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On the distance between adjacent zeros of solutions of first order differential equations with distributed delays

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Abstract. We estimate the distance between adjacent zeros of all solutions of the first order differential equation

$$x'(t) + \int_{h(t)}^{t} x(s) d_s R(t,s) = 0.$$

This form makes it possible to study equations with both discrete and continuous distributions of the delays. The obtained results are new and improve several known estimations. Some illustrative examples are given to show the advantages of our results over the known ones.

Keywords: distance between zeros, distributed delays, first order differential equation, oscillation.

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1 Introduction

In this work, we focus on an important topic of oscillation theory, namely the estimation of the distance between adjacent zeros of all solutions of a first order differential equation of the form

$$x'(t) + \int_{h(t)}^{t} x(s) d_s R(t,s) = 0, \qquad t \geq t_0, \tag{1.1}$$

where $h(t)$ is an increasing continuous function on $[t_0, \infty)$ such that $h(t) < t$, $\lim_{t\to\infty} h(t) = \infty$, and the function $R(t,s)$ is continuous with respect to t and nondecreasing with respect to $s \in [h(t), t]$ for all $t \geq t_0$.

By a solution of Eq. (1.1), we mean a continuous function $x(t)$ on $[h(t_0), \infty)$ that satisfies (1.1) for all $t \geq t_0$. A solution is called oscillatory if it has arbitrarily large zeros; otherwise it is called nonoscillatory. Equation (1.1) is called oscillatory if all its solutions are oscillatory.

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Equation (1.1) can be reduced to many forms. For example, it becomes

$$x'(t) + \sum_{k=1}^{m} [p_k(t)x(h_k(t))] + \int_{h(t)}^{t} \varphi(t,s)x(s)ds = 0, \qquad t \geq t_0, \tag{1.2}$$

when

$$R(t,s) = \sum_{k=1}^{m} \left[p_k(t)\chi_{(h_k(t),\infty)}(s) \right] + \int_{t_0}^{s} \varphi(t,\zeta)d\zeta,$$

where the kernel $\varphi(t,s)$ is a nonnegative continuous function on $[t_0, \infty) \times [h(t_0), \infty)$, $p_k(t) \in C([t_0, \infty), [0, \infty))$, $k = 1, 2, \dots, m$ and $\{h_k(t)\}_{k=1}^{m}$ is a family of continuous functions on $[t_0, \infty)$ such that $h_1 \equiv h$ and $h(t) \leq h_k(t) < t$ for $t \geq t_0$, $k = 2, 3, \dots, m$.

The oscillatory properties of several particular cases of Eq. (1.1) have attracted a great deal of attention during the last decades, see [1, 9–11]. Most efforts were directed to study the existence or nonexistence of arbitrarily large zeros. However, only a few authors were interested in investigating the location of zeros of Eq. (1.1) or any of its prototypes. For example, [2–4, 8, 16, 18, 21] obtained many interesting results for the equation

$$x'(t) + p(t)x(t-\tau) = 0, \qquad \tau > 0,\ t \geq t_0, \tag{1.3}$$

where $p(t) \in C([t_0, \infty), [0, \infty))$.

In [17, 19, 20], the authors estimated the distance between adjacent zeros of all solutions of the variable delay equation

$$x'(t) + p(t)x(h(t)) = 0, \qquad t \geq t_0. \tag{1.4}$$

The distribution of zeros of equations with distributed delays appeared in McCalla [13] for the first order initial value problem

$$\begin{aligned} x'(t) &= \sum_{k=0}^{N} A_k\, x(t+\theta_k) + \int_0^c A(\theta)\, x(t-\theta)d\theta, \qquad t > 0, \\ x(t) &= \phi(t), \qquad -c \leq t \leq 0, \end{aligned}$$

where A_k, θ_k are constants, $c > 0$, $-c = \theta_N < \cdots < \theta_1 < \theta_0 = 0$, $\phi \in L^r(-c, 0)$ for $r \geq 1$, and $A \in L^q(-c, 0)$ where $q = \frac{r}{r-1}$.

Barr [2] obtained the lowest upper bound estimate for Eq. (1.3) which equals 3τ when $P(t) = \int_{t-\tau}^{t} p(s)ds > 1$ for all $t > t_0 + \tau$. This estimate was further improved by [8, Corollary 3.2]. Also, estimates less than 3τ can be derived from [13, 18] but with different restrictions on the coefficient $p(t)$.

In this work, by improving and extending certain techniques from [2, 8, 20] to Eq. (1.1), we obtain new results which improve the 3τ-estimate when $P(t) > 1$ for all $t > t_0 + \tau$. Also, we relax the restriction $\liminf_{t\to\infty} \int_{h(t)}^{t} p(s)ds > \frac{1}{e}$ which is commonly used in the literature, see [15–21]. Moreover, some examples are given to illustrate the importance of our results.

2 Main results

In the sequel, we assume the existence of an increasing function $g(t)$ such that $h(t) \leq g(t) < t$ for all $t \geq t_1$ and some $t_1 \geq t_0$. Also, we define a function $\eta \in C([t_1, \infty), [0, \infty))$ and a sequence of functions $\{\alpha_n\}$, respectively, by $\eta(t) = R(t, g(t)) - R(t, h(t))$ and

$$\alpha_0(t) = t \quad \text{and} \quad \alpha_n(t) = g^{-n}(t), \quad \text{for all } n = 1, 2, \dots .$$

Lemma 2.1. *Assume that $T_1 \geq t_1$. If $x(t)$ is a positive solution of Eq.* (1.1) *on $[T_1, T_2]$, then*

$$x'(t) + \eta(t)x(g(t)) \leq 0, \quad \text{for all } t \in [h^{-2}(T_1),\ T_2],\ T_2 \geq h^{-2}(T_1). \tag{2.1}$$

Proof. Since $x(t) > 0$ on $[T_1,\ T_2]$, then Eq. (1.1) implies that $x'(t) \leq 0$ for all $t \in [h^{-1}(T_1),\ T_2]$. Thus

$$\int_{h(t)}^{t} x(s)d_sR(t,s) \geq \int_{h(t)}^{g(t)} x(s)d_sR(t,s) \geq \eta(t)x(g(t)),$$

for all $t \in [h^{-2}(T_1),\ T_2]$. Combining this inequality with Eq. (1.1), we obtain (2.1). □

