Electronic Journal of Qualitative Theory of Differential Equations
2016, No. 18, 1–17; doi: 10.14232/ejqtde.2016.1.18 http://www.math.u-szeged.hu/ejqtde/

Global stability of vaccine-age/staged-structured epidemic models with nonlinear incidence

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Received 16 September 2015, appeared 18 April 2016
Communicated by Péter L. Simon

Abstract. We consider two classes of infinitely dimensional epidemic models with nonlinear incidence, where one assumes that the rate of a vaccinated individual losing immunity depends on the vaccine-age and another assumes that, before the vaccine begins to wane, there is a period during which the vaccinated individuals have complete immunity against the infection. The first model is given by a coupled ordinary-hyperbolic differential system and the second class is described by a delay differential system. We calculate their respective basic reproduction numbers, and show they characterize the global dynamics by constructing the appropriate Lyapunov functionals.

Keywords: nonlinear incidence, vaccination, basic reproduction number, steady state, global stability, Lyapunov functional.

2010 Mathematics Subject Classification: 34K20, 92D30.

1 Introduction

In mathematical epidemiology, the term "incidence" is used to describe the increasing rate of the size of the subpopulation exposed or infected individuals. With $S = S(t)$ and $I = I(t)$ being used to denote the size of the susceptible and infected individuals and using β to denote a certain transmission coefficient, both bilinear and standard incidences (βSI and $\beta SI/N$ with N the total population) have been frequently used in classical epidemic models [26], but more general forms including $\beta S^p I^q$ and $\beta SI^p/(1 + \alpha I^q)$ have also been proposed (see [23, 24]), $\beta(I + \nu I^p)S$ [22, 36, 39], $Sg(I)$ [1, 9, 14, 33], and $g(S, I)$ [10, 11], as well as $g(S, I, N)$ [12]. Contrasted to models with the bilinear or standard incidence, complex dynamic behaviors, including backward bifurcation and Bogdanov–Takens bifurcation may occur when more general nonlinear incidences are used.

In epidemic models with vaccination, the vaccination strategy is usually characterized by pulse vaccination [3,20,35] (when the vaccine is given at fixed times) or continuous vaccination

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[13,15–17,21,25] (when the susceptible population moves into the vaccinated compartment at a constant rate). For vaccinated individuals, it is often assumed that vaccine wanes exponentially or in a fashion featured by a step function. The models with exponential wane of immunity are often expressed by ordinary differential equations [13,17,21,25], and the models with step function are usually described by delay differential equations [15,16].

In general, adding additional compartments involving an immunization program can further complicate the nonlinear behaviors of infection dynamics. However, as Li et al. [8,19] considered, waning of vaccine is related to the vaccine-age (i.e., the period of time elapsing after inoculation), and the effect of vaccine providing immunity within-host depends on the vaccine-age. This consideration reflects well with some current vaccination campaigns (for example, Hepatitis B vaccine and Tuberculosis vaccine).

In this paper, we consider two classes of epidemic models with vaccination. In either class of the models, the basic S–I–R and S–V compartments (with R and V denoting the removed and vaccinated compartment respectively) are involved. In both classes of models, we use the nonlinear incidence of form $Sg(I)$. Also, we explicitly consider the vaccine-age or stage in describing the waning immunity. We obtain either a general coupled system of ordinary-hyperbolic partial differential equations (with a general waning rate), or a reduced system of delay differential equations where the immunity of vaccinated individuals is either perfect or weak depending on the vaccine stage. Our goal is to construct appropriate Lyapunov functionals to show that the so-called basic reproduction number provides the threshold to classify the global dynamics of the model systems.

2 Formulation of the models

In this section we formulate two classes of SIR–SVS epidemic models with vaccination.

The population is decomposed into the susceptible, infectious, removed and vaccinated classes. Let $S = S(t)$, $I = I(t)$ and $R = R(t)$ denote the sizes of the susceptible, infectious and removed classes at time t, respectively, and $v(a,t)$ the density of vaccinated individuals with vaccine-age a at time t. An SIR–SVS vaccine-age structured model can be described by the following system:

$$\begin{aligned}
&S'(t) = A - (\mu+p)S(t) - S(t)g(I(t)) + \int_0^\infty \delta(a)v(a,t)da, \qquad t>0,\\
&I'(t) = S(t)g(I(t)) - (\mu+\varepsilon+\theta)I(t), \qquad t>0,\\
&R'(t) = \theta I(t) - \mu R(t), \qquad t>0,\\
&\frac{\partial v(a,t)}{\partial t} + \frac{\partial v(a,t)}{\partial a} = -[\mu+\delta(a)]v(a,t), \qquad t>0,\ a>0,
\end{aligned} \tag{2.1}$$

with the boundary condition

$$v(0,t) = pS(t), \tag{2.2}$$

and the initial conditions

$$S(0)=S_s>0, \quad I(0)=I_s>0, \quad R(0)=R_s\ge 0, \quad v(a,0)=v_s(a)\in L^1_+(0,\infty). \tag{2.3}$$

Here, A is the recruitment rate of susceptible individuals, μ is the per capita natural death rate, p is the vaccination rate coefficient at which the susceptible class $S(t)$ is subjected to a vaccination campaign, ε and θ denote respectively the disease-induced death rate and the removed rate for an infected individual, $\delta(a)$ is the rate of per vaccinated capita with vaccine-age a

losing immunity and returning to the susceptible class, where $\delta(a)$ is assumed to nonnegative and bounded for $a \geq 0$. Here, the term $Sg(I)$ represents the nonlinear incidence of the model, which was used in [9, 14, 33], and function $g(I)$ satisfies the following assumptions:

- (H) $g(0) = 0,\ g'(I) > 0,\ g''(I) \leq 0$ for $I \geq 0$.

We refer to [12] for further discussions of the epidemiological background of these assumptions. The existence and uniqueness of solutions for system (2.1) with conditions (2.2) and (2.3) can be established using the standard theory for age-dependent models [38].

Since it is impossible that the age of a vaccinated individual is infinite, we add a reasonable condition for $v(a,t)$

$$v(\infty, 0) = v_s(\infty) = 0. \tag{2.4}$$

And, according to the continuity of solution of system (2.1), it is necessary to require the matching $pS_s = v_s(0)$.

Notice that the variable R does not appear in other equations of system (2.1), so we will focus on the subsystem

$$\begin{aligned} &S' = A - (\mu + p)S - Sg(I) + \int_0^\infty \delta(a)v(a,t)da, \qquad t > 0, \\ &I' = Sg(I) - (\mu + \alpha)I, \qquad t > 0, \\ &\frac{\partial v(a,t)}{\partial t} + \frac{\partial v(a,t)}{\partial a} = -[\mu + \delta(a)]v(a,t), \qquad t > 0,\ a > 0, \end{aligned} \tag{2.5}$$

where $\alpha = \varepsilon + \theta$.

