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Critical point result of Schechter type in a Banach space

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Abstract. Using Ekeland's variational principle we give a critical point theorem of Schechter type for extrema on a sublevel set in a Banach space. This result can be applied to localize the solutions of PDEs which contain nonlinear homogeneous operators.

Keywords: critical point, Ekeland's variational principle, Palais–Smale type compactness condition, p-Laplacian.

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1 Introduction

Finding the set of solutions of certain PDEs is closely related to the investigation of the critical points of a certain functional defined on an appropriate Hilbert or Banach space. Mountain pass theorems, saddle point theorems, linking theorems, mountain cliff theorems give sufficient conditions for the existence of a minimizer for a certain differentiable functional defined on the whole space or on a bounded region (for example, see [3, 16–19, 21]).

In [13–15] R. Precup studies critical point theorems of Schechter type for C^1 functionals on a closed ball and also on a closed conical shell in a Hilbert space by using Palais–Smale type compactness conditions and also Leray–Schauder conditions on the boundary. These results can be used successfully to localize the solutions of PDEs involving the Laplace operator.

In our paper we improve the above mentioned Schechter type results (on a ball) for sublevel sets in locally uniformly convex Banach spaces and then apply our result for localizing the solutions for p-Laplace type equations on bounded, and also on unbounded domains.

The paper is structured as follows: Section 2 contains certain preliminaries concerning duality mappings on Banach spaces and the assumptions for the critical point problem which we are investigating. Section 3 states the main result of our paper. Section 4 presents two examples of localizing the solutions for problems containing the p-Laplacian.

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2 Preliminaries

Let X be a real Banach space, X^* its dual, $\langle \cdot, \cdot \rangle$ denotes the duality between X^* and X. The norm on X and on X^* is denoted by $\| \cdot \|$.

A continuous function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a *normalization function* if it is strictly increasing, $\varphi(0) = 0$ and $\varphi(r) \to \infty$ for $r \to \infty$.

The *duality mapping corresponding to the normalization function* φ is the set valued operator $J_\varphi : X \to \mathcal{P}(X^*)$ defined by

$$J_\varphi x = \Big\{ x^* \in X^* : \langle x^*, x \rangle = \varphi(\|x\|)\|x\|, \|x^*\| = \varphi(\|x\|) \Big\}, \qquad x \in X.$$

Assumption (A1): X and X^* are locally uniformly convex reflexive Banach spaces.

Observe that X^* is strictly convex, because a locally uniformly convex Banach space is also strictly convex, see [5, Theorem 3, p. 31]. Then it follows that $\operatorname{card}(J_\varphi x) = 1$ by [6, Proposition 1, p. 342]. Hence, $J_\varphi : X \to X^*$

$$\langle J_\varphi x, x \rangle = \varphi(\|x\|)\|x\| \quad \text{and} \quad \|J_\varphi x\| = \varphi(\|x\|).$$

The following result holds.

Theorem 2.1. [6, Theorem 5, p. 345] *Let X be a reflexive, locally uniformly convex Banach space and $J_\varphi : X \to X^*$. Then J_φ is bijective and its inverse J_φ^{-1} is bounded, continuous and monotone. Moreover, it holds $J_\varphi^{-1} = \chi^{-1} J^*_{\varphi^{-1}}$, where $\chi : X \to X^{**}$ is the canonical isomorphism between X and X^{**} and $J^*_{\varphi^{-1}} : X^* \to X^{**}$ is the duality mapping on X^* corresponding to the normalization function φ^{-1}.*

We consider $\bar{J} : X^* \to X$ defined by $\bar{J} = J_\varphi^{-1}$. By Theorem 2.1 it follows that $\bar{J}$ is bounded, continuous and monotone. For $w \in X^*$ denote $v = J_\varphi^{-1} w$ and compute

$$\langle w, \bar{J} w \rangle = \langle J_\varphi v, v \rangle = \varphi(\|v\|)\|v\| = \|J_\varphi v\| \|v\| = \|w\| \|J_\varphi^{-1} w\|$$

and

$$\|\bar{J} w\| = \|J_\varphi^{-1} w\| = \|\chi^{-1} J^*_{\varphi^{-1}} w\| = \|J^*_{\varphi^{-1}} w\|_{X^{**}} = \varphi^{-1}(\|w\|).$$

We conclude

$$\langle w, \bar{J} w \rangle = \varphi^{-1}(\|w\|)\|w\| \quad \text{and} \quad \|\bar{J} w\| = \varphi^{-1}(\|w\|) \quad \text{for each } w \in X^*. \tag{2.1}$$

Fix $0 < R$.

Assumption (A2): Let $H : X \to \mathbb{R}$ be of class C^1 such that the level set

$$N_R = \{u \in X : H(u) = R\} \text{ is non-void and bounded },$$

$$\inf_{u \in N_R} \langle H'(u), u \rangle > 0,$$

and the operator H' maps bounded sets into bounded sets.

We denote

$$X_R = \{u \in X : H(u) \leq R\}.$$

Assumption (A3): Let $F : X_R \to \mathbb{R}$ be of class C^1 such that F is bounded by below on X_R.

We introduce some auxiliary mappings:

$$D : N_R \to X^*, \qquad D(u) = F'(u) - \frac{\langle F'(u), u\rangle}{\langle H'(u), u\rangle} H'(u),$$
$$E : N_R \to X, \qquad E(u) = \bar{J}D(u) - \frac{\langle H'(u), \bar{J}Du\rangle}{\langle H'(u), u\rangle} u.$$

Lemma 2.2. *Assume that* (A1) *holds, and that* $F : X_R \to \mathbb{R}$ *and* $H : X \to \mathbb{R}$ *are* C^1 *functions. For all* $u \in N_R$ *the following properties hold:*

(1) $\langle H'(u), E(u)\rangle = 0$,

(2) $\langle F'(u), E(u)\rangle = \varphi^{-1}(\|D(u)\|)\|D(u)\|$.

