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Positive periodic solutions generated by impulses for the delay Nicholson's blowflies model

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Abstract. In this paper, by using Krasnoselskii's fixed point theorem, we study the existence and multiplicity of positive periodic solutions for the delay Nicholson's blowflies model with impulsive effects. Our results show that these positive periodic solutions are generated by impulses. To the authors' knowledge, there are no papers about positive periodic solution generated by impulses for first order delay differential equation. Our results are completely new. Finally, some examples are given to illustrate our main results.

Keywords: positive periodic solution, Nicholson's blowflies model, impulses, Krasnoselskii's fixed point theorem.

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1 Introduction

In [4], Gurney et al. proposed the following delay differential equation model

$$x'(t) = -\delta x(t) + px(t-\tau)e^{-ax(t-\tau)}, \tag{1.1}$$

to describe the population of the Australian sheep-blowfly and to agree with the experimental data obtained in [14]. Here $x(t)$ is the size of the population at time t, p is the maximum per capita daily egg production, $\frac{1}{a}$ is the size at which the blowfly population reproduces at its maximum rate, δ is the per capita daily adult death rate and τ is the generation time. Eq. (1.1) is recognized in the literature as Nicholson's blowflies model. For more details of Eq. (1.1) and its discrete analog, see [6–8, 11, 16] and their references.

In the real world phenomena, the variation of the environment plays a crucial role in many biological and ecological dynamical systems. In particular, the effects of a periodically varying environment are important for evolutionary theories, as the selective forces on systems in a fluctuating environment differ from those in a stable environment. Thus, the assumption of periodicity of the parameters of the system incorporates the periodicity of the environment. A very basic and important ecological problem associated with study of multispecies population

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interactions in a periodic environment is the existence of positive periodic solution which plays the role of the equilibrium in the autonomous models. In fact, it has been suggested by Nicholson that any periodic change of climate tends to impose its periodicity upon oscillations of internal origin or to cause such oscillations to have a harmonic relation to periodic climatic changes. In view of this, it is realistic to assume that the parameters in the models are periodic functions.

Recently, the existence of positive periodic solutions of Nicholson's blowflies model with delay has been already investigated by many authors, see, for example, [1, 9, 10, 12, 13, 15, 21], etc. In [15], the existence of positive T-periodic solutions of the following equation

$$x'(t) = -\delta(t)x(t) + p(t)x(t)e^{-ax(t)}, \tag{1.2}$$

has been researched, where a is a positive constant, δ and p are positive T-periodic functions. The result obtained is that if

$$\min_{t\in[0,T]} p(t) \geq \max_{t\in[0,T]} \delta(t) \tag{1.3}$$

holds, then Eq. (1.2) has a positive T-periodic solution.

In [9], Li and Du considered the following delay equation

$$x'(t) = -\delta(t)x(t) + p(t)x(t-\tau(t))e^{-a(t)x(t-\tau(t))}, \tag{1.4}$$

where $\delta, p, a \in C(\mathbb{R}^+, (0, +\infty))$ and $\tau \in C(\mathbb{R}^+, \mathbb{R}^+)$ are T-periodic functions. They proved that if

$$p(t) \geq \delta(t), \qquad t \in [0, T], \tag{1.5}$$

then Eq. (1.4) has at least one positive T-periodic solution.

In the real world, impulses may appear in several phenomena. For example, consider the sheep-blowfly species with the birth rate being less than the death rate. Without any regulation, the species may tend to be extinct which means the system will collapse. In order to maintain the sustainable development of the system, the appropriate amount of density for the species should be replenished, which acts instantaneously, that is, in the form of impulses. Thus, it is more appropriate to consider the Nicholson's blowflies model with impulsive effects.

In [10], Li and Fan considered the following nonlinear impulsive delay population model

$$\begin{aligned} &x'(t) = -\delta(t)x(t) + p(t)x(t-mT)e^{-a(t)x(t-mT)}, \quad \text{a.e. } t > 0,\ t \neq t_k, \\ &\Delta x(t_k) = b_k x(t_k), \qquad k = 1, 2, \ldots, \end{aligned} \tag{1.6}$$

where m is a positive integer, $\delta(t), a(t)$ and $p(t)$ are positive periodic continuous functions with periodic $T > 0$; $0 < t_1 < t_2 < \cdots$ are fixed impulsive points with $t_k \to +\infty$ as $k \to \infty$, b_k is a real sequence and $b_k > -1, k = 1, 2, \ldots$ and $\prod_{0<t_k<t}(1 + b_k)$ is a T-periodic function. They showed that Eq. (1.6) has a unique T-periodic positive solution under the condition (1.5). Their results implied that under the appropriate linear periodic impulsive perturbations, the impulsive delay equation preserves the original periodic property of the nonimpulsive delay equation.