Along the characteristic line $t - a =$constant, integrating the third equation of system (2.5) with conditions (2.2) and (2.3) gives

$$v(a,t) = \begin{cases} pS(t-a)e^{-\int_0^a [\mu + \delta(\xi)]d\xi}, & 0 < a \leq t, \\ v_s(a-t)e^{-\int_{a-t}^a [\mu + \delta(\xi)]d\xi}, & a \geq t > 0. \end{cases} \tag{2.6}$$

From equations (2.4) and (2.6) we have $v(\infty, t) = 0$.

Note that a vaccinated individual may have the perfect immunity against the infection within a period following vaccination. After the period the vaccine wanes gradually, the vaccinated individual loses the immunity finally, and becomes susceptible. As such, we divide the vaccinated class into two subclasses, one with perfect immunity and the other with weak immunity. Accordingly, we express the rate of vaccinated individuals losing immunity by the following step function:

$$\delta(a) = \begin{cases} 0, & a \leq \tau, \\ \gamma, & a > \tau, \end{cases}$$

where τ is the period of the perfect immunity, γ is the rate at which vaccine wanes. Then $P(t) = \int_0^\tau v(a,t)da$ and $V(t) = \int_\tau^\infty v(a,t)da$ are the numbers of vaccinated individuals with perfect and weak immunity at time t, respectively. From the third equation of system (2.5) we have

$$\begin{aligned} P' &= -\int_0^\tau \frac{\partial v(a,t)}{\partial a}da - \int_0^\tau [\mu + \delta(a)]v(a,t)da \\ &= v(0,\tau) - v(\tau,t) - \mu P, \\ V' &= -\int_\tau^\infty \frac{\partial v(a,t)}{\partial a}da - \int_\tau^\infty [\mu + \delta(a)]v(a,t)da \\ &= v(\tau,t) - (\mu + \gamma)V, \end{aligned}$$

where $v(\infty,t)=0$ is used. It follows from equation (2.6) that $v(\tau,t)=pS(t-\tau)e^{-\mu\tau}$ for $t\geq\tau$. Then, using the boundary condition (2.2), corresponding to system (2.5) we have the following vaccine-staged structured model

$$\begin{aligned} S' &= A-(\mu+p)S-Sg(I)+\gamma V,\\ I' &= Sg(I)-(\mu+\alpha)I,\\ P' &= pS-pS(t-\tau)e^{-\mu\tau}-\mu P,\\ V' &= pS(t-\tau)e^{-\mu\tau}-(\mu+\gamma)V. \end{aligned} \tag{2.7}$$

The variable P only appears in the third equation of system (2.7), for simplicity, we will consider dynamics of the following subsystem of system (2.7)

$$\begin{aligned} S' &= A-(\mu+p)S-Sg(I)+\gamma V,\\ I' &= Sg(I)-(\mu+\alpha)I,\\ V' &= pS(t-\tau)e^{-\mu\tau}-(\mu+\gamma)V, \end{aligned} \tag{2.8}$$

for $t\geq 0$. Without loss of generality, the initial conditions for system (2.8) take the form

$$\begin{aligned} &S(\theta)=\varphi_1(\theta),\quad I(\theta)=\varphi_2(\theta),\quad V(\theta)=\varphi_3(\theta),\\ &\varphi_i(\theta)\geq 0,\quad \theta\in[-\tau,0],\quad \varphi_i(0)>0\ (i=1,2,3), \end{aligned} \tag{2.9}$$

where $(\varphi_1,\varphi_2,\varphi_3)\in C([-\tau,0],\mathbb{R}^3_{+0})$, the Banach space of continuous functions mapping the interval $[-\tau,0]$ into $\mathbb{R}^3_{+0}:=\{(x_1,x_2,x_3):x_i\geq 0, i=1,2,3\}$. Indeed, the conditions (2.9) include the initial conditions of form

$$S(\theta)=\varphi(\theta)\in C([-\tau,0],\mathbb{R}_{+0}),\quad I(0)=I_0>0,\quad V(\theta)=V_0>0,$$

which are suitable for system (2.8), but may be not after making some transformation of variables for system (2.8). Thus in contrast the conditions (2.9) are more general.

It is well known by the fundamental theory of functional differential equations [4], that model system (2.8) has a unique solution $(S(t),I(t),V(t))$ satisfying the initial conditions (2.9). It is easy to show that all solutions of system (2.8) with initial conditions (2.9) are defined on $[0,\infty)$ and remain positive for all $t\geq 0$.

In the following we will investigate dynamical behaviors of system (2.5) with conditions (2.2) and (2.3) and system (2.8) with conditions (2.9).

3 Global dynamics of the vaccine-age-structured model

In this section, we consider the positivity of solutions of system (2.5) with conditions (2.2) and (2.3) and the existence of an endemic steady state, and prove the global stability of the disease-free and endemic steady states of system (2.5) by constructing appropriate Lyapunov functionals.

We claim that solutions of system (2.5) with conditions (2.2) and (2.3) are always positive for $t>0$. In fact, it follows from equation (2.6) that $v(a,t)$ remains positive for positive initial data. Further, suppose that there exists $t_1>0$ such that $S(t_1)=0$ and $S(t)>0$ for $0\leq t<t_1$, then $S'(t_1)\leq 0$. However, from the first equation of system (2.5) we have $S'(t_1)\geq A>0$. This contradiction implies that $S(t) > 0$ for all $t \geq 0$. Additionally, from the second equation of system (2.5) we have

$$I(t) = I(0)e^{\int_0^t \left[S(\xi)\frac{g(I(\xi))}{I(\xi)} - (\mu+\alpha)\right]d\xi}.$$

From assumption (H) it follows that $I(t) > 0$ for all $t \geq 0$ as $I(0) > 0$. Therefore, solutions of system (2.5) are always positive for $t > 0$.

Note that $e^{-\int_0^a [\mu+\delta(\xi)]d\xi}$ is the probability for an individual to stay in the vaccinated class for a time units. Denote by

$$\eta = \int_0^\infty \delta(a)e^{-\int_0^a [\mu+\delta(\xi)]d\xi}da, \tag{3.1}$$

then it is easy to see that $\eta < 1$, and it represents the probability of leaving the vaccinated class for an individual with lost immunity but alive. Further, it follows from the boundary condition $v(0,t) = pS(t)$ that $p\eta$ is the per capita rate at which the vaccinated individuals return to the susceptible class.