Proof. Let $u \in N_R$ be arbitrary. We compute

$$\langle H'(u), E(u)\rangle = \left\langle H'(u), \bar{J}D(u) - \frac{\langle H'(u), \bar{J}Du\rangle}{\langle H'(u), u\rangle} u \right\rangle = 0.$$

Observe that

$$\langle D(u), u\rangle = \left\langle F'(u) - \frac{\langle F'(u), u\rangle}{\langle H'(u), u\rangle} H'(u), u \right\rangle = 0. \tag{2.2}$$

By using the statement (1) of this lemma, by (2.2) and (2.1) we have

$$\begin{aligned}\langle F'(u), E(u)\rangle &= \left\langle F'(u) - \frac{\langle F'(u), u\rangle}{\langle H'(u), u\rangle} H'(u), E(u) \right\rangle = \langle D(u), E(u)\rangle \\ &= \left\langle D(u), \bar{J}D(u)\rangle - \frac{\langle H'(u), \bar{J}Du\rangle}{\langle H'(u), u\rangle} \langle D(u), u \right\rangle \\ &= \langle D(u), \bar{J}D(u)\rangle = \varphi^{-1}(\|D(u)\|)\|D(u)\|.\end{aligned}$$

□

3 Main result

Theorem 3.1. *Assume that* (A1), (A2) *and* (A3) *are satisfied. Then, there exists a sequence* $(x_n)_n \subset X_R$ *such that* $F(x_n) \to \inf F(X_R)$ *and one of the following statements hold*

(a) $F'(x_n) \to 0$ *as* $n \to \infty$;

(b) *for each* $n \in \mathbb{N}$ *we have* $H(x_n) = R$, $\langle H'(x_n), \bar{J}F'(x_n)\rangle \leq 0$ *and*

$$F'(x_n) - \frac{\langle F'(x_n), x_n\rangle}{\langle H'(x_n), x_n\rangle} H'(x_n) \to 0 \quad \text{as } n \to \infty.$$

If, in addition, there exists $a \in \mathbb{R}_+$ *such that*

$$\langle H'(x), \bar{J}F'(x)\rangle \geq -a \quad \text{for each } x \in N_R,$$

and F *satisfies a Palais–Smale type compactness condition (i.e. any sequence satisfying* (a) *or* (b) *has a convergent subsequence) and the following boundary condition holds*

$$F'(x) + \mu H'(x) \neq 0 \quad \text{for any } \mu > 0,\ x \in N_R, \tag{3.1}$$

then there exists $x \in X_R$ *such that*

$$F(x) = \inf F(X_R) \quad \text{and} \quad F'(x) = 0.$$

Proof. By Ekeland's variational principle, see [8, Theorem 1, p. 444], applied for X_R (we use here that H is continuous, hence X_R is a closed set), the distance $d(x,y) = \|x-y\|$, the function F (which is continuous and bounded by below, see (A3)), $\varepsilon = \frac{1}{n}$ and for $u \in X_R$ such that

$$F(u) \le \inf F(X_R) + \frac{1}{n},$$

it follows that there exists a sequence $(x_n)_n$ in X_R such that

$$F(x_n) \le F(u) \le \inf F(X_R) + \frac{1}{n},$$

and

$$F(x_n) < F(y) + \frac{1}{n}\|x_n - y\| \quad \text{for each } y \in X_R \setminus \{x_n\}. \tag{3.2}$$

This yields $F(x_n) \to \inf F(X_R)$.

Since $(x_n)_n$ belongs to X_R, we distinguish two cases:

(1) there exists a subsequence of $(x_n)_n$, still denoted by $(x_n)_n$, such that $H(x_n) < R$ for each $n \in \mathbb{N}$;

(2) there exists a subsequence of $(x_n)_n$, still denoted by $(x_n)_n$, such that $H(x_n) = R$ for each $n \in \mathbb{N}$.

Case (1) Fix $n \in \mathbb{N}$. Let $t > 0$ and $z \in X$ such that $\|z\| = 1$. Since H is a continuous function and $H(x_n) < R$, we have that there exits $\delta > 0$ (small enough) such that $H(x_n - tz) < R$ for each $t \in (0, \delta)$. Hence $x_n - tz \in X_R \setminus \{x_n\}$ for each $t \in (0, \delta)$ and by (3.2) it holds

$$F(x_n) - F(x_n - tz) < \frac{t}{n}.$$

By taking $t \searrow 0$ it follows

$$\langle F'(x_n), z\rangle \le \frac{1}{n}.$$

But $z \in X$ with $\|z\| = 1$ was arbitrary chosen, hence $\|F'(x_n)\| \le \frac{1}{n}$, which yields $F'(x_n) \to 0$ as $n \to \infty$. Hence we constructed a sequence $(x_n)_n$ which satisfies the statement (a) of this theorem.

Case (2) Fix $n \in \mathbb{N}$. We have $H(x_n) = R$. Let $z \in X$ such that $\|z\| = 1$. We use the definition of the Fréchet derivative of H: for each $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that for each $t \in (0, \delta_\varepsilon)$ we have

$$-\varepsilon t < H(x_n - tz) - H(x_n) + \langle H'(x_n), tz\rangle < \varepsilon t.$$

Hence,

$$R - \varepsilon t - t\langle H'(x_n), z\rangle < H(x_n - tz) < R + \varepsilon t - t\langle H'(x_n), z\rangle \quad \text{for each } t \in (0, \delta_\varepsilon). \tag{3.3}$$

• If $\langle H'(x_n), z\rangle > 0$: by taking $\varepsilon = \langle H'(x_n), z\rangle$ in (3.3) we get

$$H(x_n - tz) < R \quad \text{for each } t \in (0, \delta_\varepsilon).$$

Hence $x_n - tz \in X_R \setminus \{x_n\}$ for $t \in (0, \delta_\varepsilon)$. By (3.2) it follows that for $t \in (0, \delta_\varepsilon)$

$$F(x_n) - F(x_n - tz) < \frac{t}{n},$$

then by $t \searrow 0$ we get

$$\langle F'(x_n), z\rangle \leq \frac{1}{n} \quad \text{for each } z \in X \text{ with } \|z\| = 1 \quad \text{and} \quad \langle H'(x_n), z\rangle > 0. \tag{3.4}$$

• If $\langle H'(x_n), z\rangle = 0$: we approximate z by a sequence $(z_k)_k$ such that $\|z_k\| = 1$ and $\langle H'(x_n), z_k\rangle > 0$ for each $k \in \mathbb{N}$, while $\|z - z_k\| \to 0$ as $k \to \infty$. Since $\langle H'(x_n), \cdot\rangle$ is continuous, we have $\langle H'(x_n), z_k\rangle \to \langle H'(x_n), z\rangle = 0$ as $k \to \infty$.