In most of the aforementioned references, the condition (1.5) is very important to ensure the existence of positive T-periodic solutions. In fact, in [9] and [10], authors proved that if

$$p(t) \leq \delta(t), \qquad t \in [0, T], \tag{1.7}$$

then Eq. (1.4) and Eq. (1.6) have no positive periodic solutions.

In this paper, we will point out that, under the case of (1.7), if the impulses happen, for Eq. (1.4) there may exist positive periodic solutions. More precisely, we consider the following impulsive delay differential equation

$$\begin{aligned} x'(t) &= -\delta(t)x(t) + p(t)x(t-\tau(t))e^{-a(t)x(t-\tau(t))}, \quad \text{a.e. } t \geq 0,\ t \neq t_k, \\ \Delta x(t_k) &= I_k(x(t_k)), \qquad k = 0, 1, \ldots, \end{aligned} \tag{1.8}$$

where $\delta, p, a \in C(\mathbb{R}^+, (0, +\infty))$ and $\tau \in C(\mathbb{R}^+, \mathbb{R}^+)$ are T-periodic functions; $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$ with $x(t_k^\pm) = \lim_{t\to t_k^\pm} x(t)$; t_k are the instants where the impulses occur and there exists a positive integer q such that $t_{k+q} = t_k + T$ and $0 = t_0 < t_1 < \cdots < t_{q-1} < t_q = T$; $I_k : \mathbb{R} \to \mathbb{R}$ are continuous and $I_{k+q} = I_k$.

The main aim of this paper is to reveal several new existence results on the positive T-periodic solutions for the Nicholson's blowflies equation (1.8) with both delay and impulsive effects under the case of (1.7). What is worth mentioning is that these positive T-periodic solutions are generated by impulses. Here, we say that a solution is generated by impulses if this solution is non-trivial when $I_k \neq 0$ for some $0 \leq k \leq q-1$, but it is trivial when $I_k \equiv 0$ for all $0 \leq k \leq q-1$. For example, if problem (1.8) does not possess a positive periodic solution when $I_k \equiv 0$ for all $0 \leq k \leq q-1$, then positive periodic solutions of problem (1.8) with $I_k \neq 0$ for some $0 \leq k \leq q-1$ are called positive periodic solutions generated by impulses (see [2,5,17–20]). To the authors' knowledge, there are no results about positive periodic solutions generated by impulses for first order delay differential equations.

The rest of this paper is organized as follows. In Section 2, some useful lemmas are listed. And then, by using a well-known fixed point theorem in cones (Krasnoselskii's fixed point theorem), some sufficient conditions which ensure the existence and multiplicity of positive periodic solutions of Eq. (1.8) are established in Section 3. Section 4 presents two examples to illustrate our main results.

2 Preliminaries

For convenience, we introduce the notation:

$$f_* = \min_{t\in[0,T]} f(t), \qquad f^* = \max_{t\in[0,T]} f(t),$$

and

$$\bar{f} = \frac{1}{T}\int_0^T f(t)dt$$

where f is a continuous T-periodic function.

Take the initial condition

$$x(s) = \phi(s), \qquad \phi \in C([-\tau^*, 0], \mathbb{R}^+) \quad \text{and} \quad \phi(0) > 0. \tag{2.1}$$

Definition 2.1. A function $x \in ([-\tau^*, +\infty), \mathbb{R}^+)$ is said to be a solution of Eq. (1.8) on $[-\tau^*, +\infty)$ if:

(i) $x(t)$ is absolutely continuous on each interval $(0, t_1]$ and $(t_k, t_{k+1}]$, $k = 1, 2, \ldots,$

(ii) for any $t_k, k = 1, 2, \ldots, x(t_k^+)$ and $x(t_k^-)$ exist and $x(t_k^-) = x(t_k)$,

(iii) $x(t)$ satisfies the former equation of (1.8) for almost everywhere in $[0, +\infty) \setminus \{t_k\}$ and satisfies the latter equation for every $t = t_k$, $k = 1, 2, \ldots.$

It is easy to prove that the initial value problem (1.8) and (2.1) has a unique non-negative solution $x(t)$ on $[0,+\infty)$, and $x(t)>0$ for $t>\tau^*$.

