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A priori bounds and existence results for singular boundary value problems

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Abstract. This article furnishes qualitative properties for solutions to two-point boundary value problems (BVPs) which are systems of singular, second-order, nonlinear ordinary differential equations. The right-hand side of the differential equation is allowed to be unrestricted in the growth of its variables and may depend on the derivative of the solution, which incurs additional difficultly in the mathematical proofs. A new approach is introduced by using a singular differential inequality that ensures that all possible solutions satisfy certain a priori bounds, including their "derivatives", to the singular BVP under consideration. Topological methods, in particular Schauder's fixed point theorem are then applied to generate new existence results for solutions to the singular boundary value problems. Many of the results are novel for both the singular and the nonsingular cases.

Keywords: singular, boundary value problems, differential inequalities, a priori, existence of solutions, systems.

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1 Introduction

This work focuses on achieving novel *a priori* bounds and existence results to systems of nonlinear, singular, second order boundary value problems (BVPs) given by:

$$\frac{1}{p(t)}(p(t)\mathbf{y}'(t))' = q(t)\mathbf{f}(t,\mathbf{y}(t),p(t)\mathbf{y}'(t)), \qquad 0 < t < T; \tag{1.1}$$

with various forms of the boundary conditions

$$-\alpha\mathbf{y}(0) + \beta \lim_{t\to 0^+} p(t)\mathbf{y}'(t) = \mathbf{c}; \tag{1.2}$$

$$\mathbf{y}(T) = \mathbf{d}. \tag{1.3}$$

The study of these singular BVPs are partially motivated by their application in modelling a variety of physical phenomena, some examples can be found in [11, 22]. The strategy used

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to achieve *a priori* bounds herein is to consider the BVP as an integral representation and use novel differential inequalities to yield *a priori* bounds on solutions. To prove the existence of solutions, the application of Schauder's fixed point theorem [24] is utilised once these bounds are known.

In the above, $\mathbf{f} : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function, $\mathbf{c}, \mathbf{d} \in \mathbb{R}^n$ are vector valued constants and the functions p, q satisfy

$$p \in C([0,T];\mathbb{R}) \cap C^1((0,T);\mathbb{R}) \quad \text{with } p > 0 \text{ on } (0,T); \tag{1.4}$$
$$q \in C((0,T);\mathbb{R}) \quad \text{with } q > 0 \text{ on } (0,T). \tag{1.5}$$

For $\mathbf{u} \in \mathbb{R}^n$, we define $\|\mathbf{u}\| := \langle \mathbf{u}, \mathbf{u} \rangle^{1/2}$ where $\langle \cdot, \cdot \rangle$ is the usual Euclidean dot product in $\mathbb{R}^n$.

A solution to (1.1)–(1.3) is defined to be a function, $\mathbf{y} \in C^2((0,T);\mathbb{R}^n) \cap C([0,T];\mathbb{R}^n)$ and $p\mathbf{y}' \in C([0,T];\mathbb{R}^n)$ such that $\mathbf{y}$ satisfies (1.1)–(1.3).

One of the contributions of this work is that it removes the widely used, Nagumo condition [20] in obtaining *a priori* bounds of solutions to derivative dependent nonlinear systems of singular BVPs. The use of the Nagumo condition has been studied extensively by [4,6,7,12,18, 26]. In a recent paper by Fewster-Young [9], the singular vector-valued version of the Nagumo condition is introduced, that is: there is a positive continuous function $\phi : [0,\infty) \to (0,\infty)$ such that

$$\|\mathbf{u}\| \le R, \qquad \|p(t)q(t)\mathbf{f}(t,\mathbf{u},\mathbf{v})\| \le \phi(\|\mathbf{v}\|), \quad \text{and} \quad \int \frac{x}{\phi(x)}\,dx = \infty. \tag{1.6}$$

However, it is not too hard to produce an example where (1.6) does not hold like the next problem.

Example 1.1. Consider the singular BVP for $0 < t < 1$,

$$\frac{1}{t^{1/2}}(t^{1/2}\mathbf{y}'(t))' = \frac{1}{t}\left(t^{3/2}y_1(t)[y_2'(t)]^2 + y_1(t), y_2(t)e^{-t[y_1'(t)]^2} + y_2(t)\cos^2(y_2(t))\right); \tag{1.7}$$
$$\lim_{t\to 0^+} p(t)\mathbf{y}'(t) = \mathbf{1}, \qquad \mathbf{y}(1) = \mathbf{0}. \tag{1.8}$$

In this BVP, the functions $p(t) = t^{1/2}$, $q(t) = 1/t$ and

$$\mathbf{f}(t,\mathbf{u},\mathbf{v}) := \left(t^{1/2}u_1[v_2]^2 + u_1, u_2 e^{-[v_1]^2} + u_2\cos^2(u_2)\right).$$

Now, by considering (1.6) for this example, the function ϕ does not exist when $\|\mathbf{u}\| \le R$, $t \in [0,1]$ since

$$\begin{aligned}
\|p(t)q(t)\mathbf{f}(t,\mathbf{u},\mathbf{v})\| &= \frac{1}{t^{1/2}}\sqrt{(t^{1/2}u_1v_2^2 + u_1)^2 + (u_2e^{-v_1^2} + u_2\cos^2 u_2)^2} \\
&\le \frac{1}{t^{1/2}}\sqrt{u_1^2(tv_2^4 + 1 + 2t^{1/2}v_2^2) + u_2^2(e^{-v_1^2} + \cos^2 u_2)^2} \\
&\le \frac{1}{t^{1/2}}\sqrt{u_1^2(tv_2^4 + 2t^{1/2}v_2^2) + u_1^2 + 4u_2^2} \\
&\le \frac{R}{t^{1/2}}\sqrt{(tv_2^4 + 2t^{1/2}v_2^2)} + \frac{2\|\mathbf{u}\|}{t^{1/2}} \\
&\le \frac{R}{t^{1/2}}\sqrt{t(v_2^4 + 2v_1^2v_2^2 + v_1^4) + 2t^{1/2}(v_1^2 + v_2^2)} + \frac{2\|\mathbf{u}\|}{t^{1/2}} \\
&\le R\|\mathbf{v}\|^2 + \frac{\sqrt{2}R}{t^{1/4}}\|\mathbf{v}\| + \frac{2R}{t^{1/2}}.
\end{aligned}$$