Let $k \in \mathbb{N}$ be fixed. By considering (3.3) for z_k instead of z and $\varepsilon = \langle H'(x_n), z_k\rangle$ for $t \in (0, \delta_\varepsilon)$ we get

$$H(x_n - tz_k) < R.$$

Then, $x_n - tz_k \in X_R \setminus \{x_n\}$ for $t \in (0, \delta_\varepsilon)$. By (3.2) we obtain for t sufficiently small

$$F(x_n) - F(x_n - tz_k) < \frac{t}{n},$$

which yields

$$\langle F'(x_n), z_k\rangle \leq \frac{1}{n}.$$

But $\|z - z_k\| \to 0$ as $k \to \infty$, hence

$$\langle F'(x_n), z\rangle \leq \frac{1}{n}.$$

This inequality and (3.4) imply

$$\langle F'(x_n), z\rangle \leq \frac{1}{n} \quad \text{for each } z \in X \text{ with } \|z\| = 1 \quad \text{and} \quad \langle H'(x_n), z\rangle \geq 0. \tag{3.5}$$

Further we have two possible cases.

Case (2a) There exists a subsequence of $(x_n)_n$, which we still denote by $(x_n)_n$, such that $\langle H'(x_n), \bar{J}F'(x_n)\rangle > 0$: by taking $z = \frac{1}{\|\bar{J}F'(x_n)\|}\bar{J}F'(x_n)$ in (3.5) we get

$$\langle F'(x_n), \bar{J}F'(x_n)\rangle \leq \frac{1}{n}\|\bar{J}F'(x_n)\|.$$

By the property (2.1) of $\bar{J}$ it follows that

$$\langle F'(x_n), \bar{J}F'(x_n)\rangle = \varphi^{-1}(\|F'(x_n)\|)\|F'(x_n)\| \quad \text{and} \quad \|\bar{J}F'(x_n)\| = \varphi^{-1}(\|F'(x_n)\|)$$

which yields

$$\|F'(x_n)\| \leq \frac{1}{n},$$

hence $F'(x_n) \to 0$ as $n \to \infty$ and we obtained a sequence $(x_n)_n$ which satisfies the statement (a) of this theorem.

Case (2b) There exists a subsequence of $(x_n)_n$, which we still denote by $(x_n)_n$, such that $\langle H'(x_n), \bar{J}F'(x_n)\rangle \leq 0$: by taking $z = \frac{1}{\|E(x_n)\|}E(x_n)$ in (3.5) (if $\|E(x_n)\| = 0$, then by Lemma 2.2 (2) we get $\|D(x_n)\| = 0$) and by Lemma 2.2 (2) we get

$$\varphi^{-1}(\|D(x_n)\|)\|D(x_n)\| = \langle F'(x_n), E(x_n)\rangle \leq \frac{1}{n}\|E(x_n)\|.$$

Hence,

$$\varphi^{-1}(\|D(x_n)\|)\|D(x_n)\| \leq \frac{1}{n}\|E(x_n)\|. \tag{3.6}$$

Denote the kernel of $H'(x_n)$ by $\mathcal{K}_n = \{x \in X : \langle H'(x_n), x\rangle = 0\}$ and the projection mapping

$$P_n : X \to \mathcal{K}_n \quad \text{by} \quad P_n v = v - \frac{\langle H'(x_n), v\rangle}{\langle H'(x_n), x_n\rangle} x_n.$$

Since $v \in X \mapsto \langle H'(x_n), v\rangle$ is linear and continuous, it follows that P_n is also linear and continuous.

Since $(x_n)_n \subset N_R$ and the level set N_R is bounded, it follows that $(x_n)_n$ is a bounded sequence. By the assumption on H' it follows that $(H'(x_n))_n$ is also bounded and there exists

$$0 < \beta_R := \inf_{u \in N_R} \langle H'(u), u\rangle \le \langle H'(x_n), x_n\rangle \quad \text{for each } n \in \mathbb{N}.$$

We write

$$\|P_n v\| \le \|v\| + \frac{\|H'(x_n)\| \|v\|}{\inf_{n\in\mathbb{N}} \langle H'(x_n), x_n\rangle} \|x_n\| \le \left(1 + \frac{\|H'(x_n)\| \|x_n\|}{\beta_R}\right) \|v\| \quad \text{for each } v \in X.$$

Hence there exists $\alpha_R > 0$ (independent of n) such that

$$\|P_n v\| \le \alpha_R \|v\| \quad \text{for each } v \in X.$$

We take $v = \bar{J}D(x_n)$ to get $P_n \bar{J}D(x_n) = E(x_n)$ and

$$\|E(x_n)\| \le \alpha_R \|\bar{J}D(x_n)\| = \alpha_R \varphi^{-1}(\|D(x_n)\|) \quad \text{for each } n \in \mathbb{N}.$$

Then by (3.6) we have

$$\varphi^{-1}(\|D(x_n)\|)\|D(x_n)\| \le \frac{\alpha_R}{n} \varphi^{-1}(\|D(x_n)\|) \quad \text{for each } n \in \mathbb{N}.$$

This yields $D(x_n) \to 0$ as $n \to \infty$. Hence we constructed a sequence $(x_n)_n$ which satisfies statement (b) of this theorem.

If, in addition, F satisfies the (PS) type compactness condition.