Definition 2.2. Let $(X,\|\cdot\|)$ be a normed linear space, by a cone of X we mean a closed convex subset $K\subset X$ with $K\setminus\{0\}\neq\emptyset$, $\lambda K\subset K$ for every $\lambda\in\mathbb{R}^+$ and $K\cap(-K)=\{0\}$.

In order to obtain our main results, we recall the well-known Krasnoselskii's fixed point theorem.

Lemma 2.3 (Krasnoselskii, [3]). *Let X be a Banach space, and $K\subset X$ be a cone in X. Assume that Ω_1,Ω_2 are open bounded subsets of X with $0\in\Omega_1\subset\overline{\Omega}_1\subset\Omega_2$, and let $\Phi:K\cap(\overline{\Omega}_2\setminus\Omega_1)\to K$ be a completely continuous operator such that either*

(i) $\|\Phi x\|\le\|x\|$, $\forall x\in K\cap\partial\Omega_1$ *and* $\|\Phi x\|\ge\|x\|$, $\forall x\in K\cap\partial\Omega_2$; *or*

(ii) $\|\Phi x\|\ge\|x\|$, $\forall x\in K\cap\partial\Omega_1$ *and* $\|\Phi x\|\le\|x\|$, $\forall x\in K\cap\partial\Omega_2$.

Then Φ has a fixed point in $K\cap(\overline{\Omega}_2\setminus\Omega_1)$.

Set

$$PC=\left\{x:\mathbb{R}\to\mathbb{R}\mid x(t)\text{ is continuous for } t\neq t_k,\ x(t_k^{\pm})\text{ exist},\ x(t_k^-)=x(t_k)\right\}.$$

Let

$$X=\{x(t):x\in PC,\ x(t+T)=x(t)\},$$

and

$$\|x\|=\sup_{t\in[0,T]}|x(t)|,\qquad\forall x\in X.$$

Then X is a real Banach space endowed with the usual linear structure and norm $\|\cdot\|$.

Lemma 2.4. *x is an T-periodic solution of Eq.(1.8) if and only if it is an T-periodic solution of the integral equation*

$$x(t)=\int_t^{t+T}G(t,s)p(s)x(s-\tau(s))e^{-a(s)x(s-\tau(s))}ds+\sum_{t\le t_k<t+T}G(t,t_k)I_k(x(t_k)),\tag{2.2}$$

where

$$G(t,s)=\frac{e^{\int_t^s\delta(u)du}}{e^{\bar{\delta}T}-1},\qquad s\in[t,t+T].$$

Proof. If $x(t)$ is an T-periodic solution of Eq. (2.2), let $t\neq t_k$, then we have

$$\begin{aligned}&\frac{d}{dt}\left[\int_t^{t+T}G(t,s)p(s)x(s-\tau(s))e^{-a(s)x(s-\tau(s))}ds\right]\\&\quad=G(t,t+T)p(t+T)x(t+T-\tau(t+T))e^{-a(t+T)x(t+T-\tau(t+T))}\\&\qquad-G(t,t)p(t)x(t-\tau(t))e^{-a(t+)x(t-\tau(t))}-\delta(t)\int_t^{t+T}G(t,s)p(s)x(s-\tau(s))e^{-a(s)x(s-\tau(s))}ds\\&\quad=p(t)x(t-\tau(t))e^{-a(t)x(t-\tau(t))}-\delta(t)\int_t^{t+T}G(t,s)p(s)x(s-\tau(s))e^{-a(s)x(s-\tau(s))}ds.\end{aligned}$$

Similarly,

$$\frac{d}{dt}\left[\sum_{t\le t_k<t+T}G(t,t_k)I_k(x(t_k))\right]=-\delta(t)\sum_{t\le t_k<t+T}G(t,t_k)I_k(x(t_k)).$$

Hence

$$x'(t) = -\delta(t)x(t) + p(t)x(t-\tau(t))e^{-a(t)x(t-\tau(t))}, \qquad [0,+\infty)\setminus\{t_k\}.$$