Also, Fewster-Young & Tisdell [10] used a Hartman inequality [15], namely

$$\|\mathbf{f}(t,\mathbf{u},\mathbf{v})\| \le 2V\left(\langle \mathbf{u}, \mathbf{f}(t,\mathbf{u},\mathbf{v})\rangle + \|\mathbf{v}\|^2\right) + W, \quad \text{for all } (t,\mathbf{u},\mathbf{v}) \in [0,T]\times\mathbb{R}^{2n} \tag{1.9}$$

where V, W are non–negative constants to produce *a priori* bounds and existence results for singular BVPs. They showed for singular BVPs that the Nagumo condition could be replaced by the assumption introduced by Hartman [15]: $2VR < 1$ where R is a non-negative constant and all solutions satisfy $\max_{t\in[0,T]} \|\mathbf{y}(t)\| \le R$. However, the condition $2VR < 1$ is a difficult assumption to satisfy since the constant R depends on V and can be very large. The first result builds off this idea and relaxes this condition to $2V\|\mathbf{d}\| < 1$, a very manageable assumption for applications. In addition, Hartman [15] only deals with Dirichlet boundary conditions where herein a Sturm–Liouville and a Dirichlet boundary condition are dealt with.

The final new result increases the freedom of the differential inequality by stating it in a general form using Lyapunov functions. This means the growth of the function $\mathbf{f}$ can be unrestricted and leads to many more examples able to be discussed. In the example below, the inequality (1.9) is not satisfied. In addition, in the scalar case, Bobisud [4] introduced a sufficient condition on the relationship between the functions p, q to be $p^2 q \le 1$ on $[0,T]$. This new result removes this condition, thus expanding the possible scenarios. For instance, the following scalar example fails both conditions and will be used to illustrate the new result.

Example 1.2. Consider the singular BVP:

$$\frac{1}{t^{1/4}}(t^{1/4}y')' = \frac{1}{t}\left(ty'^2 + y^3\right), \qquad 0<t<1, \qquad \lim_{t\to 0^+} t^{1/4}y'(t) = 0, \qquad y(1) = 1/3. \tag{1.10}$$

In this BVP, the functions $p(t) = t^{1/4}$, $q(t) = 1/t$ and $f(t,u,v) = t^{1/2}v^2 + u^3$. Notice that the condition $p^2 q \le 1$ on $[0,T]$ does not hold. In addition, suppose that g is a positive function such that $g(u) \ge -u$ for all $u \in \mathbb{R}$. See that the inequality (1.9) does not necessarily hold because

$$\begin{aligned}|f(t,u,v)| = |t^{1/2}v^2 + u^3| &\le ut^{1/2}v^2 + u^4 + 1 + (1+g(u))t^{1/2}v^2 \\ &\le uf(t,u,v) + v^2 + g(u)t^{1/2}v^2 + 1\end{aligned}$$

for $(t,u,v) \in (0,1)\times\mathbb{R}^2$ and the term: $g(u)v^2$ can not be bounded for all $u, v \in \mathbb{R}$.

The work presented herein complements the advancements made in singular BVPs by [1, 2, 5, 21, 23]. In addition, the results herein are novel in the non-singular setting when $p \equiv 1 \equiv q$ on $[0,T]$ and extends the contributions made by [8,13,16,17,19,25].

2 Main results

In this section, the main results are presented and the approach is to provide the *a priori* bounds on all possible solutions to (1.1)–(1.3) first then to follow with the existence of at least one solution to (1.1)–(1.3). The first result was proved by Fewster-Young & Tisdell [10] and provides an *a priori* bound on solutions to (1.1)–(1.3). It can be stated as follows.

Theorem 2.1. *Let* $\mathbf{f} \in C([0,T]\times\mathbb{R}^{2n};\mathbb{R}^n)$*. Let* $\alpha/\beta \ge 0$ *(*$\beta \ne 0$*), (1.4), (1.5) hold and let*

$$K_1 := \int_0^T \frac{ds}{p(s)} < \infty, \qquad K_2 := \int_0^T p(s)q(s)\,ds < \infty \tag{2.1}$$

with $p^2 q \le 1$ *on* $[0,T]$. *If there exist non-negative constants* V, W *such that*

$$\|\mathbf{f}(t,\mathbf{u},\mathbf{v})\| \le 2V\left(\langle \mathbf{u}, \mathbf{f}(t,\mathbf{u},\mathbf{v})\rangle + \|\mathbf{v}\|^2\right) + W, \quad \text{for all } (t,\mathbf{u},\mathbf{v}) \in [0,T] \times \mathbb{R}^{2n} \tag{2.2}$$

then all solutions $\mathbf{y} = \mathbf{y}(t)$ *to the singular BVP* (1.1), (1.2), (1.3) *satisfy*

$$\max_{t\in[0,T]} \|\mathbf{y}(t)\| \le R := \|\mathbf{d}\| + A_1 + V\|\mathbf{d}\|^2 + VK_1^2\|\mathbf{c}\|^2/[\beta(\beta+2K_1\alpha)] + K_1K_2W,$$

where

$$A_1 := K_1 \frac{\|\mathbf{c}\| + |\alpha|(\|\mathbf{d}\| + V\|\mathbf{d}\|^2 + VK_1^2\|\mathbf{c}\|^2/[\beta(\beta+2K_1\alpha)] + K_1K_2W)}{\left|\alpha\int_0^T ds/p(s) + \beta\right|}.$$

Theorem 2.2. *If the conditions of Theorem 2.1 are satisfied and* $2V\|\mathbf{d}\| < 1$ *then all solutions* $\mathbf{y} = \mathbf{y}(t)$ *to the singular BVP* (1.1)–(1.3) *satisfy*

$$\sup_{t\in(0,T)} \|p(t)\mathbf{y}'(t)\| \le S := \frac{\|\mathbf{c}\| + |\alpha|(\|\mathbf{d}\| + V\|\mathbf{d}\|^2 + \frac{VK_1^2\|\mathbf{c}\|^2}{\beta(\beta+2K_1\alpha)} + K_1K_2W)}{|\alpha K_1 + \beta|} + K_2W$$
$$+\ 2V\left[\|\mathbf{d}\|\frac{R + \|\mathbf{d}\| + V[R^2 - \|\mathbf{d}\|^2] + K_1K_2W}{1 - 2V\|\mathbf{d}\|} + \frac{R\|\mathbf{c}\|}{|\beta|}\right].$$