For convenience, we define a sequence $\{q_n(t)\}_{n\geq 0}$ as follows:

$$\begin{aligned} &q_0(t) = \eta(t), \qquad t \geq t_1, \\ &q_n(t) = q_{n-1}(t)e^{\int_{g(t)}^{t} q_{n-1}(s)ds} \int_{g(t)}^{t} q_{n-1}(s)ds, \qquad t \geq \alpha_n(t_1), \quad n = 1, 2, \dots . \end{aligned} \tag{2.2}$$

Lemma 2.2. *Let $x(t)$ be a positive solution of Eq.* (1.1) *on $[T_1,\ T_2]$ where $T_1 \geq t_1$, $T_2 \geq \alpha_{n+1}(h^{-2}(T_1))$ and $n \in N$. Then*

$$\int_{g(t)}^{t} q_n(s)ds < 1, \quad \text{for all } t \in [\alpha_{n+1}(h^{-2}(T_1)),\ T_2]. \tag{2.3}$$

Proof. Since $x(t) > 0$ on $[T_1,\ T_2]$, then Lemma 2.1 yields

$$x'(t) + q_0(t)x(g(t)) \leq 0, \qquad t \in [h^{-2}(T_1),\ T_2]. \tag{2.4}$$

From Eq. (1.1), we have $x'(t) \leq 0$ on $[h^{-1}(T_1),\ T_2]$. So $x(g(t)) \geq x(t)$ on $[h^{-2}(T_1),\ T_2]$ and

$$x'(t) + q_0(t)x(t) \leq 0, \qquad t \in [h^{-2}(T_1),\ T_2]. \tag{2.5}$$

Integrating (2.4) from $g(t)$ to t,

$$x(t) - x(g(t)) + \int_{g(t)}^{t} q_0(s)x(g(s))ds \leq 0, \qquad t \in [\alpha_1(h^{-2}(T_1)),\ T_2].$$

Multiplying both sides of this inequality by $q_0(t)$ and using (2.4), we obtain

$$x'(t) + q_0(t)x(t) + q_0(t)\int_{g(t)}^{t} q_0(s)x(g(s))ds \leq 0, \qquad t \in [\alpha_1(h^{-2}(T_1)),\ T_2]. \tag{2.6}$$

The substitution $y_1(t) = e^{\int_{t_1}^{t} q_0(s)ds}x(t)$, $t \geq t_1$, yields, $y_1(t) > 0$ on $[T_1,\ T_2]$ and reduces (2.6) to the form

$$y_1'(t) + e^{\int_{t_1}^{t} q_0(s)ds} q_0(t)\int_{g(t)}^{t} q_0(s)x(g(s))ds \leq 0, \qquad t \in [\alpha_1(h^{-2}(T_1)),\ T_2],$$

which, due to the nonincreasing nature of $x(t)$ on $[h^{-1}(T_1),\ T_2]$, implies that

$$y_1'(t) + e^{\int_{g(t)}^{t} q_0(s)ds} y_1(g(t))q_0(t)\int_{g(t)}^{t} q_0(s)ds \leq 0, \qquad t \in [\alpha_1(h^{-2}(T_1)),\ T_2].$$

That is,

$$y_1'(t) + q_1(t)y_1(g(t)) \leq 0, \qquad t \in [\alpha_1(h^{-2}(T_1)),\ T_2]. \tag{2.7}$$

Since $y_1'(t) = e^{\int_{t_1}^{t} q_0(s)ds}\,[x'(t) + q_0(t)x(t)] \leq 0$ on $[h^{-2}(T_1)),\ T_2]$ by (2.5), then $y_1(t)$ is nonincreasing on $[h^{-2}(T_1)),\ T_2]$.

Now, consider that $y_n(t) = e^{\int_{\alpha_{n-1}(t_1)}^{t} q_{n-1}(s)ds}\, y_{n-1}(t)$, $t \geq \alpha_{n-1}(t_1)$ for $n \in \{2, 3, \ldots\}$ where $y_1(t)$ is defined as before. Then a simple induction leads to

$$y_n'(t) + q_n(t)y_n(g(t)) \leq 0, \qquad t \in [\alpha_n(h^{-2}(T_1)),\, T_2],$$

where $y_n'(t) \leq 0$ and $y_n(t) > 0$, for all $t \in [\alpha_{n-1}(h^{-2}(T_1)),\, T_2]$. Integrating this inequality from $g(t)$ to t,

$$y_n(t) - y_n(g(t)) + \int_{g(t)}^{t} q_n(s)y_n(g(s))ds \leq 0, \qquad t \in [\alpha_{n+1}(h^{-2}(T_1)),\, T_2],$$

which implies that

$$y_n(t) + \left[\int_{g(t)}^{t} q_n(s)ds - 1\right] y_n(g(t)) \leq 0, \qquad t \in [\alpha_{n+1}(h^{-2}(T_1)),\, T_2].$$

This proves the validity of (2.3). □

Next, we make use of a sequence $\{b_n(t)\}_{n\geq 0}$ defined as follows:

$$\begin{aligned} &b_0(t) = \eta(t), \qquad t \geq t_1, \\ &b_n(t) = b_{n-1}(t)\int_{g(t)}^{t} b_{n-1}(s)e^{\int_{g(s)}^{t} b_{n-1}(u)du}ds, \qquad t \geq \alpha_{2n}(t_1), \quad n = 1, 2, \ldots . \end{aligned} \tag{2.8}$$

Lemma 2.3. *If $x(t)$ is a positive solution of Eq. (1.1) on $[T_1,\, T_2]$ where $T_1 \geq t_1$, $T_2 \geq \alpha_{2n+1}(h^{-2}(T_1))$ and $n \in N$, then*

$$\begin{aligned} \int_{g(t)}^{t} b_1(s)e^{\int_{g(s)}^{g(t)} q_1(u)du}ds < 1, &\quad \text{for all } t \in [\alpha_3(h^{-2}(T_1)),\, T_2] \quad \text{if } n = 1, \\ \int_{g(t)}^{t} b_n(s)ds < 1, &\quad \text{for all } t \in [\alpha_{2n+1}(h^{-2}(T_1)),\, T_2] \quad \text{if } n > 1, \end{aligned} \tag{2.9}$$

where $q_1(t)$ is defined by (2.2).

