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Note on oscillation conditions for first-order delay differential equations

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Abstract. We consider explicit conditions for all solutions to linear scalar differential equations with several variable delays to be oscillatory. The considered conditions have the form of inequalities bounding the upper limit of the sum of integrals of coefficients over a subset of the real semiaxis, by the constant 1 from below. The main result is a new oscillation condition, which sharpens several known conditions of the kind. Some results are presented in the form of counterexamples.

Keywords: differential equations, delay, oscillation, sufficient conditions.

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1 Introduction

It follows from results by Ladas et al. [7] and Tramov [12] that all solutions of the equation

$$\dot{x}(t) + a(t)x(t-\tau) = 0, \quad t \geq 0, \tag{1.1}$$

where $a(t) \geq 0$ and $\tau = \text{const} > 0$, are oscillatory in case $\limsup_{t\to+\infty} \int_{t-\tau}^{t} a(s)\,ds > 1$.

For an equation with variable delay, Corollary 2.1 from [7] presents the following oscillation condition. Suppose $a \in C(\mathbb{R}_+, \mathbb{R}_+)$, $h \in C^1(\mathbb{R}_+, \mathbb{R}_+)$, $h(t) \leq t$ and $h'(t) \geq 0$ for all $t \in \mathbb{R}_+$, $\lim_{t\to\infty} h(t) = \infty$, and $\limsup_{t\to\infty} \int_{h(t)}^{t} a(s)\,ds > 1$. Then all solutions of the equation

$$\dot{x}(t) + a(t)x(h(t)) = 0, \quad t \geq 0, \tag{1.2}$$

are oscillatory. This result is extended and sharpened in many publications. In almost all of them the condition is imposed that the delay function h is nondecreasing.

The present paper is devoted to conditions for all solution of the equation

$$\dot{x}(t) + \sum_{k=1}^{m} a_k(t)x(h_k(t)) = 0, \quad t \geq 0, \tag{1.3}$$

where $a_k(t) \geq 0$, $h_k(t) \leq t$, and $h_k(t) \to \infty$ as $t \to \infty$, to be oscillatory. All new obtained oscillation conditions are generalizations of the results formulated above. We do not suppose

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that the functions h_k are necessarily nondecreasing and accompany the obtained results by a number of counterexamples in order to compare the new oscillation conditions with known ones.

In Section 2 we discuss published results concerning oscillation conditions of the considered kind. In Section 3 our main result is obtained, and it is shown that known results are its corollaries. In Section 4 equation (1.2) is discussed. In Section 5 some ideas from the previous section are extended to the case of equation (1.3). Some results in the last three sections are represented in the form of counterexamples.

2 Known oscillation conditions

Theorem 2.1.3 from the book [9] by Ladde et al. represents an oscillation condition for (1.2) that sharpens slightly the cited result from [7], as it is supposed that $h \in C(\mathbb{R}_+, \mathbb{R}_+)$, and the nonnegativity of h' is replaced by the nondecrease of h.

This result is extended to the case of equation (1.3) in Theorem 3.4.3 from the book [5] by Győri and Ladas. The basic oscillatory condition in the theorem is the inequality

$$\limsup_{t\to\infty} \int_{\max_k h_k(t)}^{t} \sum_{k=1}^{m} a_k(s)\, ds > 1.$$

It is not stated explicitly that the functions h_k are supposed to be nondecreasing, however, the authors did not mention anything to replace this condition. It is shown in Section 4 of this paper that the nondecrease is actually essential.

In [1, p. 36], there is an example showing that the inequality

$$\limsup_{t\to\infty} \int_{\min_k h_k(t)}^{t} \sum_{k=1}^{m} a_k(s)\, ds \le 1,$$

in contrast to that containing max in place of min, is not necessary for a nonoscillating solution to exist. In Section 3 of the present work we sharpen this result.