Proof. Let $\mathbf{y} = \mathbf{y}(t)$ be any solution to the singular BVP (1.1)–(1.3). From [22, p. 14], the BVP (1.1)–(1.3) has the equivalent integral equation given by

$$\mathbf{y}(t) = \mathbf{d} - \mathbf{A}\int_t^T \frac{ds}{p(s)} - \int_t^T \frac{1}{p(s)} \int_0^s p(x)q(x)\mathbf{f}(x,\mathbf{y}(x),p(x)\mathbf{y}'(x))\,dx\,ds, \tag{2.3}$$

for all $t \in [0,T]$ and where

$$\mathbf{A} := \frac{\mathbf{c} + \alpha\left(\mathbf{d} - \int_0^T \frac{1}{p(s)} \int_0^s p(x)q(x)\mathbf{f}(x,\mathbf{y}(x),p(x)\mathbf{y}'(x))\,dx\,ds\right)}{\alpha\int_0^T \frac{ds}{p(s)} + \beta}. \tag{2.4}$$

Let

$$\kappa := \left\|\int_0^T \frac{1}{p(s)} \int_0^s p(x)q(x)\mathbf{f}(x,\mathbf{y}(x),p(x)\mathbf{y}'(x))\,dx\,ds\right\|.$$

We now claim that

$$\kappa \le V\|\mathbf{d}\|^2 + \frac{VK_1^2\|\mathbf{c}\|^2}{\beta(\beta+2K_1\alpha)} + K_1K_2W. \tag{2.5}$$

If $r(t) := \|\mathbf{y}(t)\|^2$ where $\mathbf{y}$ is a solution to (1.1) then for all $t \in (0,T)$: we have

$$(p(t)r'(t))' = 2\left[\langle \mathbf{y}(t), p(t)q(t)\mathbf{f}(t,\mathbf{y}(t),p(t)\mathbf{y}'(t)\rangle + p(t)\|\mathbf{y}'(t)\|^2\right]. \tag{2.6}$$

If we estimate κ and use (2.2) then

$$\kappa \le \int_0^T \frac{1}{p(s)} \int_0^s p(x)q(x)\left[2V\left(\langle y(x), \mathbf{f}(x,\mathbf{y}(x),p(x)\mathbf{y}'(x))\rangle + p^2(x)\|\mathbf{y}'(x)\|^2\right) + W\right]dx\,ds$$

The condition $p^2q \le 1$ on $[0,T]$, the integral conditions (2.1) and (1.1) yields

$$\kappa \le \int_0^T \frac{1}{p(s)} \int_0^s 2V\left(\langle y(x), (p(x)\mathbf{y}'(x))'\rangle + p(x)\|\mathbf{y}'(x)\|^2\right)dx\,ds + K_1K_2W.$$

Substituting (2.6) gives

$$\kappa \leq V \int_0^T \frac{1}{p(s)} \int_0^s (p(x)r'(x))'\, dx\, ds + K_1K_2W.$$

Integrating yields

$$\begin{aligned}\kappa &\leq V(r(T) - r(0)) - K_1V \lim_{s\to 0^+} p(s)r'(s) + K_1K_2W \\ &= V(\|\mathbf{y}(T)\|^2 - \|\mathbf{y}(0)\|^2) - 2K_1V \left\langle \mathbf{y}(0), \lim_{s\to 0^+} p(s)\mathbf{y}'(s) \right\rangle + K_1K_2W.\end{aligned}$$

Substituting the boundary conditions (1.2), (1.3) gives

$$\kappa \leq V(\|\mathbf{d}\|^2 - \|\mathbf{y}(0)\|^2) - \frac{2K_1V\alpha\|\mathbf{y}(0)\|^2}{\beta} - 2K_1V\langle \mathbf{y}(0), \mathbf{c}/\beta\rangle + K_1K_2W.$$

If we apply the *Schwarz inequality* [14] to the last term then

$$\kappa \leq V\|\mathbf{d}\|^2 - V\|\mathbf{y}(0)\|^2 \left(\frac{2K_1\alpha}{\beta} + 1\right) + 2K_1V\|\mathbf{y}(0)\|\|\mathbf{c}/\beta\| + K_1K_2W. \tag{2.7}$$

Consider the inequality:

$$2ab \leq \epsilon a^2 + b^2/\epsilon; \quad a, b \in \mathbb{R} \quad \text{and} \quad \epsilon > 0.$$

For $a = \|\mathbf{y}(0)\|$, $b = K_1\|\mathbf{c}/\beta\|$, $\epsilon = 1 + 2K_1\alpha/\beta > 0$, we have

$$2K_1\|\mathbf{y}(0)\|\|\mathbf{c}/\beta\| \leq \|\mathbf{y}(0)\|^2 \left(\frac{2K_1\alpha}{\beta} + 1\right) + \frac{K_1^2\|\mathbf{c}\|^2}{\beta(\beta + 2K_1\alpha)}. \tag{2.8}$$

By employing the inequality (2.8) in (2.7) obtains the desired bound on κ, that is (2.5). From the integral representation of solutions to (1.1)–(1.3), that is (2.3), differentiating obtains the equivalent integral equation

$$p(t)\mathbf{y}'(t) = \mathbf{A} + \int_0^t p(x)q(x)\mathbf{f}(x, \mathbf{y}(x), p(x)\mathbf{y}'(x))\, dx, \quad \text{for all } t \in [0, T]. \tag{2.9}$$

We now claim that

$$\begin{aligned}\eta &:= \left\| \int_0^T p(x)q(x)\mathbf{f}(x, \mathbf{y}(x), p(x)\mathbf{y}'(x))\, dx \right\| \\ &\leq 2V\left[\|\mathbf{d}\| \left(\frac{R + \|\mathbf{d}\| + V[R^2 - \|\mathbf{d}\|^2] + K_1K_2W}{1 - 2V\|\mathbf{d}\|}\right) + \frac{R\|\mathbf{c}\|}{|\beta|}\right] + K_2W.\end{aligned}$$

The essence of the procedure is to show $\|\lim_{t\to T^-} p(t)\mathbf{y}'(t)\|$ is bounded. Consider the integral equation:

$$\lim_{t\to T^-} p(t)\mathbf{y}'(t) = \frac{\mathbf{y}(T) - \mathbf{y}(0) + \int_0^T \frac{1}{p(s)} \int_s^T (p(x)y'(x))'\, dx\, ds}{K_1} \tag{2.10}$$

Next, consider the equation (2.10) and apply (2.2) to see that

$$\begin{aligned}\left\| \lim_{t\to T^-} p(t)\mathbf{y}'(t) \right\| &\leq \frac{R + \|\mathbf{d}\| + V\int_0^T \frac{1}{p(s)} \int_s^T (p(x)r'(x))'\, dx\, ds + K_1K_2W}{K_1} \\ &= \frac{R + \|\mathbf{d}\| + V[r(0) - r(T)]}{K_1} + V \lim_{t\to T^-} p(t)r'(t) + K_2W.\end{aligned}$$

