Electronic Journal of Qualitative Theory of Differential Equations
2016, No. **22**, 1–28; doi: 10.14232/ejqtde.2016.1.22 http://www.math.u-szeged.hu/ejqtde/

Existence of positive solutions for a system of semipositone fractional boundary value problems

Johnny Henderson[1] and **Rodica Luca**[✉ 2]

[1]Baylor University, One Bear Place 97328, Waco, Texas, 76798-7328 USA
[2]Gheorghe Asachi Technical University, 11 Blvd. Carol I, Iași 700506, Romania

Received 30 December 2015, appeared 2 May 2016
Communicated by Paul Eloe

Abstract. We investigate the existence of positive solutions for a system of nonlinear Riemann–Liouville fractional differential equations with sign-changing nonlinearities, subject to coupled integral boundary conditions.

Keywords: Riemann–Liouville fractional differential equations, coupled integral boundary conditions, positive solutions, sign-changing nonlinearities.

2010 Mathematics Subject Classification: 34A08, 45G15.

1 Introduction

We consider the system of nonlinear fractional differential equations

$$\begin{cases} D_{0+}^{\alpha}u(t) + \lambda f(t,u(t),v(t)) = 0, & t \in (0,1), \\ D_{0+}^{\beta}v(t) + \mu g(t,u(t),v(t)) = 0, & t \in (0,1), \end{cases} \tag{S}$$

with the coupled integral boundary conditions

$$\begin{cases} u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, & u'(1) = \displaystyle\int_0^1 v(s)\,dH(s), \\ v(0) = v'(0) = \cdots = v^{(m-2)}(0) = 0, & v'(1) = \displaystyle\int_0^1 u(s)\,dK(s), \end{cases} \tag{BC}$$

where $\alpha \in (n-1,n]$, $\beta \in (m-1,m]$, n, $m \in \mathbb{N}$, n, $m \geq 3$, D_{0+}^{α} and D_{0+}^{β} denote the Riemann–Liouville derivatives of orders α and β, respectively, the integrals from (BC) are Riemann–Stieltjes integrals, and f and g are sign-changing continuous functions (that is, we have a so-called system of semipositone boundary value problems). These functions may be nonsingular or singular at $t = 0$ and/or $t = 1$. The boundary conditions above include multi-point and integral boundary conditions, as well as the sum of these in a single framework.

We present intervals for parameters λ and μ such that the above problem (S)–(BC) has at least one positive solution. By a positive solution of problem (S)–(BC) we mean a pair of

✉ Corresponding author. Email: rluca@math.tuiasi.ro

functions $(u,v) \in C([0,1]) \times C([0,1])$ satisfying (S) and (BC) with $u(t) \geq 0$, $v(t) \geq 0$ for all $t \in [0,1]$ and $u(t) > 0$, $v(t) > 0$ for all $t \in (0,1)$. The system (S) with the uncoupled integral boundary conditions

$$\begin{cases} u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \quad u(1) = \displaystyle\int_0^1 u(s)\,dH(s), \\ v(0) = v'(0) = \cdots = v^{(m-2)}(0) = 0, \quad v(1) = \displaystyle\int_0^1 v(s)\,dK(s), \end{cases} \tag{BC_1}$$

where the functions f and g are nonnegative, has been investigated in [6] and [11] by using the Guo–Krasnosel'skii fixed point theorem, and in [7] where in system (S) we have $\lambda = \mu = 1$ and $f(t,u,v)$ and $g(t,u,v)$ are replaced by $\widetilde{f}(t,v)$ and $\widetilde{g}(t,u)$, respectively, with $\widetilde{f}$ and $\widetilde{g}$ nonsingular or singular functions (denoted by ($\widetilde{S}$)). In [7] we used some theorems from the fixed point index theory and the Guo–Krasnosel'skii fixed point theorem. The semipositone case for problem (S)–(BC_1) was studied in [14] by using the nonlinear alternative of Leray–Schauder type. The systems (S) and ($\widetilde{S}$) with coupled integral boundary conditions

$$\begin{cases} u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \quad u(1) = \displaystyle\int_0^1 v(s)\,dH(s), \\ v(0) = v'(0) = \cdots = v^{(m-2)}(0) = 0, \quad v(1) = \displaystyle\int_0^1 u(s)\,dK(s), \end{cases} \tag{BC_2}$$

have been investigated in [8] and [9] (problem (S)–(BC_2) with f and g nonnegative functions), in [12] (problem ($\widetilde{S}$)–(BC_2) with f and g nonnegative functions, singular or not), and in [10] (problem (S)–(BC_2) with f, g sign-changing functions). We also mention the paper [20], where the authors studied the existence and multiplicity of positive solutions for system (S) with $\alpha = \beta$, $\lambda = \mu$ and the boundary conditions $u^{(i)}(0) = v^{(i)}(0) = 0$, $i = 0, \ldots, n-2$, $u(1) = av(\xi)$, $v(1) = bu(\eta)$, ξ, $\eta \in (0,1)$, with ξ, $\eta \in (0,1)$, $0 < ab\xi\eta < 1$, and f and g are sign-changing nonsingular or singular functions.

Fractional differential equations describe many phenomena in various fields of engineering and scientific disciplines such as physics, biophysics, chemistry, biology, economics, control theory, signal and image processing, aerodynamics, viscoelasticity, electromagnetics, and so on (see [2–4, 13, 15–19]). Integral boundary conditions arise in thermal conduction problems, semiconductor problems and hydrodynamic problems.

The paper is organized as follows. Section 2 contains some auxiliary results which investigate a nonlocal boundary value problem for fractional differential equations. In Section 3, we prove several existence theorems for the positive solutions with respect to a cone for our problem (S)–(BC). Finally in Section 4 some examples are given to illustrate our main results.

2 Auxiliary results

We present here the definitions of the Riemann–Liouville fractional integral and the Riemann–Liouville fractional derivative and then some auxiliary results that will be used to prove our main results.

Definition 2.1. The (left-sided) fractional integral of order $\alpha > 0$ of a function $f : (0,\infty) \to \mathbb{R}$ is given by

$$(I_{0+}^{\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,ds, \qquad t > 0,$$

provided the right-hand side is pointwise defined on $(0, \infty)$, where $\Gamma(\alpha)$ is the Euler gamma function defined by $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\, dt$, $\alpha > 0$.

Definition 2.2. The Riemann–Liouville fractional derivative of order $\alpha \geq 0$ for a function $f : (0, \infty) \to \mathbb{R}$ is given by

$$(D_{0+}^{\alpha} f)(t) = \left(\frac{d}{dt}\right)^n \left(I_{0+}^{n-\alpha} f\right)(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{f(s)}{(t-s)^{\alpha-n+1}}\, ds, \qquad t > 0,$$

where $n = \lfloor \alpha \rfloor + 1$, provided that the right-hand side is pointwise defined on $(0, \infty)$.

The notation $\lfloor \alpha \rfloor$ stands for the largest integer not greater than α. If $\alpha = m \in \mathbb{N}$ then $D_{0+}^m f(t) = f^{(m)}(t)$ for $t > 0$, and if $\alpha = 0$ then $D_{0+}^0 f(t) = f(t)$ for $t > 0$.

We consider now the fractional differential system

$$\begin{cases} D_{0+}^{\alpha} u(t) + \widetilde{x}(t) = 0, & t \in (0,1), \\ D_{0+}^{\beta} v(t) + \widetilde{y}(t) = 0, & t \in (0,1), \end{cases} \tag{2.1}$$

with the coupled integral boundary conditions

$$\begin{cases} u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, & u'(1) = \displaystyle\int_0^1 v(s)\, dH(s), \\ v(0) = v'(0) = \cdots = v^{(m-2)}(0) = 0, & v'(1) = \displaystyle\int_0^1 u(s)\, dK(s), \end{cases} \tag{2.2}$$

where $\alpha \in (n-1, n]$, $\beta \in (m-1, m]$, n, $m \in \mathbb{N}$, n, $m \geq 3$, and H, $K : [0,1] \to \mathbb{R}$ are functions of bounded variation.

Lemma 2.3. *If $H, K : [0,1] \to \mathbb{R}$ are functions of bounded variation, $\Delta = (\alpha-1)(\beta-1) - \left(\int_0^1 \tau^{\alpha-1}\, dK(\tau)\right)\left(\int_0^1 \tau^{\beta-1}\, dH(\tau)\right) \neq 0$ and $\widetilde{x}, \widetilde{y} \in C(0,1) \cap L^1(0,1)$, then the pair of functions $(u,v) \in C([0,1]) \times C([0,1])$ given by*

$$\begin{aligned} u(t) = {}& -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \widetilde{x}(s)\, ds + \frac{t^{\alpha-1}}{\Delta} \left[\frac{\beta-1}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} \widetilde{x}(s)\, ds \right. \\ & - \frac{1}{\Gamma(\alpha)} \left(\int_0^1 s^{\beta-1}\, dH(s) \right) \left(\int_0^1 \left(\int_s^1 (\tau-s)^{\alpha-1}\, dK(\tau) \right) \widetilde{x}(s)\, ds \right) \\ & + \frac{1}{\Gamma(\beta-1)} \left(\int_0^1 s^{\beta-1}\, dH(s) \right) \left(\int_0^1 (1-s)^{\beta-2} \widetilde{y}(s)\, ds \right) \\ & \left. - \frac{1}{\Gamma(\beta-1)} \int_0^1 \left(\int_s^1 (\tau-s)^{\beta-1}\, dH(\tau) \right) \widetilde{y}(s)\, ds \right], \qquad t \in (0,1],\ u(0) = 0, \\ v(t) = {}& -\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \widetilde{y}(s)\, ds + \frac{t^{\beta-1}}{\Delta} \left[\frac{\alpha-1}{\Gamma(\beta-1)} \int_0^1 (1-s)^{\beta-2} \widetilde{y}(s)\, ds \right. \\ & - \frac{1}{\Gamma(\beta)} \left(\int_0^1 s^{\alpha-1}\, dK(s) \right) \left(\int_0^1 \left(\int_s^1 (\tau-s)^{\beta-1}\, dH(\tau) \right) \widetilde{y}(s)\, ds \right) \\ & + \frac{1}{\Gamma(\alpha-1)} \left(\int_0^1 s^{\alpha-1}\, dK(s) \right) \left(\int_0^1 (1-s)^{\alpha-2} \widetilde{x}(s)\, ds \right) \\ & \left. - \frac{1}{\Gamma(\alpha-1)} \int_0^1 \left(\int_s^1 (\tau-s)^{\alpha-1}\, dK(\tau) \right) \widetilde{x}(s)\, ds \right], \qquad t \in (0,1],\ v(0) = 0, \end{aligned} \tag{2.3}$$

is solution of problem (2.1)–(2.2).

Proof. We denote by

$$c_1 = \frac{1}{\Delta}\left[\frac{\beta-1}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}\widetilde{x}(s)\,ds - \frac{1}{\Gamma(\beta-1)}\int_0^1\left(\int_s^1 (\tau-s)^{\beta-1}\,dH(\tau)\right)\widetilde{y}(s)\,ds \right.$$
$$+\frac{1}{\Gamma(\beta-1)}\left(\int_0^1 \tau^{\beta-1}\,dH(\tau)\right)\left(\int_0^1 (1-s)^{\beta-2}\widetilde{y}(s)\,ds\right)$$
$$\left.-\frac{1}{\Gamma(\alpha)}\left(\int_0^1 \tau^{\beta-1}\,dH(\tau)\right)\left(\int_0^1\left(\int_s^1 (\tau-s)^{\alpha-1}\,dK(\tau)\right)\widetilde{x}(s)\,ds\right)\right],$$

and

$$d_1 = \frac{1}{\Delta}\left[\frac{\alpha-1}{\Gamma(\beta-1)}\int_0^1 (1-s)^{\beta-2}\widetilde{y}(s)\,ds - \frac{1}{\Gamma(\alpha-1)}\int_0^1\left(\int_s^1 (\tau-s)^{\alpha-1}\,dK(\tau)\right)\widetilde{x}(s)\,ds \right.$$
$$+\frac{1}{\Gamma(\alpha-1)}\left(\int_0^1 \tau^{\alpha-1}\,dK(\tau)\right)\left(\int_0^1 (1-s)^{\alpha-2}\widetilde{x}(s)\,ds\right)$$
$$\left.-\frac{1}{\Gamma(\beta)}\left(\int_0^1 \tau^{\alpha-1}\,dK(\tau)\right)\left(\int_0^1\left(\int_s^1 (\tau-s)^{\beta-1}\,dH(\tau)\right)\widetilde{y}(s)\,ds\right)\right].$$