Tang [11] obtained an oscillation condition for the case of several constant delays

$$\dot{x}(t) + \sum_{k=1}^{m} a_k(t) x(t - \tau_k) = 0, \tag{2.1}$$

which is not a consequence of the above conditions for (1.3). The basic inequality

$$\limsup_{t\to\infty} \sum_{k=1}^{m} \int_{t}^{t+\tau_k} a_k(s)\, ds > 1$$

is derived from an oscillation condition obtained for an equation with distributed delay. It is shown in Section 3 that the above inequality cannot be replaced by

$$\limsup_{t\to\infty} \sum_{k=1}^{m} \int_{t-\tau_k}^{t} a_k(s)\, ds > 1.$$

There are few published extensions of the considered oscillation conditions for the case of nondecreasing delay. The following result is by Tramov [12]. If $a(t) \ge 0$, $t - h(t) \ge h_0 > 0$, $\lim_{t\to\infty} h(t) = \infty$, and

$$\limsup_{t\to\infty} \int_{t}^{t+h_0} a(s)\, ds > 1,$$

then every solution of (1.2) oscillates. In [12] the author also presented an example showing the sharpness of the constant 1: if it is diminished by arbitrary $\varepsilon > 0$, then the condition does not guarantee oscillation.

Koplatadze and Kvinikadze [6] obtained another oscillation condition for the case of non-monotone delay. Suppose $a(t) \geq 0$, $h \in C(\mathbb{R}_+, \mathbb{R}_+)$, $h(t) \leq t$, and $\lim_{t\to\infty} h(t) = \infty$. Define $\delta(t) = \max\{h(s) \mid s \in [0, t]\}$. Then the inequality

$$\limsup_{t\to\infty} \int_{\delta(t)}^{t} a(s)\, ds > 1$$

is sufficient for all solutions of (1.2) to be oscillatory.

Note that the nature of the considered oscillation conditions differs from that of the oscillation conditions of $1/e$-type. This is expressed, in particular, in the possibility to extend the above oscillation condition to equations with oscillating coefficients. Such extension was apparently first made by Ladas at al. [8], their results sharpened by Fukagai and Kusano [4]. Below we do not consider $1/e$-type theorems and the problem of 'filling the gap' between $1/e$ and 1. A detailed discussion of this subject is found in the monographs [1–3] and the review [10].

3 Main result

Let parameters of equation (1.3) satisfy the following conditions for all $k = 1, \ldots, m$:

- the functions $a_k\colon \mathbb{R}_+ \to \mathbb{R}$ are locally integrable;
- the functions $h_k\colon \mathbb{R}_+ \to \mathbb{R}$ are Lebesgue measurable;
- $a_k(t) \geq 0$ and $h_k(t) \leq t$ for all $t \in \mathbb{R}_+$.

We say that a locally absolutely continuous function $x\colon \mathbb{R}_+ \to \mathbb{R}$ is a *solution* to the equation

$$\dot{x}(t) + \sum_{k=1}^{m} a_k(t) x(h_k(t)) = 0, \quad t \geq 0, \tag{1.3}$$

if there exists a Borel *initial function* $\varphi\colon (-\infty, 0] \to \mathbb{R}$ such that the equality (1.3) takes place for almost all $t \geq 0$, where $x(\xi) = \varphi(\xi)$ for all $\xi \leq 0$.

Let us define a family of sets

$$E_k(t) = \{s \mid h_k(s) \leq t \leq s\}, \quad t \geq 0, \quad k = 1, \ldots, m.$$

It follows from the stated above that all the sets of the family are Lebesgue measurable.

Theorem 3.1. *Suppose* $\lim_{t\to\infty} h_k(t) = \infty$ *for all* $k = 1, \ldots, m$*, and*

$$\limsup_{t\to\infty} \sum_{k=1}^{m} \int_{E_k(t)} a_k(s)\, ds > 1.$$

Then every solution of equation (1.3) *is oscillatory.*

Proof. Suppose the conditions of the theorem are fulfilled and consider an arbitrary solution x of equation (1.3).

Assume that x is not oscillatory. Without loss of generality, suppose that there exists $t_0 \geq 0$ such that $x(t) > 0$ for all $t \geq t_0$. Then there exists $t_1 \geq t_0$ such that $h_k(t) \geq t_0$ for all $t \geq t_1$ and $k = 1, \dots, m$. It is obvious that $x(t)$ is nonincreasing for all $t \geq t_1$. Further, there exists $t_2 \geq t_1$ such that $x(h_k(t)) \geq x(t)$ for all $t \geq t_2$ and $k = 1, \dots, m$, and $\sum_{k=1}^m \int_{E_k(t_2)} a_k(s)\, ds > 1$.