Proof. Since $x(t) > 0$ on $[T_1,\, T_2]$, Lemma 2.1 implies

$$x'(t) + b_0(t)x(g(t)) \leq 0, \qquad t \in [h^{-2}(T_1),\, T_2]. \tag{2.10}$$

It is convenient (due to (2.9)) to complete the proof for the cases $n = 1$ and $n > 1$ separately. First, when $n = 1$, we have $x'(t) \leq 0$ for all $t \in [h^{-1}(T_1),\, T_2]$. Therefore, (2.10) yields

$$x'(t) + b_0(t)x(t) \leq 0, \qquad t \in [h^{-2}(T_1),\, T_2].$$

So, using similar reasoning as in the proof of Lemma 2.2, it is easy to obtain

$$z_1'(t) + b_0(t)\int_{g(t)}^{t} b_0(s)z_1(g(s))e^{\int_{g(s)}^{t} b_0(u)du}ds \leq 0, \qquad t \in [\alpha_1(h^{-2}(T_1)),\, T_2],$$

where $z_1(t) = e^{\int_{t_1}^{t} b_0(s)ds}x(t),\ t \geq t_1$. Since $z_1(t)$ is nonincreasing on $[h^{-2}(T_1),\, T_2]$, we obtain the inequalities

$$z_1'(t) + b_1(t)z_1(g(t)) \leq 0, \qquad t \in [\alpha_2(h^{-2}(T_1)),\, T_2], \tag{2.11}$$

and

$$z_1'(t) + b_1(t)z_1(t) \leq 0, \qquad t \in [\alpha_2(h^{-2}(T_1)),\, T_2]. \tag{2.12}$$

Inequality (2.11) leads to

$$z_1(t) - z_1(g(t)) + \int_{g(t)}^{t} b_1(s) z_1(g(s)) ds \le 0, \qquad t \in [\alpha_3(h^{-2}(T_1)), T_2]. \tag{2.13}$$

Note that $z_1(t)$ is the same as $y_1(t)$ of the proof of Lemma 2.2. Therefore, z_1 is a solution of (2.7) and hence,

$$-\frac{z_1'(u)}{z_1(u)} \ge q_1(u) \frac{z_1(g(u))}{z_1(u)}, \qquad u \in [\alpha_1(h^{-2}(T_1)), T_2]. \tag{2.14}$$

Assume that $g(t) \le s \le t$ for $t \in [\alpha_3(h^{-2}(T_1)), T_2]$ and integrate (2.14) from $g(s)$ to $g(t)$, we obtain

$$z_1(g(s)) \ge z_1(g(t)) e^{\int_{g(s)}^{g(t)} \frac{z_1(g(u))}{z_1(u)} q_1(u) du}, \qquad t \in [\alpha_3(h^{-2}(T_1)), T_2].$$

Since $z_1(t)$ is nonincreasing on $[h^{-2}(T_1), T_2]$, the above inequality yields

$$z_1(g(s)) \ge z_1(g(t)) e^{\int_{g(s)}^{g(t)} q_1(u) du}, \qquad t \in [\alpha_3(h^{-2}(T_1)), T_2] \text{ and } s \in [g(t), t].$$

Substituting into (2.13) and rearranging,

$$z_1(t) + \left[\int_{g(t)}^{t} b_1(s) e^{\int_{g(s)}^{g(t)} q_1(u) du} ds - 1 \right] z_1(g(t)) \le 0, \qquad t \in [\alpha_3(h^{-2}(T_1)), T_2].$$

Thus (2.9) holds when $n = 1$ due to the positivity of $z_1(t)$ and $z_1(g(t))$ on $[\alpha_3(h^{-2}(T_1)), T_2]$.

For the case when $n > 1$, multiplying (2.13) by $b_1(t)$ and using (2.11), we obtain

$$z_2'(t) + b_1(t) \int_{g(t)}^{t} b_1(s) z_2(g(s)) e^{\int_{g(s)}^{t} b_1(u) du} ds \le 0, \qquad t \in [\alpha_3(h^{-2}(T_1)), T_2],$$

where $z_2(t) = e^{\int_{\alpha_2(t_1)}^{t} b_1(s) ds} z_1(t)$, $t \ge \alpha_2(t_1)$. But (2.12) leads to $z_2'(t) \le 0$ on $[\alpha_2(h^{-2}(T_1)), T_2]$. Hence,

$$z_2'(t) + b_2(t) z_2(g(t)) \le 0, \qquad t \in [\alpha_4(h^{-2}(T_1)), T_2].$$

So, using induction, we can show

$$z_n'(t) + b_n(t) z_n(g(t)) \le 0, \qquad t \in [\alpha_{2n}(h^{-2}(T_1)), T_2], \tag{2.15}$$

where $z_n(t) = e^{\int_{\alpha_{2n-2}(t_1)}^{t} b_{n-1}(s) ds} z_{n-1}(t)$, $t \ge \alpha_{2n-2}(t_1)$ and $z_n'(t) \le 0$, on $[\alpha_{2n-2}(h^{-2}(T_1)), T_2]$.

Now, integrating (2.15) from $g(t)$ to t and using the nonincreasing nature of $z_n(t)$ on $[\alpha_{2n-2}(h^{-2}(T_1)), T_2]$, it follows that

$$z_n(t) + \left[\int_{g(t)}^{t} b_n(s) ds - 1 \right] z_n(g(t)) \le 0, \qquad t \in [\alpha_{2n+1}(h^{-2}(T_1)), T_2].$$

This completes the proof since $z_n(t)$ and $z_n(g(t))$ are positive on $[\alpha_{2n+1}(h^{-2}(T_1)), T_2]$. □

Following [15,19], we define a sequence $\{v_n(\rho)\}$, for $0 < \rho < 1$, as follows:

$$v_0(\rho) = 1, \qquad v_1(\rho) = \frac{1}{1-\rho}, \qquad v_n(\rho) = \frac{v_{n-2}(\rho)}{v_{n-2}(\rho) + 1 - e^{\rho v_{n-2}(\rho)}}, \qquad n = 2, 3, \ldots . \tag{2.16}$$

The following result is an extension of [19, Lemma 2.1] to Eq. (1.1).