Case (a) $F'(x_n) \to 0$ as $n \to \infty$ and there exist $x \in X_R$ and a subsequence $(x_{n_k})_k$ such that $\|x_{n_k} - x\| \to 0$ as $k \to \infty$. Since F is a C^1 function, we get $F'(x) = 0$ and by the construction of $(x_n)_n$ we have $F(x_n) \to \inf F(X_R)$, hence $F(x) = \inf F(X_R)$.

Case (b) We have $D(x_n) \to 0$ as $n \to \infty$, $H(x_n) = R$ and $\langle H'(x_n), \bar{J}F'(x_n)\rangle \le 0$ for all $n \in \mathbb{N}$ and there exist $x \in X_R$ (X_R is a closed set, since H is continuous) and a subsequence $(x_{n_k})_k$ such that $\|x_{n_k} - x\| \to 0$ as $k \to \infty$. Hence $F(x) = \lim_{k\to\infty} F(x_{n_k}) = \inf F(X_R)$, $F'(x) = \lim_{k\to\infty} F'(x_{n_k})$ and $x \in N_R$, i.e. $H(x) = R$. But $D(x_{n_k}) \to 0$ as $k \to \infty$, which implies

$$F'(x) - \frac{\langle F'(x), x\rangle}{\langle H'(x), x\rangle} H'(x) = 0. \tag{3.7}$$

Applying the operator $\bar{J}$ we get

$$\bar{J}F'(x) - \frac{\langle F'(x), x\rangle}{\langle H'(x), x\rangle} \bar{J}H'(x) = 0,$$

which yields

$$\langle H'(x), \bar{J}F'(x)\rangle - \frac{\langle F'(x), x\rangle}{\langle H'(x), x\rangle} \langle H'(x), \bar{J}H'(x)\rangle = 0,$$

$$\langle H'(x), \bar{J}F'(x)\rangle - \frac{\langle F'(x), x\rangle}{\langle H'(x), x\rangle}\varphi^{-1}(\|H'(x)\|)\|H'(x)\| = 0. \tag{3.8}$$

Since we are investigating Case (b), it follows that $(\langle H'(x_n), \bar{J}F'(x_n)\rangle)_n$ is a bounded sequence in $\mathbb{R}$, hence there exist $b \in \mathbb{R}, b \leq 0$, and a subsequence, denoted again by $(x_{n_k})_k$, such that

$$\langle H'(x_{n_k}), \bar{J}F'(x_{n_k})\rangle \to b.$$

But $\|x_{n_k} - x\| \to 0$ as $k \to \infty$, hence

$$\langle H'(x), \bar{J}F'(x)\rangle = b \leq 0.$$

Using (3.8) it follows

$$\frac{\langle F'(x), x\rangle}{\langle H'(x), x\rangle}\varphi^{-1}(\|H'(x)\|)\|H'(x)\| = b \leq 0.$$

Since $\langle H'(x), x\rangle > 0$ (assumption on H' and the fact that $x \in N_R$ as the limit of $(x_{n_k})_k$), we obtain

$$\langle F'(x), x\rangle \leq 0.$$

- If $\langle F'(x), x\rangle = 0$, then (3.7) implies $F'(x) = 0$.
- If $\langle F'(x), x\rangle < 0$, then (3.7) implies

$$F'(x) + \mu H'(x) = 0 \quad \text{where } \mu = -\frac{\langle F'(x), x\rangle}{\langle H'(x), x\rangle} > 0 \text{ and } x \in N_R,$$

which contradicts the assumption (3.1) from the statement of this theorem. □

4 Applications

4.1 Example 1

Consider the Sobolev space $W_0^{1,p}(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^k$ with Lipschitz continuous boundary and $1 < p < \infty$, equipped with the norm

$$\|u\|_{1,p} = \left(\int_\Omega |\nabla u(x)|^p dx\right)^{\frac{1}{p}},$$

where

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_k}\right), \qquad |\nabla u| = \left(\sum_{i=1}^k \left(\frac{\partial u}{\partial x_i}\right)^2\right)^{\frac{1}{2}}.$$

The Banach space $(W_0^{1,p}(\Omega), \|\cdot\|_{1,p})$ is uniformly convex, see [1, Theorem 3.6]. Moreover, it is also uniformly smooth (which is proved by using Clarkson's inequalities [1, 2.38 Theorem, p. 44] and [4, Definition 2.4., p. 13]). The dual space $(W_0^{1,p}(\Omega))^*$ will be denoted by $W^{-1,p'}(\Omega)$, where $\frac{1}{p} + \frac{1}{p'} = 1$, and by [4, Theorem 2.10] it follows that it is uniformly convex.

The Rellich–Kondrachov Theorem states that the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact for $q \in (1, p^*)$ (where $p^* = \frac{kp}{k-p}$ if $p < k$ and $p^* = \infty$, if $p \geq k$) and there exists $C_q > 0$ such that

$$\|u\|_{L^q(\Omega)} \leq C_q \|u\|_{1,p} \quad \text{for each } u \in W_0^{1,p}(\Omega). \tag{4.1}$$

In the context of our paper we choose $(X, \|\cdot\|) = (W_0^{1,p}(\Omega), \|\cdot\|_{1,p})$, $\bar{J} = J_\varphi^{-1}$, where $\varphi(t) = t^{p-1}$ for $t \in \mathbb{R}_+$ and let $H : W_0^{1,p}(\Omega) \to \mathbb{R}$ be given by $H(u) = \frac{1}{p}\|u\|_{1,p}^p$.

