Electronic Journal of Qualitative Theory of Differential Equations
2016, No. 7, 1–15; doi: 10.14232/ejqtde.2016.1.7 http://www.math.u-szeged.hu/ejqtde/

New oscillation results to fourth-order delay differential equations with damping

Jozef Džurina, **Blanka Baculíková** and **Irena Jadlovská** ✉

Department of Mathematics, Faculty of Electrical Engineering and Informatics, Technical University of Košice, Letná 9, 042 00 Košice, Slovakia

Received 10 September 2015, appeared 8 February 2016
Communicated by Josef Diblík

Abstract. This paper is concerned with the oscillation of the linear fourth-order delay differential equation with damping

$$\left(r_3(t)\left(r_2(t)\left(r_1(t)y'(t)\right)'\right)'\right)' + p(t)y'(t) + q(t)y(\tau(t)) = 0$$

under the assumption that the auxiliary third-order differential equation

$$\left(r_3(t)\left(r_2(t)z'(t)\right)'\right)' + \frac{p(t)}{r_1(t)}z(t) = 0$$

is nonoscillatory. In addition, a couple of examples is provided to illustrate the relevance of the main results.

Keywords: fourth-order, delay differential equation, oscillation, Riccati transformation, comparison theorem.

2010 Mathematics Subject Classification: 34C10, 34K11.

1 Introduction

We consider the fourth-order trinomial differential equation with delay argument

$$\left(r_3(t)\left(r_2(t)\left(r_1(t)y'(t)\right)'\right)'\right)' + p(t)y'(t) + q(t)y(\tau(t)) = 0, \quad \text{for } t \geq t_0. \tag{E}$$

Throughout the paper, the following hypotheses will be made:

(H_1) $p, q, \tau \in C([t_0, \infty), \mathbb{R})$ such that $p(t) \geq 0$, $q(t) > 0$, $\tau(t) \leq t$ for all $t \geq t_0$ and $\lim_{t\to\infty} \tau(t) = \infty$.

(H_2) $r_i(t) \in C([t_0, \infty), \mathbb{R})$, $r_i(t) > 0$, $\int_{t_0}^{\infty} \frac{ds}{r_i(s)} = \infty$, $\quad i = 1, 2, 3$.

✉ Corresponding author. Email: irena.jadlovska@student.tuke.sk

(H_3) $\lim_{t\to\infty} \dfrac{r_3(t)}{r_1(t)} > 0.$

By a solution to (E) we mean a function $y \in C([\tau(T_y),\infty))$, $T_y \in [t_0,\infty)$ which has the property r_1y', $r_2\,(r_1y')'$, $r_3\big(r_2(r_1y')'\big)' \in C^1([T_y,\infty))$ and satisfies (E) on $[T_y,\infty)$. Our attention is restricted to those solutions $y(t)$ of (E) which satisfy $\sup\{|y(t)| : t \ge T\} > 0$ for all $T \ge T_y$. We make the standing hypothesis that (E) admits such a solution. A solution of (E) is called *oscillatory* if it has arbitrarily large zeros on $[T_y,\infty)$ and otherwise it is called to be *nonoscillatory*. Equation (E) is said to be *oscillatory* if all its solutions are *oscillatory*.

Over the last few decades, we could bear witness to a great research interest in the study of oscillatory and asymptotic properties of functional differential equations of the form

$$y^{(n)} + q(t)y(\tau(t)) = 0. \tag{1.1}$$

An immense body of relevant literature has been devoted to this topic, the reader is referred to monographs [12, 15, 16] for a complex overview of many significant oscillation results. Among higher-order differential equations, those of fourth-order are generally of considerable practical importance and therefore are often investigated separately. Even though qualitative properties of solutions of a binomial differential equation related to (E), namely,

$$\Big(r_3(t)\,\Big(r_2(t)\,\big(r_1(t)y'(t)\big)'\Big)'\Big)' + q(t)y(\tau(t)) = 0$$

have been widely investigated in the literature (see, for example, [2,3,19] and references cited therein); much less is known about the asymptotic behavior of (E). So far, prototypes of higher-order trinomial differential equations with delay, which have been primarily studied in the literature are such that a difference in the derivative order between the first and the middle term differs either by one or two [4,9].

Similar problems for the third-order damped differential equations with or without deviating argument have been investigated intensively [5,7,8,18]. For a detailed survey of many known oscillation results for such equations, see the recent paper [13].

In [14], the authors initiated a study on the partial case of (E), namely on

$$y^{(4)}(t) + p(t)y'(t) + q(t)y(\tau(t)) = 0. \tag{E_0}$$

By means of the Riccati technique, they presented some sufficient conditions under which any solution of (E_0) oscillates or tends to zero as $t \to \infty$.

Their crucial "preliminary" theorem ensures a constant sign of the first-derivative $y(t)$ provided an auxiliary third-order differential equation

$$z'''(t) + p(t)z(t) = 0 \tag{1.2}$$

has an increasing solution. Some contribution to the investigation of asymptotic properties of (E_0) has been also made by the present authors, see [6].

This paper is organized as follows: in order to acquire a better insight into the solution structure of (E), we use an auxiliary transformation to the equivalent binomial form. Our method proposed in the next section employs the basic properties of a related disconjugate canonical operator so that the obtained knowledge provides a direct improvement of results stated in [14, 17]. As an application of that principle, we will use the Riccati transformation technique to establish a new sufficient condition ensuring *oscillation* of all solutions of the studied trinomial equation (E). The criterion derived directly involves a coefficient $p(t)$ pertaining to a damped term and does not depend on solutions of the auxiliary differential equation.

2 Classification of nonoscillatory solutions

For the reader's convenience, let us define the following operators

$$L_0y(t) = y(t), \qquad L_iy(t) = r_i\left(L_{i-1}y(t)\right)', \quad i = 1,2,3, \qquad L_4y(t) = \left(L_3y(t)\right)'.$$

With this notation, (E) can be rewritten as

$$L_4y(t) + \frac{p(t)}{r_1(t)}L_1y(t) + q(t)y(\tau(t)) = 0.$$

As is customary, we state here that all the functional inequalities considered in this section and in the latter parts are assumed to hold eventually, that is, they are satisfied for all t large enough.