Lemma 2.4. *Assume that*

$$\int_{g(t)}^{t} \eta(s)ds \geq \rho, \quad \text{for all } t \geq \alpha_1(t_1), \tag{2.17}$$

where $0 < \rho < 1$. *If* $x(t)$ *is a positive solution of Eq.* (1.1) *on* $[T_1, T_2]$ *where* $T_1 \geq t_1$, $T_2 \geq \alpha_n(h^{-2}(T_1))$, *and* n *is a nonnegative integer, then there exists a sequence* $\{v_n(\rho)\}$ *defined by* (2.16) *such that*

$$\frac{x(g(t))}{x(t)} \geq v_n(\rho) > 0, \quad \text{for all } t \in [\alpha_n(h^{-2}(T_1)), T_2]. \tag{2.18}$$

Proof. Since $x(t)$ is a positive solution of Eq. (1.1) on $[T_1, T_2]$, $x'(t) \leq 0$ for all $t \in [h^{-1}(T_1), T_2]$ which means that

$$\frac{x(g(t))}{x(t)} \geq 1 = v_0(\rho), \qquad t \in [\alpha_1(h^{-1}(T_1)), T_2] \subseteq [h^{-2}(T_1), T_2]. \tag{2.19}$$

Also, Lemma 2.1 leads to

$$x'(t) + \eta(t)x(g(t)) \leq 0, \qquad t \in [h^{-2}(T_1), T_2]. \tag{2.20}$$

Integrating (2.20) from $g(t)$ to t,

$$x(t) - x(g(t)) + \int_{g(t)}^{t} \eta(s)x(g(s))ds \leq 0, \qquad t \in [\alpha_1(h^{-2}(T_1)), T_2]. \tag{2.21}$$

The nondecreasing nature of $x(t)$ on $[h^{-1}(T_1), T_2]$ implies that

$$x(g(t)) \geq x(t) + \int_{g(t)}^{t} \eta(s)x(g(s))ds \geq x(t) + \rho x(g(t)), \qquad t \in [\alpha_1(h^{-2}(T_1)), T_2].$$

Therefore

$$\frac{x(g(t))}{x(t)} \geq \frac{1}{1-\rho} = v_1(\rho) > 0, \qquad t \in [\alpha_1(h^{-2}(T_1)), T_2].$$

On the other hand, dividing (2.20) by $x(t)$ and integrating from $g(s)$ to $g(t)$ where $g(t) \leq s \leq t$, we find

$$\int_{g(s)}^{g(t)} \frac{x'(u)}{x(u)}du \leq -\int_{g(s)}^{g(t)} \eta(u)\frac{x(g(u))}{x(u)}du,$$

hence

$$\ln \frac{x(g(s))}{x(g(t))} \geq \int_{g(s)}^{g(t)} \eta(u)\frac{x(g(u))}{x(u)}du, \qquad t \in [\alpha_2(h^{-2}(T_1)), T_2].$$

That is,

$$\frac{x(g(s))}{x(g(t))} \geq e^{\int_{g(s)}^{g(t)} \eta(u)\frac{x(g(u))}{x(u)}du}, \qquad t \in [\alpha_2(h^{-2}(T_1)), T_2],$$

which, according to (2.19), implies that

$$\frac{x(g(s))}{x(g(t))} \geq e^{v_0(\rho)\int_{g(s)}^{g(t)} \eta(u)du}, \tag{2.22}$$

where $g(t) \le s \le t$ and $t \in [\alpha_2(h^{-2}(T_1)), T_2]$. Combining (2.22) with (2.21),

$$\begin{aligned} x(g(t)) - x(t) &\ge x(g(t)) \int_{g(t)}^{t} \eta(s) \frac{x(g(s))}{x(g(t))} ds \\ &\ge x(g(t)) \int_{g(t)}^{t} \eta(s) e^{v_0(\rho) \int_{g(s)}^{g(t)} \eta(u)du} ds \\ &= x(g(t)) \int_{g(t)}^{t} \eta(s) e^{v_0(\rho)\left(\int_{g(s)}^{s} \eta(u)du - \int_{g(t)}^{s} \eta(u)du\right)} ds \\ &\ge x(g(t)) e^{\rho v_0(\rho)} \int_{g(t)}^{t} \eta(s) e^{-v_0(\rho) \int_{g(t)}^{s} \eta(u)du} ds \\ &= \frac{x(g(t)) e^{\rho v_0(\rho)} \left[1 - e^{-v_0(\rho) \int_{g(t)}^{t} \eta(u)du}\right]}{v_0(\rho)} \\ &\ge \frac{x(g(t)) \left(e^{\rho v_0(\rho)} - 1\right)}{v_0(\rho)}, \end{aligned}$$

for all $t \in [\alpha_2(h^{-2}(T_1)), T_2]$. Thus

$$\frac{x(g(t))}{x(t)} \ge \frac{v_0(\rho)}{v_0(\rho) + 1 - e^{\rho v_0(\rho)}} = v_2(\rho) > 0, \qquad t \in [\alpha_2(h^{-2}(T_1)), T_2].$$

Repeating this argument n times, we obtain

$$\frac{x(g(t))}{x(t)} \ge v_n(\rho) > 0, \qquad t \in [\alpha_n(h^{-2}(T_1)), T_2].$$

The proof is complete. □

In the sequel, we employ a sequence $\{c_n(s)\}_{n\ge 1}$ defined as follows:

$$\begin{aligned} &c_1(s) = \eta(s), \\ &c_n(s) = c_1(g^{n-1}(s))\gamma_{n-2}(s) \int_{g(t)}^{s} c_{n-1}(u)du, \qquad t \ge \alpha_n(t_1), \quad n = 2, 3, \ldots, \end{aligned}$$

where $g(t) \le s \le t$ for $t \ge t_1$, g^i stands for the ith composition of g and $\gamma_n(s) = \prod_{i=0}^{n} g'(g^i(s))$ for $n = 0, 1, \ldots$.

Lemma 2.5. *If $x(t)$ is a positive solution of Eq. (1.1) on $[T_1, T_2]$, where $T_1 \ge t_1$, $T_2 \ge \alpha_n(h^{-2}(T_1))$ and $n \in N$, then there exists a sequence $\{v_n(\rho)\}$ defined by (2.16) such that*

$$\sum_{r=1}^{n} \prod_{i=2}^{r} v_{n-(i-1)}(\rho) \int_{g(t)}^{t} c_r(s)ds < 1, \quad \text{for all } \ t \in [\alpha_n(h^{-2}(T_1)), T_2], \tag{2.23}$$

where $\prod_{i=2}^{1} v_{n-(i-1)}(\rho) = 1$, ρ is defined by (2.17) and $g(t)$ is continuously differentiable on $[t_1, \infty)$ when $n > 1$.