Since $r(t) = \|\mathbf{y}(t)\|^2$ and $r'(t) = 2\langle \mathbf{y}(t), \mathbf{y}'(t)\rangle$, this gives

$$\left\| \lim_{t\to T^-} p(t)\mathbf{y}'(t) \right\| \le \frac{R + \|\mathbf{d}\| + V[R^2 - \|\mathbf{d}\|^2]}{K_1} + 2V\left\langle \mathbf{y}(T), \lim_{t\to T^-} p(t)\mathbf{y}'(t)\right\rangle + K_2 W.$$

By applying the *Schwarz inequality*, we have

$$\left\| \lim_{t\to T^-} p(t)\mathbf{y}'(t) \right\| \le \frac{R + \|\mathbf{d}\| + V[R^2 - \|\mathbf{d}\|^2]}{K_1} + 2V\|\mathbf{d}\|\left\| \lim_{t\to T^-} p(t)\mathbf{y}'(t) \right\| + K_2 W.$$

By using the condition, $2V\|\mathbf{d}\| < 1$, we can rearrange this inequality to yield an *a priori* bound:

$$\left\| \lim_{t\to T^-} p(t)\mathbf{y}'(t) \right\| \le \frac{R + \|\mathbf{d}\| + V[R^2 - \|\mathbf{d}\|^2] + K_1 K_2 W}{1 - 2V\|\mathbf{d}\|}.$$

Now, if we estimate η and use (2.2), $p^2 q \le 1$ on $[0, T]$ then

$$\eta \le \int_0^T 2V\left(\langle y(x), p(x)q(x)\mathbf{f}(x, \mathbf{y}(x), p(x)\mathbf{y}'(x))\rangle + p(x)\|\mathbf{y}'(x)\|^2\right) dx\, ds + K_2 W.$$

Substituting (2.6) and integrating yields

$$\begin{aligned}\eta &\le V\left[\lim_{t\to T^-} p(t)r'(t) - \lim_{t\to 0^+} p(t)r'(t)\right] + K_2 W \\ &= 2V\left[\left\langle \mathbf{y}(T), \lim_{t\to T^-} p(t)\mathbf{y}'(t)\right\rangle - \left\langle \mathbf{y}(0), \lim_{t\to 0^+} p(t)\mathbf{y}'(t)\right\rangle\right] + K_2 W.\end{aligned}$$

By employing the boundary condition (1.2), this results in

$$\eta \le 2V\left[\left\langle \mathbf{y}(T), \lim_{t\to T^-} p(t)\mathbf{y}'(t)\right\rangle - \left\langle \mathbf{y}(0), \frac{\mathbf{c}}{\beta}\right\rangle\right] + K_2 W.$$

If we apply the *Schwarz inequality* and employ our bounds on the terms then this produces:

$$\begin{aligned}\eta &\le 2V\left[\|\mathbf{y}(T)\|\left\| \lim_{t\to T^-} p(t)\mathbf{y}'(t) \right\| + \|\mathbf{y}(0)\|\|\mathbf{c}/\beta\|\right] + K_2 W \\ &\le 2V\left[\|\mathbf{d}\|\left(\frac{R + \|\mathbf{d}\| + V[R^2 - \|\mathbf{d}\|^2] + K_1 K_2 W}{1 - 2V\|\mathbf{d}\|}\right) + \frac{R\|\mathbf{c}\|}{|\beta|}\right] + K_2 W.\end{aligned}$$

So far, we have achieved bounds on η and κ. Now, if we estimate (2.9) then the bounds on κ and η imply

$$\begin{aligned}\sup_{t\in(0,T)} \|p(t)\mathbf{y}'(t)\| &\le \frac{\|\mathbf{c}\| + |\alpha|(\|\mathbf{d}\| + \kappa)}{\left|\alpha \int_0^T \frac{ds}{p(s)} + \beta\right|} + \eta \\ &\le \frac{\|\mathbf{c}\| + |\alpha|\left[\|\mathbf{d}\| + \left(V\|\mathbf{d}\|^2 + \frac{VK_1^2\|\mathbf{c}\|^2}{\beta(\beta + 2K_1\alpha)} + K_1 K_2 W\right)\right]}{|\alpha K_1 + \beta|} + K_2 W \\ &\quad + 2V\left[\|\mathbf{d}\|\frac{R + \|\mathbf{d}\| + V[R^2 - \|\mathbf{d}\|^2] + K_1 K_2 W}{1 - 2V\|\mathbf{d}\|} + \frac{R\|\mathbf{c}\|}{|\beta|}\right].\end{aligned}$$

□

By combining the two previous results, the following result now proves the existence of solutions to the BVP (1.1)–(1.3).

Theorem 2.3. *If the conditions of Theorem 2.2 are satisfied then exists at least one solution to the singular BVP (1.1)–(1.3).*

Proof. Define the norm

$$\|\mathbf{u}\|_1 := \max\left\{ \sup_{t\in[0,T]} \|\mathbf{u}(t)\|,\ \sup_{t\in(0,T)} \|p(t)\mathbf{u}'(t)\| \right\}.$$

Recall

$$R := \|\mathbf{d}\| + A_1 + V\|\mathbf{d}\|^2 + VK_1^2\|\mathbf{c}\|^2/[\beta(\beta+2K_1\alpha)] + K_1K_2W$$

and

$$S := \frac{\|\mathbf{c}\| + |\alpha|\left[\|\mathbf{d}\| + \left(V\|\mathbf{d}\|^2 + \frac{VK_1^2\|\mathbf{c}\|^2}{\beta(\beta+2K_1\alpha)} + K_1K_2W\right)\right]}{|\alpha K_1 + \beta|} + K_2W$$
$$+ 2V\left[\|\mathbf{d}\|\frac{R + \|\mathbf{d}\| + V[R^2 - \|\mathbf{d}\|^2] + K_1K_2W}{1 - 2V\|\mathbf{d}\|} + \frac{R\|\mathbf{c}\|}{|\beta|}\right].$$

Consider the Banach space:

$$X := \{\mathbf{u} \in C([0,T];\mathbb{R}^n) :\ p\mathbf{u}' \in C([0,T];\mathbb{R}^n) \text{ with norm } \|\mathbf{u}\|_1\}$$

and the convex set in X,

$$U := \{\mathbf{y} \in X : \|\mathbf{y}\|_1 \le \max\{\mathrm{R},\mathrm{S}\}\}.$$