The essential task in the study of asymptotic properties of equations such as (E) consists in determining the sign of particular quasi-derivatives $L_iy(t)$. It follows from the familiar Kiguradze's lemma [15] that in a particular case of (E), namely,

$$L_4y(t) + q(t)y(\tau(t)) = 0, \tag{2.1}$$

the set $\mathcal{N}$ of all nonoscillatory solutions can be decomposed into two classes

$$\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_3,$$

where the nonoscillatory, say positive solution $y(t)$ satisfies

$$y(t) \in \mathcal{N}_1 \iff L_1y(t) > 0, \quad L_2y(t) < 0, \quad L_3y(t) > 0, \quad L_4y(t) < 0,$$

or

$$y(t) \in \mathcal{N}_3 \iff L_1y(t) > 0, \quad L_2y(t) > 0, \quad L_3y(t) > 0, \quad L_4y(t) < 0.$$

On the other hand, such an approach cannot be applied when $p(t)$ does not vanish identically so that the solution space of (E) is unclear. To get over difficulties caused by the presence of the middle term, we use an associated binomial form of (E) that allows us to deduce the result on the signs $L_iy(t)$, $i = 1,2,3,4$.

Since the principal theorem presented in this section, as well as the latter ones, relate properties of solutions of (E) to those of solutions to an auxiliary third-order linear ordinary differential equation

$$\left(r_3(t)\left(r_2(t)z'(t)\right)'\right)' + \frac{p(t)}{r_1(t)}z(t) = 0, \tag{P_1}$$

we summarize its asymptotic properties briefly.

By virtue of the main assumption (H_2), we note that the equation (P_1) always admits a decreasing solution $z(t)$ satisfying

$$z(t) > 0, \quad r_2(t)z'(t) < 0, \quad r_3(t)\left(r_2(t)z'(t)\right)' > 0, \quad \left(r_3(t)\left(r_2(t)z'(t)\right)'\right)' < 0, \tag{2.2}$$

while increasing solutions such that

$$z(t) > 0, \quad r_2(t)z'(t) > 0, \quad r_3(t)\left(r_2(t)z'(t)\right)' > 0, \quad \left(r_3(t)\left(r_2(t)z'(t)\right)'\right)' < 0 \tag{2.3}$$

exist only if (P_1) is nonoscillatory.

The formal adjoint to (P_1) given by

$$\left(r_2(t)\left(r_3(t)\sigma'(t)\right)'\right)' - \frac{p(t)}{r_1(t)}\sigma(t) = 0 \tag{P_1'}$$

has been shown to be important in the study of oscillatory properties to (P_1). It is well known [11] that all solutions of (P_1) are nonoscillatory if and only if all solutions of (P_1') are as well.

The next result is based on an equivalent representation for the linear differential operator

$$L_y = \left(r_3(t)\left(r_2(t)\left(r_1(t)y'(t)\right)'\right)'\right)' + p(t)y'(t) \tag{2.4}$$

in terms of a positive solution of (P_1).

Lemma 2.1. *Let $z(t)$ be a positive solution of (P_1). Then the operator (2.4) can be written as*

$$L_y = \left(\frac{r_3(t)}{z(t)}\left(r_2(t)z^2(t)\left(\frac{r_1(t)}{z(t)}y'(t)\right)'\right)'\right)' + r_3(t)\left(r_2(t)z'(t)\right)'\left(\frac{r_1(t)}{z(t)}y'(t)\right)'. \tag{2.5}$$

Proof. Simple computation shows that the right-hand side of (2.5) equals

$$\begin{aligned}
&\left[\frac{r_3(t)}{z(t)}\left(r_2(t)\left(r_1(t)y'(t)\right)'z(t) - r_2(t)r_1(t)y'(t)z'(t)\right)'\right]' + r_3(t)\left(r_2(t)z'(t)\right)'\left(\frac{r_1(t)}{z(t)}y'(t)\right)' \\
&\quad = \left[r_3(t)\left(r_2(t)\left(r_1(t)y'(t)\right)'\right)' - \frac{r_3(t)}{z(t)}r_1(t)y'(t)\left(r_2(t)z'(t)\right)'\right]' \\
&\qquad + r_3(t)\left(r_2(t)z'(t)\right)'\left(\frac{r_1(t)}{z(t)}y'(t)\right)' \\
&\quad = \left(r_3(t)\left(r_2(t)\left(r_1(t)y'(t)\right)'\right)'\right)' - r_3(t)\left(r_2(t)z'(t)\right)'\left(\frac{r_1(t)}{z(t)}y'(t)\right)' \\
&\qquad - \left(r_3(t)\left(r_2(t)z'(t)\right)'\right)'\frac{r_1(t)}{z(t)}y'(t) + r_3(t)\left(r_2(t)z'(t)\right)'\left(\frac{r_1(t)}{z(t)}y'(t)\right)' \\
&\quad = \left(r_3(t)\left(r_2(t)\left(r_1(t)y'(t)\right)'\right)'\right)' + p(t)y'(t).
\end{aligned}$$

The proof is complete. □

Lemma 2.2. *Let $z(t)$ be a positive solution of (P_1) and let the equation*

$$\left(\frac{r_3(t)}{z(t)}v'(t)\right)' + \left(\frac{r_3(t)\left(r_2(t)z'(t)\right)'}{r_2(t)z^2(t)}\right)v(t) = 0 \tag{P_2}$$

possess a positive solution. Then the operator (2.4) can be written as

$$L_y = \frac{1}{v(t)}\left(\frac{r_3(t)v^2(t)}{z(t)}\left(\frac{r_2(t)z^2(t)}{v(t)}\left(\frac{r_1(t)}{z(t)}y'(t)\right)'\right)'\right)'. \tag{2.6}$$