There also exists $t_3 > t_2$ such that for all the sets $S_k = E_k(t_2) \cap [t_2, t_3]$, $k = 1, \dots, m$, we have $\sum_{k=1}^m \int_{S_k} a_k(s)\, ds > 1$. Therefore,

$$\begin{aligned} x(t_3) &= x(t_2) + \int_{t_2}^{t_3} \dot{x}(s)\, ds = x(t_2) - \int_{t_2}^{t_3} \sum_{k=1}^{m} a_k(s) x(h_k(s))\, ds \\ &\leq x(t_2) - \int_{S_k} \sum_{k=1}^{m} a_k(s) x(h_k(s))\, ds \leq x(t_2) \left(1 - \sum_{k=1}^{m} \int_{S_k} a_k(s)\, ds\right) < 0, \end{aligned}$$

which contradicts the assumption. □

Corollary 3.2. *Suppose the functions h_k are continuous and strictly increasing, $\lim_{t\to\infty} h_k(t) = \infty$ for $k = 1, \dots, m$, and*

$$\limsup_{t\to\infty} \sum_{k=1}^{m} \int_{t}^{h_k^{-1}(t)} a_k(s)\, ds > 1. \tag{3.1}$$

Then every solution of equation (1.3) *is oscillatory.*

Proof. For each $k=1, \dots, m$ there exists the inverse function h_k^{-1}, which is defined on $[h_k(0), \infty)$ and is strictly increasing. Hence $E_k(t) = [t, h_k^{-1}(t)]$. □

Corollary 3.3 ([11])**.** *Suppose $h_k(t) = t - \tau_k$, where $\tau_k > 0$, and*

$$\limsup_{t\to\infty} \sum_{k=1}^{m} \int_{t}^{t+\tau_k} a_k(s)\, ds > 1. \tag{3.2}$$

Then every solution of equation (1.3) *is oscillatory.*

Proof. We have $h_k^{-1}(t) = t + \tau_k$ and $E_k(t) = [t, t + \tau_k]$. □

Corollary 3.4 ([5])**.** *Suppose the functions h_k is nondecreasing, $\lim_{t\to\infty} h_k(t) = \infty$ for $k = 1, \dots, m$, and*

$$\limsup_{t\to\infty} \int_{\max_k h_k(t)}^{t} \sum_{k=1}^{m} a_k(s)\, ds > 1. \tag{3.3}$$

Then every solution of equation (1.3) *is oscillatory.*

Proof. By virtue of the nondecrease of h_k we have that $[\max_k h_k(t), t] \subset E_k(\max_k h_k(t))$. Since $\lim_{t\to\infty} h_k(t) = \infty$, it follows from (3.3) that $\limsup_{t\to\infty} \sum_{k=1}^m \int_{E_k(t)} a_k(s)\, ds > 1$. □

The following example supplements Corollaries 3.2, 3.3 and 3.4.

Example 3.5. Consider the equation

$$\dot{x}(t) + a_1(t)x(t-3) + a_2(t)x(t-1) = 0, \quad t \geq 0, \tag{3.4}$$

where for $n = 0, 1, 2, \ldots$ we put

$$a_1(t) = \begin{cases} 0, & t \in [6n, 6n+3), \\ 3/4, & t \in [6n+3, 6n+4), \\ 0, & t \in [6n+4, 6(n+1)); \end{cases} \qquad a_2(t) = \begin{cases} 0, & t \in [6n, 6n+5), \\ 3/4, & t \in [6n+5, 6(n+1)). \end{cases}$$

We see that

$$\limsup_{t\to\infty} \left(\int_{t-3}^{t} a_1(s)\, ds + \int_{t-1}^{t} a_2(s)\, ds \right) = \int_{6n+3}^{6(n+1)} a_1(s)\, ds + \int_{6n+5}^{6(n+1)} a_2(s)\, ds = 3/2 > 1.$$

However, every solution x of equation (3.4) is nonincreasing on $\mathbb{R}_+$, and $x(6(n+1)) = x(6n)/16$, $n = 0, 1, 2, \ldots$, that is $x(t) > 0$ for all $t \geq 0$.

