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Limit cycles for a class of polynomial differential systems

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Abstract. In this paper, we consider the limit cycles of a class of polynomial differential systems of the form $\dot{x} = -y^{2p-1}$, $\dot{y} = x^{2mp-1} + \varepsilon(px^{2mp} + qy^{2p})(g(x,y) - A)$, where $g(x,y)$ is a polynomial. We obtain the maximum number of limit cycles that bifurcate from the periodic orbits of a center using the averaging theory of first order.

Keywords: limit cycles, polynomial differential systems, averaging theory, (p,q)-trigonometric functions.

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1 Introduction and main results

One of the main problems in the qualitative theory of real planar differential equations is to determinate the number of limit cycles for a given planar differential system. As we all know, this is a very difficult problem for a general polynomial system. Therefore, many mathematicians study some systems with special conditions. To obtain the number of limit cycles as many as possible for a planar differential system, we usually take in consideration of the bifurcation theory. In recent decades, many new results have been obtained (see [11, 13]).

Consider the following Kukles polynomial differential system

$$\begin{cases} \dot{x} = -y \\ \dot{y} = x + \sum_{d=2}^{n} g_d(x,y), \end{cases} \tag{1.1}$$

where $[\cdot] = \frac{d}{dt}$ and $g_d(x,y)$ is a homogeneous polynomial of degree n.

Several works have been done for $\sum_{d=2}^{n} g_d(x,y)$ of lower order. In [12], the author gave necessary and sufficient conditions in order that (1.1) with $n = 3$ has a center at the origin. The authors in [21] considered a reduced Kukles systems with $n = 3$ without the term y^3. The authors of [3] presented some systems that yield at most five limit cycles bifurcating from the origin. In [9], the authors studied the conditions of at most one limit cycle bifurcating from the origin for (1.1) with $n = 3$. The authors proved that the Kukles system with two fine foci

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can generate at least six limit cycles in [26]. In [22], the author proved some cubic systems of (1.1) can have seven limit cycles.

The authors in [16] introduced a class of cubic systems (1.1) with an invariant parabola which coexists with a center under given parameters. In [2], the authors described a cubic system (1.1) that has an invariant hyperbola to coexist with two limit cycles.

Afterwards, the authors' interests converted to finding maximum number of small amplitude limit cycles coexisting with invariant ellipses. In [6], the authors studied the systems of the form (1.1). For $n = 4$ and $n = 5$, they obtained the maximum numbers of small-amplitude cycles using the method of calculation of Poincaré–Liapunov constants (see [8]). The authors in [23] presented a class of quintic systems of the form (1.1) having an invariant ellipse with what small amplitude limit cycles bifurcating from the origin coexist. In [25], the authors considered another class of extended Kukles system of degree $2n + 5$ with an invariant non-degenerate conic and two invariant straight line which coexist with at least n small amplitude limit cycles for certain values of the parameters. In [24], the authors introduced the following system

$$\begin{cases} \dot{x} = -y \\ \dot{y} = x + \varepsilon(x^2 + y^2)\big(\sum_{i=1}^{n_1-2}(q_i x^i + \tilde{q}_i y^i) - A_1\big), \end{cases} \tag{1.2}$$

where $q_i, \tilde{q}_i \in R$ and $A_1 > 0$.

Regarding system (1.2) as a perturbation of a Hamiltonian system, the authors studied a class of Kukles systems having an invariant ellipse in the case of $n \in \{2k, 2k-1\}$ and obtained at most $k - 2$ limit cycles bifurcating from a unperturbed Hamiltonian center.

In [18], using the averaging theory of first and second order, the authors studied the maximum number of limit cycles bifurcating from the periodic orbits of the linear center $\dot{x} = -y$, $\dot{y} = x$ perturbed inside a class of generalized Kukles polynomial differential systems

$$\begin{cases} \dot{x} = -y \\ \dot{y} = x + \sum_{k\geq 1} \varepsilon^k (f^k_{n_1}(x) + g^k_{n_2}(x)y + h^k_{n_3}(x)y^2 + d^k_0 y^3), \end{cases}$$

where for every k, the polynomials $f^k_{n_1}(x)$, $g^k_{n_2}(x)$ and $h^k_{n_3}(x)$ have degree n_1, n_2 and n_3, respectively, $d^k_0 \neq 0$ is a real number and ε is a small parameter.

