int_{\mathbb{R}^n} \phi(|u_k - u|) dx + \int_{\mathbb{R}^n} \phi(|D(u_k - u)|) dx + \int_{\mathbb{R}^n} \phi(|D^2(u_k - u)|) dx + \int_{\mathbb{R}^n} \phi(V|u_k - u|) dx \to 0, \text{ as } k \to \infty.$$

$$(4.13)$$

Let $u_k^0 = u_k \rho^0$. Then using the properties of ρ , the monotonicity, convexity of ϕ , (4.13), (2.4) and Remark 2.7, we obtain

$$\|u_k^0 - u^0\|_{W^2L^{\phi}(\mathbb{R}^n)} + \|Vu_k^0 - Vu^0\|_{L^{\phi}(\mathbb{R}^n)} \to 0$$
, as $k \to \infty$. (4.14)

Set

$$f_k = Lu_k - \mu u_k$$
 and $f_k^0 = Lu_k^0 - \mu u_k^0$.

It follows by (H_1) and (4.12) that

$$||f_k^0 - f^0||_{L^{\phi}(\mathbb{R}^n)} \le ||Lu_k^0 - Lu^0||_{L^{\phi}(\mathbb{R}^n)} + \mu ||u_k^0 - u^0||_{L^{\phi}(\mathbb{R}^n)} \to 0, \ as \ k \to \infty.$$
(4.15)

Hence, by (4.14), (4.15), Lemma 4.1 and Remark 2.7 we have

$$\mu^{\alpha_{2}} \int_{\mathbb{R}^{n}} \phi(|u^{0}|) dx + \mu^{\alpha_{2}/2} \int_{\mathbb{R}^{n}} \phi(|Du^{0}|) dx + \int_{\mathbb{R}^{n}} \phi(|Vu^{0}|) dx + \int_{\mathbb{R}^{n}} \phi(|D^{2}u^{0}|) dx$$

$$+ \int_{\mathbb{R}^{n}} \phi(|D^{2}u^{0}|) dx$$

$$\leq C \int_{\mathbb{R}^{n}} \phi(|f^{0}|) dx$$

$$\leq C \left\{ \int_{B_{R_{0}/2}(x_{0})} \phi(|f|) dx + \int_{B_{R_{0}/2}(x_{0})} \phi(|u|) dx + \int_{B_{R_{0}/2}(x_{0})} \phi(|Du|) dx \right\}. (4.16)$$

EJQTDE, 2013 No. 78, p. 13

Note that (4.11) and (2.4) yield

$$\int_{\mathbb{R}^n} \phi\left(\left|\rho^0 D u\right|\right) dx \le C \left\{ \int_{\mathbb{R}^n} \phi\left(\left|D u^0\right|\right) dx + \int_{\mathbb{R}^n} \phi\left(\left|u D \rho^0\right|\right) dx \right\}$$
(4.17)

and

$$\int_{\mathbb{R}^{n}} \phi\left(\left|\rho^{0} D^{2} u\right|\right) dx \leq C \left\{ \int_{\mathbb{R}^{n}} \phi\left(\left|D^{2} u^{0}\right|\right) dx + \int_{\mathbb{R}^{n}} \phi\left(\left|u D^{2} \rho^{0}\right|\right) dx + \int_{\mathbb{R}^{n}} \phi\left(\left|D u \cdot D \rho^{0}\right|\right) dx \right\}.$$

$$+ \int_{\mathbb{R}^{n}} \phi\left(\left|D u \cdot D \rho^{0}\right|\right) dx \right\}.$$
(4.18)

Then combining (4.16), (4.17) and (4.18) implies that

$$\mu^{\alpha_{2}} \int_{B_{R_{0}/2}(x_{0})} \phi\left(\left|\rho^{0} u\right|\right) dx + \mu^{\alpha_{2}/2} \int_{B_{R_{0}/2}(x_{0})} \phi\left(\left|\rho^{0} D u\right|\right) dx$$

$$+ \int_{B_{R_{0}/2}(x_{0})} \phi\left(\left|\rho^{0} V u\right|\right) dx + \int_{B_{R_{0}/2}(x_{0})} \phi\left(\left|\rho^{0} D^{2} u\right|\right) dx$$

$$\leq C \left\{ \int_{B_{R_{0}/2}(x_{0})} \phi\left(\left|f\right|\right) dx + \mu^{\alpha_{2}/2} \int_{B_{R_{0}/2}(x_{0})} \phi\left(\left|u\right|\right) dx + \int_{B_{R_{0}/2}(x_{0})} \phi\left(\left|D u\right|\right) dx \right\}.$$