Then the continuous functions u and v from (2.3) can be written as

$$\begin{cases} u(t) = c_1 t^{\alpha-1} - \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1}\widetilde{x}(s)\,ds = c_1 t^{\alpha-1} - I_{0+}^{\alpha}\widetilde{x}(t), \quad t\in(0,1],\ u(0)=0, \\ v(t) = d_1 t^{\beta-1} - \dfrac{1}{\Gamma(\beta)}\displaystyle\int_0^t (t-s)^{\beta-1}\widetilde{y}(s)\,ds = d_1 t^{\beta-1} - I_{0+}^{\beta}\widetilde{y}(t), \quad t\in(0,1],\ v(0)=0. \end{cases}$$

Because $D_{0+}^{\alpha}u(t) = c_1 D_{0+}^{\alpha}(t^{\alpha-1}) - D_{0+}^{\alpha}I_{0+}^{\alpha}\widetilde{x}(t) = -\widetilde{x}(t)$ and $D_{0+}^{\beta}v(t) = d_1 D_{0+}^{\beta}(t^{\beta-1}) - D_{0+}^{\beta}I_{0+}^{\beta}\widetilde{y}(t) = -\widetilde{y}(t)$ for all $t\in(0,1)$, we deduce that u and v satisfy the system (2.1). In addition, we have $u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0$ and $v(0)=v'(0)=\cdots=v^{(m-2)}(0)=0$. A simple computation shows us that $u'(1)=\int_0^1 v(s)\,dH(s)$ and $v'(1)=\int_0^1 u(s)\,dK(s)$, that is

$$\begin{cases} c_1(\alpha-1) - \dfrac{1}{\Gamma(\alpha-1)}\displaystyle\int_0^1 (1-s)^{\alpha-2}\widetilde{x}(s)\,ds = \int_0^1\left(d_1 s^{\beta-1} - \dfrac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}\widetilde{y}(\tau)\,d\tau\right)dH(s), \\ d_1(\beta-1) - \dfrac{1}{\Gamma(\beta-1)}\displaystyle\int_0^1 (1-s)^{\beta-2}\widetilde{y}(s)\,ds = \int_0^1\left(c_1 s^{\alpha-1} - \dfrac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}\widetilde{x}(\tau)\,d\tau\right)dK(s). \end{cases}$$

Therefore we deduce that (u,v) is solution of problem (2.1)–(2.2). □

Lemma 2.4. *Under the assumptions of Lemma 2.3, the solution (u,v) of problem (2.1)–(2.2) given by (2.3) can be written as*

$$\begin{cases} u(t) = \displaystyle\int_0^1 G_1(t,s)\widetilde{x}(s)\,ds + \int_0^1 G_2(t,s)\widetilde{y}(s)\,ds, \quad t\in[0,1], \\ v(t) = \displaystyle\int_0^1 G_3(t,s)\widetilde{y}(s)\,ds + \int_0^1 G_4(t,s)\widetilde{x}(s)\,ds, \quad t\in[0,1], \end{cases} \tag{2.4}$$

where

$$\begin{cases} G_1(t,s) = g_1(t,s) + \dfrac{t^{\alpha-1}}{\Delta}\left(\displaystyle\int_0^1 \tau^{\beta-1}dH(\tau)\right)\left(\int_0^1 g_1(\tau,s)dK(\tau)\right), \\ G_2(t,s) = \dfrac{(\beta-1)t^{\alpha-1}}{\Delta}\displaystyle\int_0^1 g_2(\tau,s)dH(\tau), \\ G_3(t,s) = g_2(t,s) + \dfrac{t^{\beta-1}}{\Delta}\left(\displaystyle\int_0^1 \tau^{\alpha-1}dK(\tau)\right)\left(\int_0^1 g_2(\tau,s)dH(\tau)\right), \\ G_4(t,s) = \dfrac{(\alpha-1)t^{\beta-1}}{\Delta}\displaystyle\int_0^1 g_1(\tau,s)dK(\tau), \qquad \forall\, t,\, s\in[0,1], \end{cases} \tag{2.5}$$

and

$$\begin{cases} g_1(t,s) = \dfrac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-2} - (t-s)^{\alpha-1}, & 0 \le s \le t \le 1, \\ t^{\alpha-1}(1-s)^{\alpha-2}, & \le t \le s \le 1, \end{cases} \\ g_2(t,s) = \dfrac{1}{\Gamma(\beta)} \begin{cases} t^{\beta-1}(1-s)^{\beta-2} - (t-s)^{\beta-1}, & 0 \le s \le t \le 1, \\ t^{\beta-1}(1-s)^{\beta-2}, & 0 \le t \le s \le 1. \end{cases} \end{cases} \tag{2.6}$$

Proof. By Lemma 2.3 and relation (2.3), we conclude

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\alpha)} \left[\int_0^t [t^{\alpha-1}(1-s)^{\alpha-2} - (t-s)^{\alpha-1}] \widetilde{x}(s)\, ds + \int_t^1 t^{\alpha-1}(1-s)^{\alpha-2} \widetilde{x}(s)\, ds \right. \\ &\qquad \left. - \int_0^1 t^{\alpha-1}(1-s)^{\alpha-2}\widetilde{x}(s)\, ds \right] + \frac{(\beta-1)t^{\alpha-1}}{\Delta\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2}\widetilde{x}(s)\, ds \\ &\quad - \frac{t^{\alpha-1}}{\Delta\Gamma(\alpha)} \left(\int_0^1 \tau^{\beta-1}\, dH(\tau) \right) \left(\int_0^1 \left(\int_s^1 (\tau-s)^{\alpha-1}\, dK(\tau) \right) \widetilde{x}(s)\, ds \right) \\ &\quad + \frac{t^{\alpha-1}}{\Delta\Gamma(\beta-1)} \left[\int_0^1 \left(\int_0^1 \tau^{\beta-1}(1-s)^{\beta-2}\, dH(\tau) \right) \widetilde{y}(s)\, ds \right. \\ &\qquad \left. - \int_0^1 \left(\int_s^1 (\tau-s)^{\beta-1}\, dH(\tau) \right) \widetilde{y}(s)\, ds \right] \\ &= \frac{1}{\Gamma(\alpha)} \left[\int_0^t [t^{\alpha-1}(1-s)^{\alpha-2} - (t-s)^{\alpha-1}] \widetilde{x}(s)\, ds + \int_t^1 t^{\alpha-1}(1-s)^{\alpha-2} \widetilde{x}(s)\, ds \right] \\ &\quad - \frac{\beta-1}{\Delta\Gamma(\alpha-1)} \int_0^1 t^{\alpha-1}(1-s)^{\alpha-2}\widetilde{x}(s)\, ds + \frac{1}{\Delta\Gamma(\alpha)} \left(\int_0^1 \tau^{\alpha-1} dK(\tau) \right) \left(\int_0^1 \tau^{\beta-1}\, dH(\tau) \right) \\ &\quad \times \left(\int_0^1 t^{\alpha-1}(1-s)^{\alpha-2}\widetilde{x}(s)\, ds \right) + \frac{\beta-1}{\Delta\Gamma(\alpha-1)} \int_0^1 t^{\alpha-1}(1-s)^{\alpha-2}\widetilde{x}(s)\, ds \\ &\quad - \frac{t^{\alpha-1}}{\Delta\Gamma(\alpha)} \left(\int_0^1 \tau^{\beta-1} dH(\tau) \right) \left(\int_0^1 \left(\int_s^1 (\tau-s)^{\alpha-1} dK(\tau) \right) \widetilde{x}(s)\, ds \right) \\ &\quad + \frac{t^{\alpha-1}}{\Delta\Gamma(\beta-1)} \left[\int_0^1 \left(\int_0^1 \tau^{\beta-1}(1-s)^{\beta-2} dH(\tau) \right) \widetilde{y}(s)\, ds \right. \\ &\qquad \left. - \int_0^1 \left(\int_s^1 (\tau-s)^{\beta-1} dH(\tau) \right) \widetilde{y}(s)\, ds \right] \\ &= \frac{1}{\Gamma(\alpha)} \left[\int_0^t [t^{\alpha-1}(1-s)^{\alpha-2} - (t-s)^{\alpha-1}] \widetilde{x}(s)\, ds + \int_t^1 t^{\alpha-1}(1-s)^{\alpha-2} \widetilde{x}(s)\, ds \right] \\ &\quad + \frac{t^{\alpha-1}}{\Delta\Gamma(\alpha)} \left(\int_0^1 \tau^{\beta-1} dH(\tau) \right) \left[\int_0^1 \left(\int_0^1 \tau^{\alpha-1}(1-s)^{\alpha-2} dK(\tau) \right) \widetilde{x}(s)\, ds \right. \\ &\quad \left. - \int_0^1 \left(\int_s^1 (\tau-s)^{\alpha-1} dK(\tau) \right) \widetilde{x}(s)\, ds \right] \\ &\quad + \frac{t^{\alpha-1}}{\Delta\Gamma(\beta-1)} \left[\int_0^1 \left(\int_0^1 \tau^{\beta-1}(1-s)^{\beta-2} dH(\tau) \right) \widetilde{y}(s) ds \right. \\ &\qquad \left. - \int_0^1 \left(\int_s^1 (\tau-s)^{\beta-1} dH(\tau) \right) \widetilde{y}(s)\, ds \right]. \end{aligned}$$

Therefore, we obtain

$$
\begin{aligned}
u(t) &= \frac{1}{\Gamma(\alpha)}\left[\int_0^t [t^{\alpha-1}(1-s)^{\alpha-2}-(t-s)^{\alpha-1}]\widetilde{x}(s)\,ds + \int_t^1 t^{\alpha-1}(1-s)^{\alpha-2}\widetilde{x}(s)\,ds\right] \\
&\quad + \frac{t^{\alpha-1}}{\Delta\Gamma(\alpha)}\left(\int_0^1 \tau^{\beta-1}dH(\tau)\right)\left[\int_0^1\left(\int_0^s \tau^{\alpha-1}(1-s)^{\alpha-2}dK(\tau)\right)\widetilde{x}(s)\,ds\right. \\
&\quad \left.+\int_0^1\left(\int_s^1 \tau^{\alpha-1}(1-s)^{\alpha-2}dK(\tau)\right)\widetilde{x}(s)\,ds - \int_0^1\left(\int_s^1 (\tau-s)^{\alpha-1}dK(\tau)\right)\widetilde{x}(s)\,ds\right] \\
&\quad + \frac{t^{\alpha-1}}{\Delta\Gamma(\beta-1)}\left[\int_0^1\left(\int_0^s \tau^{\beta-1}(1-s)^{\beta-2}dH(\tau)\right)\widetilde{y}(s)\,ds\right. \\
&\quad \left.+\int_0^1\left(\int_s^1 \tau^{\beta-1}(1-s)^{\beta-2}dH(\tau)\right)\widetilde{y}(s)ds - \int_0^1\left(\int_s^1 (\tau-s)^{\beta-1}dH(\tau)\right)\widetilde{y}(s)ds\right] \\
&= \frac{1}{\Gamma(\alpha)}\left[\int_0^t [t^{\alpha-1}(1-s)^{\alpha-2}-(t-s)^{\alpha-1}]\widetilde{x}(s)\,ds + \int_t^1 t^{\alpha-1}(1-s)^{\alpha-2}\widetilde{x}(s)\,ds\right] \\
&\quad + \frac{t^{\alpha-1}}{\Delta\Gamma(\alpha)}\left(\int_0^1 \tau^{\beta-1}dH(\tau)\right)\left[\int_0^1\left(\int_0^s \tau^{\alpha-1}(1-s)^{\alpha-2}dK(\tau)\right)\widetilde{x}(s)\,ds\right. \\
&\quad \left.+\int_0^1\left(\int_s^1 [\tau^{\alpha-1}(1-s)^{\alpha-2}-(\tau-s)^{\alpha-1}]dK(\tau)\right)\widetilde{x}(s)\,ds\right] \\
&\quad + \frac{t^{\alpha-1}}{\Delta\Gamma(\beta-1)}\left[\int_0^1\left(\int_0^s \tau^{\beta-1}(1-s)^{\beta-2}dH(\tau)\right)\widetilde{y}(s)\,ds\right. \\
&\quad \left.+\int_0^1\left(\int_s^1 [\tau^{\beta-1}(1-s)^{\beta-2}-(\tau-s)^{\beta-1}]dH(\tau)\right)\widetilde{y}(s)\,ds\right] \\
&= \int_0^1 g_1(t,s)\widetilde{x}(s)\,ds + \frac{t^{\alpha-1}}{\Delta}\left(\int_0^1 \tau^{\beta-1}dH(\tau)\right)\left(\int_0^1\left(\int_0^1 g_1(\tau,s)dK(\tau)\right)\widetilde{x}(s)\,ds\right) \\
&\quad + \frac{(\beta-1)t^{\alpha-1}}{\Delta}\left(\int_0^1\left(\int_0^1 g_2(\tau,s)dH(\tau)\right)\widetilde{y}(s)\,ds\right) \\
&= \int_0^1 G_1(t,s)\widetilde{x}(s)\,ds + \int_0^1 G_2(t,s)\widetilde{y}(s)\,ds.
\end{aligned}
$$