For any $t = t_j$, $j = 0,1,\dots$, we have from (2.2) that

$$x(t_j^+) = \int_{t_j}^{t_j+T} G(t_j^+,s)p(s)x(s-\tau(s))e^{-a(s)x(s-\tau(s))}ds + \sum_{t_j^+\le t_k<t_j^++T} G(t_j^+,t_k)I_k(x(t_k)),$$

$$x(t_j) = \int_{t_j}^{t_j+T} G(t_j,s)p(s)x(s-\tau(s))e^{-a(s)x(s-\tau(s))}ds + \sum_{t_j\le t_k<t_j+T} G(t_j,t_k)I_k(x(t_k)).$$

Therefore

$$\begin{aligned} x(t_j^+) - x(t_j) &= \int_{t_j}^{t_j+T}\Big[G(t_j^+,s) - G(t_j,s)\Big]p(s)x(s-\tau(s))e^{-a(s)x(s-\tau(s))}ds \\ &\quad + \sum_{t_j^+\le t_k<t_j^++T} G(t_j^+,t_k)I_k(x(t_k)) - \sum_{t_j\le t_k<t_j+T} G(t_j,t_k)I_k(x(t_k)) \\ &= I_j(x(t_j)). \end{aligned}$$

Thus $x(t)$ is an T-periodic solution of Eq. (1.8).

Conversely, suppose that $x(t)$ is an T-periodic solution of Eq. (2.2). Then for any $t \ne t_k$, it follows from the former equation of (1.8) that

$$\begin{aligned} \Big(x(t)e^{\int_0^t \delta(s)ds}\Big)' &= x'(t)e^{\int_0^t \delta(s)ds} + \delta(t)x(t)e^{\int_0^t \delta(s)ds} \\ &= p(t)x(t-\tau(t))e^{-a(t)x(t-\tau(t))}e^{\int_0^t \delta(s)ds}. \end{aligned}$$

Integrating the above equation from t to $t+T$ and noticing that $x(t_k^+) - x(t_k) = I_k(x(t_k))$, $k = 0,1,\dots$, we have

$$\begin{aligned} &x(t+T)e^{\int_0^{t+T} \delta(s)ds} - x(t)e^{\int_0^t \delta(s)ds} \\ &\quad = \int_t^{t+T} p(s)x(s-\tau(s))e^{-a(s)x(s-\tau(s))}e^{\int_0^s \delta(u)du}ds + \sum_{t\le t_k<t+T} (x(t_k^+) - x(t_k))e^{\int_0^{t_k} \delta(s)ds}. \end{aligned}$$

Since $x(t) = x(t+T)$, we obtain

$$x(t) = \int_t^{t+T} G(t,s)p(s)x(s-\tau(s))e^{-a(s)x(s-\tau(s))}ds + \sum_{t\le t_k<t+T} G(t,t_k)I_k(x(t_k)).$$

This means that $x(t)$ is a T-periodic solution for Eq. (1.8). The proof of Lemma 2.4 is complete. □

Clearly, $G(t+T,s+T) = G(t,s)$, and

$$0 < \frac{1}{e^{\bar\delta T}-1} = G(t,t) \le G(t,s) \le G(t,t+T) = \frac{e^{\bar\delta T}}{e^{\bar\delta T}-1}, \qquad s \in [t,t+T].$$

Let

$$M = \frac{e^{\bar\delta T}}{e^{\bar\delta T}-1}, \qquad N = \frac{1}{e^{\bar\delta T}-1}.$$

Then, we have

$$N \le G(t,s) \le M, \quad \text{for } s \in [t, t+T], \tag{2.3}$$

and

$$0 < \rho \triangleq \frac{N}{M} < 1.$$

Now, choose a cone defined by

$$K = \{x \in X : x(t) \ge \rho\|x\|,\ t \in [0,T]\},$$

and define an operator $\Phi : X \to X$ by

$$(\Phi x)(t) = \int_t^{t+T} G(t,s)p(s)x(s-\tau(s))e^{-a(s)x(s-\tau(s))}ds + \sum_{t \le t_k < t+T} G(t,t_k)I_k(x(t_k)). \tag{2.4}$$

Lemma 2.5. $\Phi K \subset K$.