Define the operator $T : U \to X$ by

$$T\mathbf{y} := \mathbf{d} - \mathbf{A}\int_t^T \frac{ds}{p(s)} - \int_t^T \frac{1}{p(s)}\int_0^s p(x)q(x)\mathbf{f}(x,\mathbf{y}(x),p(x)\mathbf{y}'(x))\,dx\,ds,$$

for $t \in [0,T]$ and where $\mathbf{A}$ is given in (2.4). Consequently the solutions to (1.1)–(1.3) are the fixed points of the operator T. The aim now is to use the Schauder fixed point theorem [24] to prove the existence of fixed points of T. This requires T to be a continuous compact mapping from U to U. Now, the integral conditions (2.1) and $\mathbf{f}$ is continuous on X implies that T is a continuous mapping. To prove that T is compact, see that for any bounded set $V \subset X$, T maps V to a bounded set in X by the assumptions (2.1) and $\mathbf{f}$ is continuous. Furthermore, this means there is a positive constant M such that

$$\max_{(t,\mathbf{u},\mathbf{v})\in\Omega} \|\mathbf{f}(t,\mathbf{u},\mathbf{v})\| \le M$$

where

$$\Omega := \{(t,\mathbf{u},\mathbf{v}) \in [0,T]\times\mathbb{R}^{2n}\ :\ \|\mathbf{u}\| \le \max\{R,S\},\ \|\mathbf{v}\| \le \max\{R,S\}\}.$$

Furthermore, consider $\mathbf{y} \in U$, $r,t \in [0,T]$ where $t \ge r$; so this gives

$$\|T\mathbf{y}(t) - T\mathbf{y}(r)\| \le \|\mathbf{A}\|\int_r^t \frac{ds}{p(s)} + \int_r^t \frac{1}{p(s)}\int_0^s p(x)q(x)\|\mathbf{f}(x,\mathbf{y}(x),p(x)\mathbf{y}'(x))\|\,dx\,ds.$$

Also, this gives

$$\|p(t)(T\mathbf{y})'(t) - p(t)(T\mathbf{y})'(r)\| \le \int_r^t p(x)q(x)\|\mathbf{f}(x,\mathbf{y}(x),p(x)\mathbf{y}'(x))\|\,dx\,ds.$$

The condition (2.1) implies that the functions

$$\psi_1(t) = \int_0^t \frac{ds}{p(s)}, \qquad \psi_2(t) := \int_0^t p(s)q(s)\,ds \quad \text{and} \quad \psi_3(t) = \int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)\,dx\,ds$$

are continuous for $t \in [0, T]$. This means that there is a $\delta \geq |t - r|$ such that

$$|\psi_1(t) - \psi_1(r)| \leq \frac{\epsilon(|\alpha K_1 + \beta|)}{2(\|\mathbf{c}\| + |\alpha|(\|\mathbf{d}\| + K_1K_2M))},$$
$$|\psi_2(t) - \psi_2(r)| \leq \frac{\epsilon}{M}, \qquad |\psi_3(t) - \psi_3(r)| \leq \frac{\epsilon}{2M},$$

where $\epsilon > 0$. Thus, it follows that

$$\|T\mathbf{y}(t) - T\mathbf{y}(r)\|_1 \leq \epsilon, \quad \text{whenever } |t - r| \leq \delta$$

and so T is equicontinuous. Consequently, the *Arzelà–Ascoli* theorem [3] implies that $T : U \to X$ is a continuous compact mapping. We now show that for any $\mathbf{y} \in U$, $T\mathbf{y} \in U$, that is $T(U) \subset U$. The assumptions of Theorem 2.2 hold, this means by the proof of that Theorem 2.2 that

$$\kappa := \left\| \int_0^T \frac{1}{p(s)} \int_0^s p(x)q(x)\mathbf{f}(x, \mathbf{y}(x), p(x)\mathbf{y}'(x))\,dx\,ds \right\|$$
$$\leq V\|\mathbf{d}\|^2 + \frac{VK_1^2\|\mathbf{c}\|^2}{\beta(\beta + 2K_1\alpha)} + K_1K_2W,$$

and

$$\eta := \left\| \int_0^T p(x)q(x)\mathbf{f}(x, \mathbf{y}(x), p(x)\mathbf{y}'(x))\,dx \right\|$$
$$\leq 2V\left[\|\mathbf{d}\| \left(\frac{R + \|\mathbf{d}\| + V[R^2 - \|\mathbf{d}\|^2] + K_1K_2W}{1 - 2V\|\mathbf{d}\|} \right) + \frac{R\|\mathbf{c}\|}{|\beta|}\right] + K_2W.$$

By estimating $\|T\mathbf{y}\|_1$, we obtain

$$\sup_{t \in [0,T]} \|T\mathbf{y}(t)\| \leq \|\mathbf{d}\| + K_1\frac{(\|\mathbf{c}\| + \alpha(\|\mathbf{d}\| + \kappa))}{|\alpha K_1 + \beta|} + \kappa$$

and

$$\sup_{t \in (0,T)} \|p(t)(T\mathbf{y})'(t)\| \leq \frac{(\|\mathbf{c}\| + \alpha(\|\mathbf{d}\| + \kappa))}{|\alpha K_1 + \beta|} + \eta.$$

By using the bounds on κ, η, it follows that

$$\|T\mathbf{y}(t)\|_1 \leq \max\{R, S\}.$$

This means for any $\mathbf{y} \in U$, $T\mathbf{y} \in U$. By applying Schauder's fixed point theorem, the operator T has at least one fixed point. Moreover, the integral representation implies that $\mathbf{y} \in C^2((0, T); \mathbb{R}^n)$, $p\mathbf{y}' \in C([0, T]; \mathbb{R}^n)$ and $\mathbf{y}$ satisfies the boundary conditions (1.2), (1.3). This proves that there is at least one solution to the BVP (1.1)–(1.3). □

The Example 1.1 is next examined by applying the previous results to show the existence of at least one solution to the singular BVP (1.7), (1.8).