In a similar manner, we deduce

$$
\begin{aligned}
v(t) &= \int_0^1 g_2(t,s)\widetilde{y}(s)\,ds + \frac{t^{\beta-1}}{\Delta}\left(\int_0^1 \tau^{\alpha-1}dK(\tau)\right)\left(\int_0^1\left(\int_0^1 g_2(\tau,s)dH(\tau)\right)\widetilde{y}(s)\,ds\right) \\
&\quad + \frac{(\alpha-1)t^{\beta-1}}{\Delta}\left(\int_0^1\left(\int_0^1 g_1(\tau,s)dK(\tau)\right)\widetilde{x}(s)\,ds\right) \\
&= \int_0^1 G_3(t,s)\widetilde{y}(s)\,ds + \int_0^1 G_4(t,s)\widetilde{x}(s)\,ds.
\end{aligned}
$$

Therefore, we obtain the expression (2.4) for the solution (u,v) of problem (2.1)–(2.2) given by relations (2.3). □

Lemma 2.5. *The functions g_1, g_2 given by (2.6) have the properties*

a) $g_1, g_2 : [0,1]\times[0,1] \to \mathbb{R}_+$ *are continuous functions, and* $g_1(t,s) > 0$, $g_2(t,s) > 0$ *for all* $(t,s) \in (0,1]\times(0,1)$.

b) $g_1(t,s) \le h_1(s)$, $g_2(t,s) \le h_2(s)$ *for all* $(t,s) \in [0,1]\times[0,1]$, *where* $h_1(s) = \frac{s(1-s)^{\alpha-2}}{\Gamma(\alpha)}$ *and* $h_2(s) = \frac{s(1-s)^{\beta-2}}{\Gamma(\beta)}$ *for all* $s \in [0,1]$.

c) $g_1(t,s) \geq t^{\alpha-1}h_1(s)$, $g_2(t,s) \geq t^{\beta-1}h_2(s)$ *for all* $(t,s) \in [0,1] \times [0,1]$.

d) $g_1(t,s) \leq \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, $g_2(t,s) \leq \frac{t^{\beta-1}}{\Gamma(\beta)}$, *for all* $(t,s) \in [0,1] \times [0,1]$.

Proof. Part a) of this lemma is evident.

b) The function g_1 is nondecreasing in the first variable. Indeed, for $s \leq t$, we have

$$\begin{aligned}\frac{\partial g_1}{\partial t}(t,s) &= \frac{1}{\Gamma(\alpha)}[(\alpha-1)t^{\alpha-2}(1-s)^{\alpha-2} - (\alpha-1)(t-s)^{\alpha-2}] \\ &= \frac{1}{\Gamma(\alpha-1)}[(t-ts)^{\alpha-2} - (t-s)^{\alpha-2}] \geq 0.\end{aligned}$$

Then, $g_1(t,s) \leq g_1(1,s)$ for all $(t,s) \in [0,1] \times [0,1]$ with $s \leq t$.

For $s \geq t$, we obtain

$$\frac{\partial g_1}{\partial t}(t,s) = \frac{1}{\Gamma(\alpha-1)}t^{\alpha-2}(1-s)^{\alpha-2} \geq 0.$$

Hence, $g_1(t,s) \leq g_1(s,s)$ for all $(t,s) \in [0,1] \times [0,1]$ with $s \geq t$.

Therefore, we deduce that $g_1(t,s) \leq h_1(s)$ for all $(t,s) \in [0,1] \times [0,1]$, where $h_1(s) = g_1(1,s) = \frac{1}{\Gamma(\alpha)}s(1-s)^{\alpha-2}$ for $s \in [0,1]$.

c) For $s \leq t$, we have

$$\begin{aligned}g_1(t,s) &= \frac{1}{\Gamma(\alpha)}[t^{\alpha-1}(1-s)^{\alpha-2} - (t-s)^{\alpha-1}] \\ &\geq \frac{1}{\Gamma(\alpha)}[t^{\alpha-1}(1-s)^{\alpha-2} - (t-ts)^{\alpha-1}] = t^{\alpha-1}h_1(s).\end{aligned}$$

For $s \geq t$, we obtain

$$g_1(t,s) = \frac{1}{\Gamma(\alpha)}t^{\alpha-1}(1-s)^{\alpha-2} \geq t^{\alpha-1}h_1(s).$$

Hence, we conclude that $g_1(t,s) \geq t^{\alpha-1}h_1(s)$ for all $(t,s) \in [0,1] \times [0,1]$.

d) For all $(t,s) \in [0,1] \times [0,1]$ we have

$$g_1(t,s) \leq \frac{1}{\Gamma(\alpha)}t^{\alpha-1}(1-s)^{\alpha-2} \leq \frac{t^{\alpha-1}}{\Gamma(\alpha)}.$$

In a similar manner we obtain the corresponding inequalities for g_2, with $h_2(s) = \frac{s(1-s)^{\beta-2}}{\Gamma(\beta)}$ for $s \in [0,1]$. □

Lemma 2.6. *If* H, $K : [0,1] \to \mathbb{R}$ *are nondecreasing functions, and* $\Delta > 0$, *then* G_i, $i = 1,\ldots,4$, *given by* (2.5) *are continuous functions on* $[0,1] \times [0,1]$ *and satisfy* $G_i(t,s) \geq 0$ *for all* $(t,s) \in [0,1] \times [0,1]$, $i = 1,\ldots,4$. *Moreover, if* $\widetilde{x}$, $\widetilde{y} \in C(0,1) \cap L^1(0,1)$ *satisfy* $\widetilde{x}(t) \geq 0$, $\widetilde{y}(t) \geq 0$ *for all* $t \in (0,1)$, *then the solution* (u,v) *of problem* (2.1)–(2.2) *given by* (2.4) *satisfies* $u(t) \geq 0$, $v(t) \geq 0$ *for all* $t \in [0,1]$.

Proof. By using the assumptions of this lemma, we have $G_i(t,s) \geq 0$ for all $(t,s) \in [0,1] \times [0,1]$, $i = 1,\ldots,4$, and so $u(t) \geq 0$, $v(t) \geq 0$ for all $t \in [0,1]$. □

Lemma 2.7. *Assume that $H, K : [0,1] \to \mathbb{R}$ are nondecreasing functions, $\Delta > 0$, and that $\int_0^1 \tau^{\alpha-1}\,dK(\tau) > 0$, $\int_0^1 \tau^{\beta-1}\,dH(\tau) > 0$. Then the functions G_i, $i = 1,\dots,4$ satisfy the inequalities*

a_1) $G_1(t,s) \le \sigma_1 h_1(s)$, $\forall\,(t,s) \in [0,1]\times[0,1]$, *where*

$$\sigma_1 = 1 + \frac{1}{\Delta}(K(1)-K(0))\int_0^1 \tau^{\beta-1}\,dH(\tau) > 0.$$

a_2) $G_1(t,s) \le \delta_1 t^{\alpha-1}$, $\forall\,(t,s) \in [0,1]\times[0,1]$, *where*

$$\delta_1 = \frac{1}{\Gamma(\alpha)}\left[1 + \frac{1}{\Delta}\left(\int_0^1 \tau^{\beta-1}\,dH(\tau)\right)\left(\int_0^1 \tau^{\alpha-1}\,dK(\tau)\right)\right] > 0.$$

a_3) $G_1(t,s) \ge \varrho_1 t^{\alpha-1} h_1(s)$, $(t,s) \in [0,1]\times[0,1]$, *where*

$$\varrho_1 = 1 + \frac{1}{\Delta}\left(\int_0^1 \tau^{\beta-1}\,dH(\tau)\right)\left(\int_0^1 \tau^{\alpha-1}\,dK(\tau)\right) > 0.$$

b_1) $G_2(t,s) \le \sigma_2 h_2(s)$, $\forall\,(t,s) \in [0,1]\times[0,1]$, *where* $\sigma_2 = \frac{\beta-1}{\Delta}(H(1)-H(0)) > 0$.
b_2) $G_2(t,s) \le \delta_2 t^{\alpha-1}$, $\forall\,(t,s) \in [0,1]\times[0,1]$, *where* $\delta_2 = \frac{1}{\Delta\Gamma(\beta-1)}\int_0^1 \tau^{\beta-1}\,dH(\tau) > 0$.
b_3) $G_2(t,s) \ge \varrho_2 t^{\alpha-1} h_2(s)$, $\forall\,(t,s) \in [0,1]\times[0,1]$, *where* $\varrho_2 = \frac{\beta-1}{\Delta}\int_0^1 \tau^{\beta-1}\,dH(\tau) > 0$.
c_1) $G_3(t,s) \le \sigma_3 h_2(s)$, $\forall\,(t,s) \in [0,1]\times[0,1]$, *where*

$$\sigma_3 = 1 + \frac{1}{\Delta}(H(1)-H(0))\int_0^1 \tau^{\alpha-1}\,dK(\tau) > 0.$$

c_2) $G_3(t,s) \le \delta_3 t^{\beta-1}$, $\forall\,(t,s) \in [0,1]\times[0,1]$, *where*

$$\delta_3 = \frac{1}{\Gamma(\beta)}\left[1 + \frac{1}{\Delta}\left(\int_0^1 \tau^{\alpha-1}\,dK(\tau)\right)\left(\int_0^1 \tau^{\beta-1}\,dH(\tau)\right)\right] > 0.$$

c_3) $G_3(t,s) \ge \varrho_3 t^{\beta-1} h_2(s)$, $\forall\,(t,s) \in [0,1]\times[0,1]$, *where*

$$\varrho_3 = 1 + \frac{1}{\Delta}\left(\int_0^1 \tau^{\alpha-1}\,dK(\tau)\right)\left(\int_0^1 \tau^{\beta-1}\,dH(\tau)\right) = \varrho_1 > 0.$$

d_1) $G_4(t,s) \le \sigma_4 h_1(s)$, $\forall\,(t,s) \in [0,1]\times[0,1]$, *where* $\sigma_4 = \frac{\alpha-1}{\Delta}(K(1)-K(0)) > 0$.
d_2) $G_4(t,s) \le \delta_4 t^{\beta-1}$, $\forall\,(t,s) \in [0,1]\times[0,1]$, *where* $\delta_4 = \frac{1}{\Delta\Gamma(\alpha-1)}\int_0^1 \tau^{\alpha-1}\,dK(\tau) > 0$.
d_3) $G_4(t,s) \ge \varrho_4 t^{\beta-1} h_1(s)$, $\forall\,(t,s) \in [0,1]\times[0,1]$, *where* $\varrho_4 = \frac{\alpha-1}{\Delta}\int_0^1 \tau^{\alpha-1}\,dK(\tau) > 0$.

Proof. From the assumptions of this lemma, we obtain

$$K(1)-K(0) = \int_0^1 dK(\tau) \ge \int_0^1 \tau^{\alpha-1}\,dK(\tau) > 0,$$
$$H(1)-H(0) = \int_0^1 dH(\tau) \ge \int_0^1 \tau^{\beta-1}\,dH(\tau) > 0.$$

By using Lemma 2.4 and Lemma 2.5, we deduce for all $(t,s) \in [0,1]\times[0,1]$

$a_1)$

$$\begin{aligned}G_1(t,s) &= g_1(t,s) + \frac{t^{\alpha-1}}{\Delta}\left(\int_0^1 \tau^{\beta-1}\,dH(\tau)\right)\left(\int_0^1 g_1(\tau,s)\,dK(\tau)\right)\\ &\le h_1(s) + \frac{1}{\Delta}\left(\int_0^1 \tau^{\beta-1}\,dH(\tau)\right)\left(\int_0^1 h_1(s)\,dK(\tau)\right)\\ &= h_1(s)\left[1+\frac{1}{\Delta}(K(1)-K(0))\int_0^1 \tau^{\beta-1}\,dH(\tau)\right] = \sigma_1 h_1(s).\end{aligned}$$

$a_2)$

$$\begin{aligned}G_1(t,s) &\le \frac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^{\alpha-1}}{\Delta}\left(\int_0^1 \tau^{\beta-1}\,dH(\tau)\right)\left(\int_0^1 \frac{\tau^{\alpha-1}}{\Gamma(\alpha)}dK(\tau)\right)\\ &= t^{\alpha-1}\frac{1}{\Gamma(\alpha)}\left[1+\frac{1}{\Delta}\left(\int_0^1 \tau^{\beta-1}\,dH(\tau)\right)\left(\int_0^1 \tau^{\alpha-1}\,dK(\tau)\right)\right] = \delta_1 t^{\alpha-1}.\end{aligned}$$