Proof. In view of (2.3) and (2.4), for any $x \in K$, we have

$$\|\Phi x\| \le M\left[\int_0^T p(s)x(s-\tau(s))e^{-a(s)x(s-\tau(s))}ds + \sum_{k=0}^{q-1} I_k(x(t_k))\right],$$

and

$$(\Phi x)(t) \ge N\left[\int_0^T p(s)x(s-\tau(s))e^{-a(s)x(s-\tau(s))}ds + \sum_{k=0}^{q-1} I_k(x(t_k))\right] \ge \rho\|\Phi x\|.$$

Hence, $\phi K \subset K$. The proof of Lemma 2.5 is completed. □

Lemma 2.6. *$\Phi : K \to K$ is completely continuous.*

Proof. We omit the proof of this lemma since it is a very well known fact. □

3 Main results

In this section, by using Krasnoselskii's fixed point theorem, we investigate the existence and multiplicity of positive periodic solutions for Eq. (1.8). Our main results are presented as follows.

Theorem 3.1. *Assume that the condition (1.7) holds and I_k satisfy the following.*

(I_1) There exist constants $b_k \in (0, \frac{1}{qM})$ and $0 < m_1 \le m_2$ such that

$$m_1 \le I_k(x) \le m_2 + b_k x, \qquad \forall x \ge 0,\ k = 0, 1, \dots, q-1.$$

Then problem (1.8) possesses at least one positive T-periodic solution.

Proof. Set

$$\Omega_1 = \{x \in X,\ \|x\| < qNm_1\}.$$

If $x \in K \cap \partial\Omega_1$, then

$$\begin{aligned}(\Phi x)(t) &= \int_t^{t+T} G(t,s)p(s)x(s-\tau(s))e^{-a(s)x(s-\tau(s))}ds + \sum_{t\le t_k<t+T} G(t,t_k)I_k(x(t_k)) \\ &\ge N\int_0^T p(s)x(s-\tau(s))e^{-a^*x(s-\tau(s))}ds + N\sum_{k=0}^{q-1} I_k(x(t_k)) \\ &\ge N\bar{p}T\rho\|x\|e^{-a^*\|x\|} + Nqm_1 \ge Nqm_1 = \|x\|,\end{aligned}$$

which implies that $\|\Phi x\| \ge \|x\|$, for all $x \in K \cap \partial\Omega_1$. Now we define $b = \max_{0\le k\le q-1} b_k$. Then $0 < qMb < 1$. Set

$$\Omega_2 = \{x \in X,\ \|x\| < d\},$$

where $d = \frac{\frac{1}{a_* e} + Mqm_2}{1-qMb}$. If $x \in K \cap \partial\Omega_2$, then $\|x\| = d$. By the conditions (1.7) and (I_1), we have

$$\begin{aligned}(\Phi x)(t) &= \int_t^{t+T} G(t,s)p(s)x(s-\tau(s))e^{-a(s)x(s-\tau(s))}ds + \sum_{t\le t_k<t+T} G(t,t_k)I_k(x(t_k)) \\ &\le \int_0^T \frac{\delta(s)e^{\int_0^s \delta(u)du}}{e^{\bar{\delta}T}-1} x(s-\tau(s))e^{-a_*x(s-\tau(s))}ds + M\sum_{k=0}^{q-1} I_k(x(t_k)) \\ &\le \frac{1}{a_* e} + qMm_2 + qMbd = d = \|x\|,\end{aligned}$$

which implies that $\|\Phi x\| \le \|x\|$, for all $x \in K \cap \partial\Omega_2$.

From $0 < qMb < 1$, we have

$$qNm_1 < qMm_2 < \frac{\frac{1}{a_* e} + qMm_2}{1-qMb} = d.$$

Therefore, Ω_1 and Ω_2 are open bounded subsets of X with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$. In addition, $\Phi : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ is a completely continuous operator. By Lemma 2.3, there exists one positive T-periodic solution $x \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$. The proof of Theorem 3.1 is complete. □

Theorem 3.2. *Assume that the condition (1.7) holds and I_k satisfy the following assertions.*

- (I_2) $I_k(x) \le \frac{1}{qMa_* e}$ *for* $x \in [0, c_1]$, *where* $c_1 = \frac{2}{a_* e}$.
- (I_3) *There exist constant* $c_2 > \frac{2}{a_* e\rho}$, *such that* $I_k(x) \ge \frac{c_2}{qN}$ *for* $x \in [\rho c_2, c_2]$.
- (I_4) *There exist constant* $c_3 \ge \frac{2c_2}{\rho}$, *such that* $I_k(x) \le \frac{c_3}{2qM}$ *for* $x \in [\rho c_3, c_3]$.

Then problem (1.8) possesses at least two positive T-periodic solutions.