Obviously, system (2.5) with conditions (2.2) and (2.3) always has a disease-free steady state $P_{01}(S_{01}, 0, v_0(a))$, where

$$S_{01} = \frac{A}{\mu + (1-\eta)p}, \qquad v_0(a) = pS_{01}e^{-\int_0^a [\mu+\delta(\xi)]d\xi}.$$

Denote

$$R_{01} = \frac{g'(0)S_{01}}{\mu+\alpha} = \frac{Ag'(0)}{(\mu+\alpha)[\mu+(1-\eta)p]}.$$

An endemic steady state $P_1^*(S_1^*, I_1^*, v^*(a))$ with $I_1^* > 0$ of system (2.5) with conditions (2.2) and (2.3) satisfies the following steady state equations

$$\begin{aligned} A - (\mu+p)S_1^* - S_1^*g(I_1^*) + \int_0^\infty \delta(a)v^*(a)da &= 0,\\ S_1^*g(I_1^*) - (\mu+\alpha)I_1^* &= 0,\\ \frac{dv^*(a)}{da} &= -[\mu+\delta(a)]v^*(a),\\ v^*(0) &= pS_1^*. \end{aligned} \tag{3.2}$$

From the last two equations of steady state equations (3.2) we have

$$v^*(a) = pS_1^*e^{-\int_0^a [\mu+\delta(\xi)]d\xi}.$$

Substituting it into the first equation of steady state equations (3.2) gives

$$\begin{aligned} A - [\mu+(1-\eta)p]S_1^* - S_1^*g(I_1^*) &= 0,\\ S_1^*g(I_1^*) - (\mu+\alpha)I_1^* &= 0, \end{aligned} \tag{3.3}$$

which is equivalent to steady state equations

$$\begin{aligned} A - [\mu+(1-\eta)p]S_1^* - (\mu+\alpha)I_1^* &= 0,\\ S_1^*g(I_1^*) - (\mu+\alpha)I_1^* &= 0. \end{aligned} \tag{3.4}$$

From the first equation of steady state equations (3.4) we know $I_1^* < A/(\mu+\alpha)$, and from the last equation of steady state equations (3.4) we have

$$S_1^* = \frac{(\mu+\alpha)I_1^*}{g(I_1^*)}.$$

Substituting it into the first equation of steady state equations (3.4) gives

$$g(I_1^*) = \frac{[\mu+(1-\eta)p](\mu+\alpha)I_1^*}{A-(\mu+\alpha)I_1^*} =: h_1(I_1^*). \tag{3.5}$$

Notice that $I = A/(\mu+\alpha)$ is a vertical asymptote of function $h_1(I)$. And for $0 < I < A/(\mu+\alpha)$, we have

$$h_1'(I) = \frac{[\mu+(1-\eta)p](\mu+\alpha)A}{[A-(\mu+\alpha I)]^2} > 0, \qquad h_1''(I) = \frac{2[\mu+(1-\eta)p](\mu+\alpha)^2 A}{[A-(\mu+\alpha I)]^3} > 0,$$

i.e., $h_1(I)$ passes point $(0,0)$, and is increasing and concave in $(0, A/(\mu+\alpha))$. Thus, according to the assumption for function $g(I)$, when $g'(0) > h_1'(0) = [\mu+(1-\eta)p](\mu+\alpha)/A$ (i.e., $R_{01} > 1$), equation (3.5) has a unique root I_1^* in the interval $(0, A/(\mu+\alpha))$. It implies that steady state equations (3.4) (i.e., steady state equations (3.3)) has a unique positive solution (S_1^*, I_1^*) when $R_{01} > 1$, where $S_1^* = (\mu+\alpha)I_1^*/g(I_1^*)$. Correspondingly, system (2.5) has a unique endemic steady state $P_1^*(S_1^*, I_1^*, v^*(a))$ when $R_{01} > 1$. Therefore, with respect to the existence of steady states of system (2.5) we have the following theorem.

Theorem 3.1. *System (2.5) with conditions (2.2) and (2.3) always has the disease-free steady state $P_{01}(S_{01}, 0, v_0(a))$. When $R_{01}>1$, besides P_{01}, it also has a unique endemic steady state $P_1^*(S_1^*, I_1^*, v^*(a))$, where*

$$S_{01} = \frac{A}{\mu+(1-\eta)p}, \qquad v_0(a) = pS_{01}e^{-\int_0^a[\mu+\delta(\xi)]d\xi},$$
$$S_1^* = \frac{(\mu+\alpha)I_1^*}{g(I_1^*)}, \qquad v^*(a) = pS_1^*e^{-\int_0^a[\mu+\delta(\xi)]d\xi},$$

and I_1^* is determined by equation (3.5).

Much has been achieved for the global stability of epidemic age-structured models [6,27,28,31,34]. The establishment of the global stability mainly depends on construction of appropriate Lyapunov functionals when the incidence is bilinear. Before constructing a generalized Lyapunov functional for the general incidence, we start with a few technical lemmas.

Lemma 3.2. *For solution $v(a,t)$ of system (2.5) and function $v_0(a)$, define a functional*

$$\bar{L}_1(t) = \int_0^\infty q(a)\left[\int_{v_0(a)}^{v(a,t)} \frac{u-v_0(a)}{u}du\right]da,$$

where $q(\cdot) \in C^1[0,\infty)$. Then the derivative of functional $\bar{L}_1$ with respect to t can be expressed as

$$\bar{L}_1'(t) = [q(0)v_0(0)]\int_1^{pS(t)/v_0(0)} \frac{u-1}{u}du + \int_0^\infty \frac{d[q(a)v_0(a)]}{da}\left[\int_1^{v(a,t)/v_0(a)} \frac{u-1}{u}du\right]da.$$

Proof. Applying expression equation (2.6), functional $\bar{L}_1$ can be rewritten as

$$\begin{aligned}\bar{L}_1 =& \int_0^t q(a)\left[\int_{v_0(a)}^{v(a,t)} \frac{u-v_0(a)}{u}du\right]da + \int_t^\infty q(a)\left[\int_{v_0(a)}^{v(a,t)} \frac{u-v_0(a)}{u}du\right]da\\ =& \int_0^t q(a)\left[\int_{v_0(a)}^{pS(t-a)e^{-\int_0^a[\mu+\delta(\xi)]d\xi}} \frac{u-v_0(a)}{u}du\right]da\\ &+ \int_t^\infty q(a)\left[\int_{v_0(a)}^{v_s(a-t)e^{-\int_{a-t}^a[\mu+\delta(\xi)]d\xi}} \frac{u-v_0(a)}{u}du\right]da\\ =& \int_0^t q(t-b)\left[\int_{v_0(t-b)}^{pS(b)e^{-\int_0^{t-b}[\mu+\delta(\xi)]d\xi}} \frac{u-v_0(t-b)}{u}du\right]db\\ &+ \int_0^\infty q(t+b)\left[\int_{v_0(t+b)}^{v_s(b)e^{-\int_b^{t+b}[\mu+\delta(\xi)]d\xi}} \frac{u-v_0(t+b)}{u}du\right]db.\end{aligned}$$

Then

$$\begin{aligned}\bar{L}_1' =& q(0)\int_{v_0(0)}^{pS(t)} \frac{u-v_0(0)}{u}du + \int_0^t \frac{dq(t-b)}{dt}\left[\int_{v_0(t-b)}^{pS(b)e^{-\int_0^{t-b}[\mu+\delta(\xi)]d\xi}} \frac{u-v_0(t-b)}{u}du\right]db\\ &+ \int_0^t q(t-b)\left\{\frac{d}{dt}\left[\int_{v_0(t-b)}^{pS(b)e^{-\int_0^{t-b}[\mu+\delta(\xi)]d\xi}} \frac{u-v_0(t-b)}{u}du\right]\right\}db\\ &+ \int_0^\infty \frac{dq(t+b)}{dt}\left[\int_{v_0(t+b)}^{v_s(b)e^{-\int_b^{t+b}[\mu+\delta(\xi)]d\xi}} \frac{u-v_0(t+b)}{u}du\right]db\\ &+ \int_0^\infty q(t+b)\left\{\frac{d}{dt}\left[\int_{v_0(t+b)}^{v_s(b)e^{-\int_b^{t+b}[\mu+\delta(\xi)]d\xi}} \frac{u-v_0(t+b)}{u}du\right]\right\}db.\end{aligned}$$