Example 3.5 shows that inequality (3.1) cannot be replaced by

$$\limsup_{t\to\infty} \sum_{k=1}^{m} \int_{h_k(t)}^{t} a_k(s) ds > 1. \tag{3.5}$$

In particular, this means that inequality (3.2) cannot be replaced by

$$\limsup_{t\to\infty} \sum_{k=1}^{m} \int_{t-\tau_k}^{t} a_k(s) ds > 1.$$

Inequality (3.3) also cannot be replaced by (3.5). This strengthens the result from [1, p. 36] cited in Section 2, since

$$\sum_{k=1}^{m} \int_{h_k(t)}^{t} a_k(s)\, ds \leq \int_{\min_k h_k(t)}^{t} \sum_{k=1}^{m} a_k(s)\, ds.$$

4 Equation with single delay

Consider the equation with single delay

$$\dot{x}(t) + a(t)x(h(t)) = 0, \quad t \geq 0, \tag*{(1.2),}$$

which is a special case of equation (1.3).

Define $E(t) = \{s \mid h(s) \leq t \leq s\}$.

By Theorem 2.1.3 from [9], if h is nondecreasing, $\lim_{t\to\infty} h(t) = \infty$ and

$$\limsup_{t\to\infty} \int_{h(t)}^{t} a(s)\, ds > 1,$$

then all solutions of (1.2) are oscillatory. The monotonicity of h is here essential. This fact can be shown by a very simple example in case the measure $\mu \left\{ t \mid \int_{E(t)} a(s)\, ds > 1 \right\} = 0$. The last is not assumed in the following example.

Example 4.1. Consider equation (1.2), where $a(t) \equiv \alpha > 1$. Put $\varepsilon \in (0, 1)$ and

$$h(t) = \begin{cases} t, & t \in [n, n+1-\varepsilon), \\ n, & t \in [n+1-\varepsilon, n+1), \end{cases}$$

for $n = 0, 1, 2, \dots$ Consider the solution of (1.2) determined by an initial value $x(0) = x_0 > 0$.

One may choose ε so that the solution is positive. Indeed, fix an arbitrary positive integer n and consider $x(t)$ for $t \in [n, n+1)$. We have

$$x(t) = \begin{cases} x(n)e^{-\alpha(t-n)}, & t \in [n, n+1-\varepsilon); \\ x(n)e^{-\alpha(1-\varepsilon)} - \alpha x(n)(t-(n+1-\varepsilon)), & t \in [n+1-\varepsilon, n+1). \end{cases} \tag{4.1}$$

Thus, $x(n+1) = x(n)(e^{-\alpha(1-\varepsilon)} - \alpha\varepsilon)$. To provide that $x(n)$ is positive for all n it is sufficient to choose ε so that $\varepsilon < (e^{-\alpha(1-\varepsilon)})/\alpha$. Obviously, for some $\varepsilon_0 > 0$ the inequality is valid for all $\varepsilon \in (0, \varepsilon_0)$. Further, it follows from (4.1) that $x(n+1) \le x(t) \le x(n)$ for $t \in (n, n+1)$, hence for the chosen ε we have $x(t) > 0$ for all $t \in \mathbb{R}_+$.

On the other hand, $\limsup_{t\to\infty} \int_{h(t)}^{t} a(s)\,ds = \int_n^{n+1} a(s)\,ds = \alpha > 1$.

It is obvious that Example 4.1 may be modified for the case that h is continuous.

Consider Theorem 3.1 for the case $m = 1$.

Corollary 4.2. *Suppose* $\lim_{t\to\infty} h(t) = \infty$ *and* $\limsup_{t\to\infty} \int_{E(t)} a(s)\,ds > 1$. *Then every solution of equation* (1.2) *is oscillatory.*

The function h is not supposed to be nondecreasing in Corollary 4.2. The following corollaries represent an idea that to prove that all solutions to equation (1.2) are oscillatory it may be sufficient to consider an auxiliary equation with nondecreasing delay. In particular, this allows to establish oscillation in case the function h is not defined precisely.