We consider the p-Laplacian operator $-\Delta_p : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ defined by

$$\langle -\Delta_p(u), v\rangle = \int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x)dx \text{ for all } u, v \in W_0^{1,p}(\Omega).$$

It is known that the functional H is continuously Fréchet differentiable on $W_0^{1,p}(\Omega)$ and $H' = -\Delta_p$. The operator $-\Delta_p$ is in fact the duality mapping $J_\varphi : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ corresponding to the normalization function $\varphi(t) = t^{p-1}$ for $t \in \mathbb{R}_+$, i.e. $H' = J_\varphi$, for details consult [6, Theorem 7 and Theorem 9]. In our example we have

$$N_R = \left\{ v \in W_0^{1,p}(\Omega) : \frac{1}{p}\|v\|_{1,p}^p = R \right\}$$

and

$$X_R = \left\{ v \in W_0^{1,p}(\Omega) : \frac{1}{p}\|v\|_{1,p}^p \leq R \right\}.$$

Assume that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(x, 0) \neq 0$ for a.e. $x \in \Omega$ and

$$|f(x,s)| \leq a(x)|s|^{q-1} + b(x) \quad \text{for } x \in \Omega,\ s \in \mathbb{R},$$

where $a \in L^\infty(\Omega), b \in L^{\frac{q}{q-1}}(\Omega)$ are positive functions and $q \in (1, p^*)$. Define the Nemytskii operator $N_f : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ by

$$N_f(u)(x) = f(x, u(x)).$$

We have $N_f(W_0^{1,p}(\Omega)) \hookrightarrow N_f(L^q(\Omega)) \subset L^{\frac{q}{q-1}}(\Omega) = (L^q(\Omega))^* \hookrightarrow W^{-1,p'}(\Omega)$ and N_f is a continuous function which maps bounded sets into bounded sets (see [9]).

Consider the following Dirichlet problem involving the p-Laplacian:

$$-\Delta_p u = f(x, u) \quad \text{a.e. } x \in \Omega \quad \text{and} \quad u\big|_{\partial\Omega} = 0. \tag{4.2}$$

We call $u \in W_0^{1,p}(\Omega)$ a *weak solution* of (4.2) if for each $v \in W_0^{1,p}(\Omega)$ it holds

$$\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x)dx = \int_\Omega f(x, u(x))v(x)dx. \tag{4.3}$$

We introduce $F : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$F(u) = \frac{1}{p}\|u\|_{1,p}^p - \int_\Omega h(x, u)\, dx,$$

where $h : \Omega \times \mathbb{R} \to \mathbb{R}$ is $h(x, t) = \int_0^t f(x, s)\, ds$. We have (see [9, Theorem 7])

$$F'(u) = H'(u) - N_f(u).$$

The critical points of F are the solutions of (4.3).

(A4) Assumptions for R: denote by C an upper bound for C_q and suppose that one of the following three assumptions is satisfied.

(1) If $p > q$: let $R > 0$ be a solution of the inequality in $\mathbb{R}$

$$R^{\frac{p-1}{p}} > C^q p^{\frac{q-p}{p}} \|a\|_{L^\infty(\Omega)} R^{\frac{q-1}{p}} + Cp^{\frac{1-p}{p}} \|b\|_{L^{\frac{q}{q-1}}(\Omega)}.$$

(2) If $p = q$: assume $1 > C^p \|a\|_{L^\infty(\Omega)}$ and let R be such that

$$R > \left(\frac{Cp^{\frac{1-p}{p}} \|b\|_{L^{\frac{p}{p-1}}(\Omega)}}{1 - C^p \|a\|_{L^\infty(\Omega)}} \right)^{\frac{p}{p-1}}.$$

(3) If $q > p$: assume that $1 > C^q p^{\frac{q-p}{p}} \|a\|_{L^\infty(\Omega)} + Cp^{\frac{1-p}{p}} \|b\|_{L^{\frac{q}{q-1}}(\Omega)}$ and let $R > 0$ to be a solution of the inequality in $\mathbb{R}$

$$R^{\frac{p-1}{p}} - C^q p^{\frac{q-p}{p}} \|a\|_{L^\infty(\Omega)} R^{\frac{q-1}{p}} > Cp^{\frac{1-p}{p}} \|b\|_{L^{\frac{q}{q-1}}(\Omega)}.$$

Proposition 4.1. *The following relation holds*

$$F'(u) + \mu H'(u) \neq 0 \quad \textit{for any } \mu > 0,\ u \in N_R,$$

where R satisfies one of the three conditions mentioned in (A4)*.*

Proof. We reason by contradiction: assume that there exist $u \in N_R$ and $\mu > 0$ such that $F'(u) + \mu H'(u) = 0$, which implies

$$(1 + \mu)\langle J_\varphi(u), u\rangle = \langle N_f(u), u\rangle. \tag{4.4}$$

By our assumptions

$$\begin{aligned}\langle N_f(u), u\rangle &= \int_\Omega f(x, u(x))u(x)dx \\ &\leq \int_\Omega a(x)|u(x)|^q + b(x)|u(x)|dx \leq \|a\|_{L^\infty(\Omega)} \|u\|^q_{L^q(\Omega)} + \|b\|_{L^{\frac{q}{q-1}}(\Omega)} \|u\|_{L^q(\Omega)}.\end{aligned}$$

Using (4.1) and (4.4) we get

$$\begin{aligned}\|u\|^p_{1,p} \leq (1 + \mu)\|u\|^p_{1,p} &\leq \|a\|_{L^\infty(\Omega)} \|u\|^q_{L^q(\Omega)} + \|b\|_{L^{\frac{q}{q-1}}(\Omega)} \|u\|_{L^q(\Omega)} \\ &\leq C^q \|a\|_{L^\infty(\Omega)} \|u\|^q_{1,p} + C\|b\|_{L^{\frac{q}{q-1}}(\Omega)} \|u\|_{1,p}.\end{aligned}$$

But $u \in N_R$ implies $\|u\|_{1,p} = (pR)^{\frac{1}{p}}$, and we obtain

$$pR \leq C^q \|a\|_{L^\infty(\Omega)} (pR)^{\frac{q}{p}} + C\|b\|_{L^{\frac{q}{q-1}}(\Omega)} (pR)^{\frac{1}{p}},$$

which yields

$$R^{\frac{p-1}{p}} \leq C^q p^{\frac{q-p}{p}} \|a\|_{L^\infty(\Omega)} R^{\frac{q-1}{p}} + Cp^{\frac{1-p}{p}} \|b\|_{L^{\frac{q}{q-1}}(\Omega)}. \tag{4.5}$$

The assumptions in (A4) imply that (4.5) cannot be satisfied. □

Proposition 4.2. *Suppose that R satisfies one of the three conditions mentioned in* (A4)*. Then F satisfies the following Palais–Smale type compactness condition: if $(u_n)_n$ is a sequence from X_R such that one of the following statements hold*

(a) $F'(u_n) \to 0$ *as* $n \to \infty$*;*

(b) *for each* $n \in \mathbb{N}$ *we have* $H(u_n) = R$, $\langle H'(u_n), \bar{J}F'(u_n)\rangle \leq 0$ *and*

$$F'(u_n) - \frac{\langle F'(u_n), u_n\rangle}{\langle H'(u_n), u_n\rangle} H'(u_n) \to 0 \text{ as } n \to \infty,$$

then $(u_n)_n$ admits a convergent subsequence.