Example 2.4. Consider the singular BVP: (1.7), (1.8). Notice that the functions here are $p(t) = t^{1/2}$, $q(t) = 1/t$ and

$$\mathbf{f}(t, \mathbf{u}, \mathbf{v}) := \left(t^{1/2} u_1 [v_2]^2 + u_1, u_2 e^{-[v_1]^2} + u_2 \cos^2(u_2)\right) \quad \text{for } (t, \mathbf{u}, \mathbf{v}) \in [0, 1] \times \mathbb{R}^2 \times \mathbb{R}^2.$$

See that the conditions relating to the functions p, q are satisfied since

$$K_1 = 2 = K_2 \quad \text{and} \quad p^2(t) q(t) = 1 \quad \text{for } t \in [0, 1].$$

In next part of the proof, the following inequalities are used:

$$a e^{-b^2} \leq a^2 e^{-b^2} + \frac{1}{4}, \quad \text{for } a, b \in \mathbb{R} \quad \text{and} \quad |x| \leq x^2 + \frac{1}{4}, \quad \text{for } x \in \mathbb{R}.$$

The next condition to check is the inequality (2.2), if we choose $V = 1/2$ and $W = 3/4$ then

$$\begin{aligned}
\|\mathbf{f}(t, \mathbf{u}, \mathbf{v})\| &\leq |f_1(t, u_1, u_2, v_1, v_2)| + |f_2(t, u_1, u_2, v_1, v_2)| \\
&\leq |t^{1/2} u_1 [v_2]^2 + u_1| + \left| u_2 e^{-[v_1]^2} + u_2 \cos^2(u_2) \right| \\
&\leq [v_2]^2 \left(|u_1| t^{1/2} + 1 - \frac{t^{1/2}}{4} \right) + |u_1| + |u_2| e^{-[v_1]^2} + |u_2| \cos^2(u_2) \\
&\leq [v_2]^2 (|u_1|^2 t^{1/2} + 1) + |u_1|^2 + \frac{1}{4} + |u_2|^2 e^{-[v_1]^2} + \frac{1}{4} + \cos^2(u_2)(|u_2|^2 + \frac{1}{4}) \\
&\leq [v_2]^2 (u_1^2 t^{1/2} + 1) + u_1^2 + u_2^2 e^{-[v_1]^2} + u_2^2 \cos^2(u_2) + [v_1]^2 + \frac{3}{4} \\
&= u_1 f_1(t, u_1, u_2, v_1, v_2) + u_2 f_2(t, u_1, u_2, v_1, v_2) + [v_1]^2 + [v_2]^2 + \frac{3}{4} \\
&= 2V \left(\langle \mathbf{u}, \mathbf{f}(t, \mathbf{u}, \mathbf{v}) \rangle + \|\mathbf{v}\|^2 \right) + W.
\end{aligned}$$

In this example, $\|\mathbf{d}\| = 0$, so the condition $2V\|\mathbf{d}\| < 1$ is satisfied and finally by applying Theorem 2.3 there exists at least one solution to the singular BVP (1.7), (1.8) and they satisfy

$$\max_{t \in [0,1]} \|\mathbf{y}(t)\| \leq 4\sqrt{2} + 3, \quad \text{and} \quad \sup_{t \in (0,1)} \|p(t)\mathbf{y}'(t)\| \leq 4\sqrt{2} + \frac{19}{2}.$$

The next result removes the condition, $p^2 q \leq 1$ on $[0, T]$ and generalises the differential inequality (2.2) in the previous result by using a general Lyapunov function. The Lyapunov function is of two variables t and $u := \|\mathbf{y}(t)\|^2$, namely it is $r(t, u) \in C^2((0, T) \times \mathbb{R}; \mathbb{R}) \cap C([0, T] \times \mathbb{R}; \mathbb{R})$, where $pr_t, r_u \in C([0, T] \times \mathbb{R}; \mathbb{R})$. A condition is imposed for all functions in the solution space, $\mathbf{y} \in C^2((0, T); \mathbb{R}^n) \cap C([0, T]; \mathbb{R}^n)$ and $p\mathbf{y}' \in C([0, T]; \mathbb{R}^n)$ of which

$$\|(p(t)\mathbf{y}'(t))'\| \leq (p(t) r'(t, \|\mathbf{y}(t)\|^2))' \quad \text{for all } t \in (0, T). \tag{2.11}$$

Theorem 2.5. *Let R_0, $\tilde{R}$, $\tilde{S}$ be non-negative constants. Let $\mathbf{y} \in C^2((0, T); \mathbb{R}^n) \cap C([0, T]; \mathbb{R}^n)$ and $p\mathbf{y}' \in C([0, T]; \mathbb{R}^n)$ and let (1.4), (1.5), (2.1) hold. If $r(t, u) \in C^2((0, T) \times \mathbb{R}; \mathbb{R}) \cap C([0, T] \times \mathbb{R}; \mathbb{R})$, $pr_t, r_u \in C([0, T] \times \mathbb{R}; \mathbb{R})$ where $u := \|\mathbf{y}(t)\|^2$ and B is a non-negative constant such that (2.11) holds,*

$$K_1 \lim_{t \to 0^+} p(t) r'(t, \|\mathbf{y}(t)\|^2) + r(0, \|\mathbf{y}(0)\|^2) + B \geq 0 \tag{2.12}$$

and

$$2\|\mathbf{d}\| |r_u(T, \|\mathbf{d}\|^2)| < 1 \tag{2.13}$$

then all possible solutions $\mathbf{y} = \mathbf{y}(t)$ *for* $t \in [0, T]$ *to the singular BVP* (1.1)–(1.3) *satisfy*

$$\max_{t\in[0,T]} \|\mathbf{y}(t)\| \leq \tilde{R} := \|\mathbf{d}\| + \frac{\|\mathbf{c}\| + |\alpha| \left[\|\mathbf{d}\| + (r(T, \|\mathbf{d}\|^2) + B)\right]}{|\alpha K_1 + \beta|} + r(T, \|\mathbf{d}\|^2) + B,$$

$$|r(0, \|\mathbf{y}(0)\|^2)| \leq R_0, \quad \textit{when} \quad \|\mathbf{y}(0)\| \leq \tilde{R}$$

and

$$\sup_{t\in(0,T)} \|p(t)\mathbf{y}'(t)\| \leq \tilde{S} := 2\|\mathbf{d}\| \left[\frac{\tilde{R} + \|\mathbf{d}\| + R_0 - r(T, \|\mathbf{d}\|^2) + K_1 L + K_1 K_2 W}{K_1(1 - 2\|\mathbf{d}\| |r_u(T, \|\mathbf{d}\|^2)|)}\right] + \frac{\|\mathbf{c}\| + |\alpha| \left[\|\mathbf{d}\| + (r(T, \|\mathbf{d}\|^2) + B)\right]}{\left|\alpha \int_0^T \frac{ds}{p(s)} + \beta\right|} + L + \frac{R_0 + B}{K_1},$$

where $L := \lim_{t\to T^-} p(t) r_t(t, \|\mathbf{y}(t)\|^2)$.