$a_3)$

$$\begin{aligned}G_1(t,s) &\ge t^{\alpha-1}h_1(s) + \frac{t^{\alpha-1}}{\Delta}\left(\int_0^1 \tau^{\beta-1}\,dH(\tau)\right)\left(\int_0^1 \tau^{\alpha-1}h_1(s)\,dK(\tau)\right)\\ &= t^{\alpha-1}h_1(s)\left[1+\frac{1}{\Delta}\left(\int_0^1 \tau^{\beta-1}\,dH(\tau)\right)\left(\int_0^1 \tau^{\alpha-1}\,dK(\tau)\right)\right] = \varrho_1 t^{\alpha-1}h_1(s).\end{aligned}$$

$b_1)$

$$\begin{aligned}G_2(t,s) &= \frac{(\beta-1)t^{\alpha-1}}{\Delta}\int_0^1 g_2(\tau,s)\,dH(\tau) \le \frac{\beta-1}{\Delta}\int_0^1 h_2(s)\,dH(\tau)\\ &= \frac{\beta-1}{\Delta}(H(1)-H(0))h_2(s) = \sigma_2 h_2(s).\end{aligned}$$

$b_2)$

$$G_2(t,s) \le \frac{(\beta-1)t^{\alpha-1}}{\Delta}\int_0^1 \frac{\tau^{\beta-1}}{\Gamma(\beta)}dH(\tau) = \delta_2 t^{\alpha-1}.$$

$b_3)$

$$G_2(t,s) \ge \frac{(\beta-1)t^{\alpha-1}}{\Delta}\int_0^1 \tau^{\beta-1}h_2(s)\,dH(\tau) = \varrho_2 t^{\alpha-1}h_2(s).$$

$c_1)$

$$\begin{aligned}G_3(t,s) &= g_2(t,s) + \frac{t^{\beta-1}}{\Delta}\left(\int_0^1 \tau^{\alpha-1}\,dK(\tau)\right)\left(\int_0^1 g_2(\tau,s)\,dH(\tau)\right)\\ &\le h_2(s) + \frac{1}{\Delta}\left(\int_0^1 \tau^{\alpha-1}\,dK(\tau)\right)\left(\int_0^1 h_2(s)\,dH(\tau)\right)\\ &= h_2(s)\left[1+\frac{1}{\Delta}(H(1)-H(0))\int_0^1 \tau^{\alpha-1}\,dK(\tau)\right] = \sigma_3 h_2(s).\end{aligned}$$

$c_2)$

$$\begin{aligned}G_3(t,s) &\le \frac{t^{\beta-1}}{\Gamma(\beta)} + \frac{t^{\beta-1}}{\Delta}\left(\int_0^1 \tau^{\alpha-1}\,dK(\tau)\right)\left(\int_0^1 \frac{\tau^{\beta-1}}{\Gamma(\beta)}dH(\tau)\right)\\ &= \frac{t^{\beta-1}}{\Gamma(\beta)}\left[1+\frac{1}{\Delta}\left(\int_0^1 \tau^{\alpha-1}dK(\tau)\right)\left(\int_0^1 \tau^{\beta-1}dH(\tau)\right)\right] = \delta_3 t^{\beta-1}.\end{aligned}$$

c_3)

$$G_3(t,s) \geq t^{\beta-1}h_2(s) + \frac{t^{\beta-1}}{\Delta}\left(\int_0^1 \tau^{\alpha-1}dK(\tau)\right)\left(\int_0^1 \tau^{\beta-1}h_2(s)\,dH(\tau)\right)$$
$$= t^{\beta-1}h_2(s)\left[1+\frac{1}{\Delta}\left(\int_0^1 \tau^{\alpha-1}dK(\tau)\right)\left(\int_0^1 \tau^{\beta-1}\,dH(\tau)\right)\right] = \varrho_3 t^{\beta-1}h_2(s).$$

d_1)

$$G_4(t,s) = \frac{(\alpha-1)t^{\beta-1}}{\Delta}\int_0^1 g_1(\tau,s)\,dK(\tau) \leq \frac{\alpha-1}{\Delta}\int_0^1 h_1(s)dK(\tau)$$
$$= h_1(s)\frac{\alpha-1}{\Delta}(K(1)-K(0)) = \sigma_4 h_1(s).$$

d_2)

$$G_4(t,s) \leq \frac{(\alpha-1)t^{\beta-1}}{\Delta}\int_0^1 \frac{\tau^{\alpha-1}}{\Gamma(\alpha)}dK(\tau) = \delta_4 t^{\beta-1}.$$

d_3)

$$G_4(t,s) \geq \frac{(\alpha-1)t^{\beta-1}}{\Delta}\int_0^1 \tau^{\alpha-1}h_1(s)\,dK(\tau) = \varrho_4 t^{\beta-1}h_1(s).$$

□

Remark 2.8. From Lemma 2.7, we obtain for all t, t', $s \in [0,1]$ the following inequalities:

$$G_1(t,s) \geq \varrho_1 t^{\alpha-1}h_1(s) \geq \frac{\varrho_1}{\sigma_1}t^{\alpha-1}G_1(t',s), \qquad G_2(t,s) \geq \varrho_2 t^{\alpha-1}h_2(s) \geq \frac{\varrho_2}{\sigma_2}t^{\alpha-1}G_2(t',s),$$
$$G_3(t,s) \geq \varrho_3 t^{\beta-1}h_2(s) \geq \frac{\varrho_3}{\sigma_3}t^{\beta-1}G_3(t',s), \qquad G_4(t,s) \geq \varrho_4 t^{\beta-1}h_1(s) \geq \frac{\varrho_4}{\sigma_4}t^{\beta-1}G_4(t',s),$$
$$G_1(t,s) \geq \varrho_1 t^{\alpha-1}h_1(s) \geq \frac{\varrho_1}{\sigma_4}t^{\alpha-1}G_4(t',s), \qquad G_4(t,s) \geq \varrho_4 t^{\beta-1}h_1(s) \geq \frac{\varrho_4}{\sigma_1}t^{\beta-1}G_1(t',s),$$
$$G_3(t,s) \geq \varrho_3 t^{\beta-1}h_2(s) \geq \frac{\varrho_3}{\sigma_2}t^{\beta-1}G_2(t',s), \qquad G_2(t,s) \geq \varrho_2 t^{\alpha-1}h_2(s) \geq \frac{\varrho_2}{\sigma_3}t^{\alpha-1}G_3(t',s).$$

Lemma 2.9. *Assume that $H, K : [0,1] \to \mathbb{R}$ are nondecreasing functions, $\Delta > 0$, and that $\int_0^1 \tau^{\alpha-1}\,dK(\tau) > 0$, $\int_0^1 \tau^{\beta-1}\,dH(\tau) > 0$, and $\widetilde{x}, \widetilde{y} \in C(0,1) \cap L^1(0,1)$, $\widetilde{x}(t) \geq 0$, $\widetilde{y}(t) \geq 0$ for all $t \in (0,1)$. Then the solution $(u(t), v(t))$, $t \in [0,1]$ of problem (2.1)–(2.2) given by (2.4) satisfies the inequalities $u(t) \geq \gamma t^{\alpha-1}u(t')$, $u(t) \geq \gamma t^{\alpha-1}v(t')$, $v(t) \geq \gamma t^{\beta-1}v(t')$, $v(t) \geq \gamma t^{\beta-1}u(t')$, for all t, $t' \in [0,1]$, where $\gamma = \min\left\{\frac{\varrho_1}{\sigma_1}, \frac{\varrho_2}{\sigma_2}, \frac{\varrho_3}{\sigma_3}, \frac{\varrho_4}{\sigma_4}, \frac{\varrho_1}{\sigma_4}, \frac{\varrho_4}{\sigma_1}, \frac{\varrho_3}{\sigma_2}, \frac{\varrho_2}{\sigma_3}\right\} > 0$.*

Proof. By using Lemma 2.7 and Remark 2.8, we obtain for all t, $t' \in [0,1]$ the following inequalities

$$u(t) = \int_0^1 G_1(t,s)\widetilde{x}(s)\,ds + \int_0^1 G_2(t,s)\widetilde{y}(s)\,ds$$
$$\geq \int_0^1 \frac{\varrho_1}{\sigma_1}t^{\alpha-1}G_1(t',s)\widetilde{x}(s)\,ds + \int_0^1 \frac{\varrho_2}{\sigma_2}t^{\alpha-1}G_2(t',s)\widetilde{y}(s)\,ds$$
$$\geq \gamma t^{\alpha-1}\left(\int_0^1 G_1(t',s)\widetilde{x}(s)\,ds + \int_0^1 G_2(t',s)\widetilde{y}(s)\,ds\right) = \gamma t^{\alpha-1}u(t'),$$

$$
\begin{aligned}
u(t) &= \int_0^1 G_1(t,s)\widetilde{x}(s)\,ds + \int_0^1 G_2(t,s)\widetilde{y}(s)\,ds \\
&\geq \int_0^1 \frac{\varrho_1}{\sigma_4} t^{\alpha-1} G_4(t',s)\widetilde{x}(s)\,ds + \int_0^1 \frac{\varrho_2}{\sigma_3} t^{\alpha-1} G_3(t',s)\widetilde{y}(s)\,ds \\
&\geq \gamma t^{\alpha-1} \left(\int_0^1 G_4(t',s)\widetilde{x}(s)\,ds + \int_0^1 G_3(t',s)\widetilde{y}(s)\,ds \right) = \gamma t^{\alpha-1} v(t'), \\
v(t) &= \int_0^1 G_3(t,s)\widetilde{y}(s)\,ds + \int_0^1 G_4(t,s)\widetilde{x}(s)\,ds \\
&\geq \int_0^1 \frac{\varrho_3}{\sigma_3} t^{\beta-1} G_3(t',s)\widetilde{y}(s)\,ds + \int_0^1 \frac{\varrho_4}{\sigma_4} t^{\beta-1} G_4(t',s)\widetilde{x}(s)\,ds \\
&\geq \gamma t^{\beta-1} \left(\int_0^1 G_3(t',s)\widetilde{y}(s)\,ds + \int_0^1 G_4(t',s)\widetilde{x}(s)\,ds \right) = \gamma t^{\beta-1} v(t'), \\
v(t) &= \int_0^1 G_3(t,s)\widetilde{y}(s)\,ds + \int_0^1 G_4(t,s)\widetilde{x}(s)\,ds \\
&\geq \int_0^1 \frac{\varrho_3}{\sigma_2} t^{\beta-1} G_2(t',s)\widetilde{y}(s)\,ds + \int_0^1 \frac{\varrho_4}{\sigma_1} t^{\beta-1} G_1(t',s)\widetilde{x}(s)\,ds \\
&\geq \gamma t^{\beta-1} \left(\int_0^1 G_2(t',s)\widetilde{y}(s)\,ds + \int_0^1 G_1(t',s)\widetilde{x}(s)\,ds \right) = \gamma t^{\beta-1} u(t'),
\end{aligned}
$$

where $\gamma = \min\left\{\frac{\varrho_1}{\sigma_1}, \frac{\varrho_2}{\sigma_2}, \frac{\varrho_3}{\sigma_3}, \frac{\varrho_4}{\sigma_4}, \frac{\varrho_1}{\sigma_4}, \frac{\varrho_4}{\sigma_1}, \frac{\varrho_3}{\sigma_2}, \frac{\varrho_2}{\sigma_3}\right\} > 0$. □

In the proof of our main results we shall use the nonlinear alternative of Leray–Schauder type and the Guo–Krasnosel'skii fixed point theorem presented, respectively, below (see [1,5]).

Theorem 2.10. *Let X be a Banach space with $\Omega \subset X$ closed and convex. Assume U is a relatively open subset of Ω with $0 \in U$, and let $S : \overline{U} \to \Omega$ be a completely continuous operator (continuous and compact, that is it maps bounded sets into relatively compact sets, and it is continuous). Then either*

1) *S has a fixed point in $\overline{U}$, or*

2) *there exists $u \in \partial U$ and $\nu \in (0,1)$ such that $u = \nu S u$.*

Theorem 2.11. *Let X be a Banach space and let $C \subset X$ be a cone in X. Assume Ω_1 and Ω_2 are bounded open subsets of X with $0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$ and let $\mathcal{A} : C \cap (\overline{\Omega_2} \setminus \Omega_1) \to C$ be a completely continuous operator such that, either*

i) *$\|\mathcal{A}u\| \leq \|u\|$, $u \in C \cap \partial\Omega_1$, and $\|\mathcal{A}u\| \geq \|u\|$, $u \in C \cap \partial\Omega_2$, or*

ii) *$\|\mathcal{A}u\| \geq \|u\|$, $u \in C \cap \partial\Omega_1$, and $\|\mathcal{A}u\| \leq \|u\|$, $u \in C \cap \partial\Omega_2$.*

Then $\mathcal{A}$ has a fixed point in $C \cap (\overline{\Omega_2} \setminus \Omega_1)$.