Therefore, by the above inequality and Lemma 4.2 we deduce that

$$\mu^{\alpha_{2}} \int_{\mathbb{R}^{n}} \phi(|u|) dx + \mu^{\alpha_{2}/2} \int_{\mathbb{R}^{n}} \phi(|Du|) dx + \int_{\mathbb{R}^{n}} \phi(|Vu|) dx + \int_{\mathbb{R}^{n}} \phi(|D^{2}u|) dx$$

$$\leq \sum_{i=1}^{\infty} \left\{ \mu^{\alpha_{2}} \int_{B_{R_{1}}(x_{i})} \phi(|\rho^{0}u|) dx + \mu^{\alpha_{2}/2} \int_{B_{R_{1}}(x_{i})} \phi(|\rho^{0}Du|) dx + \int_{B_{R_{1}}(x_{i})} \phi(|\rho^{0}D^{2}u|) dx \right\}$$

$$\leq C \sum_{i=1}^{\infty} \left\{ \int_{B_{R_{0}/2}(x_{i})} \phi(|f|) dx + \mu^{\alpha_{2}/2} \int_{B_{R_{0}/2}(x_{i})} \phi(|u|) dx + \int_{B_{R_{0}/2}(x_{i})} \phi(|Du|) dx \right\}$$

$$\leq C \left\{ \int_{\mathbb{R}^{n}} \phi(|f|) dx + \mu^{\alpha_{2}/2} \int_{\mathbb{R}^{n}} \phi(|u|) dx + \int_{\mathbb{R}^{n}} \phi(|Du|) dx \right\}.$$

(1.3) is obtained by taking $\mu \gg 1$ large enough. The theorem is proved. \square

Acknowledgments. The author thanks the anonymous referee for offering valuable suggestions which have improved the presentation.

References

- [1] Acerbi E., Mingione G., Gradient estimates for a class of parabolic systems, Duke Math. J., 136 (2007) 285–320.
- [2] Adams R. A., Fournier J. J. F., Sobolev spaces, Academic Press, New York, 2003.
- [3] Agmon S., On the eigenfunctions and on the eigenvalues of general elliptic boundary value problems, Comm. Pure Appl. Math., 15 (1962) 119–147.
- [4] Bramanti M., Brandolini L., Harboure E., Viviani B., Global $W^{2,p}$ estimates for nondivergence elliptic operators with potentials satisfying a reverse Hölder condition, Annali di Mathematica, 191 (2012) 339–362.
- [5] Bramanti M., Cupini G., Lanconelli E., Priola E., Global L^p estimates for degenerate Ornstein-Uhlenbeck operators, Math. Z., 266 (2010) 789–816.
- [6] Gilbarg D., Trudinger N. S., Elliptic Partial Differential Equations of Second Order, Springer Series "Classics in Mathematics", 1976.
- [7] Jia H., Li D., Wang L., Regularity theory in Orlicz spaces for the Poissson equation, Manuscripta Math., 122 (2007) 265–275.
- [8] Rao M., Ren Z., Applications of Orlicz spaces, Marcel Dekker Inc., New York, 2000.
- [9] Shen Z., On the Neumann problem for Schrödinger operators in Lipschitz domains, Indiana Univ. Math. J., 43 (1994) 143–176.
- [10] Shen Z., L^p estimates for Schrödinger operators with certain potentials, Ann. Inst. Fourier (Grenoble), 45 (1995) 513–546.
- [11] Thangavelu S., Riesz transforms and the wave equation for the Hermite operator. Comm. Partial Differ. Equ., 15 (1990) 1199–1215.
- [12] Vitanza C., A new contribution to the $W^{2,p}$ regularity for a class of elliptic second order equations with discontinuous coefficients. Le Matematiche, 48 (1993) 287–296.
- [13] Wang L., Yao F., Zhou S., Jia H., Optimal regularity for the poisson equation, Proc. Amer. Math. Soc., 137 (2009) 2037–2047.

- [14] Yao F., Optimal regularity for Schrödinger equations, Nonlinear Anal., 71 (2009) 5144–5150.
- [15] Yao F., Regularity theory for the uniformly elliptic operators in Orlicz spaces, Computers Math. Applic., 60 (2010) 3098–3104.
- [16] Yao F., Second order elliptic equations of nondivergence form with small BMO coefficients in \mathbb{R}^n , Potential Anal., 36 (2012) 557–568.

(Received June 27, 2013)