Using the averaging theory of first, second order and third order, the author in [20] studied the limit cycles of the following differential systems obtained by polynomial perturbations with arbitrary degree on the second component of the standard linear center

$$\begin{cases} \dot{x} = -y \\ \dot{y} = x + (x^2 + y^2) \sum_{l\geq 1} \varepsilon^l (g_l(x, y) - A_l), \end{cases}$$

where the degree of polynomial $g_l(x, y)$ with $g_l(0, 0) = 0$ is $n_l - 2$ which is much greater than 1 and $A_l > 0$, and gave an accurate upper bound of the maximum number of limit cycles that the perturbed system can have bifurcating from the periodic orbits of linear center:

(a) $k_1 - 2$ for first order;

(b) $\{k_2 - 2,\ 2[\frac{n_1-2}{2}] - 2\}$ for second order;

(c) $\{k_3 - 2,\ [\frac{n_2-2}{2}] - 1\}$ for third order.

In [17], the authors proved that the maximum number of limit cycles of the following generalized *Liénard* polynomial differential system

$$\begin{cases} \dot{x} = -y^{2p-1} \\ \dot{y} = x^{2q-1} - \varepsilon f(x) y^{2n-1}, \end{cases}$$

is at most $[\frac{m}{2}]$ where p, q and n are positive integers, m is the degree of the polynomial $f(x)$.

Consider the following system

$$\begin{cases} \dot{x} = -y^{2p-1} \\ \dot{y} = x^{2mp-1} + \varepsilon(px^{2mp} + mpy^{2p})(g(x,y) - A), \end{cases} \tag{1.3}$$

where the degree of polynomial $g(x,y)$ with $g(0,0) = 0$ is n which is much greater than or equal to 1, p is a positive integer and $A > 0$. Our main result is the following.

Theorem 1.1. *Assume that $A > 0$, the degree of polynomial $g(x,y)$ with $g(0,0) = 0$ is n. Let $n \in \{2k, 2k-1\}$ and $k \geq 1$, then for ε sufficiently small, the maximum number of limit cycles of (1.3) bifurcating from the periodic orbits of the center $\dot{x} = -y^{2p-1}$, $\dot{y} = x^{2mp-1}$ using the averaging theory of first order is*

(a) $\dfrac{(k+2)(k-1)}{2}$, *if* $k \leq m$,

(b) $mk - \dfrac{m(m-1)+2}{2}$, *if* $k \geq m+1$.

System (1.3) with $p = m = 1$ was studied by [20].

In [19], the authors studied the maximum numbers of limit cycles that can bifurcate from an integrable nonlinear quadratic isochronous center. They proved that the number of limit cycles in a Liénard-like perturbation of a quadratic nonlinear center is always greater or equal than a Liénard-like perturbation of a linear center.

Our results indicate that the number of limit cycles in a perturbation of a nonlinear center is always greater or equal than a perturbation of a linear center.

2 Preliminaries

Averaging theory of first order

In order to obtain the existence of periodic orbits, we introduce the averaging theory found in lots of literatures, such as [5,7,10,14] and so on. In short, the method gives a quantitative relation between the solutions of a non-autonomous periodic system and the solutions of its averaged system, which is autonomous. It is summarized as follows.

Consider the differential system

$$\dot{x}(t) = \varepsilon F_1(x,t) + \varepsilon^2 R(x,t,\varepsilon),$$

where $F_1 : R \times D \to R$ and $R : R \times D \times (-\varepsilon_0, \varepsilon_0) \to R$ are continuous functions whose first variable is T-periodic and $D \in R$ is an open set. Assume that the following hypotheses (1) and (2) hold.

(1) F_1 and R are locally Lipschitz with respect to x. We define $F_{1,0} : D \to R$ as

$$F_{1,0}(z) = \frac{1}{T}\int_0^T F_1(s,z)ds,$$

(2) $V \subset D$ is an open and bounded set and for each $\varepsilon \in (-\varepsilon_0, \varepsilon_0)\backslash\{0\}$, there exists $a_\epsilon \in V$ such that $F_{1,0}(a_\varepsilon)$ and $d_B(F_{1,0}, V, a_\varepsilon) \neq 0$.

Then for $\varepsilon > 0$ sufficiently small there exists a T-periodic solution $\varphi(\cdot, a_\varepsilon)$ of the system such that $\varphi(0, a_\varepsilon) \to a_\varepsilon$ when $\varepsilon \to 0$.

The expression $d_B(F_{1,0}, V, a_\varepsilon) \neq 0$ means that the Brouwer degree of the function $F_{1,0} : V \to R$ at the point a_ε is not zero. A sufficient condition in order that this inequality holds is that the Jacobian of the function $F_{1,0}$ at a_ε is different from zero.