Proof. From (I_2), we define

$$\Omega_3 = \{x \in X,\ \|x\| < c_1\}.$$

By (1.7) and (I_2), if $x \in K \cap \partial\Omega_3$, we obtain

$$\begin{aligned}(\Phi x)(t) &\le \int_0^T \frac{\delta(s)e^{\int_0^s \delta(u)du}}{e^{\bar{\delta}T}-1} x(s-\tau(s))e^{-a(s)x(s-\tau(s))}ds + M\sum_{k=0}^{q-1} I_k(x(t_k)) \\ &\le \frac{1}{a_* e} + M\sum_{k=0}^{q-1}\frac{1}{qMa_* e} \\ &= \frac{1}{a_* e} + \frac{1}{a_* e} = \|x\|,\end{aligned}$$

which implies that $\|\Phi x\| \le \|x\|$, for all $x \in K \cap \partial\Omega_3$.

Moreover, from (I_3), we define

$$\Omega_4 = \{x \in X,\ \|x\| < c_2\}.$$

If $x \in K \cap \partial\Omega_4$, then $\rho c_2 \le x(t) \le c_2$ and $\|x\| = c_2$. Therefore, we have

$$\begin{aligned}(\Phi x)(t) &\ge N\int_0^T p(s)x(s-\tau(s))e^{-a(s)x(s-\tau(s))}ds + N\sum_{k=0}^{q-1} I_k(x(t_k))\\ &> N\sum_{k=0}^{q-1} I_k(x(t_k))\\ &\ge N\sum_{k=0}^{q-1}\frac{c_2}{Nq} = c_2 = \|x\|,\end{aligned}$$

which implies that $\|\Phi x\| > \|x\|$, for all $x \in K \cap \partial\Omega_4$.

Next, by (I_4) we define

$$\Omega_5 = \{x \in X,\ \|x\| < c_3\}.$$

If $x \in K \cap \partial\Omega_5$, we have

$$x(t) \ge \rho c_3 \ge 2c_2 > \frac{4}{a_* e\rho} > \frac{1}{a_*}\ln 2.$$

This implies

$$e^{-a(t)x} \le e^{-a_* x} \le \frac{1}{2}$$

for all $x \in K \cap \partial\Omega_5$. Combining this inequality with (1.7) and (I_4), we have

$$\begin{aligned}(\Phi x)(t) &\le \int_0^T \frac{\delta(s)e^{\int_0^s \delta(u)du}}{e^{\bar{\delta}T}-1}x(s-\tau(s))e^{-a(s)x(s-\tau(s))}ds + M\sum_{k=0}^{q-1} I_k(x(t_k))\\ &\le \frac{1}{2}\|x\| + Mq\frac{c_3}{2Mq}\\ &= \frac{1}{2}\|x\| + \frac{c_3}{2} = \|x\|.\end{aligned}$$

Therefore, $\|\Phi x\| \le \|x\|$, for all $x \in K \cap \partial\Omega_5$.

It is easy to show that $0 \in \Omega_3 \subset \overline{\Omega}_3 \subset \Omega_4 \subset \overline{\Omega}_4 \subset \Omega_5$ and $\Phi : K \cap (\overline{\Omega}_4 \setminus \Omega_3) \to K$ and $\Phi : K \cap (\overline{\Omega}_5 \setminus \Omega_4) \to K$ are completely continuous. By Lemma 2.3, there exist two positive T-periodic solutions $x_1 \in \overline{\Omega}_4 \setminus \Omega_3$ and $x_2 \in \overline{\Omega}_5 \setminus \Omega_4$ satisfying $0 < c_1 < \|x_1\| < c_2 < \|x_2\| < c_3$. This completes the proof of Theorem 3.2. □

4 Examples

In this section, we give two examples to illustrate the results obtained in the previous section.