3 Main results

In this section, we investigate the existence and multiplicity of positive solutions for our problem (*S*)–(*BC*). We present now the assumptions that we shall use in the sequel.

(*H*1) $H, K : [0,1] \to \mathbb{R}$ are nondecreasing functions, $\Delta = (\alpha-1)(\beta-1) - \left(\int_0^1 \tau^{\alpha-1}\,dK(\tau)\right) \times \left(\int_0^1 \tau^{\beta-1}\,dH(\tau)\right) > 0$ and $\int_0^1 \tau^{\alpha-1}\,dK(\tau) > 0$, $\int_0^1 \tau^{\beta-1}\,dH(\tau) > 0$.

$(H2)$ The functions $f, g \in C([0,1] \times [0,\infty) \times [0,\infty), (-\infty,+\infty))$ and there exist functions $p_1, p_2 \in C([0,1],[0,\infty))$ such that $f(t,u,v) \geq -p_1(t)$ and $g(t,u,v) \geq -p_2(t)$ for any $t \in [0,1]$ and $u, v \in [0,\infty)$.

$(H3)$ $f(t,0,0) > 0$, $g(t,0,0) > 0$ for all $t \in [0,1]$.

$(H4)$ The functions $f, g \in C((0,1) \times [0,\infty) \times [0,\infty), (-\infty,+\infty))$, f, g may be singular at $t = 0$ and/or $t = 1$, and there exist functions $p_1, p_2 \in C((0,1),[0,\infty))$, $\alpha_1, \alpha_2 \in C((0,1),[0,\infty))$, $\beta_1, \beta_2 \in C([0,1] \times [0,\infty) \times [0,\infty),[0,\infty))$ such that

$$-p_1(t) \leq f(t,u,v) \leq \alpha_1(t)\beta_1(t,u,v), \qquad -p_2(t) \leq g(t,u,v) \leq \alpha_2(t)\beta_2(t,u,v),$$

for all $t \in (0,1)$, $u, v \in [0,\infty)$, with $0 < \int_0^1 p_i(s)\,ds < \infty$, $0 < \int_0^1 \alpha_i(s)\,ds < \infty$, $i = 1,2$.

$(H5)$ There exists $c \in (0,1/2)$ such that

$$f_\infty = \lim_{u+v\to\infty} \min_{t\in[c,1-c]} \frac{f(t,u,v)}{u+v} = \infty \quad \text{or} \quad g_\infty = \lim_{u+v\to\infty} \min_{t\in[c,1-c]} \frac{g(t,u,v)}{u+v} = \infty.$$

$(H6)$ $\beta_{i\infty} = \lim\limits_{u+v\to\infty} \max\limits_{t\in[0,1]} \dfrac{\beta_i(t,u,v)}{u+v} = 0, \;\; i = 1,2.$

We consider the system of nonlinear fractional differential equations

$$\begin{cases} D_{0+}^{\alpha} x(t) + \lambda(f(t,[x(t)-q_1(t)]^*,[y(t)-q_2(t)]^*) + p_1(t)) = 0, & 0 < t < 1, \\ D_{0+}^{\beta} y(t) + \mu(g(t,[x(t)-q_1(t)]^*,[y(t)-q_2(t)]^*) + p_2(t)) = 0, & 0 < t < 1, \end{cases} \tag{3.1}$$

with the integral boundary conditions

$$\begin{cases} x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0, & x'(1) = \displaystyle\int_0^1 y(s)dH(s), \\ y(0) = y'(0) = \cdots = y^{(m-2)}(0) = 0, & y'(1) = \displaystyle\int_0^1 x(s)dK(s), \end{cases} \tag{3.2}$$

where $z(t)^* = z(t)$ if $z(t) \geq 0$, and $z(t)^* = 0$ if $z(t) < 0$. Here (q_1,q_2) with

$$q_1(t) = \lambda \int_0^1 G_1(t,s)p_1(s)\,ds + \mu \int_0^1 G_2(t,s)p_2(s)\,ds, \qquad t \in [0,1],$$
$$q_2(t) = \mu \int_0^1 G_3(t,s)p_2(s)\,ds + \lambda \int_0^1 G_4(t,s)p_1(s)\,ds, \qquad t \in [0,1],$$

is solution of the system of fractional differential equations

$$\begin{cases} D_{0+}^{\alpha} q_1(t) + \lambda p_1(t) = 0, & 0 < t < 1, \\ D_{0+}^{\beta} q_2(t) + \mu p_2(t) = 0, & 0 < t < 1, \end{cases} \tag{3.3}$$

with the integral boundary conditions

$$\begin{cases} q_1(0) = q_1'(0) = \cdots = q_1^{(n-2)}(0) = 0, & q_1'(1) = \displaystyle\int_0^1 q_2(s)dH(s), \\ q_2(0) = q_2'(0) = \cdots = q_2^{(m-2)}(0) = 0, & q_2'(1) = \displaystyle\int_0^1 q_1(s)dK(s). \end{cases} \tag{3.4}$$

Under the assumptions $(H1)$ and $(H2)$, or $(H1)$ and $(H4)$, we have $q_1(t) \geq 0$, $q_2(t) \geq 0$ for all $t \in [0,1]$.

We shall prove that there exists a solution (x,y) for the boundary value problem (3.1)–(3.2) with $x(t) \geq q_1(t)$ and $y(t) \geq q_2(t)$ on $[0,1]$, $x(t) > q_1(t)$, $y(t) > q_2(t)$ on $(0,1)$. In this case (u,v) with $u(t) = x(t) - q_1(t)$ and $v(t) = y(t) - q_2(t)$ for all $t \in [0,1]$ represents a positive solution of the boundary value problem (S)–(BC). Indeed, by (3.1)–(3.4), we have

$$\begin{aligned} D^{\alpha}_{0+}u(t) &= D^{\alpha}_{0+}x(t) - D^{\alpha}_{0+}q_1(t) = -\lambda f(t,[x(t)-q_1(t)]^*,[y(t)-q_2(t)]^*) \\ &\quad -\lambda p_1(t) + \lambda p_1(t) = -\lambda f(t,u(t),v(t)), \qquad \forall\, t \in (0,1), \\ D^{\beta}_{0+}v(t) &= D^{\beta}_{0+}y(t) - D^{\beta}_{0+}q_2(t) = -\mu g(t,[x(t)-q_1(t)]^*,[y(t)-q_2(t)]^*) \\ &\quad -\mu p_2(t) + \mu p_2(t) = -\mu g(t,u(t),v(t)), \qquad \forall\, t \in (0,1), \end{aligned}$$

and

$$\begin{aligned} &u(0) = x(0) - q_1(0) = 0, \ldots, u^{(n-2)}(0) = x^{(n-2)}(0) - q_1^{(n-2)}(0) = 0, \\ &v(0) = y(0) - q_2(0) = 0, \ldots, v^{(m-2)}(0) = y^{(m-2)}(0) - q_2^{(m-2)}(0) = 0, \\ &u'(1) = x'(1) - q_1'(1) = \int_0^1 y(s)\,dH(s) - \int_0^1 q_2(s)\,dH(s) = \int_0^1 v(s)\,dH(s), \\ &v'(1) = y'(1) - q_2'(1) = \int_0^1 x(s)\,dK(s) - \int_0^1 q_1(s)\,dK(s) = \int_0^1 u(s)\,dK(s). \end{aligned}$$

Therefore, in what follows, we shall investigate the boundary value problem (3.1)–(3.2).

By using Lemma 2.4 (relations (2.4)), a solution of the system

$$\begin{cases} x(t) = \lambda \displaystyle\int_0^1 G_1(t,s)(f(s,[x(s)-q_1(s)]^*,[y(s)-q_2(s)]^*) + p_1(s))\,ds \\ \qquad + \mu \displaystyle\int_0^1 G_2(t,s)(g(s,[x(s)-q_1(s)]^*,[y(s)-q_2(s)]^*) + p_2(s))\,ds, \quad t \in [0,1], \\ y(t) = \mu \displaystyle\int_0^1 G_3(t,s)(g(s,[x(s)-q_1(s)]^*,[y(s)-q_2(s)]^*) + p_2(s))\,ds \\ \qquad + \lambda \displaystyle\int_0^1 G_4(t,s)(f(s,[x(s)-q_1(s)]^*,[y(s)-q_2(s)]^*) + p_1(s))\,ds, \quad t \in [0,1], \end{cases}$$

is a solution for problem (3.1)–(3.2).

We consider the Banach space $X = C([0,1])$ with the supremum norm $\|\cdot\|$, $\|u\| = \sup_{t\in[0,1]} u(t)$, and the Banach space $Y = X \times X$ with the norm $\|(u,v)\|_Y = \max\{\|u\|,\|v\|\}$. We define the cone

$$P = \{(x,y) \in Y,\ x(t) \geq \gamma t^{\alpha-1}\|(x,y)\|_Y,\ y(t) \geq \gamma t^{\beta-1}\|(x,y)\|_Y,\ \forall\, t \in [0,1]\},$$

where γ is defined in Section 2 (Lemma 2.9).

For λ, $\mu > 0$, we introduce the operators $T_1, T_2 : Y \to X$ and $\mathcal{T} : Y \to Y$ defined by $\mathcal{T}(x,y) = (T_1(x,y), T_2(x,y))$, $(x,y) \in Y$ with

$$\begin{aligned} T_1(x,y)(t) &= \lambda \int_0^1 G_1(t,s)(f(s,[x(s)-q_1(s)]^*,[y(s)-q_2(s)]^*) + p_1(s))\,ds \\ &\quad + \mu \int_0^1 G_2(t,s)(g(s,[x(s)-q_1(s)]^*,[y(s)-q_2(s)]^*) + p_2(s))\,ds, \qquad t \in [0,1], \\ T_2(x,y)(t) &= \mu \int_0^1 G_3(t,s)(g(s,[x(s)-q_1(s)]^*,[y(s)-q_2(s)]^*) + p_2(s))\,ds \\ &\quad + \lambda \int_0^1 G_4(t,s)(f(s,[x(s)-q_1(s)]^*,[y(s)-q_2(s)]^*) + p_1(s))\,ds, \qquad t \in [0,1]. \end{aligned}$$

It is clear that if (x, y) is a fixed point of operator $\mathcal{T}$, then (x, y) is a solution of problem (3.1)–(3.2).

Lemma 3.1. *If $(H1)$ and $(H2)$, or $(H1)$ and $(H4)$ hold, then operator $\mathcal{T} : P \to P$ is a completely continuous operator.*

Proof. The operators T_1 and T_2 are well-defined. To prove this, let $(x, y) \in P$ be fixed with $\|(x, y)\|_Y = \widetilde{L}$. Then, we have

$$[x(s) - q_1(s)]^* \le x(s) \le \|x\| \le \|(x, y)\|_Y = \widetilde{L}, \qquad \forall\, s \in [0, 1],$$
$$[y(s) - q_2(s)]^* \le y(s) \le \|y\| \le \|(x, y)\|_Y = \widetilde{L}, \qquad \forall\, s \in [0, 1].$$

If $(H1)$ and $(H2)$ hold, we obtain

$$\begin{aligned} T_1(x, y)(t) &\le \lambda\sigma_1 \int_0^1 h_1(s)(f(s, [x(s) - q_1(s)]^*, [y(s) - q_2(s)]^*) + p_1(s))\, ds \\ &\quad + \mu\sigma_2 \int_0^1 h_2(s)(g(s, [x(s) - q_1(s)]^*, [y(s) - q_2(s)]^*) + p_2(s))\, ds \\ &\le 2M \left(\lambda\sigma_1 \int_0^1 h_1(s)\, ds + \mu\sigma_2 \int_0^1 h_2(s)\, ds \right) < \infty, \qquad \forall\, t \in [0, 1], \\ T_2(x, y)(t) &\le \mu\sigma_3 \int_0^1 h_2(s)(g(s, [x(s) - q_1(s)]^*, [y(s) - q_2(s)]^*) + p_2(s))\, ds \\ &\quad + \lambda\sigma_4 \int_0^1 h_1(s)(f(s, [x(s) - q_1(s)]^*, [y(s) - q_2(s)]^*) + p_1(s))\, ds \\ &\le 2M \left(\mu\sigma_3 \int_0^1 h_2(s)\, ds + \lambda\sigma_4 \int_0^1 h_1(s)\, ds \right) < \infty, \qquad \forall\, t \in [0, 1], \end{aligned}$$

where

$$M = \max \left\{ \max_{t \in [0,1],\, u,v \in [0, \widetilde{L}]} f(t, u, v), \max_{t \in [0,1],\, u,v \in [0, \widetilde{L}]} g(t, u, v), \max_{t \in [0,1]} p_1(t), \max_{t \in [0,1]} p_2(t) \right\}.$$