Proof. Since the sequence $(u_n)_n$ is bounded in $W_0^{1,p}(\Omega)$ (it belongs to X_R) and since the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact for $q \in (1, p^*)$, there exist $u \in W_0^{1,p}(\Omega)$ and a subsequence of $(u_n)_n$, which we denote again by $(u_n)_n$, which converges weakly in $W_0^{1,p}(\Omega)$ to u and strongly in $L^q(\Omega)$ to u. Then by Hölder's inequality we have

$$\langle N_f(u_n), u_n - u\rangle \leq \|f(\cdot, u_n)\|_{L^{\frac{q}{q-1}}(\Omega)} \|u_n - u\|_{L^q(\Omega)} \to 0. \tag{4.6}$$

Case (a): $F'(u_n) \to 0$ as $n \to \infty$. Then,

$$\langle H'(u_n), u_n - u\rangle = \langle F'(u_n), u_n - u\rangle + \langle N_f(u_n), u_n - u\rangle \to 0.$$

The (S_+) property of $H' = J_\varphi$ (see [6, Theorem 10]) implies $(u_n)_n$ converges strongly to u.

Case (b): For each $n \in \mathbb{N}$ we have $H(u_n) = R$, $\langle H'(u_n), \bar{J}F'(u_n)\rangle \leq 0$ and

$$F'(u_n) - \frac{\langle F'(u_n), u_n\rangle}{\langle H'(u_n), u_n\rangle} H'(u_n) \to 0 \quad \text{as } n \to \infty.$$

We denote

$$\mu = \lim_{n\to\infty} \frac{\langle F'(u_n), u_n\rangle}{\langle H'(u_n), u_n\rangle} \in \mathbb{R}.$$

Therefore,

$$F'(u_n) - \mu H'(u_n) \to 0 \quad \text{as } n \to \infty. \tag{4.7}$$

If $\mu = 0$, the above convergence implies

$$F'(u_n) \to 0 \quad \text{as } n \to \infty.$$

As in Case (a) it follows that there exist $u \in W_0^{1,p}(\Omega)$ and a subsequence of $(u_n)_n$, which we denote again by $(u_n)_n$, which converges strongly in $W_0^{1,p}(\Omega)$ to u. Since F' is continuous, we have $F'(u) = 0$, which implies $\langle H'(u), u\rangle = \langle N_f(u), u\rangle$. But $u \in N_R$ (since $(u_n)_n$ belongs to the closed set N_R), which yields

$$pR = \langle N_f(u), u\rangle \leq C^q \|a\|_{L^\infty(\Omega)} (pR)^{\frac{q}{p}} + C\|b\|_{L^{\frac{q}{q-1}}(\Omega)} (pR)^{\frac{1}{p}}.$$

This contradicts the assumption on R from (A4). Hence, the case $\mu = 0$ is not possible.
For $\mu \neq 0$ we have by (4.7)

$$\langle F'(u_n) - \mu H'(u_n), u_n - u\rangle \to 0 \quad \text{as } n \to \infty.$$

$$(1-\mu)\langle H'(u_n), u_n - u\rangle + \langle N_f(u_n), u_n - u\rangle \to 0 \quad \text{as } n \to \infty.$$

If $\mu \neq 1$ we get by (4.6)

$$(1-\mu)\langle H'(u_n), u_n - u\rangle \to 0 \quad \text{as } n \to \infty.$$

The (S_+) property of $H' = J_\varphi$ implies $(u_n)_n$ converges strongly to u. The convergence (4.7) and the strong convergence $u_n \to u$ implies $F'(u) - \mu H'(u) = 0$.

$$\langle F'(u), \bar{J}F'(u)\rangle = \mu\langle H'(u), \bar{J}F'(u)\rangle.$$

But $\langle H'(u), \bar{J}F'(u)\rangle \leq 0$ and $\langle F'(u), \bar{J}F'(u)\rangle \geq 0$, therefore $\mu < 0$ and the relation $F'(u) - \mu H'(u) = 0$ contradicts the statement of Proposition 4.1. Hence, the case $\mu \neq 1$ is not possible.

For $\mu = 1$: by (4.7) it follows that

$$\langle F'(u_n), \bar{J}F'(u_n)\rangle - \langle H'(u_n), \bar{J}F'(u_n)\rangle \to 0 \quad \text{as } n \to \infty.$$

Using (2.1) we have

$$\langle F'(u_n), \bar{J}F'(u_n)\rangle = \varphi^{-1}(\|F'(u_n)\|)\|F'(u_n)\| \geq 0$$

and by the assumptions of Case (b) we have

$$-\langle H'(u_n), \bar{J}F'(u_n)\rangle \geq 0$$

Then $F'(u_n) \to 0$ as $n \to \infty$. As in Case (a) it follows that there exist $u \in W_0^{1,p}(\Omega)$ and a subsequence of $(u_n)_n$, which we denote again by $(u_n)_n$, which converges strongly in $W_0^{1,p}(\Omega)$ to u. This yields $u \in N_R$, because $(u_n)_n$ belongs to the closed set N_R. Since F' and H' are continuous, we have by the convergence (4.7) that $F'(u) - H'(u) = 0$,

$$\langle F'(u), \bar{J}F'(u)\rangle = \langle H'(u), \bar{J}F'(u)\rangle.$$

But $\langle H'(u), \bar{J}F'(u)\rangle \leq 0$ (by the assumption of Case (b) and by the strong convergence $u_n \to u$) and $\langle F'(u), \bar{J}F'(u)\rangle \geq 0$, hence $F'(u) = 0$, which implies $H'(u) = 0$ and then $u = 0$. But $0 \notin N_R$, contradicts $u \in N_R$. Hence the case $\mu = 1$ is not possible. □

We apply Theorem 3.1 in order to localize the solution of (4.3).