Proof. It is straightforward to verify that the right-hand side of (2.6) equals

$$\begin{aligned}
&\frac{1}{v(t)}\left[\left(r_2(t)z^2(t)\left(\frac{r_1(t)}{z(t)}y'(t)\right)'\right)'\frac{r_3(t)}{z(t)}v(t)-r_2(t)z^2(t)\left(\frac{r_1(t)}{z(t)}y'(t)\right)'\frac{r_3(t)}{z(t)}v'(t)\right]'\\
&\quad=\left(\frac{r_3(t)}{z(t)}\left(r_2(t)z^2(t)\left(\frac{r_1(t)}{z(t)}y'(t)\right)'\right)'\right)'+\left(r_2(t)z^2(t)\left(\frac{r_1(t)}{z(t)}y'(t)\right)'\right)'\frac{r_3(t)}{z(t)}\frac{v'(t)}{v(t)}\\
&\qquad-\left(r_2(t)z^2(t)\left(\frac{r_1(t)}{z(t)}y'(t)\right)'\right)'\frac{r_3(t)}{z(t)}\frac{v'(t)}{v(t)}-\frac{r_2(t)z^2(t)}{v(t)}\left(\frac{r_1(t)}{z(t)}y'(t)\right)'\left(\frac{r_3(t)}{z(t)}v'(t)\right)'\\
&\quad=\left(\frac{r_3(t)}{z(t)}\left(r_2(t)z^2(t)\left(\frac{r_1(t)}{z(t)}y'(t)\right)'\right)'\right)'-\frac{r_2(t)z^2(t)}{v(t)}\left(\frac{r_1(t)}{z(t)}y'(t)\right)'\left(\frac{r_3(t)}{z(t)}v'(t)\right)'\\
&\quad=\diamondsuit.
\end{aligned}$$

Applying (2.5) from Lemma 2.1, we get

$$\begin{aligned}
\diamondsuit&=\left(r_3(t)\left(r_2(t)\left(r_1(t)y'(t)\right)'\right)'\right)'+p(t)y'(t)\\
&\quad-r_3(t)\left(r_2(t)z'(t)\right)'\left(\frac{r_1(t)}{z(t)}y'(t)\right)'-\frac{r_2(t)z^2(t)}{v(t)}\left(\frac{r_1(t)}{z(t)}y'(t)\right)'\left(\frac{r_3(t)}{z(t)}v'(t)\right)'\\
&=\left(r_3(t)\left(r_2(t)\left(r_1(t)y'(t)\right)'\right)'\right)'+p(t)y'(t)\\
&\quad-\left(\frac{r_1(t)}{z(t)}y'(t)\right)'\frac{r_2(t)z^2(t)}{v(t)}\left[\left(\frac{r_3(t)}{z(t)}v'(t)\right)'+\frac{r_3(t)\left(r_2(t)z'(t)\right)'}{r_2(t)z^2(t)}v(t)\right].
\end{aligned}$$

Since $v(t)$ is a solution of (P_2), the previous equality yields

$$\begin{aligned}
&\frac{1}{v(t)}\left(\frac{r_3(t)v^2(t)}{z(t)}\left(\frac{r_2(t)z^2(t)}{v(t)}\left(\frac{r_1(t)}{z(t)}y'(t)\right)'\right)'\right)'\\
&\quad=\left(r_3(t)\left(r_2(t)\left(r_1(t)y'(t)\right)'\right)'\right)'+p(t)y'(t)=L_y.
\end{aligned}\tag{2.7}$$

□

Lemma 2.1 and Lemma 2.2 permit us to rewrite (E) into its binomial form

$$\frac{1}{v(t)}\left(\frac{r_3(t)v^2(t)}{z(t)}\left(\frac{r_2(t)z^2(t)}{v(t)}\left(\frac{r_1(t)}{z(t)}y'(t)\right)'\right)'\right)'+q(t)y(\tau(t))=0,\tag{E_c}$$

where we assume that $z(t)$ and $v(t)$ are positive solutions of (P_1) and (P_2), respectively. Now, it naturally follows to derive criterion for (P_2) to have a positive solution.

Lemma 2.3. *If $z(t)$ is a positive decreasing solution of (P_1) and $z_*(t)$ is any solution of (P_1), then*

$$v(t)=r_2(t)\left[z(t)z_*'(t)-z'(t)z_*(t)\right]\tag{2.8}$$

is a solution of (P_2).

Proof. Direct computation shows that (2.8) satisfies (P_2) so we omit the proof. □

Lemma 2.4. *Let* (P_1) *be nonoscillatory and* $z(t)$ *be its positive decreasing solution. Then* (P_2) *admits a nonoscillatory solution* $v(t)$ *such that*

$$v(t) > 0, \qquad v'(t) > 0, \qquad \left(\frac{r_3(t)}{z(t)} v'(t)\right)' < 0. \tag{2.9}$$

Proof. Since (P_1) is nonoscillatory, it possesses a positive increasing solution $z_*(t)$. By Lemma 2.3, $v(t)$ given by (2.8) is a positive solution of (P_2). Moreover, it can be directly verified that $v(t)$ satisfies the adjoint equation (P_1'). Hence it follows from (P_1') that $\left(r_2(t)(r_3(t)v'(t))'\right)' > 0$ and in view of Kiguradze's lemma [15], we conclude $v'(t) > 0$. □

Remark 2.5. We recall from [10] that condition

$$\int_{t_0}^{\infty} \frac{p(t)}{r_1(t)} \int_{t_0}^{t} \frac{1}{r_2(s)} \int_{t_0}^{s} \frac{1}{r_3(u)} \, \mathrm{d}u \, \mathrm{d}s \, \mathrm{d}t < \infty$$

is sufficient for (P_1) to be nonoscillatory.