Proof. Let $\mathbf{y} = \mathbf{y}(t)$ be any solution to (1.1)–(1.3), which has the integral representation given by (2.3). We now wish to estimate the terms κ and η as in Theorem 2.1. See that by the differential equation (1.1), we have

$$\kappa \leq \int_0^T \frac{1}{p(s)} \int_0^s \|p(t)q(t)\mathbf{f}(t, \mathbf{y}(t), p(t)\mathbf{y}'(t))\| \, dx \, ds$$
$$= \int_0^T \frac{1}{p(s)} \int_0^s \|(p(x)\mathbf{y}'(x))'\| \, dx \, ds.$$

By using (2.11) instead of (2.2), we have

$$\kappa \leq \int_0^T \frac{1}{p(s)} \int_0^s (p(x) r'(x, \|\mathbf{y}(x)\|^2))' \, dx \, ds$$
$$\leq r(T, \|\mathbf{y}(T)\|^2) - r(0, \|\mathbf{y}(0)\|^2) - K_1 \lim_{t\to 0^+} p(t) r'(t, \|\mathbf{y}(t)\|^2).$$

The condition (2.12) and the boundary condition (1.2) implies

$$\kappa \leq r(T, \|\mathbf{d}\|^2) + B.$$

Thus, if we estimate the integral representation (2.3) then we obtain

$$\max_{t\in[0,T]} \|\mathbf{y}(t)\| \leq \tilde{\mathrm{R}} := \|\mathbf{d}\| + \frac{\|\mathbf{c}\| + |\alpha| \left[\|\mathbf{d}\| + (r(T, \|\mathbf{d}\|^2) + B)\right]}{|\alpha K_1 + \beta|} + r(T, \|\mathbf{d}\|^2) + B. \tag{2.14}$$

Notice that the *a priori* bound (2.14) and because the function r is continuous, implies there is non–negative constant R_0 such that

$$|r(0, \|\mathbf{y}(0)\|^2)| \leq R_0, \quad \text{when} \quad \|\mathbf{y}(0)\| \leq \tilde{\mathrm{R}}.$$

Differentiating (2.3) gives an equivalent integral representation for the derivative of a solution to (1.1)–(1.3) given by equation (2.9). It now suffices to find an estimate for η, namely:

$$\eta := \left\| \int_0^T p(x)q(x)\mathbf{f}(x, \mathbf{y}(x), p(x)\mathbf{y}'(x)) \, dx \right\|.$$

Consider the integral equation (2.10) and see that for a solution $\mathbf{y}$ to our singular BVP (1.1)–(1.3), we have

$$\left\| \lim_{t \to T^-} p(t)\mathbf{y}'(t) \right\| \leq \frac{\tilde{R} + \|\mathbf{d}\| + \int_0^T \frac{1}{p(s)} \int_s^T \|(p(x)\mathbf{y}'(x))'\| \, dx \, ds}{K_1}.$$

If we apply (2.11) and integrate then we obtain

$$\begin{aligned} \left\| \lim_{t \to T^-} p(t)\mathbf{y}'(t) \right\| \leq & \frac{\tilde{R} + \|\mathbf{d}\| + [r(0, \|\mathbf{y}(0)\|^2) - r(T, \|\mathbf{y}(T)\|^2)]}{K_1} \\ & + \lim_{t \to T^-} p(t) r'(t, \|\mathbf{y}(t)\|^2) + K_2 W. \end{aligned}$$

Let $u := \|\mathbf{y}(t)\|^2$ and notice that by the chain rule,

$$r'(t, u) = r_t(t, u) + 2 \langle \mathbf{y}(t), \mathbf{y}'(t) \rangle \, r_u(t, u). \tag{2.15}$$

By substituting (2.15) and applying *Schwarz's inequality* gives

$$\begin{aligned} \left\| \lim_{t \to T^-} p(t)\mathbf{y}'(t) \right\| \leq & \frac{\tilde{R} + \|\mathbf{d}\| + [R_0 - r(T, \|\mathbf{y}(T)\|^2)]}{K_1} + \lim_{t \to T^-} p(t) r_t(t, \|\mathbf{y}(t)\|^2) \\ & + 2\|\mathbf{d}\| \left\| \lim_{t \to T^-} p(t)\mathbf{y}'(t) \right\| |r_u(T, \|\mathbf{d}\|^2)| + K_2 W. \end{aligned}$$

Due to condition (2.13), we can rearrange and this yields

$$\left\| \lim_{t \to T^-} p(t)\mathbf{y}'(t) \right\| \leq \frac{\tilde{R} + \|\mathbf{d}\| + R_0 - r(T, \|\mathbf{d}\|^2) + K_1 L + K_1 K_2 W}{K_1 (1 - 2\|\mathbf{d}\| |r_u(T, \|\mathbf{d}\|^2)|)} \tag{2.16}$$

where

$$L := \lim_{t \to T^-} p(t) r_t(t, \|\mathbf{y}(t)\|^2).$$

If we estimate η and use (2.11) then

$$\eta \leq \int_0^T p(x) q(x) \|\mathbf{f}(x, \mathbf{y}(x), p(x)\mathbf{y}'(x))\| \, dx = \int_0^T \|(p(x)\mathbf{y}'(x))'\| \, dx \leq \int_0^T (p(x) r'(x))' \, dx.$$

By integrating and applying the condition (2.12) to η, we obtain

$$\begin{aligned} \eta & \leq \lim_{t \to T^-} p(t) r'(t, \|\mathbf{y}(t)\|^2) - \lim_{t \to 0^+} p(t) r'(t, \|\mathbf{y}(t)\|^2) \\ & \leq \lim_{t \to T^-} p(t) r'(t, \|\mathbf{y}(t)\|^2) + \frac{r(0, \|\mathbf{y}(0)\|^2) + B}{K_1}. \end{aligned}$$