Theorem 4.3. *Suppose that R satisfies one of the three conditions mentioned in* (A4). *Then, equation* (4.3) *admits a weak solution $u \in X_R$, which minimizes F on X_R.*

In what follows we discuss situations when the best Sobolev constant C_q admits an upper estimate which can be computed:

Denote the first eigenvalue of the p-Laplace operator by

$$\lambda_p(\Omega) = \min_{v \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla v(x)|^p dx}{\int_\Omega |v(x)|^p dx}.$$

Then,

$$\|u\|^p_{L^p(\Omega)} \leq \frac{1}{\lambda_p(\Omega)}\|u\|^p_{1,p} \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Hence the best embedding constant of $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is $C_p = \left(\frac{1}{\lambda_p(\Omega)}\right)^{1/p}$, while for $q < p$ the best embedding constant of $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ verifies (via Hölder's inequality)

$C_q \leq |\Omega|^{\frac{p-q}{pq}} \left(\frac{1}{\lambda_p(\Omega)}\right)^{1/p}$ (here $|\Omega|$ denotes the Lebesgue measure, i.e. the k-dimensional volume, of the set Ω). In order to obtain upper bounds for C_q ($q \leq p$) we need lower bounds for $\lambda_p(\Omega)$.

By using the Faber–Krahn inequality [2, Theorem 1] it holds

$$\lambda_p(\Omega) \geq \lambda_p(\Omega^*),$$

where Ω^* is the k-dimensional ball centered at the origin having the same volume as Ω. So it has the radius $r = \frac{1}{\sqrt{\pi}}\left(|\Omega|\Gamma(\frac{k}{2}+1)\right)^{1/k}$.

By [10] we have for the ball $\Omega^* = B_r \subset \mathbb{R}^k$ of radius r the inequality

$$\lambda_p(B_r) \geq \left(\frac{k}{rp}\right)^p.$$

Then the best Sobolev constant has the following upper estimate, which can be computed:

$$C_p \leq \frac{p}{k\sqrt{\pi}}\left(|\Omega|\Gamma\left(\frac{k}{2}+1\right)\right)^{\frac{1}{k}},$$

and for $1 < q < p$

$$C_q \leq \frac{p}{k\sqrt{\pi}}\left(|\Omega|^{\frac{k(p-q)}{pq}+1}\Gamma\left(\frac{k}{2}+1\right)\right)^{\frac{1}{k}}.$$

For $k = 1$ and $\Omega = (0, T) \subset \mathbb{R}$ the value of the first eigenvalue is known (see [7])

$$\lambda_p(\Omega) = (p-1)\left(\frac{2\pi}{Tp\sin(\frac{\pi}{p})}\right)^p,$$

hence

$$C_p = \frac{Tp\sin(\frac{\pi}{p})}{2\pi(p-1)^{\frac{1}{p}}}.$$

For the case $k = 1$ and $\Omega = (0, T)$ the sharp Poincaré inequality is known (see [20], p. 357): for each $p > 1, q > 1$ and $u \in W_0^{1,p}(0, T)$ it holds

$$\|u\|_{L^q(\Omega)} \leq C_q\|u\|_{1,p},$$

where the embedding constant is given by

$$C_q = \frac{T^{\frac{1}{q}+\frac{1}{p'}}}{2B(\frac{1}{q}, \frac{1}{p'})}(p')^{\frac{1}{q}}\, q^{\frac{1}{p'}}\,(p'+q)^{\frac{1}{p}-\frac{1}{q}},$$

$p' = \frac{p}{p-1}$ and B is the Beta function.

4.2 Example 2

For $1 < p < \infty$ we define the following subspace of radially symmetric functions of $W^{1,p}(\mathbb{R}^k)$

$$W_r^{1,p}(\mathbb{R}^k) = \{u \in W^{1,p}(\mathbb{R}^k) : u(x) = u(x')\ \forall\, x, x' \in \mathbb{R}^k, |x| = |x'|\},$$

endowed with the norm induced from $W^{1,p}(\mathbb{R}^k)$

$$\|u\|^p = \int_{\mathbb{R}^k} |\nabla u(z)|^p + |u(z)|^p dz.$$

The space $W^{1,p}(\mathbb{R}^k)$ is a separable, reflexive and uniformly convex Banach space [1, 3.6 Theorem, p. 61]; moreover, it is also uniformly smooth (which is proved by using Clarkson's inequalities [1, 2.38 Theorem, p. 44] and [4, Definition 2.4., p. 13]).

In the context of Section 2 and Section 3 we consider $X = W_r^{1,p}(\mathbb{R}^k)$ endowed with the above norm $\|\cdot\|$, X is a closed subspace of $W^{1,p}(\mathbb{R}^k)$. Hence it is also uniformly smooth and by [4, Theorem 2.10] it follows that its dual X^* is uniformly convex.

Let $J_\varphi : X \to X^*$ be the duality mapping corresponding to the weight function $\varphi(t) = t^{p-1}$, $t \in \mathbb{R}_+$ where $p \in (1, +\infty)$ (see [3, Proposition 2.2.4]). It is well known that the duality mapping J_φ satisfies the following conditions:

$$\|J_\varphi u\| = \varphi(\|u\|) \quad \text{and} \quad \langle J_\varphi u, u\rangle = \|J_\varphi u\|\|u\| \quad \text{for all } u \in X.$$

Moreover, the functional $H : X \to \mathbb{R}$ defined by $H(u) = \frac{1}{p}\|u\|^p$ is convex and Fréchet differentiable with $H' = J_\varphi$. We take $\bar{J} = J_\varphi^{-1}$.