For our next purposes, it is desirable for (E_c) to be in the canonical form, i.e. following conditions

$$\int_{t_0}^{\infty} \frac{z(s)}{r_3(s)v^2(s)} \, \mathrm{d}s = \infty, \tag{2.10}$$

$$\int_{t_0}^{\infty} \frac{v(s)}{r_2(s)z^2(s)} \, \mathrm{d}s = \infty, \tag{2.11}$$

$$\int_{t_0}^{\infty} \frac{z(s)}{r_1(s)} \, \mathrm{d}s = \infty, \tag{2.12}$$

are required to hold.

Lemma 2.6. *Let* (P_1) *be nonoscillatory. Then there exist positive solutions* $z(t)$ *and* $v(t)$ *of* (P_1) *and* (P_2)*, respectively, such that* (2.10), (2.11) *and* (2.12) *are satisfied.*

Proof. Suppose that $z(t)$ is a positive decreasing solution of (P_1). The existence of a positive solution $v(t)$ of (P_2) follows from Lemma 2.4. Assume that $v(t)$ does not satisfy (2.10), then it is easy to see that $v_*(t)$ given by

$$v_*(t) = v(t) \int_{t}^{\infty} \frac{z(s)}{r_3(s)v^2(s)} \, \mathrm{d}s \tag{2.13}$$

satisfies

$$\begin{aligned}
\left(\frac{r_3(t)}{z(t)} v_*'(t)\right)' &= \left(\frac{r_3(t)}{z(t)} v'(t)\right)' \int_{t}^{\infty} \frac{z(s)}{r_3(s)v^2(s)} \, \mathrm{d}s \\
&= -\left(\frac{r_3(t)\left(r_2(t)z'(t)\right)'}{r_2(t)z^2(t)}\right) v(t) \int_{t}^{\infty} \frac{z(s)}{r_3(s)v^2(s)} \, \mathrm{d}s \\
&= -\left(\frac{r_3(t)\left(r_2(t)z'(t)\right)'}{r_2(t)z^2(t)}\right) v_*(t).
\end{aligned}$$

Thus $v_*(t)$ is another positive solution of (P_2). Moreover, $v_*(t)$ meets (2.10) by now. To see this, let us denote

$$\mathcal{V}(t) = \int_{t}^{\infty} \frac{z(s)}{r_3(s)v^2(s)} \, \mathrm{d}s,$$

then $\lim_{t\to\infty} \mathcal{V}(t) = 0$ and

$$\int_{t_0}^{\infty} \frac{z(t)}{r_3(t)v_*^2(t)}\,\mathrm{d}t = -\int_{t_0}^{\infty} \frac{\mathcal{V}'(t)}{\mathcal{V}^2(t)}\,\mathrm{d}t = \lim_{t\to\infty}\left(\frac{1}{\mathcal{V}(t)} - \frac{1}{\mathcal{V}(t_0)}\right) = \infty.$$

Moreover, noting (H_3) and (2.9), the last equality implies (2.12). On the other hand, taking $v_*(t) > c_1$ and $z(t) < c_2$ into account, we see that in view of (H_2), condition (2.11) is satisfied. The proof is complete. □

Remark 2.7. If (P_1) possesses such solution $z(t)$ that the condition (2.12) holds, we can relax assumption (H_3).

In view of the canonical representation of (E_c) ensured by Lemma 2.6, we get immediately the lemma below.

Lemma 2.8. *Let (P_1) be nonoscillatory and $z(t)$ and $v(t)$ be needed solutions of (P_1) and (P_2), respectively. Assume that $y(t)$ is a positive solution of (E), then either*

$$y'(t) > 0, \qquad \left(\frac{r_1(t)}{z(t)}y'(t)\right)' < 0, \qquad \left(\frac{r_2(t)z^2(t)}{v(t)}\left(\frac{r_1(t)}{z(t)}y'(t)\right)'\right)' > 0,$$

or

$$y'(t) > 0, \qquad \left(\frac{r_1(t)}{z(t)}y'(t)\right)' > 0, \qquad \left(\frac{r_2(t)z^2(t)}{v(t)}\left(\frac{r_1(t)}{z(t)}y'(t)\right)'\right)' > 0,$$

eventually.

Now, we are able to state the final result on the sign properties of possible nonoscillatory solutions for (E).

Theorem 2.9. *Let (P_1) be nonoscillatory. Then any positive solution $y(t)$ of (E) satisfies either*

$$y(t) \in \mathcal{N}_1 \iff L_1y(t) > 0, \quad L_2y(t) < 0, \quad L_3y(t) > 0, \quad L_4y(t) < 0,$$

or

$$y(t) \in \mathcal{N}_3 \iff L_1y(t) > 0, \quad L_2y(t) > 0, \quad L_3y(t) > 0, \quad L_4y(t) < 0,$$

eventually.

Proof. Assume that $y(t)$ is an eventually positive solution of (E). Since we have $y'(t) > 0$, it follows from (E) that $L_4(t) < 0$. The rest signs properties of derivatives of $y(t)$ follows from Kiguradze's lemma. □

3 Main results

To start with, we first derive some useful estimates that will be needed in establishing our main results.

For the simplicity of notation, let us define the functions

$$J_1(t) = \int_{t_1}^{t} \frac{\mathrm{d}s}{r_3(s)}, \qquad J_k(t) = \int_{t_1}^{t} \frac{1}{r_{4-k}(s)} J_{k-1}(s)\,\mathrm{d}s, \qquad k = 2,3,$$

$$R_i(t) = \int_{t_1}^{t} \frac{\mathrm{d}s}{r_i(s)}, \qquad i = 1,2,$$

$$I_2(t) = \int_{t_1}^{t} \frac{1}{r_1(s)} R_2(s)\,\mathrm{d}s,$$

where t_1 is sufficiently large.

Theorem 3.1. *Assume that* (P_1) *is nonoscillatory. Let* $y(t)$ *be a positive solution of* (E)*. If*

(i) $y(t) \in \mathcal{N}_1$*, then* $\dfrac{y(t)}{R_1(t)}$ *is decreasing.*

(ii) $y(t) \in \mathcal{N}_3$*, then* $\dfrac{y(t)}{J_3(t)}$ *is decreasing and* $L_1 y(t) \geq J_2(t) L_3 y(t)$.