Since function $v_0(a)$ satisfies the following differential equation

$$v_0'(a) = -[\mu+\delta(a)]v_0(a), \tag{3.6}$$

we have

$$\begin{aligned}&\frac{d}{dt}\left[\int_{v_0(t-b)}^{pS(b)e^{-\int_0^{t-b}[\mu+\delta(\xi)]d\xi}} \frac{u-v_0(t-b)}{u}du\right]\\ &= -[\mu+\eta(t-b)]\left\{pS(b)e^{-\int_0^{t-b}[\mu+\delta(\xi)]d\xi} - v_0(t-b)\right\}\\ &\quad+ \int_{v_0(t-b)}^{pS(b)e^{-\int_0^{t-b}[\mu+\eta(\xi)]d\xi}} \frac{[\mu+\delta(t-b)]v_0(t-b)}{u}du\\ &= -[\mu+\eta(t-b)]\int_{v_0(t-b)}^{pS(b)e^{-\int_0^{t-b}[\mu+\delta(\xi)]d\xi}} \frac{u-v_0(t-b)}{u}du.\end{aligned} \tag{3.7}$$

Similarly,

$$\frac{d}{dt}\left[\int_{v_0(t+b)}^{v_s(b)e^{-\int_b^{t+b}[\mu+\delta(\xi)]d\xi}} \frac{u-v_0(t+b)}{u}du\right] = -[\mu+\eta(t+b)]\int_{v_0(t+b)}^{v_s(b)e^{-\int_b^{t+b}[\mu+\delta(\xi)]d\xi}} \frac{u-v_0(t+b)}{u}du. \tag{3.8}$$

Using equations (3.7) and (3.8) gives

$$
\begin{aligned}
\bar{L}_1' &= q(0)\int_{v_0(0)}^{pS(t)} \frac{u-v_0(0)}{u}du + \int_0^t \frac{dq(t-b)}{dt}\left[\int_{v_0(t-b)}^{pS(b)e^{-\int_0^{t-b}[\mu+\delta(\xi)]d\xi}} \frac{u-v_0(t-b)}{u}du\right] db \\
&\quad - \int_0^t q(t-b)[\mu+\delta(t-b)]\left[\int_{v_0(t-b)}^{pS(b)e^{-\int_0^{t-b}[\mu+\delta(\xi)]d\xi}} \frac{u-v_0(t-b)}{u}du\right] db \\
&\quad + \int_0^\infty \frac{dq(t+b)}{dt}\left[\int_{v_0(t+b)}^{v_s(b)e^{-\int_b^{t+b}[\mu+\delta(\xi)]d\xi}} \frac{u-v_0(t+b)}{u}du\right] db \\
&\quad - \int_0^\infty q(t+b)[\mu+\delta(t+b)]\left[\int_{v_0(t+b)}^{v_s(b)e^{-\int_b^{t+b}[\mu+\delta(\xi)]d\xi}} \frac{u-v_0(t+b)}{u}du\right] db \\
&= q(0)\int_{v_0(0)}^{pS(t)} \frac{u-v_0(0)}{u}du \\
&\quad + \int_0^t \left\{\frac{dq(t-b)}{dt} - q(t-b)[\mu+\delta(t-b)]\right\}\left[\int_{v_0(t-b)}^{pS(b)e^{-\int_0^{t-b}[\mu+\delta(\xi)]d\xi}} \frac{u-v_0(t-b)}{u}du\right] db \\
&\quad + \int_0^\infty \left\{\frac{dq(t+b)}{dt} - q(t+b)[\mu+\delta(t+b)]\right\}\left[\int_{v_0(t+b)}^{v_s(b)e^{-\int_b^{t+b}[\mu+\delta(\xi)]d\xi}} \frac{u-v_0(t+b)}{u}du\right] db.
\end{aligned}
$$

From equation (3.6) we have

$$
\frac{dq(a)}{da} - q(a)[\mu+\delta(a)] = \frac{d[q(a)v_0(a)]}{v_0(a)}.
$$

Then

$$
\begin{aligned}
\bar{L}_1' &= q(0)\int_{v_0(0)}^{pS(t)} \frac{u-v_0(0)}{u}du \\
&\quad + \int_0^t \frac{d[q(t-b)v_0(t-b)]}{dt}\left[\int_1^{pS(b)e^{-\int_0^{t-b}[\mu+\delta(\xi)]d\xi}/v_0(t-b)} \frac{u-1}{u}du\right] db \\
&\quad + \int_0^\infty \frac{d[q(t+b)v_0(t+b)]}{dt}\left[\int_1^{v_s(b)e^{-\int_b^{t+b}[\mu+\delta(\xi)]d\xi}/v_0(t+b)} \frac{u-1}{u}du\right] db \\
&= q(0)\int_{v_0(0)}^{pS(t)} \frac{u-v_0(0)}{u}du \\
&\quad + \int_0^t \frac{d[q(a)v_0(a)]}{da}\left[\int_1^{pS(t-a)e^{-\int_0^{a}[\mu+\delta(\xi)]d\xi}/v_0(a)} \frac{u-1}{u}du\right] da \\
&\quad + \int_t^\infty \frac{d[q(a)v_0(a)]}{da}\left[\int_1^{v_s(a-t)e^{-\int_{a-t}^{a}[\mu+\delta(\xi)]d\xi}/v_0(a)} \frac{u-1}{u}du\right] da.
\end{aligned}
$$

Further, using equation (2.6), we have

$$
\bar{L}_1' = [q(0)v_0(0)]\int_1^{pS(t)/v_0(0)} \frac{u-1}{u}du + \int_0^\infty \frac{d[q(a)v_0(a)]}{da}\left[\int_1^{v(a,t)/v_0(a)} \frac{u-1}{u}du\right] da.
$$

The proof of Lemma 3.2 is complete. □

Similar to the argument for Lemma 3.2, we also have the following.