Proof. As in the proof of Lemma 2.3, we distinguish between two cases: $n = 1$ and $n > 1$. First, we assume that $n = 1$. A direct application of Lemma 2.1 yields

$$x'(t) + c_1(t)x(g(t)) \le 0, \qquad t \in [h^{-2}(T_1), T_2]. \tag{2.24}$$

Integrating the above inequality from $g(t)$ to t, we get

$$x(t) - x(g(t)) + \int_{g(t)}^{t} c_1(s)x(g(s))ds \le 0, \qquad t \in [\alpha_1(h^{-2}(T_1)), T_2], \tag{2.25}$$

which gives

$$x(t) + \left[\int_{g(t)}^{t} c_1(s)ds - 1\right] x(g(t)) \le 0, \qquad t \in [\alpha_1(h^{-2}(T_1)), T_2].$$

Due to the positivity of $x(t)$ and $x(g(t))$ on $[\alpha_1(h^{-2}(T_1)), T_2]$, we obtain $\int_{g(t)}^{t} c_1(s)ds < 1$. Hence, the proof of this case is complete.

Now, assume that $n > 1$. Using integration by parts and (2.24), we have

$$\begin{aligned}
\int_{g(t)}^{t} c_1(s)x(g(s))ds &= \int_{g(t)}^{t} x(g(s))d\left(\int_{g(t)}^{s} c_1(u)du\right) \\
&\ge \left(\int_{g(t)}^{t} c_1(s)ds\right) x(g(t)) + \int_{g(t)}^{t} x(g^2(s))c_1(g(s))g'(s)\int_{g(t)}^{s} c_1(u)du\, ds \\
&= \left(\int_{g(t)}^{t} c_1(s)ds\right) x(g(t)) + \int_{g(t)}^{t} c_2(s)x(g^2(s))ds, \qquad t \in [\alpha_2(h^{-2}(T_1)), T_2].
\end{aligned}$$

Again, applying integration by parts to the integral $\int_{g(t)}^{t} c_2(s)x(g^2(s))ds$, making use of (2.24) and substituting into the above inequality, we arrive at the inequality

$$\begin{aligned}
\int_{g(t)}^{t} c_1(s)x(g(s))ds \ge{}& \left(\int_{g(t)}^{t} c_1(s)ds\right) x(g(t)) + \left(\int_{g(t)}^{t} c_2(s)ds\right) x(g^2(t)) \\
&+ \int_{g(t)}^{t} x(g^3(s))c_1(g^2(s))g'(g(s))g'(s)\int_{g(t)}^{s} c_2(u)du\, ds \\
={}& \left(\int_{g(t)}^{t} c_1(s)ds\right) x(g(t)) + \left(\int_{g(t)}^{t} c_2(s)ds\right) x(g^2(t)) \\
&+ \int_{g(t)}^{t} c_3(s)x(g^3(s))ds,
\end{aligned}$$

for all $t \in [\alpha_3(h^{-2}(T_1)), T_2]$. Repeating this process, it follows that

$$\begin{aligned}
\int_{g(t)}^{t} c_1(s)x(g(s))ds \ge{}& \left(\int_{g(t)}^{t} c_1(s)ds\right) x(g(t)) + \left(\int_{g(t)}^{t} c_2(s)ds\right) x(g^2(t)) \\
&+ \left(\int_{g(t)}^{t} c_3(s)ds\right) x(g^3(t)) + \cdots + \int_{g(t)}^{t} c_n(s)x(g^n(s))ds,
\end{aligned} \tag{2.26}$$

for all $t \in [\alpha_n(h^{-2}(T_1)), T_2]$. From Eq. (1.1), we infer that $x'(t) \le 0$ for all $t \in [h^{-1}(T_1), T_2]$. Assume that $s \in [g(t), t]$ for all $t \in [\alpha_n(h^{-2}(T_1)), T_2]$, then $g^n(s) \in [g(h^{-2}(T_1)), T_2] \subseteq [h^{-1}(T_1), T_2]$ and hence $x(g^n(s)) \ge x(g^n(t))$. Thus

$$\int_{g(t)}^{t} c_n(s)x(g^n(s))ds \ge x(g^n(t))\int_{g(t)}^{t} c_n(s)ds, \qquad t \in [\alpha_n(h^{-2}(T_1)), T_2].$$

Combining this inequality with (2.26), it follows that

$$\int_{g(t)}^{t} c_1(s)x(g(s))ds \ge \sum_{r=1}^{n}\left(\int_{g(t)}^{t} c_r(s)ds\right) x(g^r(t)), \qquad t \in [\alpha_n(h^{-2}(T_1)), T_2]. \tag{2.27}$$

It is clear, for $t \in [\alpha_n(h^{-2}(T_1)), T_2]$, that $x(t) > 0$ and

$$g^{i-1}(t) \in [\alpha_{n-i+1}(h^{-2}(T_1)), T_2], \qquad i = 2, 3, \ldots, n.$$

Therefore, (2.18) implies that

$$\frac{x(g^i(t))}{x(g^{i-1}(t))} \geq v_{n-(i-1)}(\rho), \qquad i = 2, 3, \ldots, n.$$

Thus,

$$x(g^r(t)) = \left(\prod_{i=2}^{r} \frac{x(g^i(t))}{x(g^{i-1}(t))}\right) x(g(t)) \geq \left(\prod_{i=2}^{r} v_{n-(i-1)}(\rho)\right) x(g(t)), \qquad r = 1, 2, \ldots, n.$$

Consequently, (2.27) leads to,

$$\int_{g(t)}^{t} c_1(s)x(g(s))ds \geq \left(\sum_{r=1}^{n}\left(\prod_{i=2}^{r} v_{n-(i-1)}(\rho)\right)\int_{g(t)}^{t} c_r(s)ds\right) x(g(t)),$$

for all $t \in [\alpha_n(h^{-2}(T_1)), T_2]$. So (2.25) yields

$$x(t) + \left[\sum_{r=1}^{n}\prod_{i=2}^{r} v_{n-(i-1)}(\rho)\int_{g(t)}^{t} c_r(s)ds - 1\right] x(g(t)) \leq 0,$$

for all $t \in [\alpha_n(h^{-2}(T_1)), T_2]$. This inequality leads to (2.23) due to the positivity of $x(t)$ on $[T_1, T_2]$. $\square$

Now, we come into our main results. They are the contrapositive of the previous lemmas and hence will be given without proofs. Next, D_a stands for the upper bound between adjacent zeros of all solutions of Eq. (1.1) on $[a, \infty)$.