Proof. Assume that $y(t)$ is a positive solution of (E) and $y(t) \in \mathcal{N}_1$. It follows from the monotonicity of $L_1 y(t)$ that

$$y(t) > y(t) - y(t_1) = \int_{t_1}^{t} \frac{1}{r_1(s)} L_1 y(s)\, ds \geq L_1 y(t) \int_{t_1}^{t} \frac{1}{r_1(s)}\, ds.$$

Therefore,

$$\left(\frac{y(t)}{R_1(t)}\right)' = \frac{y'(t) R_1(t) - \frac{1}{r_1(t)} y(t)}{R_1^2(t)} < 0$$

and part (i) is proved. Now assume that $y(t) \in \mathcal{N}_3$. Since

$$L_2 y(t) = L_2 y(t_1) + \int_{t_1}^{t} \frac{1}{r_3(s)} L_3 y(s)\, ds > L_3 y(t) \int_{t_1}^{t} \frac{1}{r_3(s)}\, ds,$$

then

$$\left(\frac{L_2 y(t)}{J_1(t)}\right)' = \frac{J_1(t) L_2' y(t) - \frac{1}{r_3(t)} L_2 y(t)}{J_1^2(t)} < 0.$$

Thus $L_2 y(t)/J_1(t)$ is decreasing. Moreover,

$$L_1 y(t) = L_1 y(t_1) + \int_{t_1}^{t} \frac{J_1(s))}{r_2(s)} \frac{L_2 y(s)}{J_1(s)}\, ds > \frac{L_2 y(t)}{J_1(t)} J_2(t),$$

Picking up the previous inequalities, we see that $L_1 y(t) \geq J_2(t) L_3 y(t)$ and

$$\left(\frac{L_1 y(t)}{J_2(t)}\right)' = \frac{(L_1 y(t))' J_2(t) - \frac{1}{r_2(t)} L_1 y(t) J_1(t)}{J_2^2(t)} < 0,$$

and we conclude that $L_1 y(t)/J_2(t)$ is decreasing. On the other hand,

$$y(t) = y(t_1) + \int_{t_1}^{t} \frac{J_2(s)}{r_1(s)} \frac{L_1 y(s)}{J_2(s)}\, ds > \frac{L_1 y(t)}{J_2(t)} J_3(t),$$

which implies

$$\left(\frac{y(t)}{J_3(t)}\right)' = \frac{y'(t) J_3(t) - \frac{1}{r_1(t)} y(t) J_2(t)}{J_3^2(t)} < 0.$$

So that $y(t)/J_3(t)$ is decreasing. The proof is complete now. □

Let us denote the function

$$Q(t) = \frac{1}{r_1(t)} \left(\int_{t}^{\tau^{-1}(t)} \frac{1}{r_2(s)} \int_{s}^{\tau^{-1}(t)} \frac{1}{r_3(v)}\, dv\, ds \int_{\tau^{-1}(t)}^{\infty} q(s)\, ds \right).$$

Theorem 3.2. *Assume that* (P_1) *is nonoscillatory. Let* $y(t)$ *be a positive solution of* (E)*. If*

(i) $y(t) \in \mathcal{N}_1$*, then* $y'(t) \geq Q(t)y(t)$.

(ii) $y(t) \in \mathcal{N}_3$*, then* $y'(t) \geq \dfrac{1}{r_1(t)R_1(t)}y(t)$.

Proof. Assume that $y(t)$ is a positive solution of (E) and $y(t) \in \mathcal{N}_1$. For any $u > t$, we have

$$-L_2y(t) \geq L_2y(u) - L_2y(t) = \int_t^u \frac{1}{r_3(s)} L_3y(s)\,\mathrm{d}s \geq L_3y(u) \int_t^u \frac{1}{r_3(s)}\,\mathrm{d}s$$

Multiplying by $1/r_2(t)$ and then integrating from t to u, one gets

$$\begin{aligned} L_1y(t) &\geq \int_t^u L_3y(t)\frac{1}{r_2(s)} \int_s^u \frac{1}{r_3(v)}\,\mathrm{d}v\,\mathrm{d}s \\ &\geq L_3y(u) \int_t^u \frac{1}{r_2(s)} \int_s^u \frac{1}{r_3(v)}\,\mathrm{d}v\,\mathrm{d}s. \end{aligned} \tag{3.1}$$

On the other hand, an integration of (E) from u to ∞, yields

$$\begin{aligned} L_3y(u) &\geq \int_u^\infty p(s)y'(s)\,\mathrm{d}s + \int_u^\infty q(s)y(\tau(s))\,\mathrm{d}s \\ &\geq y(\tau(u)) \int_u^\infty q(s)\,\mathrm{d}s. \end{aligned} \tag{3.2}$$

Combining (3.1) together with (3.2) and setting $u = \tau^{-1}(t)$, we obtain

$$y'(t) \geq \frac{1}{r_1(t)} \left(\int_t^{\tau^{-1}(t)} \frac{1}{r_2(s)} \int_s^{\tau^{-1}(t)} \frac{1}{r_3(v)}\,\mathrm{d}v\,\mathrm{d}s \int_{\tau^{-1}(t)}^\infty q(s)\,\mathrm{d}s \right) y(t).$$

Now assume that $y(t) \in \mathcal{N}_3$. Employing (H_2), the monotonicity of $L_1y(t)$ and the fact that $L_1y(t) \to \infty$ as $t \to \infty$, we see that

$$\begin{aligned} y(t) &= y(t_1) + \int_{t_1}^t \frac{1}{r_1(s)} L_1y(s)\mathrm{d}s \leq y(t_1) + L_1y(t) \int_{t_1}^t \frac{1}{r_1(s)}\,\mathrm{d}s \\ &= y(t_1) - L_1y(t) \int_{t_0}^{t_1} \frac{1}{r_1(s)}\,\mathrm{d}s + L_1y(t) \int_{t_0}^t \frac{1}{r_1(s)}\,\mathrm{d}s \\ &\leq L_1y(t) \int_{t_0}^t \frac{1}{r_1(s)}\,\mathrm{d}s. \end{aligned}$$