Corollary 4.3. *Let* $h_0 = 0$, $h_{n+1} > h_n$ *for* $n = 0, 1, 2, \dots$, *and* $\lim_{n\to\infty} h_n = \infty$. *Suppose* $h(t) \le h_n$ *for* $t \in [h_n, h_{n+1})$ *and*

$$\limsup_{n\to\infty} \int_{h_n}^{h_{n+1}} a(s)\,ds > 1.$$

Then every solution of (1.2) *is oscillatory.*

Proof. It is readily seen that for $n = 0, 1, 2 \dots$ and $t \in [h_n, h_{n+1})$ we have $[t, h_{n+1}) \subset E(t)$. Therefore,

$$\int_{h_n}^{h_{n+1}} a(s)\,ds \le \int_{E(h_n)} a(s)\,ds.$$

Hence $\limsup_{t\to\infty} \int_{E(t)} a(s)\,ds \ge \limsup_{n\to\infty} \int_{h_n}^{h_{n+1}} a(s)\,ds$.

It remains to apply Corollary 4.2. □

Corollary 4.4 ([6])**.** *Put* $g(t) = \sup\{h(s) \mid s < t\}$. *Suppose* $\lim_{t\to\infty} h(t) = \infty$ *and*

$$\limsup_{t\to\infty} \int_{g(t)}^{t} a(s)\,ds > 1.$$

Then every solution of (1.2) *is oscillatory.*

Proof. We have $[g(t), t) \subset E(g(t))$. Indeed, if $r \in [g(t), t)$, then

$$h(r) \le \sup\{h(s) \mid s < t\} = g(t),$$

and hence,

$$r \in \{s \ge g(t) \mid h(s) \le g(t)\} = E(g(t)).$$

Obviously, $g(t) \to \infty$ as $t \to \infty$, therefore,

$$\limsup_{t\to\infty} \int_{g(t)}^{t} a(s)\,ds \le \limsup_{t\to\infty} \int_{E(t)} a(s)\,ds.$$

It remains to apply Corollary 4.2. □

Corollary 4.5. *Put* $G(t) = \inf\{s \mid h(s) > t\}$. *Suppose* $\lim_{t\to\infty} h(t) = \infty$ *and*

$$\limsup_{t\to\infty} \int_{t}^{G(t)} a(s)\,ds > 1.$$

Then every solution of (1.2) *is oscillatory.*

Proof. It is not hard to see that $[t, G(t)) \subset E(t)$. Hence the result follows from Corollary 4.2. □

Note that both the functions g and G defined in Corollaries 4.4 and 4.5, respectively are nondecreasing. In Figure 4.1 the graphs of some delay h and the corresponding g and G are represented. The sections of the graph of $g(t)$, where it differs from that of $h(t)$, are coloured red. The set $E(T)$ is marked green in the axis Ot.