Theorem 2.6. *If there exists $n \in N$ such that*

$$\int_{g(t)}^{t} q_n(s)ds \geq 1, \quad \textit{for all } t \geq \alpha_{n+1}(h^{-2}(t_1)), \tag{2.28}$$

then Eq. (1.1) is oscillatory and $D_{t_1} \leq \sup\{\alpha_{n+1}(h^{-2}(t)) - t : t \geq t_1\}$.

Theorem 2.7. *If there exists $n \in N$ such that*

$$\begin{aligned} \int_{g(t)}^{t} b_1(s)e^{\int_{g(s)}^{g(t)} q_1(u)du}ds \geq 1, &\quad \textit{for all } t \geq \alpha_3(h^{-2}(t_1)), \quad \textit{if } n = 1,\\ \int_{g(t)}^{t} b_n(s)ds \geq 1, &\quad \textit{for all } t \geq \alpha_{2n+1}(h^{-2}(t_1)), \quad \textit{if } n > 1, \end{aligned} \tag{2.29}$$

where $q_1(t)$ is defined by (2.2), then Eq. (1.1) is oscillatory and $D_{t_1} \leq \sup\{\alpha_{2n+1}(h^{-2}(t)) - t : t \geq t_1\}$.

Remark 2.8. There are major differences between Theorems 2.6 and 2.7. Indeed, (2.29) is stronger than (2.28), while Theorem 2.6 provides smaller estimates than Theorem 2.7.

Theorem 2.9. *If there exists $n \in N$ such that $v_i(\rho) > 0$ for all $i = 1, 2, \ldots, n-1$ and*

$$\sum_{r=1}^{n}\prod_{i=2}^{r} v_{n-(i-1)}(\rho)\int_{g(t)}^{t} c_r(s)ds \geq 1, \quad \textit{for all } t \geq \alpha_n(h^{-2}(t_1)), \tag{2.30}$$

where $\prod_{i=2}^{1} v_{n-(i-1)}(\rho) = 1$, ρ is defined by (2.17) and $g(t)$ is continuously differentiable on $[t_1, \infty)$ when $n > 1$, then Eq.(1.1) is oscillatory and $D_{t_1} \leq \sup\{\alpha_n(h^{-2}(t)) - t : t \geq t_1\}$.

Generally, $v_n(\rho)$ could be negative or undefined for some values of ρ and n. For example $v_2(\rho) < 0$ for $\rho > \ln 2$, while it is undefined at $\rho = \ln 2$. For such values of ρ and n, Lemma 2.4 implies that $x(t)$ cannot be positive on $[T_1, T_2]$, $T_1 \geq t_1$, $T_2 \geq \alpha_n(h^{-2}(T_1))$. So, taking the linearity of Eq. (1.1) into consideration, we conclude that $x(t)$ has at least one zero on $[T_1, T_2]$. This leads to the following result.

Theorem 2.10. *If there exists $n \in N$ such that*

$$n = \min_{i \geq 1}\{i : v_i(\rho) < 0 \text{ or } v_i(\rho) \text{ is undefined}\},$$

where ρ is defined by (2.17), then Eq. (1.1) is oscillatory and $D_{t_1} \leq \sup\{\alpha_n(h^{-2}(t)) - t : t \geq t_1\}$.

Example 2.11. Consider the first order integro-differential equation

$$x'(t) + \int_{t-\tau}^{t} x(s)d(k^2 s) = 0, \qquad t \geq 0,$$

where $1.532 \leq k\tau < \frac{\pi}{2}$. This equation has the form of Eq. (1.1) with $R(t,s) = k^2 s$ and $h(t) = t - \tau$. Let $g(t) = t - \delta\tau$, $\delta = 0.48$. Then $\eta(t) = R(t, g(t)) - R(t, h(t)) = k^2\tau(1-\delta)$ and

$$\int_{t-\delta\tau}^{t} \eta(s)ds = k^2\tau^2\delta(1-\delta) = \rho.$$

Calculating (2.30) when $n = 2$, it follows that

$$\sum_{r=1}^{2}\prod_{i=2}^{r} v_{2-(i-1)}(\rho)\int_{t-\delta\tau}^{t} c_r(s)ds = \rho + \frac{\rho^2}{2(1-\rho)} > 1.$$

Therefore, Theorem 2.9 implies that $D_0 \leq 2\tau + 2\delta\tau = 2.96\tau$. It is worth noting that all results in [2,4,8,15–21] cannot be applied in this case. The only known result for us that can be used to estimate D_0 is [13, Theorem 1] which gives the estimation $D_0 < 3\tau$.

Notice that, the form of Eq. (1.1) produces Eq. (1.4) when $R(t, s) = p(t)\chi_{(h(t),\infty)}(s)$. In this case, if we assume that $h = g$ then $\eta(t) \equiv 0$ and hence our preceding results fail to apply. Fortunately, the techniques used to prove all above results can be applied verbatim to Eq. (1.2). The following result corresponds to Lemma 2.1.

Lemma 2.12. *Assume that $T_1 \geq t_1$ and $h_k(t) \leq g(t)$, $t \geq t_1$, $k = 2, 3, \ldots, m$. If Eq. (1.2) has a positive solution $x(t)$ on $[T_1, T_2]$, $T_2 \geq h^{-2}(T_1)$, then $x(t)$ is also a solution of the inequality*

$$x'(t) + Q(t)x(g(t)) \leq 0, \quad \text{for all } t \in [h^{-2}(T_1), T_2],$$

where $Q(t) = \sum_{k=1}^{m}[p_k(t)] + \int_{h(t)}^{g(t)} \varphi(t,s)ds$.

Now using Lemma 2.12 instead of Lemma 2.1 in the proofs of Theorems 2.6, 2.7 and 2.9, we obtain the next theorems for Eq. (1.2).