If $(H1)$ and $(H4)$ hold, we deduce for all $t \in [0, 1]$

$$\begin{aligned} T_1(x, y)(t) &\le \lambda\sigma_1 \int_0^1 h_1(s)(f(x, [x(s) - q_1(s)]^*, [y(s) - q_2(s)]^*) + p_1(s))\, ds \\ &\quad + \mu\sigma_2 \int_0^1 h_2(s)(g(s, [x(s) - q_1(s)]^*, [y(s) - q_2(s)]^*) + p_2(s))\, ds \\ &\le \lambda\sigma_1 \int_0^1 h_1(s)[\alpha_1(s)\beta_1(s, [x(s) - q_1(s)]^*, [y(s) - q_2(s)]^*) + p_1(s)]\, ds \\ &\quad + \mu\sigma_2 \int_0^1 h_2(s)[\alpha_2(s)\beta_2(s, [x(s) - q_1(s)]^*, [y(s) - q_2(s)]^*) + p_2(s)]\, ds \\ &\le \widetilde{M} \left(\lambda\sigma_1 \int_0^1 h_1(s)(\alpha_1(s) + p_1(s))\, ds + \mu\sigma_2 \int_0^1 h_2(s)(\alpha_2(s) + p_2(s))\, ds \right) < \infty, \\ T_2(x, y)(t) &\le \mu\sigma_3 \int_0^1 h_2(s)(g(x, [x(s) - q_1(s)]^*, [y(s) - q_2(s)]^*) + p_2(s))\, ds \\ &\quad + \lambda\sigma_4 \int_0^1 h_1(s)(f(s, [x(s) - q_1(s)]^*, [y(s) - q_2(s)]^*) + p_1(s))\, ds \end{aligned}$$

$$
\begin{aligned}
&\le \mu\sigma_3 \int_0^1 h_2(s)[\alpha_2(s)\beta_2(s,[x(s)-q_1(s)]^*,[y(s)-q_2(s)]^*)+p_2(s)]\,ds \\
&\quad + \lambda\sigma_4 \int_0^1 h_1(s)[\alpha_1(s)\beta_1(s,[x(s)-q_1(s)]^*,[y(s)-q_2(s)]^*)+p_1(s)]\,ds \\
&\le \widetilde{M}\left(\mu\sigma_3 \int_0^1 h_2(s)(\alpha_2(s)+p_2(s))\,ds + \lambda\sigma_4 \int_0^1 h_1(s)(\alpha_1(s)+p_1(s))\,ds\right) < \infty,
\end{aligned}
$$

where

$$
\widetilde{M} = \max\left\{ \max_{t\in[0,1],\, u,v\in[0,\widetilde{L}]} \beta_1(t,u,v),\ \max_{t\in[0,1],\, u,v\in[0,\widetilde{L}]} \beta_2(t,u,v), 1 \right\}.
$$

Besides, by Lemma 2.9, we conclude that

$$
\begin{aligned}
&T_1(x,y)(t) \ge \gamma t^{\alpha-1} T_1(x,y)(t'),\ \ T_1(x,y)(t) \ge \gamma t^{\alpha-1} T_2(x,y)(t'), \\
&T_2(x,y)(t) \ge \gamma t^{\beta-1} T_2(x,y)(t'),\ \ T_2(x,y)(t) \ge \gamma t^{\beta-1} T_1(x,y)(t'), \qquad \forall\, t,\, t' \in [0,1],
\end{aligned}
$$

and so

$$
\begin{aligned}
&T_1(x,y)(t) \ge \gamma t^{\alpha-1} \|T_1(x,y)\|,\ \ T_1(x,y)(t) \ge \gamma t^{\alpha-1} \|T_2(x,y)\|, \\
&T_2(x,y)(t) \ge \gamma t^{\beta-1} \|T_2(x,y)\|,\ \ T_2(x,y)(t) \ge \gamma t^{\beta-1} \|T_1(x,y)\|, \qquad \forall\, t \in [0,1].
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&T_1(x,y)(t) \ge \gamma t^{\alpha-1} \|(T_1(x,y),T_2(x,y))\|_Y, \\
&T_2(x,y)(t) \ge \gamma t^{\beta-1} \|(T_1(x,y),T_2(x,y))\|_Y, \qquad \forall\, t \in [0,1].
\end{aligned}
$$

We deduce that $(T_1(x,y),T_2(x,y)) \in P$, and hence $T(P) \subset P$.

By using standard arguments, we deduce that operator $\mathcal{T} : P \to P$ is a completely continuous operator. □

Theorem 3.2. *Assume that $(H1)$–$(H3)$ hold. Then there exist constants $\lambda_0 > 0$ and $\mu_0 > 0$ such that for any $\lambda \in (0,\lambda_0]$ and $\mu \in (0,\mu_0]$, the boundary value problem (S)–(BC) has at least one positive solution.*

Proof. Let $\delta \in (0,1)$ be fixed. From $(H2)$ and $(H3)$, there exists $R_0 \in (0,1]$ such that

$$
f(t,u,v) \ge \delta f(t,0,0), \qquad g(t,u,v) \ge \delta g(t,0,0), \qquad \forall\, t \in [0,1],\ \ u,\, v \in [0,R_0]. \tag{3.5}
$$

We define

$$
\begin{aligned}
\bar{f}(R_0) &= \max_{t\in[0,1],\, u,v\in[0,R_0]} \{f(t,u,v)+p_1(t)\} \ge \max_{t\in[0,1]} \{\delta f(t,0,0)+p_1(t)\} > 0, \\
\bar{g}(R_0) &= \max_{t\in[0,1],\, u,v\in[0,R_0]} \{g(t,u,v)+p_2(t)\} \ge \max_{t\in[0,1]} \{\delta g(t,0,0)+p_2(t)\} > 0,
\end{aligned}
$$

$$
c_1 = \sigma_1 \int_0^1 h_1(s)\,ds, \quad c_2 = \sigma_2 \int_0^1 h_2(s)\,ds, \quad c_3 = \sigma_3 \int_0^1 h_2(s)\,ds, \quad c_4 = \sigma_4 \int_0^1 h_1(s)\,ds,
$$

$$
\lambda_0 = \max\left\{ \frac{R_0}{4c_1\bar{f}(R_0)}, \frac{R_0}{4c_4\bar{f}(R_0)} \right\}, \qquad \mu_0 = \max\left\{ \frac{R_0}{4c_2\bar{g}(R_0)}, \frac{R_0}{4c_3\bar{g}(R_0)} \right\}.
$$

We will show that for any $\lambda \in (0,\lambda_0]$ and $\mu \in (0,\mu_0]$, problem (3.1)–(3.2) has at least one positive solution.

So, let $\lambda \in (0,\lambda_0]$ and $\mu \in (0,\mu_0]$ be arbitrary, but fixed for the moment. We define the set $U = \{(x,y) \in P,\ \|(u,v)\|_Y < R_0\}$. We suppose that there exist $(x,y) \in \partial U$ ($\|(x,y)\|_Y = R_0$ or $\max\{\|x\|,\|y\|\} = R_0$) and $\nu \in (0,1)$ such that $(x,y) = \nu\mathcal{T}(x,y)$ or $x = \nu T_1(x,y)$, $y = \nu T_2(x,y)$.

We deduce that

$$
\begin{aligned}
&[x(t) - q_1(t)]^* = x(t) - q_1(t) \le x(t) \le R_0, \quad \text{if } x(t) - q_1(t) \ge 0,\\
&[x(t) - q_1(t)]^* = 0, \quad \text{for } x(t) - q_1(t) < 0,\ \forall\, t \in [0,1],\\
&[y(t) - q_2(t)]^* = y(t) - q_2(t) \le y(t) \le R_0, \quad \text{if } y(t) - q_2(t) \ge 0,\\
&[y(t) - q_2(t)]^* = 0, \quad \text{for } y(t) - q_2(t) < 0,\ \forall\, t \in [0,1].
\end{aligned}
$$

Then by Lemma 2.7, for all $t \in [0,1]$, we obtain

$$
\begin{aligned}
x(t) &= \nu T_1(x,y)(t) \le T_1(x,y)(t)\\
&= \lambda \int_0^1 G_1(t,s)(f(s,[x(s)-q_1(s)]^*,[y(s)-q_2(s)]^*) + p_1(s))\, ds\\
&\quad + \mu \int_0^1 G_2(t,s)(g(s,[x(s)-q_1(s)]^*,[y(s)-q_2(s)]^*) + p_2(s))\, ds\\
&\le \lambda\sigma_1 \int_0^1 h_1(s)\bar{f}(R_0)\, ds + \mu\sigma_2 \int_0^1 h_2(s)\bar{g}(R_0)\, ds \le \lambda_0 c_1 \bar{f}(R_0) + \mu_0 c_2 \bar{g}(R_0) \le \frac{R_0}{4} + \frac{R_0}{4} = \frac{R_0}{2},\\
y(t) &= \nu T_2(x,y)(t) \le T_2(x,y)(t)\\
&= \mu \int_0^1 G_3(t,s)(g(s,[x(s)-q_1(s)]^*,[y(s)-q_2(s)]^*) + p_2(s))\, ds\\
&\quad + \lambda \int_0^1 G_4(t,s)(f(s,[x(s)-q_1(s)]^*,[y(s)-q_2(s)]^*) + p_1(s))\, ds\\
&\le \mu\sigma_3 \int_0^1 h_2(s)\bar{g}(R_0)\, ds + \lambda\sigma_4 \int_0^1 h_1(s)\bar{f}(R_0)\, ds \le \mu_0 c_3 \bar{g}(R_0) + \lambda_0 c_4 \bar{f}(R_0) \le \frac{R_0}{4} + \frac{R_0}{4} = \frac{R_0}{2}.
\end{aligned}
$$

Hence $\|x\| \le \frac{R_0}{2}$ and $\|y\| \le \frac{R_0}{2}$. Then $R_0 = \|(x,y)\|_Y = \max\{\|x\|,\|y\|\} \le \frac{R_0}{2}$, which is a contradiction.

Therefore, by Theorem 2.10 (with $\Omega = P$), we deduce that $\mathcal{T}$ has a fixed point $(x_0,y_0) \in \bar{U} \cap P$. That is, $(x_0,y_0) = \mathcal{T}(x_0,y_0)$ or $x_0 = T_1(x_0,y_0)$, $y_0 = T_2(x_0,y_0)$, and $\max\{\|x_0\|,\|y_0\|\} \le R_0$ with $x_0(t) \ge \gamma t^{\alpha-1}\|(x_0,y_0)\|_Y$ and $y_0(t) \ge \gamma t^{\beta-1}\|(x_0,y_0\|_Y$ for all $t \in [0,1]$.

Moreover, by (3.5), we conclude

$$
\begin{aligned}
x_0(t) &= T_1(x_0,y_0)(t) \ge \lambda \int_0^1 G_1(t,s)(\delta f(t,0,0) + p_1(s))ds + \mu \int_0^1 G_2(t,s)(\delta g(t,0,0) + p_2(s))ds\\
&\ge \lambda \int_0^1 G_1(t,s)p_1(s)\, ds + \mu \int_0^1 G_2(t,s)p_2(s)\, ds = q_1(t), \qquad \forall\, t \in [0,1],\\
x_0(t) &> \lambda \int_0^1 G_1(t,s)p_1(s)\, ds + \mu \int_0^1 G_2(t,s)p_2(s)\, ds = q_1(t), \qquad \forall\, t \in (0,1),\\
y_0(t) &= T_2(x_0,y_0)(t) \ge \mu \int_0^1 G_3(t,s)(\delta g(t,0,0) + p_2(s))ds + \lambda \int_0^1 G_4(t,s)(\delta f(t,0,0) + p_1(s))ds\\
&\ge \mu \int_0^1 G_3(t,s)p_2(s)\, ds + \lambda \int_0^1 G_4(t,s)p_1(s)\, ds = q_2(t), \qquad \forall\, t \in [0,1],\\
y_0(t) &> \mu \int_0^1 G_3(t,s)p_2(s)\, ds + \lambda \int_0^1 G_4(t,s)p_1(s)\, ds = q_2(t), \qquad \forall\, t \in (0,1).
\end{aligned}
$$

Therefore $x_0(t) \ge q_1(t)$, $y_0(t) \ge q_2(t)$ for all $t \in [0,1]$, and $x_0(t) > q_1(t)$, $y_0(t) > q_2(t)$ for all $t \in (0,1)$. Let $u_0(t) = x_0(t) - q_1(t)$ and $v_0(t) = y_0(t) - q_2(t)$ for all $t \in [0,1]$. Then

$u_0(t) \geq 0$, $v_0(t) \geq 0$ for all $t \in [0,1]$, $u_0(t) > 0$, $v_0(t) > 0$ for all $t \in (0,1)$. Therefore (u_0, v_0) is a positive solution of (S)–(BC). □

Theorem 3.3. *Assume that* ($H1$), ($H4$) *and* ($H5$) *hold. Then there exists* $\lambda^* > 0$ *and* $\mu^* > 0$ *such that for any* $\lambda \in (0, \lambda^*]$ *and* $\mu \in (0, \mu^*]$, *the boundary value problem* (S)–(BC) *has at least one positive solution.*

Proof. We choose a positive number

$$R_1 > \max\left\{1, \frac{1}{\gamma}\int_0^1 (\delta_1 p_1(s) + \delta_2 p_2(s))\, ds, \frac{1}{\gamma}\int_0^1 (\delta_3 p_2(s) + \delta_4 p_1(s))\, ds, \right\}$$

and we define the set $\Omega_1 = \{(x,y) \in P,\ \|(x,y)\|_Y < R_1\}$.