Descartes Theorem

In order to confirm the number of zeros of certain real polynomial, we will make use of the following Descartes Theorem [1,4].

Consider the real polynomial $p(x) = a_{i_1}x^{i_1} + a_{i_2}x^{i_2} + \cdots + a_{i_k}x^{i_k}$ with $0 \le i_1 < i_2 < \cdots < i_k$ and $i_k \neq 0$ real constants for $j \in \{1, 2, \cdots, k\}$. When $a_{i_j}a_{i_{j+1}} < 0$, we say that a_{i_j} and $a_{i_{j+1}}$ have a variation of sign. If the number of variations of signs is m, then $p(x)$ has at most m positive real roots. Moreover, it is always possible to choose the coefficients of $p(x)$ in such a way that $p(x)$ has exactly $k-1$ positive real roots.

3 Proof of of Theorem 1.1

The proof of Theorem 1.1 depends on the first order averaging theory presented in Section 2.

Obviously, system (1.3) with $\varepsilon = 0$ is a Hamiltonian system with Hamiltonian function

$$H(x,y) = \frac{1}{2mp}x^{2mp} + \frac{1}{2p}y^{2p}.$$

This system has a global center (i.e. the periodic orbits surrounding the origin filled the whole plane $\mathbb{R}$ at the origin of coordinates). In this paper we want to study how many periodic orbits persist after perturbing the periodic orbits of this center in the system (1.3) for $\varepsilon \neq 0$ sufficiently small.

Applying the following (p,q)-trigonometric function

$$x(\theta) = \operatorname{Cs}\theta, \qquad y(\theta) = \operatorname{Sn}\theta. \tag{3.1}$$

We regard it as the solution of the following initial value problem

$$\dot{x} = -y^{2p-1}, \qquad \dot{y} = x^{2mp-1}, \qquad x(0) = p^{-\frac{1}{2mp}}, \qquad y(0) = 0.$$

Clearly, we can verify that the following equality holds

$$p\operatorname{Cs}^{2mp}\theta + mp\operatorname{Sn}^{2p}\theta = 1.$$

Referring to [15], by means of easy computation, it follows that $\operatorname{Cs}\theta$ and $\operatorname{Sn}\theta$ are T-periodic functions with period

$$T = 2p^{-\frac{1}{2mp}}mp^{-\frac{1}{2p}}\frac{\Gamma(\frac{1}{2p})\Gamma(\frac{1}{2mp})}{\Gamma(\frac{1}{2p}+\frac{1}{2mp})}. \tag{3.2}$$

We will take the following generalized Liapunov polar coordinate change of variables

$$x = r^{p} \operatorname{Cs}\theta, \qquad y = r^{mp} \operatorname{Sn}\theta. \tag{3.3}$$

Let

$$\begin{aligned} g(x,y) &= \sum_{d=1}^{n} \left(\sum_{j=0}^{d} a_{j,d} x^{d-j} y^{j} \right) \\ &= A_g(x,y) + B_g(x,y) + C_g(x,y) + D_g(x,y), \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} A_g(x,y) &= \sum_{l=1}^{[\frac{n}{2}]} \left(\sum_{i=1}^{l} a_{2i-1,2l} x^{2l-2i+1} y^{2i-1} \right), \\ B_g(x,y) &= \sum_{l=1}^{k} \left(\sum_{i=0}^{l} a_{2i,2l-1} x^{2l-2i-1} y^{2i} \right), \\ C_g(x,y) &= \sum_{l=1}^{[\frac{n}{2}]} \left(\sum_{i=0}^{i} a_{2i,2l} x^{2l-2i} y^{2i} \right), \\ D_g(x,y) &= \sum_{l=1}^{k} \left(\sum_{i=1}^{i} a_{2i-1,2l-1} x^{2l-2i} y^{2i-1} \right). \end{aligned}$$

For simplicity of computation, we will use the following formula

$$\int_0^T \operatorname{Cs}^{2a+1}\theta \operatorname{Sn}^{b}\theta d\theta = \int_0^T \operatorname{Cs}^{a}\theta \operatorname{Sn}^{2b+1}\theta d\theta = 0, \qquad \forall\, a,b \in N, \tag{3.5}$$

where N denotes a nonnegative integer.