Theorem 2.13. *Assume that $h_k(t) \leq g(t)$, $t \geq t_1$, $k = 2, 3, \ldots, m$. If there exists $n \in N$ such that condition (2.28) is satisfied, with $q_0(t) = \sum_{k=1}^{m}[p_k(t)] + \int_{h(t)}^{g(t)} \varphi(t,s)ds$, $t \geq t_1$, then Eq. (1.2) is oscillatory and $D_{t_1} \leq \sup\{\alpha_{n+1}(h^{-2}(t)) - t : t \geq t_1\}$.*

Corollary 2.14. *If there exists $n \in N$ such that condition (2.28) is satisfied, with $q_0(t) = p(t)$, for $t \geq t_1$, then Eq. (1.4) is oscillatory and $D_{t_1} \leq \sup\{\alpha_{n+1}(h^{-2}(t)) - t : t \geq t_1\}$.*

Corollary 2.15. *Assume, in* (2.2)*, that* $q_0(t) = p(t)$ *and* $g(t) = t - \delta\tau$ *on* $[t_1, \infty)$*, for some* $\delta \in (0, 1]$*. If there exists* $n \in N$ *such that*

$$\int_{t-\delta\tau}^{t} q_{n-1}(s) e^{\int_{s-\delta\tau}^{s} q_{n-1}(u)du} \int_{s-\delta\tau}^{s} q_{n-1}(u)du\, ds \geq 1,$$

for all $t \geq t_1 + ((n+1)\delta + 2)\tau$*, then Eq.* (1.3) *is oscillatory and* $D_{t_1} \leq ((n+1)\delta + 2)\tau$*.*

Remark 2.16. It can be seen that Corollary 2.15 gives an estimate less than 3τ, when $n = 1$ and $\delta < \frac{1}{2}$. We refer that all estimates in [2,16,20,21] are integer multiples of τ greater than or equal 3τ.

Theorem 2.17. *Assume that* $h_k(t) \leq g(t)$*,* $t \geq t_1$*,* $k = 2, 3, \dots, m$*. If there exists* $n \in N$ *such that condition* (2.29) *is satisfied with* $b_0(t) = \sum_{k=1}^{m} [p_k(t)] + \int_{h(t)}^{g(t)} \varphi(t,s)ds$*,* $t \geq t_1$ *and* q_1 *is defined by* (2.2) *with* $q_0 = b_0$*, then Eq.* (1.2) *is oscillatory and* $D_{t_1} \leq \sup\{\alpha_{2n+1}(h^{-2}(t)) - t : t \geq t_1\}$*.*

Corollary 2.18. *If there exists* $n \in N$ *such that condition* (2.29) *is satisfied, with* $b_0(t) = p(t)$*,* $t \geq t_1$ *and* q_1 *is defined by* (2.2) *with* $q_0 = b_0$*, then Eq.* (1.4) *is oscillatory and* $D_{t_1} \leq \sup\{\alpha_{2n+1}(h^{-2}(t)) - t : t \geq t_1\}$*.*

The following corollary improves [8, Theorem 2.3].

Corollary 2.19. *Assume that* $b_0(t) = q_0(t) = p(t)$*,* $g(t) = t - \delta\tau$ *on* $[t_1, \infty)$*, for some* $\delta \in (0, 1]$ *and* q_1 *is defined by* (2.2)*. If there exists* $n \in N$ *such that*

$$\int_{t-\delta\tau}^{t} b_1(s) e^{\int_{s-\delta\tau}^{t-\delta\tau} q_1(u)du} ds \geq 1, \quad \text{for all } t \geq t_1 + (2 + 3\delta)\tau, \quad \text{if } n = 1,$$

$$\int_{t-\delta\tau}^{t} b_n(s) ds \geq 1, \quad \text{for all } t \geq t_1 + (2 + (2n+1)\delta)\tau, \quad \text{if } n > 1,$$

then Eq. (1.3) *is oscillatory and* $D_{t_1} \leq ((2n+1)\delta + 2)\tau$*.*

Corollary 2.20. *Assume that* $p(t) \geq p$*,* $t \geq t_1$*. If there exists* $\delta \in (0, 1]$ *such that* $p\tau\delta = \beta$ *and*

$$\frac{1}{\beta}\left(e^{\beta} - 1\right)\left(e^{\beta^2 e^{\beta}} - 1\right) \geq 1,$$

then Eq. (1.3) *is oscillatory and* $D_{t_1} \leq (3\delta + 2)\tau$*.*

Proof. Since

$$\begin{aligned}
\int_{t-\delta\tau}^{t} b_1(s) e^{\int_{s-\delta\tau}^{t-\delta\tau} q_1(u)du} ds &\geq \int_{t-\delta\tau}^{t} e^{p\beta e^{\beta}(t-s)} p \int_{s-\delta\tau}^{s} p e^{(s-u+\delta\tau)p} du\, ds \\
&= \left(e^{2\beta} - e^{\beta}\right) \int_{t-\delta\tau}^{t} e^{p\beta e^{\beta}(t-s)} p\, ds \\
&= \frac{1}{\beta e^{\beta}}\left(e^{2\beta} - e^{\beta}\right)\left(e^{\beta^2 e^{\beta}} - 1\right) \\
&= \frac{1}{\beta}\left(e^{\beta} - 1\right)\left(e^{\beta^2 e^{\beta}} - 1\right) \geq 1.
\end{aligned}$$

Then Corollary 2.19 implies that $D_{t_1} \leq (3\delta + 2)\tau$. □

The proof of the following corollary is the same as [8, Corollary 2.4].

Corollary 2.21. *Assume that* $\int_{t-\delta\tau}^{t} b_k(s)ds \geq \lambda > 0$, $t \geq t_1 + (2 + (2k+1)\delta)\tau$, *where* $\delta \in (0, 1]$, k *is a nonnegative integer and* $b_n(t)$ *is defined by* (2.8) *with* $b_0(t) = p(t)$ *and* $g(t) = t - \delta\tau$ *on* $[t_1, \infty)$. *Assume, further, that the sequence* $\{\lambda_n\}_{n\geq 0}$ *is defined by*

$$\lambda_n = \lambda_{n-1}(e^{2\lambda_{n-1}} - e^{\lambda_{n-1}}), \qquad n \geq 1, \text{ and } \lambda_0 = \lambda. \tag{2.31}$$

If there exists a positive integer n_0 *such that* $\lambda_{n_0} \geq 1$, *then Eq.* (1.3) *oscillates and*

$$D_{t_1} \leq ((2(n_0 + k) + 1)\delta + 2)\tau.$$

Corollary 2.21 is reduced to [8, Corollary 2.4], when $k = 0$ and $\delta = 1$. It is shown, in [8], that the recurrence relation defined by (2.31) diverges to infinity when $\lambda > \ln \frac{1+\sqrt{5}}{2}$ and converges to zero when $\lambda < \ln \frac{1+\sqrt{5}}{2}$. Therefore, in the latter case, [8, Corollary 2.4] cannot be applied. The following example shows that the conditions of Corollary 2.21 can be satisfied, even if λ is so small that many other approaches fail to apply.