We introduce

$$\lambda^* = \min\left\{1, \tfrac{R_1}{2\sigma_1 M_1}\left(\int_0^1 h_1(s)(\alpha_1(s) + p_1(s))\, ds\right)^{-1}, \tfrac{R_1}{2\sigma_4 M_1}\left(\int_0^1 h_1(s)(\alpha_1(s) + p_1(s))\, ds\right)^{-1}\right\},$$

$$\mu^* = \min\left\{1, \tfrac{R_1}{2\sigma_2 M_2}\left(\int_0^1 h_2(s)(\alpha_2(s) + p_2(s))\, ds\right)^{-1}, \tfrac{R_1}{2\sigma_3 M_2}\left(\int_0^1 h_2(s)(\alpha_2(s) + p_2(s))\, ds\right)^{-1}\right\},$$

with

$$M_1 = \max\left\{\max_{\substack{t\in[0,1],\, u,v\geq 0,\\ u+v\leq R_1}} \beta_1(t,u,v), 1\right\}, \qquad M_2 = \max\left\{\max_{\substack{t\in[0,1],\, u,v\geq 0,\\ u+v\leq R_1}} \beta_2(t,u,v), 1\right\}.$$

Let $\lambda \in (0, \lambda^*]$ and $\mu \in (0, \mu^*]$. Then for any $(x,y) \in P \cap \partial\Omega_1$ and $s \in [0,1]$, we have

$$[x(s) - q_1(s)]^* \leq x(s) \leq \|x\| \leq R_1, \qquad [y(s) - q_2(s)]^* \leq y(s) \leq \|y\| \leq R_1.$$

Then for any $(x,y) \in P \cap \partial\Omega_1$, we obtain

$$\begin{aligned}
\|T_1(x,y)\| &\leq \lambda\sigma_1 \int_0^1 h_1(s)[\alpha_1(s)\beta_1(s, [x(s) - q_1(s)]^*, [y(s) - q_2(s)]^*) + p_1(s)]\, ds \\
&\quad + \mu\sigma_2 \int_0^1 h_2(s)[\alpha_2(s)\beta_2(s, [x(s) - q_1(s)]^*, [y(s) - q_2(s)]^*) + p_2(s)]\, ds \\
&\leq \lambda^*\sigma_1 M_1 \int_0^1 h_1(s)(\alpha_1(s) + p_1(s))\, ds + \mu^*\sigma_2 M_2 \int_0^1 h_2(s)(\alpha_2(s) + p_2(s))\, ds \\
&\leq \frac{R_1}{2} + \frac{R_1}{2} = R_1 = \|(x,y)\|_Y, \\
\|T_2(x,y)\| &\leq \mu\sigma_3 \int_0^1 h_2(s)[\alpha_2(s)\beta_2(s, [x(s) - q_1(s)]^*, [y(s) - q_2(s)]^*) + p_2(s)]\, ds \\
&\quad + \lambda\sigma_4 \int_0^1 h_1(s)[\alpha_1(s)\beta_1(s, [x(s) - q_1(s)]^*, [y(s) - q_2(s)]^*) + p_1(s)]\, ds \\
&\leq \mu^*\sigma_3 M_2 \int_0^1 h_2(s)(\alpha_2(s) + p_2(s))\, ds + \lambda^*\sigma_4 M_1 \int_0^1 h_1(s)(\alpha_1(s) + p_1(s))\, ds \\
&\leq \frac{R_1}{2} + \frac{R_1}{2} = R_1 = \|(x,y)\|_Y.
\end{aligned}$$

Therefore

$$\|\mathcal{T}(x,y)\|_Y = \max\{\|T_1(x,y)\|, \|T_2(x,y)\|\} \leq \|(x,y)\|_Y, \qquad \forall\, (x,y) \in P \cap \partial\Omega_1. \tag{3.6}$$

On the other hand, for c given in $(H5)$, we choose a constant $L > 0$ such that

$$\lambda L \varrho_1 \gamma c^{2(\alpha-1)} \int_c^{1-c} h_1(s)\, ds \geq 2, \qquad \mu L \varrho_2 \gamma c^{2(\alpha-1)} \int_c^{1-c} h_2(s)\, ds \geq 2.$$

From $(H5)$, we deduce that there exists a constant $M_0 > 0$ such that

$$f(t,u,v) \geq L(u+v) \quad \text{or} \quad g(t,u,v) \geq L(u+v), \quad \forall\, t \in [c, 1-c],\ u,v \geq 0,\ u+v \geq M_0. \tag{3.7}$$

Now we define

$$R_2 = \max\left\{2R_1, \frac{2M_0}{\gamma c^{\alpha-1}}, \frac{2}{\gamma}\int_0^1 (\delta_1 p_1(s) + \delta_2 p_2(s))\, ds\right\} > 0,$$

and let $\Omega_2 = \{(x,y) \in P,\ \|(x,y)\|_Y < R_2\}$.

We suppose that $f_\infty = \infty$, that is $f(t,u,v) \geq L(u+v)$ for all $t \in [c, 1-c]$ and $u,v \geq 0$, $u+v \geq M_0$. Then for any $(x,y) \in P \cap \partial\Omega_2$, we have $\|(x,y)\|_Y = R_2$ or $\max\{\|x\|, \|y\|\} = R_2$. In addition, for any $(x,y) \in P \cap \partial\Omega_2$, we obtain

$$\begin{aligned} x(t) - q_1(t) &= x(t) - \lambda \int_0^1 G_1(t,s) p_1(s)\, ds - \mu \int_0^1 G_2(t,s) p_2(s)\, ds \\ &\geq x(t) - t^{\alpha-1}\left(\delta_1 \int_0^1 p_1(s)\, ds + \delta_2 \int_0^1 p_2(s)\, ds\right) \\ &\geq x(t) - \frac{x(t)}{\gamma \|(x,y)\|_Y} \int_0^1 (\delta_1 p_1(s) + \delta_2 p_2(s))\, ds \\ &= x(t)\left[1 - \frac{1}{\gamma R_2}\int_0^1 (\delta_1 p_1(s) + \delta_2 p_2(s))\, ds\right] \geq \frac{1}{2}x(t) \geq 0, \qquad \forall\, t \in [0,1]. \end{aligned}$$

Therefore, we conclude

$$\begin{aligned} [x(t) - q_1(t)]^* = x(t) - q_1(t) &\geq \frac{1}{2}x(t) \geq \frac{1}{2}\gamma t^{\alpha-1}\|(x,y)\|_Y \\ &= \frac{1}{2}\gamma t^{\alpha-1} R_2 \geq \frac{1}{2}\gamma c^{\alpha-1} R_2 \geq M_0, \qquad \forall\, t \in [c, 1-c]. \end{aligned}$$

Hence

$$[x(t) - q_1(t)]^* + [y(t) - q_2(t)]^* \geq [x(t) - q_1(t)]^* = x(t) - q_1(t) \geq M_0, \qquad \forall\, t \in [c, 1-c]. \tag{3.8}$$

Then, for any $(x,y) \in P \cap \partial\Omega_2$ and $t \in [c, 1-c]$, by (3.7) and (3.8), we deduce

$$\begin{aligned} f(t, [x(t) - q_1(t)]^*, [y(t) - q_2(t)]^*) &\geq L([x(t) - q_1(t)]^* + [y(t) - q_2(t)]^*) \\ &\geq L[x(t) - q_1(t)]^* \geq \frac{L}{2}x(t), \qquad \forall\, t \in [c, 1-c]. \end{aligned}$$

It follows that for any $(x,y) \in P \cap \partial\Omega_2$, $t \in [c, 1-c]$, we obtain

$$\begin{aligned} T_1(x,y)(t) &\geq \lambda \int_0^1 G_1(t,s)(f(s, [x(s) - q_1(s)]^*, [y(s) - q_2(s)]^*) + p_1(s))\, ds \\ &\geq \lambda \int_c^{1-c} G_1(t,s)(f(s, [x(s) - q_1(s)]^*, [y(s) - q_2(s)]^*) + p_1(s))\, ds \\ &\geq \lambda \int_c^{1-c} G_1(t,s) L([x(s) - q_1(s)]^*)\, ds \geq \lambda \int_c^{1-c} G_1(t,s) \frac{1}{2} L \gamma c^{\alpha-1} R_2\, ds \\ &\geq \lambda \int_c^{1-c} \varrho_1 t^{\alpha-1} h_1(s) \frac{1}{2} L \gamma c^{\alpha-1} R_2\, ds \geq \lambda c^{2(\alpha-1)} \frac{1}{2} \varrho_1 L \gamma R_2 \int_c^{1-c} h_1(s)\, ds \geq R_2. \end{aligned}$$

Then $\|T_1(x,y)\| \geq \|(x,y)\|_Y$ and

$$\|\mathcal{T}(x,y)\|_Y \geq \|(x,y)\|_Y, \qquad \forall\,(x,y) \in P \cap \partial\Omega_2. \tag{3.9}$$

We suppose now that $g_\infty = \infty$, that is $g(t,u,v) \geq L(u+v)$, for all $t \in [c, 1-c]$ and $u, v \geq 0$, $u + v \geq M_0$. Then for any $(x,y) \in P \cap \partial\Omega_2$, we have $\|(x,y)\|_Y = R_2$ or $\max\{\|x\|, \|y\|\} = R_2$. In addition, for any $(x,y) \in P \cap \partial\Omega_2$, we deduce in a similar manner as above that $x(t) - q_1(t) \geq \frac{1}{2}x(t)$ for all $t \in [0,1]$, $[x(t) - q_1(t)]^* \geq \frac{1}{2}\gamma c^{\alpha-1}R_2$ for all $t \in [c, 1-c]$, and

$$\begin{aligned}
T_1(x,y)(t) &\geq \mu \int_0^1 G_2(t,s)(g(s,[x(s)-q_1(s)]^*,[y(s)-q_2(s)]^*) + p_2(s))\,ds \\
&\geq \mu \int_c^{1-c} G_2(t,s)(g(s,[x(s)-q_1(s)]^*,[y(s)-q_2(s)]^*) + p_2(s))\,ds \\
&\geq \mu \int_c^{1-c} G_2(t,s)L([x(s)-q_1(s)]^*)\,ds \geq \mu \int_c^{1-c} G_2(t,s)\frac{1}{2}L\gamma c^{\alpha-1}R_2\,ds \\
&\geq \mu \int_c^{1-c} \varrho_2 t^{\alpha-1}h_2(s)\frac{1}{2}L\gamma c^{\alpha-1}R_2\,ds \\
&\geq \mu c^{2(\alpha-1)}\frac{1}{2}\varrho_2 L\gamma R_2 \int_c^{1-c} h_2(s)\,ds \geq R_2, \qquad \forall\, t \in [c, 1-c].
\end{aligned}$$

Hence we obtain relation (3.9).

Therefore, by Theorem 2.11, relations (3.6) and (3.9), we conclude that $\mathcal{T}$ has a fixed point $(x_1,y_1) \in P \cap (\overline{\Omega_2} \setminus \Omega_1)$, that is $R_1 \leq \|(x_1,y_1)\|_Y \leq R_2$.

Then we deduce

$$\begin{aligned}
x_1(t) - q_1(t) &= x_1(t) - \lambda \int_0^1 G_1(t,s)p_1(s)\,ds - \mu \int_0^1 G_2(t,s)p_2(s)\,ds \\
&\geq x_1(t) - t^{\alpha-1}\left(\delta_1 \int_0^1 p_1(s)\,ds + \delta_2 \int_0^1 p_2(s)\,ds\right) \\
&\geq x_1(t) - \frac{x_1(t)}{\gamma\|(x_1,y_1)\|_Y} \int_0^1 (\delta_1 p_1(s) + \delta_2 p_2(s))\,ds \\
&\geq \left[1 - \frac{1}{\gamma R_1}\int_0^1 (\delta_1 p_1(s) + \delta_2 p_2(s))\,ds\right] x_1(t) \\
&\geq \left[1 - \frac{1}{\gamma R_1}\int_0^1 (\delta_1 p_1(s) + \delta_2 p_2(s))\,ds\right] \gamma t^{\alpha-1}\|(x_1,y_1)\|_Y \\
&\geq R_1\left[1 - \frac{1}{\gamma R_1}\int_0^1 (\delta_1 p_1(s) + \delta_2 p_2(s))\,ds\right] \gamma t^{\alpha-1} = \Lambda_1 t^{\alpha-1}, \qquad \forall\, t \in [0,1],
\end{aligned}$$

and so $x_1(t) \geq q_1(t) + \Lambda_1 t^{\alpha-1}$ for all $t \in [0,1]$, where $\Lambda_1 = \gamma R_1 - \int_0^1 (\delta_1 p_1(s) + \delta_2 p_2(s))\,ds > 0$.