It is known [11, Théorème II. 1] that the embedding $W_r^{1,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ is compact for $q \in (p, p^*)$ (where $k \geq 2$, $p^* = \frac{kp}{k-p}$ if $p < k$ and $p^* = \infty$, if $p \geq k$) there exists $C_q > 0$ (the best embedding constant) such that

$$\|u\|_{L^q(\mathbb{R}^k)} \leq C_q\|u\| \quad \text{for each } u \in W_r^{1,p}(\mathbb{R}^k). \tag{4.8}$$

Let $f : \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}$ be a Carathéodory function such that $f(x, 0) \neq 0$ for a.e. $x \in \mathbb{R}$ which satisfies

$$|f(x,s)| \leq a(x)|s|^{q-1} + b(x) \quad \text{for } (x,s) \in \mathbb{R}^k \times \mathbb{R},$$

where $a \in L^\infty(\mathbb{R}^k), b \in L^{\frac{q}{q-1}}(\mathbb{R}^k)$ are positive functions and $q \in (p, p^*)$ and $f(x, \cdot) = f(x', \cdot)$ for all $x, x' \in \mathbb{R}^n$, $|x| = |x'|$ (f is radially symmetric in the first variable).

Consider the following problem involving the p-Laplacian:

$$-\Delta_p u + |u|^{p-2}u = f(x,u) \quad \text{a.e. } x \in \mathbb{R}^k. \tag{4.9}$$

We call $u \in W^{1,p}(\mathbb{R}^k)$ a *weak solution* of (4.9) if for each $v \in W^{1,p}(\mathbb{R}^k)$ it holds

$$\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x) + |u(x)|^{p-2}u(x)v(x)dx = \int_\Omega f(x,u(x))v(x)dx. \tag{4.10}$$

Define the Nemytskii operator $N_f : X \to X^*$ by $N_f(u)(x) = f(x, u(x))$ and $F : X \to \mathbb{R}$ by

$$F(u) = \frac{1}{p}\|u\|_{1,p}^p - \int_{\mathbb{R}^n} h(x,u)\, dx,$$

where $h : \Omega \times \mathbb{R} \to \mathbb{R}$ is $h(x,t) = \int_0^t f(x,s)\, ds$. We have

$$F'(u) = H'(u) - N_f(u).$$

Let $G = O(\mathbb{R}^k)$ be the set of all rotations on $\mathbb{R}^k$. Observe that the elements of G leave $\mathbb{R}^k$ invariant, i.e. $g(\mathbb{R}^k) = \mathbb{R}^k$ for all $g \in G$. G induces an isometric linear action over $W^{1,p}(\mathbb{R}^k)$ by

$$(gu)(z) = u(g^{-1}z), \quad g \in G, \quad u \in W^{1,p}(\mathbb{R}^k), \quad \text{a.e. } z \in \mathbb{R}^k.$$

A function ϕ defined on $W^{1,p}(\mathbb{R}^k)$ is said to be G-invariant if

$$\phi(gu) = \phi(u) \quad \text{for all } g \in G,\ u \in W^{1,p}(\mathbb{R}^k).$$

In fact $W_r^{1,p}(\mathbb{R}^k)$ is the fixed point set of $W^{1,p}(\mathbb{R}^k)$ under G and the norm

$$\|u\| = \left\{ \int_{\mathbb{R}^k} (|\nabla u(z)|^p + |u(z)|^p) dz \right\}^{\frac{1}{p}}$$

is G-invariant on $W^{1,p}(\mathbb{R}^k)$.

Observe that by the assumption on f and the above remark, the functional F is G-invariant and then by the principle of symmetric criticality [12] every critical point of F is also a solution of (4.10).

Consider

$$X_R = \left\{ v \in W_r^{1,p}(\mathbb{R}^k) : \frac{1}{p}\|v\|^p \leq R \right\}.$$

Reasoning as in Section 4.1 one has the following result.

Theorem 4.4. *Suppose that R satisfies one of the three conditions mentioned in* (A4). *Then, F admits a critical point $u \in X_R$, which minimizes F on X_R. Moreover, this critical point is also a weak solution of* (4.10).

We discuss situations when the Sobolev constant C_q admits an upper estimate which can be computed: by [20] we have that for $1 < p < k$ and $p^* = \frac{kp}{k-p}$ it holds for all $u \in W^{1,p}(\mathbb{R}^k)$

$$\|u\|_{L^{p^*}(\mathbb{R}^k)} \leq C_{\mathbb{R}} \left(\int_{\mathbb{R}^k} |\nabla u(x)|^p dx \right)^{\frac{1}{p}},$$

where

$$C_{\mathbb{R}} = \frac{1}{\sqrt{\pi} k^{\frac{1}{p}}} \left(\frac{p-1}{k-p} \right)^{1-\frac{1}{p}} \left(\frac{\Gamma(1+\frac{k}{2})\Gamma(k)}{\Gamma(\frac{k}{p})\Gamma(1+k-\frac{k}{p})} \right)^{\frac{1}{k}}.$$

Obviously this implies

$$\|u\|_{L^{p^*}(\mathbb{R}^k)} \leq C_{\mathbb{R}} \|u\| \text{ for each } u \in W^{1,p}(\mathbb{R}^k).$$

For any $q \in (p, p^*)$ there exists $\theta \in (0,1)$ such that $q = \theta p + (1-\theta)p^*$, then by Hölder's inequality

$$\|u\|^q_{L^q(\mathbb{R}^k)} \leq \|u\|^{\theta p}_{L^p(\mathbb{R}^k)} \|u\|^{(1-\theta)p^*}_{L^{p^*}(\mathbb{R}^k)} \leq C_{\mathbb{R}}^{kq\left(\frac{1}{p}-\frac{1}{q}\right)} \|u\|^q,$$

for each $u \in W^{1,p}(\mathbb{R}^k)$. Then the Sobolev constant has the following upper estimate, which can be computed:

$$C_q \leq C_{\mathbb{R}}^{k\left(\frac{1}{p}-\frac{1}{q}\right)} \quad \text{for } q \in \left(p, \frac{kp}{k-p}\right) \text{ and } 1 < p < k.$$

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