Example 2.22. Consider the first order delay differential equation

$$x'(t) + \left(1 + b\cos(t) - \frac{1}{2}\sin(t)\right) x\left(t - \frac{\pi}{2}\right) = 0, \qquad t \geq 0,$$

which has the form (1.3) with $p(t) = 1 + b\cos(t) - \frac{1}{2}\sin(t)$ and $\tau = \frac{\pi}{2}$ such that $b = 0.735$. Thus

$$\int_{t-\frac{\pi}{2}}^{t} p(s)ds = \frac{\pi}{2} + b(\cos(t) + \sin(t)) + \frac{1}{2}(\cos(t) - \sin(t)).$$

Then,

$$\min_{t\geq\frac{\pi}{2}} \int_{t-\frac{\pi}{2}}^{t} p(s)ds = \frac{\pi}{2} - \frac{4b^2+1}{\sqrt{8b^2+2}} \simeq 0.3136 < \frac{1}{e}.$$

This means that [8, Corollary 2.4], [15, Theorem 2], [17, Theorem 2.1], [19, Theorem 2.1], [20, Theorems 1–2] and all results in [16,18,21] fail to apply. However,

$$\begin{aligned}\int_{t-\frac{\pi}{2}}^{t} b_1(s)ds &= \int_{t-\frac{\pi}{2}}^{t} p(s) \int_{s-\frac{\pi}{2}}^{s} p(u) e^{\int_{u-\frac{\pi}{2}}^{s} p(v)dv} du\, ds,\\ &\geq e \int_{t-\frac{\pi}{2}}^{t} p(s) \int_{s-\frac{\pi}{2}}^{s} p(u) \int_{u-\frac{\pi}{2}}^{s} p(v)dv\, du\, ds > 0.23e \approx 0.62521.\end{aligned}$$

Thus $\lambda_0 \approx 0.62521$, and (2.31) implies that $\lambda_1 \approx 1.0148$. Now, applying Corollary 2.21 with $\delta = k = n_0 = 1$, it follows that $D_0 \leq 7\tau$.

Theorem 2.23. *Assume that* $h_k(t) \leq g(t) < t$, $t \geq t_1$, $k = 2, 3, ..., m$. *If there exists* $n \in N$ *such that* $v_i(\rho) > 0$ *for all* $i = 1, 2, ..., n-1$ *and*

$$\sum_{r=1}^{n} \prod_{i=2}^{r} v_{n-(i-1)}(\rho) \int_{g(t)}^{t} c_r(s)ds \geq 1, \quad \text{for all } t \geq \alpha_n(h^{-2}(t_1)),$$

where $c_1(t) = \sum_{k=1}^{m} [p_k(t)] + \int_{h(t)}^{g(t)} \varphi(t,s)ds$, $t \geq t_1$, $g(t)$ *is continuously differentiable on* $[t_1, \infty)$ *when* $n > 1$, $\prod_{i=2}^{1} v_{n-(i-1)}(\rho) = 1$ *and* ρ *is defined by* (2.17) *with* $\eta(t)$ *is replaced by* $c_1(t)$, *then Eq.* (1.2) *is oscillatory and* $D_{t_1} \leq \sup\{\alpha_n(h^{-2}(t)) - t : t \geq t_1\}$.

The following two particular cases of Theorem 2.23 improve [8, Theorem 2.5] and [20, Theorem 1], respectively.

Corollary 2.24. *Assume that $c_1(t) = p(t)$ and $g(t) = t - \delta\tau$, $\delta \in (0, 1]$, $t \geq t_1$. If there exists $n \in N$ such that $v_i(\rho) > 0$ for all $i = 1, 2, \ldots, n-1$ and*

$$\sum_{r=1}^{n} \prod_{i=2}^{r} v_{n-(i-1)}(\rho) \int_{t-\delta\tau}^{t} c_r(s)ds \geq 1, \quad \text{for all } t \geq t_1 + (n\delta + 2)\tau,$$

where $\prod_{i=2}^{1} v_{n-(i-1)}(\rho) = 1$ and ρ satisfies (2.17) with $\eta(t)$ is replaced by $c_1(t)$, then Eq. (1.3) is oscillatory and $D_{t_1} \leq (n\delta + 2)\tau$.

Corollary 2.25. *If there exists $n \in N$ such that $v_i(\rho) > 0$ for all $i = 1, 2, \ldots, n-1$ and*

$$\sum_{r=1}^{n} \prod_{i=2}^{r} v_{n-(i-1)}(\rho) \int_{g(t)}^{t} c_r(s)ds \geq 1, \quad \text{for all } t \geq \alpha_n(h^{-2}(t_1)),$$

where $c_1(t) = p(t)$, $t \geq t_1$, $g(t)$ is continuously differentiable function on $[t_1, \infty)$ when $n > 1$, $\prod_{i=2}^{1} v_{n-(i-1)}(\rho) = 1$ and ρ is defined by (2.17) with $\eta(t)$ is replaced by $c_1(t)$, then Eq. (1.4) is oscillatory and $D_{t_1} \leq \sup\{\alpha_n(h^{-2}(t)) - t : t \geq t_1\}$.

Remark 2.26.

1- There are many alternatives to the sequence $\{v_n(\rho)\}_{n\geq 0}$ (see [7, 8]). We employed $\{v_n(\rho)\}_{n\geq 0}$ since it gives a good lower bound of the quotient $\frac{x(h(t))}{x(t)}$. Theorems 2.9 and 2.23 can be improved if one uses a better sequence than $\{v_n(\rho)\}_{n\geq 0}$. A possible choice of such sequence can be found in [12, 14].

2- Also, it is very interesting if one could adapt the approaches of [5, 6, 13] to study the distribution of zeros of all solutions of Eq. (1.1), particularly if $R(t, s)$ is not increasing in s.

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