The proof is complete now. □

Now, we are prepared to apply the results of previous sections to obtain a new oscillation criterion for studied trinomial differential equation (E). We denote

$$\begin{aligned} Q_1(t) &= p(t)Q(t) + q(t)\frac{R_1(\tau(t))}{R_1(t)}, \\ Q_2(t) &= \frac{p(t)}{r_1(t)R_1(t)} + q(t)\frac{J_3(\tau(t))}{J_3(t)}. \end{aligned}$$

Theorem 3.3. *Assume that* (P_1) *is nonoscillatory and there exists a positive continuously differentiable function* $\rho(t)$ *such that*

$$\limsup_{t\to\infty} \int_{t_1}^{\infty} \left(\frac{\rho(v)}{r_2(v)} \int_v^{\infty} \frac{1}{r_3(u)} \int_u^{\infty} Q_1(s)\, \mathrm{d}s\, \mathrm{d}u - \frac{r_1(v)\,(\rho'(v))^2}{4\rho(v)} \right) \mathrm{d}v = \infty, \tag{3.3}$$

and a positive continuously differentiable function $\gamma(t)$ *such that*

$$\limsup_{t\to\infty} \int_{t_1}^{\infty} \left(Q_2(v)\gamma(s) - \frac{r_1(s)\,(\gamma'(s))^2}{4\gamma(s)J_2(s)} \right) \mathrm{d}s = \infty. \tag{3.4}$$

Then (E) *is oscillatory.*

Proof. Assume that $y(t)$ is a positive solution of (E). Then either $y(t) \in \mathcal{N}_1$ or $y(t) \in \mathcal{N}_3$. At first assume that $y(t) \in \mathcal{N}_1$. Theorem 3.1 implies that

$$y(\tau(t)) \geq \frac{R_1(\tau(t))}{R_1(t)}\, y(t).$$

On the other hand, it follows from Theorem 3.2 that

$$y'(t) \geq Q(t)y(t).$$

Setting both estimates into (E), we are led to

$$L_4 y(t) + Q_1(t)y(t) \leq 0.$$

Integrating the last inequality from t to ∞, one gets

$$-\, L_3 y(t) \geq \int_t^{\infty} Q_1(s)y(s)\, \mathrm{d}s \geq y(t) \int_t^{\infty} Q_1(s)\, \mathrm{d}s. \tag{3.5}$$

Integrating once more, we have

$$L_2 y(t) + \left(\int_t^{\infty} \frac{1}{r_3(u)} \int_u^{\infty} Q_1(s)\, \mathrm{d}s\, \mathrm{d}u \right) y(t) \leq 0. \tag{3.6}$$

Let us define the function $\omega(t)$

$$\omega(t) = \rho(t) \frac{L_1 y(t)}{y(t)} > 0.$$

We easily verify that

$$\begin{aligned} \omega'(t) &= \rho'(t) \frac{L_1 y(t)}{y(t)} + \frac{\rho(t)}{r_2(t)} \frac{L_2 y(t)}{y(t)} - \rho(t) \frac{L_1 y(t)}{y(t)} \frac{y'(t)}{y(t)} \\ &\leq -\frac{\rho(t)}{r_2(t)} \int_t^{\infty} \frac{1}{r_3(u)} \int_u^{\infty} Q_1(s)\, \mathrm{d}s\, \mathrm{d}u + \frac{\rho'(t)}{\rho(t)} \omega(t) - \frac{\omega^2(t)}{r_1(t)\rho(t)} \\ &\leq -\frac{\rho(t)}{r_2(t)} \int_t^{\infty} \frac{1}{r_3(u)} \int_u^{\infty} Q_1(s)\, \mathrm{d}s\, \mathrm{d}u + \frac{r_1(t)\,(\rho'(t))^2}{4\rho(t)}. \end{aligned} \tag{3.7}$$

Integration of the previous inequality yields

$$\int_{t_1}^{t} \left[\frac{\rho(v)}{r_2(v)} \int_v^{\infty} \frac{1}{r_3(u)} \int_u^{\infty} Q_1(s)\, \mathrm{d}s\, \mathrm{d}u - \frac{r_1(v)\,(\rho'(v))^2}{4\rho(v)} \right] \mathrm{d}v \leq \omega(t_1),$$

which contradicts with (3.3) as $t \to \infty$. Now assume that $y(t) \in \mathcal{N}_3$. Theorems 3.1 and 3.2 guarantee that

$$y(\tau(t)) \geq \frac{J_3(\tau(t))}{J_3(t)} y(t), \qquad y'(t) \geq \frac{1}{r_1(t)R_1(t)} y(t), \qquad L_1 y(t) \geq J_2(t) L_3 y(t),$$

what in view of (*E*) provides

$$L_4 y(t) + Q_2(t) y(t) \leq 0.$$

Now the suitable Riccati transformation is

$$\omega_*(t) = \gamma(t) \frac{L_3 y(t)}{y(t)} > 0.$$

Then,

$$\begin{aligned} \omega_*'(t) &= \gamma'(t) \frac{L_3 y(t)}{y(t)} + \gamma(t) \frac{L_4 y(t)}{y(t)} - \gamma(t) \frac{L_3 y(t) y'(t)}{y^2(t)} \\ &\leq -\gamma(t) Q_2(t) + \frac{\gamma'(t)}{\gamma(t)} \omega_*(t) - \frac{J_2(t)}{\gamma(t) r_1(t)} \omega_*^2(t) \\ &\leq -\gamma(t) Q_2(t) + \frac{r_1(t) \left(\gamma'(t)\right)^2}{4\gamma(t) J_2(t)}. \end{aligned} \tag{3.8}$$

Integrating the last inequality from t_1 to t and letting $t \to \infty$, we get

$$\int_{t_1}^{\infty} \left[\gamma(s) Q_2(s) - \frac{r_1(s) \left(\gamma'(s)\right)^2}{4\gamma(s) J_2(s)} \right] \mathrm{d}s \leq \omega_*(t_1),$$

which contradicts with (3.4) and the proof is complete now. □

Setting

$$\rho(t) = R_1(t) \quad \text{and} \quad \gamma(t) = J_3(t),$$

we immediately get the following oscillatory criterion.