Lemma 3.3. *For solution $v(a,t)$ of system (2.5) and function $v^*(a)$, define a functional*

$$\bar{L}_1^* = \int_0^\infty q(a)\left[\int_{v^*(a)}^{v(a,t)} \frac{u-v^*(a)}{u}du\right]da,$$

where $q(\cdot) \in C^1[0,\infty)$. Then the derivative of functional $\bar{L}_1^*$ with respect to t can be expressed as

$$\bar{L}_1^{*\prime} = [q(0)v^*(0)]\int_1^{pS(t)/v^*(0)} \frac{u-1}{u}du + \int_0^\infty \frac{d[q(a)v^*(a)]}{da}\left[\int_1^{v(a,t)/v^*(a)} \frac{u-1}{u}du\right]da.$$

Lemma 3.4. *Assume that function $f(x)$ satisfies the condition (H), then we have the following statements:*

- *(i) $f(x) < f'(0)x$ for $x > 0$.*
- *(ii) For an arbitrary positive number x^* the following inequality holds:*

$$\left[1-\frac{f(x)}{f(x^*)}\right]\left[\frac{xf(x^*)}{x^*f(x)}-1\right] < 0 \quad \textit{for } x>0 \textit{ and } x \neq x^*.$$

Proof. This Lemma can easily be illustrated geometrically. In the following, we give an analytical proof.

(i) $f''(x) < 0$ implies that function $f(x)$ is convex in $(0,\infty)$. Then any tangent line of the function $f(x)$ is above the graph of $f(x)$. Since both the graph of $f(x)$ and its tangent line at $(0, f(0)) = (0,0)$ pass the origin, we know $f(x) < f'(0)x$ for $x > 0$.

(ii) Define a function

$$h(x) = f(x) - \frac{f(x^*)}{x^*}x,$$

then $h(0) = h(x^*) = 0$. By Rolle's Theorem, there is a $\xi \in (0, x^*)$ such that $h'(\xi) = 0$. Since $h''(x) = f''(x) < 0$ for $x > 0$, i.e., $h(x)$ is convex, it follows that $h'(x)$ is decreasing in $(0,\infty)$, so ξ is the only zero of $h'(x)$ in $(0,\infty)$. It implies that $h'(x) < 0$ for $x > x^* > \xi$, then from $h(x^*) = 0$ we know that $h(x) < 0$ for $x > x^*$. Further, from $h(0) = h(x^*) = 0$ we get $h(x) > 0$ for $0 < x < x^*$. From the above argument it follows that $h(x) > 0$ for $0 < x < x^*$ and that $h(x) < 0$ for $x > x^*$, that is, $f(x) > \frac{f(x^*)}{x^*}x$ for $0 < x < x^*$, and $f(x) < \frac{f(x^*)}{x^*}x$ for $x > x^*$. Again, $f'(x) > 0$ implies that $f(x) < f(x^*)$ for $0 < x < x^*$ and $f(x) > f(x^*)$ for $x > x^* > 0$. Thus Lemma 3.4 (ii) is true. This completes the proof. □

Applying the above lemmas, we have the following statements with respect to the global stability of steady states of system (2.5).

Theorem 3.5. *For system (2.5), the disease-free steady state P_{01} is globally stable as $R_{01} \leq 1$; the endemic steady state P_1^* is globally stable as $R_{01} > 1$.*

Proof. We first prove the global stability of the disease-free steady state $P_{01}(S_{01}, 0, v_0(a))$.

Define a Lyapunov functional

$$L_{11} = \int_{S_{01}}^{S} \frac{u-S_{01}}{u}du + I + \int_0^\infty q(a)\left[\int_{v_0(a)}^{v(a,t)} \frac{u-v_0(a)}{u}du\right]da,$$

where function $q(\cdot) \in C^1[0,\infty)$ is to be determined below, then, applying the equalities $A = (\mu+p)S_{01} - \eta p S_{01}$ and $v_0(0) = pS_{01}$, and Lemma 3.2, the derivative of L_{11} with respect to time t along solution of system (2.5) is given by

$$
\begin{aligned}
L'_{11} = & \left(1 - \frac{S_{01}}{S}\right)\left[-(\mu+p)(S - S_{01}) - Sg(I) + \int_0^\infty \delta(a)v(a,t)da - p\eta S_{01}\right] + [Sg(I) - (\mu+\alpha)I] \\
& + [q(0)v_0(0)] \int_1^{pS(t)/v_0(0)} \frac{u-1}{u} du + \int_0^\infty \frac{d[q(a)v_0(a)]}{da}\left[\int_1^{v(a,t)/v_0(a)} \frac{u-1}{u} du\right] da \\
= & -(\mu+p)\frac{(S-S_{01})^2}{S} + [S_{01}g(I) - (\mu+\alpha)I] + \left(1 - \frac{S_{01}}{S}\right)\left[\int_0^\infty \delta(a)v(a,t)da - p\eta S_{01}\right] \\
& + [q(0)v_0(0)]\left(\frac{S}{S_{01}} - 1 - \ln\frac{S}{S_{01}}\right) + \int_0^\infty \frac{d[q(a)v_0(a)]}{da}\left[\frac{v(a,t)}{v_0(a)} - 1 - \ln\frac{v(a,t)}{v_0(a)}\right] da.
\end{aligned}
$$

In order to eliminate the term $\int_0^\infty \delta(a)v(a,t)da$ in L'_{11}, we choose $q(a)$ such that

$$
\frac{d}{da}\left(q(a)v_0(a)\right) = -\delta(a)v_0(a) \quad \text{and} \quad q(0) = \eta,
$$

that is,

$$
q(a) = \frac{\eta - \int_0^a \delta(\xi) e^{-\int_0^\xi [\mu+\delta(\zeta)]d\zeta} d\xi}{e^{-\int_0^a [\mu+\delta(\zeta)]d\zeta}} = \frac{\int_a^\infty \delta(\xi) e^{-\int_0^\xi [\mu+\delta(\zeta)]d\zeta} d\xi}{e^{-\int_0^a [\mu+\delta(\zeta)]d\zeta}}. \tag{3.9}
$$

Noting that $\eta v_0(0) = p\eta S_{01} = \int_0^\infty \delta(a)v_0(a)da$, then we have

$$
\begin{aligned}
L'_{11} = & -(\mu+p)\frac{(S-S_{01})^2}{S} + [S_{01}g(I) - (\mu+\alpha)I] \\
& + \int_0^\infty \delta(a)v_0(a)\left[\frac{S_{01}}{S} + \frac{S}{S_{01}} - 1 - \frac{S_{01}v(a,t)}{S(t)v_0(a)} + \ln\frac{S_{01}v(a,t)}{S(t)v_0(a)}\right] da \\
= & -[\mu + (1-\eta)p]\frac{(S-S_{01})^2}{S} + [S_{01}g(I) - (\mu+\alpha)I] \\
& + \int_0^\infty \delta(a)v_0(a)\left[1 - \frac{S_{01}v(a,t)}{S(t)v_0(a)} + \ln\frac{S_{01}v(a,t)}{S(t)v_0(a)}\right] da.
\end{aligned}
$$

According to assumption (H) for function $g(I)$, from Lemma 3.4 (i) we have that $g(I) < g'(0)I$ for $I > 0$. Hence,

$$
\begin{aligned}
L'_{11} \le & -[\mu + (1-\eta)p]\frac{(S-S_{01})^2}{S} + (\mu+\alpha)(R_{01} - 1)I \\
& + \int_0^\infty \delta(a)v_0(a)\left[1 - \frac{S_{01}v(a,t)}{S(t)v_0(a)} + \ln\frac{S_{01}v(a,t)}{S(t)v_0(a)}\right] da.
\end{aligned}
$$

Notice that $1 - x + \ln x \le 0$ for $x > 0$ and the equality holds if and only if $x = 1$, then $L'_{11} \le 0$ as $R_{01} \le 1$ and $L'_{11} = 0$ if and only if $(S(t), I(t), v(a,t)) = (S_{01}, 0, v_0(a))$. Therefore, by the Lyapunov asymptotic stability theorem (see [?]) the disease-free steady state P_{01} is globally stable if $R_{01} \le 1$.