By the assumptions $pr_t, r_u \in C([0, T] \times \mathbb{R}; \mathbb{R})$, (2.14) and (2.16), we have

$$\begin{aligned} \eta & \leq 2 \left\langle \mathbf{y}(T), \lim_{t \to T^-} p(t)\mathbf{y}'(t) \right\rangle r_u(T, \|\mathbf{y}(T)\|^2) + L + \frac{R_0 + B}{K_1}. \\ & \leq 2\|\mathbf{d}\| \left[\frac{\tilde{R} + \|\mathbf{d}\| + R_0 - r(T, \|\mathbf{d}\|^2) + K_1 L + K_1 K_2 W}{K_1 (1 - 2\|\mathbf{d}\| |r_u(T, \|\mathbf{d}\|^2)|)} \right] |r_u(T, \|\mathbf{d}\|^2)| \\ & \quad + L + \frac{R_0 + B}{K_1}. \end{aligned}$$

Finally, we can estimate (2.9) with our estimates for κ and η to obtain

$$\begin{aligned}\sup_{t\in(0,T)} \|p(t)\mathbf{y}'(t)\| &\le \eta + \|\mathbf{A}\| \\ &\le 2\|\mathbf{d}\| \left[\frac{\tilde{\mathrm{R}} + \|\mathbf{d}\| + R_0 - r(T,\|\mathbf{d}\|^2) + K_1 L + K_1 K_2 W}{K_1(1-2\|\mathbf{d}\||r_u(T,\|\mathbf{d}\|^2)|)}\right. \\ &\quad \left. + \frac{\|\mathbf{c}\| + |\alpha|\left[\|\mathbf{d}\| + (r(T,\|\mathbf{d}\|^2)+B)\right]}{\left|\alpha\int_0^T \frac{ds}{p(s)} + \beta\right|} + L + \frac{R_0 + B}{K_1}\right].\end{aligned}$$

□

Remark 2.6. To obtain the same inequality as in Theorem 2.2, the condition $p^2 q \le 1$ on $[0,T]$ is imposed, the function $r(t,u) := Vu + W\int_0^t \frac{1}{p(s)}\int_0^s p(x)q(x)\,dx\,ds$, and the constant

$$B := \frac{VK_1^2\|\mathbf{c}\|^2}{\beta(\beta + 2K_1\alpha)}.$$

The final result is the accompanying existence result to the previous theorem.

Theorem 2.7. *If the conditions of Theorem 2.5 are satisfied then the singular BVP (1.1)–(1.3) has at least one solution.*

Proof. The proof is identical to Theorem 2.3 except the convex set in X is

$$U := \{\mathbf{y} \in X : \|\mathbf{y}\|_1 \le \max\{\tilde{\mathrm{R}}, \tilde{\mathrm{S}}\}\}.$$

The remainder of the proof is therefore omitted. □

The end of this paper is finished with applying the last two results to the Example 1.2.

Example 2.8. Consider the singular BVP (1.10). The functions relating to theory in this paper are $p(t) = t^{1/4}$, $q(t) = \frac{1}{t}$ and $f(t,w,z) = t^{1/2}z^2 + w^3$. To check the conditions of Theorem 2.5 hold, choose the function $r(t,u) := Ve^{au} + W\int_0^t \frac{1}{p(s)}\int_0^s p(x)q(x)\,dx\,ds$. See that the integral conditions in (2.1) are satisfied since $K_1 = 4/5$ and $K_2 = 4/7$. It now suffices to show that the inequality (2.5) is satisfied, that is

$$|(p(t)y'(t))'| \le 2aVe^{a|y(t)|^2}\left[y(t)(p(t)y'(t))' + p(t)|y'(t)|^2 + 2ap(t)\left(y(t)y'(t)\right)^2\right] + Wp(t)q(t)$$

for all $t \in (0,1)$. Furthermore, for solutions to (1.1), it suffices to show that

$$|p^2(t)q(t)f(t,w,z)| \le 2aVe^{a|w|^2}\left[wp^2(t)q(t)f(t,w,z) + |z|^2 + 2a\,(wz)^2\right] + Wp^2(t)q(t)$$

for $(t,w,z) \in (0,T)\times\mathbb{R}\times\mathbb{R}$. To show this holds for the desired functions, see that the following inequality holds for all $x \in \mathbb{R}$:

$$1 + x + x^2 \ge \frac{3}{4}$$

and choose $V = 2$, $W = 1$, $a = 1/2$ such that

$$
\begin{aligned}
|p^2(t)q(t)f(t,w,z)| &= \frac{|t^{1/2}z^2 + w^3|}{t^{1/2}} \\
&\leq z^2 + \frac{|w|^3}{t^{1/2}} \\
&\leq z^2 \left[2e^{w^2/2}\left(1 + w + w^2\right)\right] + \frac{|w|^4 + 1}{t^{1/2}} \\
&\leq 2\left[z^2\left(w + 1 + w^2\right) + \frac{w^4}{t^{1/2}}\right] e^{|w|^2/2} + \frac{1}{t^{1/2}} \\
&\leq 2aVe^{a|w|^2}\left[wp^2(t)q(t)f(t,w,z) + |z|^2 + 2a\left(wz\right)^2\right] + Wp^2(t)q(t).
\end{aligned}
$$

Next is to show that the conditions (2.12), (2.13) are satisfied. Notice that

$$p(t)r'(t,|y(t)|^2) = 2aV\left(y(t)p(t)y'(t)e^{a|y(t)|^2}\right) + W\int_0^t p(s)q(s)\,ds.$$

The condition (2.12) is satisfied with $B = 0$ since by substituting the boundary condition,

$$\lim_{t\to 0^+} p(t)y'(t) = 0$$

results in

$$K_1 \lim_{t\to 0^+} p(t)r'(t,|y(t)|^2) + r(0,|y(0)|^2) + B = Ve^{a|y(0)|^2} > 0.$$

Also, the condition (2.13) is satisfied since from the boundary condition, $y(1) = 1/3 = d$ and thus

$$2|d|r_u(1,|d|^2) = 2aV|d|e^{a|d|^2} = \frac{2}{3}e^{1/18} < 1.$$

Thus, all the conditions of Theorem 2.5 are satisfied and so Theorem 2.7 implies there exists at least one solution to the singular BVP 1.10 and they satisfy

$$\max_{t\in[0,1]} \|\mathbf{y}(t)\| \leq \tilde{\mathrm{R}} := \frac{37}{63} + 2e^{1/18},$$

and

$$\sup_{t\in(0,1)} \|p(t)\mathbf{y}'(t)\| \leq \frac{5}{2}\left[\frac{\tilde{\mathrm{R}} + 313/315 + 2e^{\tilde{\mathrm{R}}/2} - 2e^{1/18}}{(3 - 2e^{1/18})}\right] + \frac{4}{7} + \frac{5e^{\tilde{\mathrm{R}}/2}}{2}.$$

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