Corollary 3.4. *Assume that* (P_1) *is nonoscillatory and*

$$\limsup_{t \to \infty} \int_{t_1}^{\infty} \left(\frac{R_1(v)}{r_2(v)} \int_v^{\infty} \frac{1}{r_3(u)} \int_u^{\infty} Q_1(s) \, \mathrm{d}s \, \mathrm{d}u - \frac{1}{4 r_1(v) R_1(v)} \right) \mathrm{d}v = \infty, \tag{3.9}$$

$$\limsup_{t \to \infty} \int_{t_1}^{\infty} \left(\frac{p(s) J_3(s)}{r_1(s) R_1(s)} + q(s) J_3(\tau(s)) - \frac{J_2(s)}{4 r_1(s) J_3(s)} \right) \mathrm{d}s = \infty. \tag{3.10}$$

Then (*E*) *is oscillatory.*

Another results for oscillation of (*E*) can be obtained by comparison with ordinary differential equations of the same or a lower order. We offer a comparison theorem that relates properties of solutions of (*E*) with those of second-order differential equations. It is well known that equation

$$\left(a(t) x'(t)\right)' + b(t) x(t) = 0, \qquad t \geq t_0, \tag{3.11}$$

where $a, b \in C([t_0, \infty), \mathbb{R})$, $a(t) > 0$, $b(t) > 0$, is nonoscillatory if and only if there exists a function $u(t) \in C^1([t_0, \infty), \mathbb{R})$, which satisfies the inequality

$$u'(t) + \frac{u^2(t)}{a(t)} + b(t) \leq 0.$$

Lemma 3.5 ([1]). *Let*

$$\int_{t_0}^{\infty} \frac{1}{a(s)}\, \mathrm{d}s = \infty.$$

Then the condition

$$\liminf_{t\to\infty} \left(\int_{t_0}^{t} \frac{1}{a(s)}\, \mathrm{d}s \right) \int_{t}^{\infty} b(s)\, \mathrm{d}s > \frac{1}{4}$$

guarantees oscillation of (3.11).

Theorem 3.6. *Let* (P_1) *be nonoscillatory. Assume both equations*

$$\big(r_1(t)x'(t)\big)' + \left(\frac{1}{r_2(t)} \int_{t}^{\infty} \frac{1}{r_3(u)} \int_{u}^{\infty} Q_1(s)\, \mathrm{d}s\, \mathrm{d}u \right) x(t) = 0 \tag{3.12}$$

and

$$\left(\frac{r_1(t)}{J_2(t)} x'(t) \right)' + Q_2(t)x(t) = 0 \tag{3.13}$$

are oscillatory. Then (E) *is oscillatory.*

Proof. Similarly as in the proof of Theorem 3.3, we obtain (3.7) and (3.8). Setting $\rho(t) = 1$ in (3.7) and $\gamma(t) = 1$ in (3.8), we get

$$\omega'(t) + \frac{1}{r_1(t)}\omega^2(t) + \frac{1}{r_2(t)} \int_{t}^{\infty} \frac{1}{r_3(u)} \int_{u}^{\infty} Q_1(s)\, \mathrm{d}s\, \mathrm{d}u \le 0, \tag{3.14}$$

and

$$\omega_*'(t) + \frac{J_2(t)}{r_1(t)} \omega_*^2(t) + Q_2(t) \le 0. \tag{3.15}$$

Hence, it is clear that equations (3.12) and (3.13) are nonoscillatory. A contradiction completes the proof. □

In view of Lemma 3.5, oscillation criteria for (E) of Hille–Nehari-type are easily acquired. Note that

$$\int_{t_0}^{\infty} \frac{1}{r_1(s)}\, \mathrm{d}s = \infty \quad \text{and} \quad \int_{t_0}^{\infty} \frac{J_2(s)}{r_1(s)}\, \mathrm{d}s = \infty.$$

Corollary 3.7. *Assume that*

$$\liminf_{t\to\infty} R_1(t) \int_{t}^{\infty} \frac{1}{r_2(v)} \int_{v}^{\infty} \frac{1}{r_3(u)} \int_{u}^{\infty} Q_1(s)\, \mathrm{d}s\, \mathrm{d}u\, \mathrm{d}v > \frac{1}{4},$$
$$\liminf_{t\to\infty} \left(\int_{t_0}^{t} \frac{J_2(s)}{r_1(s)}\, \mathrm{d}s \right) \int_{t}^{\infty} Q_2(s)\, \mathrm{d}s > \frac{1}{4}.$$

Then every solution of (E) *is oscillatory.*

4 Examples

An application of our main results is provided on Euler-type differential equations.

Example 4.1. We consider the trinomial delay differential equation

$$\left(t^{1/2}\left(t^{1/2}y'(t)\right)''\right)' + \frac{a}{t^2}\,y'(t) + \frac{b}{t^3}\,y(\lambda t) = 0, \tag{E_{x1}}$$

where $a > 0$, $b > 0$, $\lambda \in (0,1)$. It is easy to verify that the corresponding third-order differential equation (P_1), namely

$$\left(t^{1/2}z''(t)\right)' + \frac{a}{t^{5/2}}z(t) = 0$$

is nonoscillatory iff

$$a \le (7\sqrt{7} - 10)/108.$$

Some computation shows that conditions (3.9) and (3.10) reduce to

$$b\lambda^{1/2}\left[a\left(\frac{2}{3} - 2\lambda + \frac{4}{3}\lambda^{3/2}\right) + 2\right] > \frac{3}{8}$$

and

$$a + 2b\lambda^2 > \frac{3}{4},$$

respectively. By Corollary 3.4, these three conditions guarantee oscillation of (E_{x1}).