Next, we prove the global stability of the endemic steady state $P_1^*(S_1^*, I_1^*, v^*(a))$.

Define another Lyapunov functional

$$
L_{12} = \int_{S_1^*}^{S} \frac{u - S_1^*}{u} du + \int_{I_1^*}^{I} \frac{g(u) - g(I_1^*)}{g(u)} du + \int_0^\infty q(a)\left[\int_{v^*(a)}^{v(a,t)} \frac{u - v^*(a)}{u} du\right] da,
$$

where $q(a)$ is a positive and continuous function to be determined later, and the monotonicity of $g(I)$ ensures that the integral $\int_{I_1^*}^{I} \frac{g(u)-g(I_1^*)}{g(u)}du$ in $(0,\infty)$ has only one extremum which is a global minimum at I_1^*, then L_{12} is positive definite with respect to $(S_1^*, I_1^*, v^*(a))$. Applying Lemma 3.3, the derivative of L_{12} with respect to time t along solution of system (2.5) is given by

$$\begin{aligned}
L_{12}' =& \left(1-\frac{S_1^*}{S}\right)\left\{-(\mu+p)(S-S_1^*) - [Sg(I) - S_1^* g(I_1^*)] + \int_0^\infty \delta(a)v(a,t)da - p\eta S_1^*\right\} \\
&+ \left[1-\frac{g(I_1^*)}{g(I)}\right]\left[Sg(I) - \frac{S_1^* g(I_1^*)}{I_1^*}I\right] + [q(0)v^*(0)]\int_1^{pS(t)/v^*(0)} \frac{u-1}{u}du \\
&+ \int_0^\infty \frac{d[q(a)v^*(a)]}{da}\left[\int_1^{v(a,t)/v^*(a)} \frac{u-1}{u}du\right]da \\
=& -[\mu+p+g(I_1^*)]\frac{(S-S_1^*)^2}{S} + S_1^* g(I_1^*)\left[1-\frac{g(I)}{g(I_1^*)}\right]\left[\frac{Ig(I_1^*)}{I_1^* g(I)} - 1\right] \\
&+ \left(1-\frac{S_1^*}{S}\right)\left[\int_0^\infty \delta(a)v(a,t)da - p\eta S_1^*\right] + q(0)v^*(0)\left(\frac{S}{S_1^*} - 1 - \ln\frac{S}{S_1^*}\right) \\
&+ \int_0^\infty \frac{d}{da}(q(a)v^*(a))\left[\frac{v(a,t)}{v^*(a)} - 1 - \ln\frac{v(a,t)}{v^*(a)}\right]da,
\end{aligned}$$

where the following equalities are used,

$$A = (\mu+p)S_1^* + S_1^* g(I_1^*) - \eta p S_1^*, \qquad \mu+\alpha = S_1^* g(I_1^*)/I_1^*, \qquad v^*(0) = pS_1^*.$$

In order to eliminate the term $\int_0^\infty \delta(a)v(a,t)da$, we choose the function $q(a)$ defined in equation (3.9) to obtain

$$\frac{d}{da}[q(a)v^*(a)] = -\delta(a)v^*(a) \quad \text{and} \quad q(0) = \eta.$$

Noting that $pv^*(0) = p\eta S_1^* = \int_0^\infty \delta(a)v^*(a)da$, we get

$$\begin{aligned}
L_{12}' =& -[\mu+p+g(I_1^*)]\frac{(S-S_1^*)^2}{S} + S_1^* g(I_1^*)\left[1-\frac{g(I)}{g(I_1^*)}\right]\left[\frac{Ig(I_1^*)}{I_1^* g(I)} - 1\right] \\
&+ \int_0^\infty \delta(a)v^*(a)\left[\frac{S_1^*}{S} + \frac{S}{S_1^*} - 1 - \frac{S_1^* v(a,t)}{S(t)v^*(a)} + \ln\frac{S_1^* v(a,t)}{S(t)v^*(a)}\right]da \\
=& -A\frac{(S-S_1^*)^2}{SS_1^*} + S_1^* g(I_1^*)\left[1-\frac{g(I)}{g(I_1^*)}\right]\left[\frac{Ig(I_1^*)}{I_1^* g(I)} - 1\right] \\
&+ \int_0^\infty \delta(a)v^*(a)\left[1 - \frac{S_1^* v(a,t)}{S(t)v^*(a)} + \ln\frac{S_1^* v(a,t)}{S(t)v^*(a)}\right]da,
\end{aligned}$$

where $\mu+p+g(I_1^*) - p\eta = A/S_1^*$ is used. Notice that $1-x+\ln x \le 0$ for $x>0$ and the equality holds if and only if $x=1$. It follows from Lemma 3.4 (ii) that $dL_{12}/dt \le 0$ and $dL_{12}/dt = 0$ if and only if $(S(t), I(t), v(a,t)) = (S_1^*, I_1^*, v^*(a))$. Therefore, by the Lyapunov asymptotic stability theorem [5] the endemic steady state P_1^* is globally stable if $R_{01} > 1$. This completes the proof. □

4 Global dynamics of the vaccine-staged model

In this section, we consider the existence of the endemic steady state of system (2.8), and prove the global stability of the disease-free and endemic steady states by constructing Lyapunov functionals.

It is easy to see that system (2.8) always has the disease-free steady state $P_{02}(S_{02}, 0, V_0)$, where

$$S_{02} = \frac{A}{\mu + p[1 - \gamma e^{-\mu\tau}/(\mu+\gamma)]}, \qquad V_0 = \frac{pe^{-\mu\tau}}{\mu+\gamma} S_{02}.$$

Denote

$$R_{02} = \frac{g'(0)S_{02}}{\mu+\alpha} = \frac{Ag'(0)}{(\mu+\alpha)\{\mu + p[1-\gamma e^{-\mu\tau}/(\mu+\gamma)]\}}.$$

An endemic steady state $P_2^*(S_2^*, I_2^*, V^*)$ with $I_2^* > 0$ of system (2.8) is given by the following steady state equations

$$\begin{aligned} A - (\mu+p)S_2^* - S_2^* g(I_2^*) + \gamma V^* &= 0, \\ S_2^* g(I_2^*) - (\mu+\alpha) I_2^* &= 0, \\ pe^{-\mu\tau} S_2^* - (\mu+\gamma)V^* &= 0. \end{aligned} \tag{4.1}$$

From the last two equations of steady state equations (4.1) we have

$$S_2^* = \frac{(\mu+\alpha)I_2^*}{g(I_2^*)}, \qquad V^* = \frac{pe^{-\mu\tau}}{\mu+\gamma} \cdot \frac{(\mu+\alpha)I_2^*}{g(I_2^*)}. \tag{4.2}$$

Substituting them into the first equation of steady state equations (4.1) can get

$$g(I_2^*) = \left[\mu + p\left(1 - \frac{\gamma e^{-\mu\tau}}{\mu+\gamma}\right)\right] \frac{(\mu+\alpha)I_2^*}{A - (\mu+\alpha)I_2^*} =: h_2(I_2^*). \tag{4.3}$$

Similar to the analysis for equation (3.5) in Section 3, we know that equation (4.3) has a unique root I_2^* in the interval $(0, A/(\mu+\alpha))$ as $R_{02} > 1$. Further, it follows from equations (4.2) that system (2.8) has a unique endemic steady state $P_2^*(S_2^*, I_2^*, V^*)$ as $R_{02} > 1$. Therefore, with respect to the existence of steady states of system (2.8) we have the following theorem.