Example 4.1. Consider the following impulsive delay differential equation

$$\begin{aligned} x'(t) &= -0.5x(t) + 0.25x(t-1)e^{-1.5x(t-1)}, \quad \text{a.e. } t \ge 0,\ t \ne t_k,\\ \Delta x(t_k) &= 2 + \frac{e^{0.5}-1}{6e^{0.5}}x(t_k), \qquad k = 0,1,\dots, \end{aligned} \tag{4.1}$$

where $t_0 = 0 < t_1 = \frac{1}{3}$ and $t_{k+1} = t_k + \frac{1}{3}$ $(k = 0, 1, \ldots)$. We have

$$\delta(t) = 0.5 > p(t) = 0.25, \qquad t \in [0, 1].$$

Obviously, $M = \frac{e^{0.5}}{e^{0.5}-1}$, $N = \frac{1}{e^{0.5}-1}$, $\rho = \frac{1}{e^{0.5}}$. Take $m_1 = 1$, $m_2 = 2$, $b_k \equiv b = \frac{e^{0.5}-1}{6e^{0.5}}$, $(k = 0, 1, \ldots)$. Thus all conditions in Theorem 3.1 are satisfied. By Theorem 3.1, Eq. (4.1) has at least one positive $\frac{1}{3}$-periodic solution (see the red line of Fig. 4.1).

According to the result in [9], we know that the non-impulsive delay differential equation

$$x'(t) = -0.5x(t) + 0.25x(t-1)e^{-1.5x(t-1)}$$

has no positive periodic solutions and the solution will eventually tend to zero (see the blue line of Fig. 4.1).

The above example shows that, under the condition of sublinear impulses, Eq. (4.1) has at least one positive $\frac{1}{3}$-periodic solution. This positive $\frac{1}{3}$-periodic solution is generated by impulses.

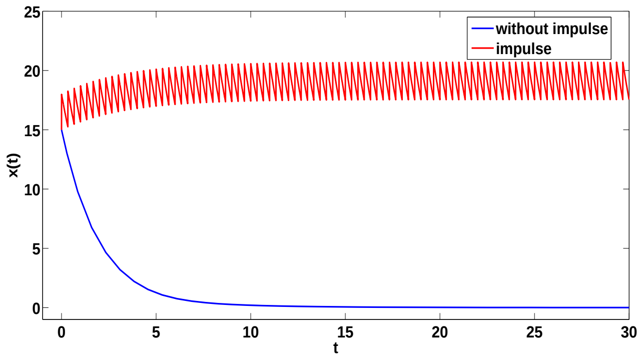

Figure 4.1: The phase trajectories for Eq. (4.1).

Example 4.2. Consider the following impulsive delay differential equation

$$\begin{aligned} x'(t) &= -(3+\cos 2\pi t)x(t) \\ &\quad + (3-\sqrt{2}+\sin 2\pi t)x(t-e^{\sin 2\pi t})e^{-(2+\sin 2\pi t)x(t-e^{\sin 2\pi t})}, \quad \text{a.e. } t \geq 0,\ t \neq t_k, \\ \Delta x(t_k) &= I(x(t_k)), \qquad k = 0, 1, \ldots, \end{aligned} \tag{4.2}$$

where $t_0 = 0 < t_1 = \frac{1}{2} < t_2 = 1$ and $t_{k+2} = t_k + 1$ $(k = 0, 1, \ldots)$,

$$I(x) = \begin{cases} \frac{e^3-1}{2e^4}, & x \in [0, \frac{2}{e}), \\ \left(\frac{3}{2}e^3 - \frac{1}{2e^3}\right)(e^3-1)\left(x - \frac{2}{e}\right) + \frac{e^3-1}{2e^4}, & x \in [\frac{2}{e}, \frac{3}{e}), \\ \frac{3}{2}e^2(e^3-1), & x \in [\frac{3}{e}, 3e^2], \\ \frac{1}{16}(e^3-1)(x-3e^2) + \frac{3}{2}e^2(e^3-1), & x \in [3e^2, 7e^2), \\ \frac{7}{4}e^2(e^3-1), & x \in [7e^2, 7e^5), \\ x - \frac{21}{4}e^5 - \frac{7}{4}e^2, & x \in [7e^5, +\infty). \end{cases}$$

We have

$$\delta(t) = 3 + \cos 2\pi t > p(t) = 3 - \sqrt{2} + \sin 2\pi t, \qquad t \in [0, 1].$$

It is easy to check that

$$M = \frac{e^3}{e^3 - 1}, \qquad N = \frac{1}{e^3 - 1}, \qquad \rho = \frac{1}{e^3},$$

and

$$a_* = 1, \qquad a^* = 3, \qquad c_1 = \frac{2}{e}.$$

Choose $c_2 = 3e^2$, $c_3 = 7e^5$. Then, all conditions of Theorem 3.2 hold. According to Theorem 3.2, Eq. (4.2) has at least two positive 1-periodic solutions generated by impulses.

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