Example 4.2. We consider

$$y^{(4)}(t) + \frac{a}{t^{3.5}}\,y'(t) + \frac{b}{t^4}\,y(\lambda t) = 0, \tag{E_{x2}}$$

where $a > 0$, $b > 0$, $\lambda \in (0,1)$. Now (P_1) reduces to

$$z'''(t) + \frac{a}{t^{3.5}}z(t) = 0,$$

which is nonoscillatory for all $a > 0$. On the other hand, conditions (3.9) and (3.10) takes the form

$$b\lambda > \frac{3}{2} \quad \text{and} \quad b\lambda^3 > 18,$$

respectively. Thus, by Corollary 3.4, (E_{x2}) is oscillatory if $b\lambda^3 > 18$.

5 Summary

There has been an open problem regarding the study of sufficient conditions ensuring oscillation of all solutions of fourth-order differential equation with damping. The present paper aims to fill this gap. In the first part, we have established a new approach for investigation of a general class of fourth-order trinomial differential equations by employing its binomial canonical representation. We utilize a couple of positive solutions of the corresponding third-order auxiliary differential equation, which allows to recognize signs properties of particular quasi-derivatives. Thereafter, we suggest a new oscillation criterion for the fourth-order delay differential equation (E) using the Riccati transformation technique. Alternatively, a comparison with a couple of second-order differential equations is also formulated.

Acknowledgements

This work was supported by Slovak Research and Development Agency under contracts No. APVV-0404-12.

References

[1] R. Agarwal, S. R. Grace, D. O'Regan, *Oscillation theory for second order linear, half-linear, superlinear and sublinear dynamic equations*, Kluwer Academic Publishers, Dordrecht, 2002. MR2091751

[2] R. Agarwal, S. R. Grace, P. Wong, On the bounded oscillation of certain fourth order functional differential equations, *Nonlinear Dyn. Syst. Theory* **5**(2005), 215–227. MR2160591

[3] R.Agarwal, S. R. Grace, J. V. Manojlovic, Oscillation criteria for certain fourth order nonlinear functional differential equations, *Math. Comput. Modelling* **44**(2006), No. 1–2, 163–187. MR2230441

[4] R.Agarwal, M. Bohner, T. Li, C. Zhang, Oscillation theorems for fourth-order half-linear delay dynamic equations with damping, *Mediterr. J. Math.* **11**(2014), No. 2, 463–475. MR3198619

[5] R. Agarwal, M. Aktas, A. Tiryaki, On oscillation criteria for third order nonlinear delay differential equations, *Arch. Math. (Brno)* **45**(2009), No. 1, 1–18. MR2591657

[6] B. Baculíková, J. Džurina, I. Jadlovská, Oscillation of solutions to fourth-order trinomial delay differential equations, *Electron. J. Differential Equations* **2015**, No. 70, 1–10. MR3337847

[7] B. Baculíková, J. Džurina, Comparison theorems for the third-order delay trinomial differential equations, *Adv. Difference Equ.* **2010**, Art. ID 160761, 12 pp. MR2739752

[8] M. Bartušek, Z. Došlá, M. Marini, Oscillation for third-order nonlinear differential equations with deviating argument, *Abstr. Appl. Anal.* **2010**, Art. ID 278962, 19 pp. MR2587610

[9] M. Bartušek, Z. Došlá, Asymptotic problems for fourth-order nonlinear differential equations, *Bound. Value Probl.* **2013**, No. 89, 15 pp. MR3070574

[10] M. Cecchi, Z. Došlá, M. Marini, G. Villari, On the qualitative behavior of solutions of third order differential equations, *J. Math. Anal. Appl.* **197**(1999), No. 3, 749–766. MR1373077

[11] M. Cecchi, Z. Došlá, M. Marini, On third order differential equations with property A and B, *J. Math. Anal. Appl.* **231**(1999), No. 2, 509–525. MR1669163

[12] U. Elias, *Oscillation theory of two-term differential equations*, Mathematics and its Applications, Vol. 396, Kluwer Academic Publishers Group, Dordrecht, 1997. MR1445292

[13] J. Graef, S. H. Saker, Oscillation theory of third-order nonlinear functional differential equations, *Hiroshima Math. J.* **43**(2013), No. 1, 49–72. MR3066525

[14] Ch. Hou, S. S. Cheng, Asymptotic dichotomy in a class of fourth-order nonlinear delay differential equations with damping, *Abstr. Appl. Anal.* **2009**, 7 pp. MR2516005

[15] I. T. Kiguradze, T. A. Chaturia, *Asymptotic properties of solutions of nonautonomous ordinary differential equations*, Kluwer Academic Publishers Group, Dordrecht, 1993. MR1220223

[16] G. S. Ladde, V. Lakshmikantham, B. G. Zhang, *Oscillation theory of differential equations with deviating arguments*, Monographs and Textbooks in Pure and Applied Mathematics, Vol. 110, Marcel Dekker, New York, 1987. MR1017244

[17] H. Liang, Asymptotic behavior of solutions to higher order nonlinear delay differential equations, *Electron. J. Differential Equations* **2014**, No. 186, 1–12. MR3262057

[18] A. Tiryaki, M. F. Aktas, Oscillation criteria of a certain class of third order nonlinear delay differential equations with damping, *J. Math. Anal. Appl.* **325**(2007), No. 1, 54–68. MR2273028

[19] Ch. Zhang, T. Li, R. Agarwal, M. Bohner, Oscillation results for fourth-order nonlinear dynamic equations, *Appl. Math. Lett.* **25**(2012), No. 12, 2058–2065. MR2967789