Theorem 4.1. *System (2.8) always has the disease-free steady state $P_{02}(S_{02}, 0, V_0)$. When $R_{02} > 1$, besides P_{02}, it also has a unique endemic steady state $P_2^*(S_2^*, I_2^*, V^*)$, where*

$$\begin{aligned} S_{02} &= \frac{A}{\mu + p[1 - \gamma e^{-\mu\tau}/(\mu+\gamma)]}, & V_0 &= \frac{pe^{-\mu\tau}}{\mu+\alpha} S_{02}, \\ S_2^* &= \frac{(\mu+\alpha)I_2^*}{g(I_2^*)}, & V^* &= \frac{pe^{-\mu\tau}}{\mu+\gamma} \cdot \frac{(\mu+\alpha)I_2^*}{g(I_2^*)}, \end{aligned}$$

and I_2^* is determined by equation (4.3).

Similarly to the works [7,29,30,32] that extend the respective results to models with discrete delay, we can now introduce a Lyapunov functional, extending the $x - x^* - x^* \ln \frac{x}{x^*} = \int_{x*}^{x} (u - x^*)/u du$ $(x^* > 0)$ into $\int_{x^*}^{x} [g(u) - g(x^*)]/g(u) du$, to obtain the global stability of steady states of system (2.8).

Theorem 4.2. *For system (2.8), the disease-free steady state P_{02} is globally stable as $R_{02} \le 1$; the endemic steady state P_2^* is globally stable as $R_{02} > 1$.*

Proof. We first prove the global stability of the disease-free steady state $P_{02}(S_{02}, 0, V_0)$.

Define a Lyapunov functional

$$L_{21} = \int_{S_{02}}^{S} \frac{u - S_{02}}{u} du + I + \frac{\gamma}{\mu+\gamma} \int_{V_0}^{V} \frac{u - V_0}{u} du + \frac{\gamma V_0}{S_{02}} \int_{t-\tau}^{t} \int_{S_{02}}^{S(\theta)} \frac{S(\xi) - S_{02}}{S(\xi)} d\xi d\theta.$$

Then, applying the equalities $A = (\mu + p)S_{02} - \gamma V_0$ and $pe^{-\mu\tau} = (\mu+\gamma)V_0/S_{02}$, the derivative of L_{21} with respect to time t along solutions of system (2.8) is given by

$$\begin{aligned} L_{21}' =& \left(1 - \frac{S_{02}}{S}\right)[-(\mu+p)(S - S_{02}) - Sg(I) + \gamma(V - V_0)] + [Sg(I) - (\mu+\alpha)I] \\ &+ \gamma\left(1 - \frac{V_0}{V}\right)\left[\frac{S(t-\tau)}{S_{02}}V_0 - V\right] + \gamma V_0\left[\frac{S(t)}{S_{02}} - \frac{S(t-\tau)}{S_{02}} + \ln\frac{S(t-\tau)}{S(t)}\right] \\ =& -(\mu+p)\frac{(S-S_{02})^2}{S} + [S_{02}g(I) - (\mu+\alpha)I] \\ &+ \gamma V_0\left[\frac{S_{02}}{S} + \frac{S}{S_{02}} - \frac{S_{02}V(t)}{S(t)V_0} - \frac{V_0 S(t-\tau)}{V(t)S_{02}} + \ln\frac{S(t-\tau)}{S(t)}\right] \\ =& -(\mu+p)\frac{(S-S_{02})^2}{S} + [S_{02}g(I) - (\mu+\alpha)I] \\ &+ \frac{\gamma V_0}{S_{02}}\frac{(S-S_{02})^2}{S} + \gamma V_0\left[2 - \frac{S_{02}V(t)}{S(t)V_0} - \frac{V_0 S(t-\tau)}{V(t)S_{02}} + \ln\frac{S(t-\tau)}{S(t)}\right]. \end{aligned}$$

Applying Lemma 3.4 (i) and the equality $V_0/S_{02} = pe^{-\mu\tau}/(\mu+\gamma)$ yields

$$\begin{aligned} L_{21}' \le& -\left[\mu + p\left(1 - \frac{\gamma e^{-\mu\tau}}{\mu+\gamma}\right)\right]\frac{(S-S_{02})^2}{S} + \left[S_0 g'(0) - (\mu+\alpha)\right] I \\ &+ \gamma V_0\left[2 - \frac{S_{02}V(t)}{S(t)V_0} - \frac{V_0 S(t-\tau)}{V(t)S_{02}} + \ln\frac{S(t-\tau)}{S(t)}\right] \\ =& -\left[\mu + p\left(1 - \frac{\gamma e^{-\mu\tau}}{\mu+\gamma}\right)\right]\frac{(S-S_{02})^2}{S} + (\mu+\alpha)(R_{02} - 1)I \\ &+ \gamma V_0\left[2 - \frac{S_{02}V(t)}{S(t)V_0} - \frac{V_0 S(t-\tau)}{V(t)S_{02}} + \ln\frac{S(t-\tau)}{S(t)}\right]. \end{aligned}$$

According to Lemma 3.1 in [18], $2 - \frac{S_{02}V(t)}{S(t)V_0} - \frac{V_0 S(t-\tau)}{V(t)S_{02}} + \ln\frac{S(t-\tau)}{S(t)} \le 0$ and the equality holds if and only if $V(t)/V_0 = S(t)/S_{02} = S(t-\tau)/S_{02}$. Then $L_{21}' \le 0$ as $R_{02} \le 1$ and $L_{21}' = 0$ if and only if $(S(t), I(t), V(t)) = (S_{02}, 0, V_0)$. It follows from the Lyapunov asymptotic stability theorem [5] that the endemic steady state P_{02} is globally stable if $R_{02} \le 1$.

Next, we prove the global stability of the endemic steady state $P_2^*(S_2^*, I_2^*, V^*)$.