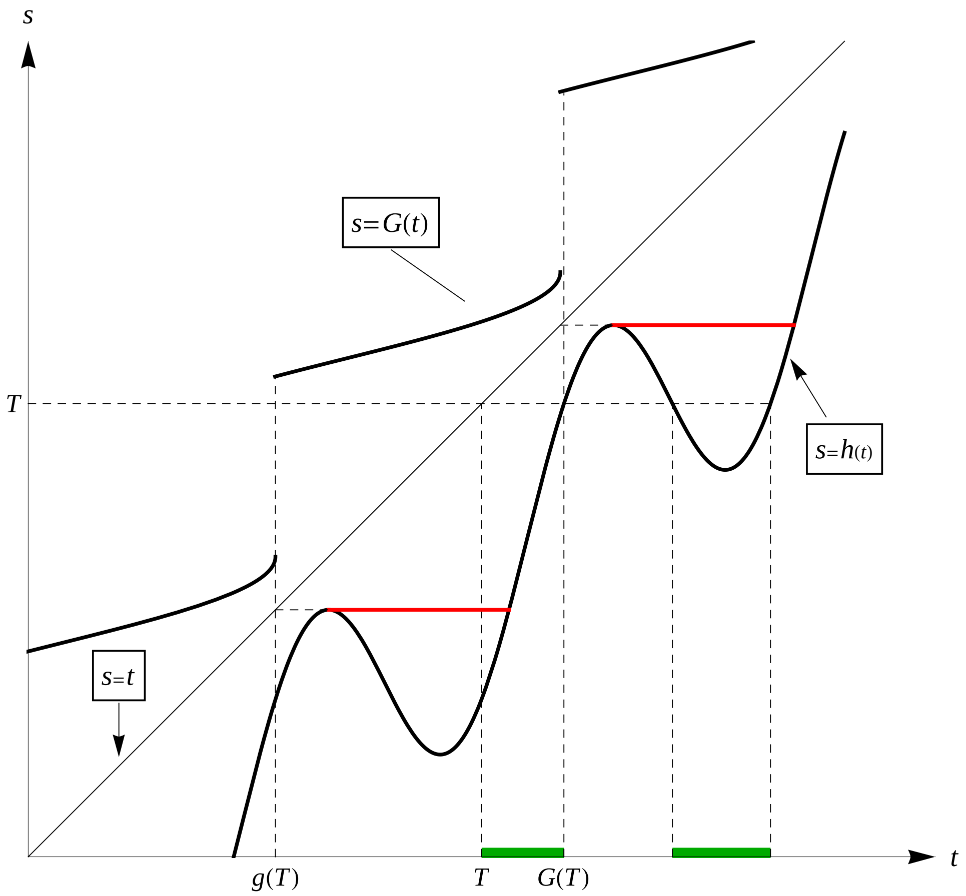

Figure 4.1: The graphs of the functions h, g and G, and the set $E(T)$.

Let us show that the oscillation conditions of Corollaries 4.4 and 4.5 are equipotent. Indeed,

$$G(g(t)) = \inf\{s \mid h(s) > \sup\{h(r) \mid r < t\}\} \ge t,$$

and since $g(t) \to \infty$ as $t \to \infty$, we have that

$$\limsup_{t\to\infty} \int_{g(t)}^{t} a(s)\,ds \le \limsup_{t\to\infty} \int_{g(t)}^{G(g(t))} a(s)\,ds \le \limsup_{t\to\infty} \int_{t}^{G(t)} a(s)\,ds.$$

On the other hand,

$$g(G(t)) = \sup\{h(s) \mid s < \inf\{r \mid h(r) > t\}\} \le t,$$

and $G(t) \to \infty$ as $t \to \infty$, hence,

$$\limsup_{t\to\infty} \int_{t}^{G(t)} a(s)\,ds \le \limsup_{t\to\infty} \int_{g(G(t))}^{G(t)} a(s)\,ds \le \limsup_{t\to\infty} \int_{g(t)}^{t} a(s)\,ds.$$

The application of Corollaries 4.4 and 4.5 is illustrated by the following example.

Example 4.6. Consider equation (1.2), where $a(t) \equiv \alpha > 0$. Suppose there exists a sequence $\{t_n\}_{n=1}^{\infty}$ such that $t_n \to \infty$ as $n \to \infty$ and $h(t) \le t_n$ for all $t \in [t_n, t_n + 1/\alpha]$.

We have $G(t_n) \ge t_n + 1/\alpha$. Hence, $\int_{t_n}^{G(t_n)} a(s)\,ds \ge \int_{t_n}^{t_n+1/\alpha} a(s)\,ds > 1$. By Corollary 4.5 every solution is oscillatory.

We also have $g(t_n + 1/\alpha) \le t_n$. Hence, $\int_{g(t_n+1/\alpha)}^{t_n+1/\alpha} a(s)\,ds > 1$, and by Corollary 4.4 every solution is oscillatory.

The next example shows that Corollaries 4.4 and 4.5 are weaker than Corollary 4.2.