By means of the change of coordinates (3.3), we rewrite the polynomial $g(x,y)$ as

$$g(r,\theta) = A_g(r,\theta) + B_g(r,\theta) + C_g(r,\theta) + D_g(r,\theta), \tag{3.6}$$

where

$$\begin{aligned} A_g(r,\theta) &= \sum_{l=1}^{[\frac{n}{2}]} \left(\sum_{i=1}^{l} a_{2i-1,2l} \operatorname{Cs}^{2l-2i+1}\theta \operatorname{Sn}^{2i-1}\theta \right) r^{p(2l-2i+1)+mp(2i-1)}, \\ B_g(r,\theta) &= \sum_{l=1}^{k} \left(\sum_{i=0}^{l} a_{2i,2l-1} \operatorname{Cs}^{2l-2i-1}\theta \operatorname{Sn}^{2i}\theta \right) r^{p(2l-2i-1)+2mpi}, \\ C_g(r,\theta) &= \sum_{l=1}^{[\frac{n}{2}]} \left(\sum_{i=0}^{i} a_{2i,2l} \operatorname{Cs}^{2l-2i}\theta \operatorname{Sn}^{2i}\theta \right) r^{p(2l-2i)+2mpi}, \\ D_g(r,\theta) &= \sum_{l=1}^{k} \left(\sum_{i=1}^{i} a_{2i-1,2l-1} \operatorname{Cs}^{2l-2i}\theta \operatorname{Sn}^{2i-1}\theta \right) r^{p(2l-2i)+mp(2i-1)}, \end{aligned}$$

and the system of the form of (1.3) can be written as

$$\begin{aligned} \dot{r} &= \varepsilon r^{2mp^2-mp+1} \operatorname{Sn}^{2p-1}\theta \cdot [g(r,\theta) - A], \\ \dot{\theta} &= r^{2mp^2-p-mp} + \varepsilon p r^{2mp^2-mp} \operatorname{Cs}\theta \cdot [g(r,\theta) - A]. \end{aligned} \tag{3.7}$$

Now taking θ as independent variable, system (3.7) becomes

$$\frac{dr}{d\theta} = \varepsilon F_1(r,\theta) + \mathcal{O}(\epsilon^2),$$

where

$$F_1(r,\theta) = r^{p+1}\,\mathrm{Sn}^{2p-1}\,\theta \cdot [g(r,\theta) - A], \tag{3.8}$$

which is in the standard form for applying the averaging theory.

Then from Section 2 we obtain that

$$F_{1,0}(r) = \frac{1}{T}\int_0^T r^{p+1}\,\mathrm{Sn}^{2p-1}\,\theta[g(r,\theta) - A]d\theta. \tag{3.9}$$

Using (3.6) in (3.9), we have in light of (3.5) that

$$F_{1,0}(r) = \frac{r^{p-mp+1}}{T}\sum_{l=1}^{k}\left(\sum_{i=1}^{l} a_{2i-1,2l-1}b_{i,l}r^{2(mp-p)i}\right)\cdot r^{2pl}, \tag{3.10}$$

where $b_{i,l} = \int_0^T \mathrm{Cs}^{2l-2i}\,\theta\,\mathrm{Sn}^{2p+2i-2}\,\theta d\theta \neq 0$.

For the simplicity of calculation, Let $c_{i,l} = a_{2i-1,2l-1}b_{i,l}$, $t = r^{2p}$, therefore, (3.10) can be reduced to

$$F_{1,0}(t) = \frac{t^{\frac{p-mp+1}{2p}}}{T}\sum_{l=1}^{k}\left(\sum_{i=1}^{l} c_{i,l}\right)\cdot t^{(m-1)i+l}. \tag{3.11}$$

As we all know, the number of positive roots of $F_{1,0}(t)$ is equal to that of

$$G_{1,0}(t) = \sum_{l=1}^{k}\left(\sum_{i=1}^{l} c_{i,l}\right)\cdot t^{(m-1)i+l}. \tag{3.12}$$

Now we expand the polynomial (3.12) as follows:

$$\begin{aligned}
G_{1,0}(t) &= \sum_{l=1}^{k}\left(\sum_{i=1}^{l} c_{i,l}\right)\cdot t^{(m-1)i+l}\\
&= \sum_{l=1}^{k}(c_{1,l}t^{m-1+l} + c_{2,l}t^{2(m-1)+l} + c_{3,l}t^{3(m-1)+l} + \cdots + c_{l-1,l}t^{(l-1)(m-1)+l} + c_{l,l}t^{lm})\\
&= c_{1,1}t^{m} + (c_{1,2}t^{m+1} + c_{2,2}t^{2m}) + (c_{1,3}t^{m+2} + c_{2,3}t^{2m+1} + c_{3,3}t^{3m})\\
&\quad + \cdots\\
&\quad + (c_{1,k-2}t^{m+k-3} + c_{2,k-2}t^{2m+k-4} + \cdots + c_{k-2,k-2}t^{m(k-2)}\\
&\quad + (c_{1,k-1}t^{m+k-2} + c_{2,k-1}t^{2m+k-3} + c_{3,k-1}t^{3m+k-4} + \cdots + c_{k-1,k-1}t^{m(k-1)})\\
&\quad + (c_{1,k}t^{m+k-1} + c_{2,k}t^{2m+k-2} + c_{3,k}t^{3m+k-3} + \cdots + c_{k,k}t^{mk-1}).
\end{aligned} \tag{3.13}$$

For the convenience of considering the above polynomial, we can further rewrite (3.13) as:

$$\begin{aligned} G_{1,0}(t) = {} & (c_{1,1}t^{m} + c_{1,2}t^{m+1} + c_{1,3}t^{m+2} + c_{1,4}t^{m+3} + \cdots + c_{1,k-1}t^{m+k-2} + c_{1,k}t^{m+k-1}) \\ & + (c_{2,2}t^{2m} + c_{2,3}t^{2m+1} + c_{2,4}t^{2m+2} + c_{2,5}t^{2m+3} + \cdots + c_{2,k-1}t^{2m+k-3} + c_{2,k}t^{2m+k-2}) \\ & + (c_{3,3}t^{3m} + c_{3,4}t^{3m+1} + c_{3,5}t^{3m+2} + \cdots + c_{3,k-1}t^{3m+k-4} + c_{3,k}t^{3m+k-3}) \\ & + \cdots \\ & + (c_{k-3,k-3}t^{(k-3)m} + c_{k-3,k-2}t^{(k-3)m+1} + c_{k-3,k-1}t^{(k-3)m+2} + c_{k-3,k}t^{(k-3)m+3}) \\ & + (c_{k-2,k-2}t^{(k-2)m} + c_{k-2,k-1}t^{(k-2)m+1} + c_{k-2,k}t^{(k-2)m+2}) \\ & + (c_{k-1,k-1}t^{(k-1)m} + c_{k-1,k}t^{(k-1)m+1}) \\ & + c_{k,k}t^{km}. \end{aligned} \tag{3.14}$$

Now we consider the number of positive roots of polynomial (3.14).

Statement (a) of Theorem 1.1

Suppose that $k \leq m$, then the number of positive roots of polynomial (3.14) is at most

$$\frac{(k-1)(k+2)}{2}.$$

Proof. From (3.14), it is not difficult to find that the degree of the latter item is greater than the one of the former item in each line. Since $k \leq m$, it follows that the degree of first item in the latter line is greater than the one of the last item in the former line. Therefore, the number of terms in polynomial (3.14) is

$$k + k - 1 + k - 2 + k - 3 + \cdots + 3 + 2 + 1 = \frac{k(k+1)}{2}.$$

Consequently, according to the Descartes Theorem stated in Section 2, we can choose the appropriate coefficients $c_{i,j}$ in order that the simple positive roots number of $G_{1,0}(r)$ is at most

$$\frac{k(k+1)}{2} - 1 = \frac{(k-1)(k+2)}{2}.$$

This completes the proof of statement (a). □

Statement (b) of Theorem 1.1

Suppose that $k \geq m+1$, then the number of positive roots of polynomial (3.14) is at most

$$mk - \frac{m(m-1)+2}{2}.$$

Proof. In the same way, let $i = k - m$, where $1 \leq i \leq k$. In this condition, we find that after the collection of terms of the polynomial (3.14), the number of terms in polynomial (3.14) is

$$k + k - 1 + k - 2 + \cdots + 2 + 1 - i - (i-1) - (i-2) - \cdots - 2 - 1 = \frac{(k-i)(k+i+1)}{2}.$$

Similarly, according to the Descartes Theorem stated in Section 2, we can choose the appropriate coefficients $c_{i,j}$ in order that maximum number of the simple positive roots of $G_{1,0}(r)$ is

$$\begin{aligned} &k+k-1+k-2+\cdots+2+1-i-(i-1)-(i-2)-\cdots-2-1-1 \\ &\quad=\frac{(k-i)(k+i+1)}{2}-1. \end{aligned} \tag{3.15}$$

Finally, substitute $i=k-m$ to (3.15), it is now obvious that the statement (b) of the theorem holds, this completes the proof of statement (b). □

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