Define another Lyapunov functional

$$\begin{aligned} L_{22} =& \int_{S_2^*}^{S}\frac{u - S_2^*}{u}du + \int_{I_2^*}^{I}\frac{g(u) - g(I_2^*)}{g(u)}du \\ &+ \frac{\gamma}{\mu+\gamma}\int_{V^*}^{V}\frac{u - V^*}{u}du + \frac{\gamma V^*}{S_2^*}\int_{t-\tau}^{t}\int_{S_2^*}^{S(\theta)}\frac{S(\xi) - S_2^*}{S(\xi)}d\xi d\theta. \end{aligned}$$

The monotonicity of $g(I)$ ensures that the integral $\int_{I_2^*}^{I}\frac{g(u)-g(I_2^*)}{g(u)}du$ in $(0,\infty)$ has only one extremum which is a global minimum at I_2^*, then L_{22} is positive definite with respect to (S_2^*, I_2^*, V^*). Applying the equalities

$$A = (\mu+p)S_2^* + S_2^* g(I_2^*) - \gamma V^*, \qquad \mu + \alpha = \frac{S_2^* g(I_2^*)}{I_2^*}, \qquad pe^{-\mu\tau} = \frac{(\mu+\gamma)V^*}{S_2^*},$$

the derivative of L_{22} with respect to time t along solutions of system (2.8) is given by

$$\begin{aligned} L_{22}' = & \left(1-\frac{S_2^*}{S}\right)\{-(\mu+p)(S-S_2^*)-[Sg(I)-S_2^*g(I_2^*)]+\gamma(V-V^*)\} \\ & + \left[1-\frac{g(I_2^*)}{g(I)}\right]\left[Sg(I)-\frac{S_2^*g(I_2^*)}{I_2^*}I\right]+\gamma\left(1-\frac{V^*}{V}\right)\left[\frac{S(t-\tau)}{S_2^*}V^*-V\right] \\ & + \gamma V^*\left[\frac{S(t)}{S_2^*}-\frac{S(t-\tau)}{S_2^*}+\ln\frac{S(t-\tau)}{S(t)}\right] \\ = & -(\mu+p)\frac{(S-S_2^*)^2}{S}+S_2^*g(I_2^*)\left[1-\frac{S_2^*}{S}-\frac{S}{S_2^*}+\frac{g(I)}{g(I_2^*)}-\frac{I}{I_2^*}+\frac{g(I_2^*)I}{g(I)I_2^*}\right] \\ & + \gamma V^*\left[\frac{S_2^*}{S}+\frac{S}{S_2^*}-\frac{S^*V(t)}{S(t)V^*}-\frac{V^*S(t-\tau)}{V(t)S_2^*}+\ln\frac{S(t-\tau)}{S(t)}\right] \\ = & -[\mu+p+g(I_2^*)]\frac{(S-S_2^*)^2}{S}+S_2^*g(I_2^*)\left[1-\frac{g(I)}{g(I_2^*)}\right]\left[\frac{Ig(I_2^*)}{I_2^*g(I)}-1\right] \\ & + \frac{\gamma V^*}{S_2^*}\frac{(S-S_2^*)^2}{S}+\gamma V^*\left[2-\frac{S^*V(t)}{S(t)V^*}-\frac{V^*S(t-\tau)}{V(t)S_2^*}+\ln\frac{S(t-\tau)}{S(t)}\right]. \end{aligned}$$

Notice that $V^*/S_2^* = pe^{-\mu\tau}/(\mu+\gamma)$, we get

$$\begin{aligned} L_{22}' = & -\left[\mu+p\left(1-\frac{\gamma e^{-\mu\tau}}{\mu+\gamma}\right)+g(I_2^*)\right]\frac{(S-S_2^*)^2}{S}+S_2^*g(I_2^*)\left[1-\frac{g(I)}{g(I_2^*)}\right]\left[\frac{Ig(I_2^*)}{I_2^*g(I)}-1\right] \\ & + \gamma V^*\left[2-\frac{S_2^*V(t)}{S(t)V^*}-\frac{V^*S(t-\tau)}{V(t)S_2^*}+\ln\frac{S(t-\tau)}{S(t)}\right]. \end{aligned}$$

Since $2-\frac{S_2^*V(t)}{S(t)V^*}-\frac{V^*S(t-\tau)}{V(t)S_2^*}+\ln\frac{S(t-\tau)}{S(t)} \le 0$ and the equality holds if and only if $V(t)/V^* = S(t)/S_2^* = S(t-\tau)/S_2^*$, it follows from Lemma 3.4 (ii) that $L_{22}' \le 0$ and $L_{22}' = 0$ if and only if $(S(t), I(t), V(t)) = (S_2^*, I_2^*, V^*)$. Therefore, by the Lyapunov asymptotic stability theorem [5] the endemic steady state P_2^* is globally stable if $R_{02} > 1$, competing the proof. □

5 Discussion

In this paper, we considered two classes of epidemic models with vaccination by incorporating the vaccine-age into an SIR epidemic model with nonlinear incidence. One takes the form of a coupled system of ordinary differential equations and hyperbolic differential equations structured by the vaccine-age, and another is a coupled ordinary-functional differential equations with a vaccine stage. The thresholds (i.e. R_{01} and R_{02}) were found and proved to determine the global dynamical behaviors.

For epidemic models, the basic reproduction number is defined as the expected number of secondary cases produced by one infective host in an entirely susceptible population [2,37]. It is easy to see that, in the absence of vaccination, both the basic reproduction numbers of systems (2.5) and (2.8) are $Ag'(0)/\mu(\mu+\alpha)$ [10, 12], where A/μ represents the size of the entirely susceptible population. When vaccination is incorporated, from the epidemiological interpretation of η we know that $p\eta$ is the rate at which a vaccinated individuals returns to the susceptible class. Then the fraction, $(1-\eta)p/[\mu+(1-\eta)p]$, of susceptible individuals is transferred to vaccinated class, and can not be infected by the infection. So the size of the entirely susceptible population becomes

$$\frac{A}{\mu}\left[1-\frac{(1-\eta)p}{\mu+(1-\eta)p}\right]=\frac{A}{\mu+(1-\eta)p}.$$

Therefore, the threshold

$$R_{01} = \frac{Ag'(0)}{(\mu+\alpha)[\mu+(1-\eta)p]}$$

is the basic reproduction number of system (2.5). Substituting the step function $\delta(a)$ into equation (3.1) yields

$$\begin{aligned}\eta &= \int_0^\tau \delta(a)e^{-\int_0^a[\mu+\delta(\xi)]d\xi}da + \int_\tau^\infty \delta(a)e^{-\int_0^a[\mu+\delta(\xi)]d\xi}da \\ &= \gamma\int_\tau^\infty e^{-[(\mu+\gamma)a-\gamma\tau]}da = \frac{\gamma e^{-\mu\tau}}{\mu+\gamma}.\end{aligned}$$

From this, we naturally conclude that R_{02} is the basic reproduction number of system (2.8).

Acknowledgements

This work is partially supported by the National Natural Science Foundation of China (Nos. 11371369, 11301320, 11471201), the China Postdoctoral Science Foundation Funded Project (No. 2013M532016), the Postdoctoral Science Foundation in Shaanxi of China, the State Scholarship Fund of China (No. 201407820120), and the International Development Research Centre of Canada.

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