Example 4.7. For $n = 0, 1, 2, \dots$ put in equation (1.2)

$$a(t) = \begin{cases} 1/4, & t \in [2n, 2n+1), \\ 2/3, & t \in [2n+1, 2n+2); \end{cases} \qquad h(t) = \begin{cases} 2n, & t \in [2n, 2n+1), \\ 2n-1, & t \in [2n+1, 2n+2). \end{cases}$$

We have $\limsup_{t\to\infty} \int_t^{G(t)} a(s)\,ds = \int_{2n}^{G(2n)} a(s)\,ds = 1/4 + 2/3 < 1$. Therefore, Corollary 4.5 (and Corollary 4.4 as well) does not allow to determine if there exists a nonoscillating solution.

In fact $E(2n+1) = [2n+1, 2n+2) \cup [2n+3, 2n+4)$,

$$\limsup_{t\to\infty} \int_{E(t)} a(s)\,ds = \int_{E(2n+1)} a(s)\,ds = 4/3 > 1,$$

and by Corollary 4.2 every solution is oscillatory.

5 Generalization

Below we extend Corollaries 4.4 and 4.5 to the case of equation (1.3).

For all $k = 1, \dots, m$ put $g_k(t) = \sup\{h_k(s) \mid s < t\}$ and $G_k(t) = \inf\{s \mid h_k(s) > t\}$.

Corollary 5.1. *Suppose* $\lim_{t\to\infty} h_k(t) = \infty$ *for* $k = 1, \dots, m$, *and*

$$\limsup_{t\to\infty} \sum_{k=1}^{m} \int_{t}^{G_k(t)} a_k(s)\,ds > 1. \tag{5.1}$$

Then every solution of equation (1.3) *is oscillatory.*

Proof. It is not hard to see that $[t, G_k(t)) \subset E_k(t)$. □

Corollary 5.2. *Suppose* $\lim_{t\to\infty} h_k(t) = \infty$ *for* $k = 1, \dots, m$, *and*

$$\limsup_{t\to\infty} \int_{\max_k g_k(t)}^{t} \sum_{k=1}^{m} a_k(s)\, ds > 1. \tag{5.2}$$

Then every solution of equation (1.3) *is oscillatory.*

Proof. Analogously to the case $m = 1$ considered in section 4, we have $G_k(g_k(t)) \ge t$. So,

$$\limsup_{t\to\infty} \sum_{k=1}^{m} \int_{t}^{G_k(t)} a_k(s)\, ds \ge \limsup_{t\to\infty} \sum_{k=1}^{m} \int_{g_k(t)}^{G_k(g_k(t))} a_k(s)\, ds \ge \limsup_{t\to\infty} \int_{\max_k g_k(t)}^{t} \sum_{k=1}^{m} a_k(s)\, ds.$$

Thus, Corollary 5.2 follows from Corollary 5.1. □

The following example shows that in case $m > 1$ Corollary 5.1 is sharper than Corollary 5.2.

Example 5.3. Consider the equation

$$\dot{x}(t) + \frac{1}{2}x(t-1) + \frac{1}{3}x(t-2) = 0, \quad t \ge 0. \tag{5.3}$$

We have $g_1(t) = t - 1$, $g_2(t) = t - 2$, $G_1(t) = t + 1$, $G_2(t) = t + 2$. Further,

$$\limsup_{t\to\infty} \int_{\max_k g_k(t)}^{t} \sum_{k=1}^{m} a_k(s)\, ds = \int_{t-1}^{t} (a_1(s) + a_2(s))\, ds = 1/2 + 1/3 < 1;$$

and

$$\limsup_{t\to\infty} \sum_{k=1}^{m} \int_{t}^{G_k(t)} a_k(s)\, ds = \int_{t}^{t+1} a_1(s)\, ds + \int_{t}^{t+2} a_2(s))\, ds = 1/2 + 2/3 > 1.$$

Thus, Corollary 5.1 does allow to establish that all solutions of (5.3) are oscillatory, while Corollary 5.2 does not.

At last, note that Example 3.5 shows that inequality (5.2) cannot be replaced by

$$\limsup_{t\to\infty} \sum_{k=1}^{m} \int_{g_k(t)}^{t} a_k(s)\, ds > 1.$$

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