We also obtain

$$\begin{aligned}
y_1(t) - q_2(t) &= y_1(t) - \mu \int_0^1 G_3(t,s)p_2(s)\,ds - \lambda \int_0^1 G_4(t,s)p_1(s)\,ds \\
&\geq y_1(t) - t^{\beta-1}\left(\delta_3 \int_0^1 p_2(s)\,ds + \delta_4 \int_0^1 p_1(s)\,ds\right) \\
&\geq y_1(t) - \frac{y_1(t)}{\gamma\|(x_1,y_1)\|_Y} \int_0^1 (\delta_3 p_2(s) + \delta_4 p_1(s))\,ds \\
&\geq \left[1 - \frac{1}{\gamma R_1}\int_0^1 (\delta_3 p_2(s) + \delta_4 p_1(s))\,ds\right] y_1(t)
\end{aligned}$$

$$\geq \left[1 - \frac{1}{\gamma R_1}\int_0^1 (\delta_3 p_2(s) + \delta_4 p_1(s))\,ds\right]\gamma t^{\beta-1}\|(x_1,y_1)\|_Y$$
$$\geq R_1\left[1 - \frac{1}{\gamma R_1}\int_0^1 (\delta_3 p_2(s) + \delta_4 p_1(s))\,ds\right]\gamma t^{\beta-1} = \Lambda_2 t^{\beta-1}, \qquad \forall\, t \in [0,1],$$

and so $y_1(t) \geq q_2(t) + \Lambda_2 t^{\beta-1}$ for all $t \in [0,1]$, where $\Lambda_2 = \gamma R_1 - \int_0^1 (\delta_3 p_2(s) + \delta_4 p_1(s))\,ds > 0$.

Let $u_1(t) = x_1(t) - q_1(t)$ and $v_1(t) = y_1(t) - q_2(t)$ for all $t \in [0,1]$. Then (u_1, v_1) is a positive solution of (S)–(BC) with $u_1(t) \geq \Lambda_1 t^{\alpha-1}$ and $v_1(t) \geq \Lambda_2 t^{\beta-1}$ for all $t \in [0,1]$. This completes the proof of Theorem 3.3. □

Theorem 3.4. *Assume that* $(H1)$, $(H3)$, $(H5)$ *and*

$(H4')$ *The functions* $f, g \in C([0,1] \times [0,\infty) \times [0,\infty), (-\infty,+\infty))$ *and there exist functions* $p_1, p_2, \alpha_1, \alpha_2 \in C([0,1],[0,\infty))$, $\beta_1, \beta_2 \in C([0,1] \times [0,\infty) \times [0,\infty), [0,\infty))$ *such that*

$$-p_1(t) \leq f(t,u,v) \leq \alpha_1(t)\beta_1(t,u,v), \ \ -p_2(t) \leq g(t,u,v) \leq \alpha_2(t)\beta_2(t,u,v),$$

for all $t \in [0,1]$, $u, v \in [0,\infty)$, *with* $\int_0^1 p_i(s)\,ds > 0$, $i = 1,2$,

hold. Then the boundary value problem (S)–(BC) *has at least two positive solutions for* $\lambda > 0$ *and* $\mu > 0$ *sufficiently small.*

Proof. Because assumption $(H4')$ imply assumptions $(H2)$ and $(H4)$, we can apply Theorems 3.2 and 3.3. Therefore, we deduce that for $0 < \lambda \leq \min\{\lambda_0, \lambda^*\}$ and $0 < \mu \leq \min\{\mu_0, \mu^*\}$, problem (S)–(BC) has at least two positive solutions (u_0, v_0) and (u_1, v_1) with $\|(u_0 + q_1, v_0 + q_2)\|_Y \leq 1$ and $\|(u_1 + q_1, v_1 + q_2)\|_Y > 1$. □

Theorem 3.5. *Assume that* $\lambda = \mu$, *and* $(H1)$, $(H4)$ *and* $(H6)$ *hold. In addition if*

$(H7)$ *There exists* $c \in (0,1/2)$ *such that*

$$f_\infty^i = \liminf_{\substack{u+v\to\infty \\ u,v\geq 0}} \min_{t\in[c,1-c]} f(t,u,v) > L_0 \quad or \quad g_\infty^i = \liminf_{\substack{u+v\to\infty \\ u,v\geq 0}} \min_{t\in[c,1-c]} g(t,u,v) > L_0,$$

where

$$L_0 = \max\left\{\frac{2}{\gamma}\int_0^1 (\delta_1 p_1(s) + \delta_2 p_2(s))ds, \frac{2}{\gamma}\int_0^1 (\delta_3 p_2(s) + \delta_4 p_1(s))ds\right\}$$
$$\times \left(\min\left\{c^{\alpha-1}\varrho_1\int_c^{1-c} h_1(s)\,ds, c^{\alpha-1}\varrho_2\int_c^{1-c} h_2(s)\,ds\right\}\right)^{-1},$$

then there exists $\lambda_* > 0$ *such that, for any* $\lambda \geq \lambda_*$, *problem* (S)–(BC) *(with* $\lambda = \mu$*) has at least one positive solution.*

Proof. By $(H7)$ we conclude that there exists $M_3 > 0$ such that

$$f(t,u,v) \geq L_0 \ \text{ or } \ g(t,u,v) \geq L_0, \ \ \forall\, t \in [c,1-c], \ \ u,v \geq 0, \ \ u+v \geq M_3.$$

We define

$$\lambda_* = \max\left\{\frac{M_3}{c^{\alpha-1}}\left(\int_0^1 (\delta_1 p_1(s) + \delta_2 p_2(s))\,ds\right)^{-1}, \frac{M_3}{c^{\beta-1}}\left(\int_0^1 (\delta_3 p_2(s) + \delta_4 p_1(s))\,ds\right)^{-1}\right\}.$$

We assume now $\lambda \geq \lambda_*$. Let

$$R_3 = \max\left\{\frac{2\lambda}{\gamma}\int_0^1 (\delta_1 p_1(s) + \delta_2 p_2(s))ds,\ \frac{2\lambda}{\gamma}\int_0^1 (\delta_3 p_2(s) + \delta_4 p_1(s))ds\right\}$$

and $\Omega_3 = \{(x,y) \in P,\ \|(x,y)\|_Y < R_3\}$.

We suppose first that $f_\infty^i > L_0$, that is $f(t,u,v) \geq L_0$ for all $t \in [c, 1-c]$ and $u, v \geq 0$, $u + v \geq M_3$. Let $(x,y) \in P \cap \partial\Omega_3$, that is $\|(x,y)\|_Y = R_3$. Then for all $t \in [0,1]$, we deduce

$$\begin{aligned}
x(t) - q_1(t) &\geq \gamma t^{\alpha-1}\|(x,y)\|_Y - \lambda t^{\alpha-1}\delta_1 \int_0^1 p_1(s)\,ds - \lambda t^{\alpha-1}\delta_2 \int_0^1 p_2(s)\,ds \\
&\geq t^{\alpha-1}\left[\gamma R_3 - \lambda\int_0^1 (\delta_1 p_1(s) + \delta_2 p_2(s))\,ds\right] \\
&\geq t^{\alpha-1}\left[2\lambda\int_0^1 (\delta_1 p_1(s) + \delta_2 p_2(s))\,ds - \lambda\int_0^1 (\delta_1 p_1(s) + \delta_2 p_2(s))\,ds\right] \\
&= t^{\alpha-1}\lambda\int_0^1 (\delta_1 p_1(s) + \delta_2 p_2(s))\,ds \\
&\geq t^{\alpha-1}\lambda_*\int_0^1 (\delta_1 p_1(s) + \delta_2 p_2(s))\,ds \geq \frac{M_3}{c^{\alpha-1}}t^{\alpha-1} \geq 0.
\end{aligned}$$

Therefore for any $(x,y) \in P \cap \partial\Omega_3$ and $t \in [c, 1-c]$, we have

$$[x(t) - q_1(t)]^* + [y(t) - q_2(t)]^* \geq [x(t) - q_1(t)]^* = x(t) - q_1(t) \geq \frac{M_3}{c^{\alpha-1}}t^{\alpha-1} \geq M_3. \tag{3.10}$$

Hence, for any $(x,y) \in P \cap \partial\Omega_3$ and $t \in [c, 1-c]$, we conclude

$$\begin{aligned}
T_1(x,y)(t) &\geq \lambda\int_0^1 G_1(t,s)[f(s,[x(s) - q_1(s)]^*,[y(s) - q_2(s)]^*) + p_1(s)]\,ds \\
&\geq \lambda\varrho_1 t^{\alpha-1}\int_c^{1-c} h_1(s)f(s,[x(s) - q_1(s)]^*,[y(s) - q_2(s)]^*)\,ds \\
&\geq \lambda L_0\varrho_1 t^{\alpha-1}\int_c^{1-c} h_1(s)\,ds \geq \lambda L_0\varrho_1 c^{\alpha-1}\int_c^{1-c} h_1(s)\,ds \geq R_3 = \|(x,y)\|_Y.
\end{aligned}$$

Therefore we obtain $\|T_1(x,y)\| \geq R_3$ for all $(x,y) \in P \cap \partial\Omega_3$, and so

$$\|\mathcal{T}(x,y)\|_Y \geq R_3 = \|(x,y)\|_Y,\ \ \forall\,(x,y) \in P \cap \partial\Omega_3. \tag{3.11}$$

We suppose now that $g_\infty^i > L_0$, that is $g(t,u,v) \geq L_0$ for all $t \in [c, 1-c]$ and $u, v \geq 0$, $u + v \geq M_3$. Let $(x,y) \in P \cap \partial\Omega_3$, that is $\|(x,y)\|_Y = R_3$. Then we obtain in a similar manner as in the first case above ($f_\infty^i > L_0$) that $x(t) - q_1(t) \geq \frac{M_3}{c^{\alpha-1}}t^{\alpha-1} \geq 0$ for all $t \in [0,1]$.

Therefore for any $(x,y) \in P \cap \partial\Omega_3$ and $t \in [c, 1-c]$, we deduce inequalities (3.10).

Hence, for any $(x,y) \in P \cap \partial\Omega_3$ and $t \in [c, 1-c]$, we conclude

$$\begin{aligned}
T_1(x,y)(t) &\geq \lambda\int_0^1 G_2(t,s)[g(s,[x(s) - q_1(s)]^*,[y(s) - q_2(s)]^*) + p_2(s)]\,ds \\
&\geq \lambda\varrho_2 t^{\alpha-1}\int_c^{1-c} h_2(s)g(s,[x(s) - q_1(s)]^*,[y(s) - q_2(s)]^*)\,ds \\
&\geq \lambda L_0\varrho_2 t^{\alpha-1}\int_c^{1-c} h_2(s)\,ds \geq \lambda L_0\varrho_2 c^{\alpha-1}\int_c^{1-c} h_2(s)\,ds \geq R_3 = \|(x,y)\|_Y.
\end{aligned}$$

Therefore we obtain $\|T_1(x,y)\| \geq R_3$ for all $(x,y) \in P \cap \partial\Omega_3$, and so we have again relation (3.11).

On the other hand, we consider the positive number

$$\varepsilon = \min\left\{ \frac{1}{4\lambda\sigma_1}\left(\int_0^1 h_1(s)\alpha_1(s)\,ds\right)^{-1}, \frac{1}{4\lambda\sigma_2}\left(\int_0^1 h_2(s)\alpha_2(s)\,ds\right)^{-1}, \right.$$
$$\left. \frac{1}{4\lambda\sigma_3}\left(\int_0^1 h_2(s)\alpha_2(s)\,ds\right)^{-1}, \frac{1}{4\lambda\sigma_4}\left(\int_0^1 h_1(s)\alpha_1(s)\,ds\right)^{-1} \right\}.$$

Then by $(H6)$ we deduce that there exists $M_4 > 0$ such that

$$\beta_i(t,u,v) \le \varepsilon(u+v) \ \ \forall\, t \in [0,1],\ \ u,\, v \ge 0,\ \ u+v \ge M_4,\ \ i = 1,2.$$

Therefore we obtain

$$\beta_i(t,u,v) \le M_5 + \varepsilon(u+v),\ \ \forall\, t \in [0,1],\ \ u,\, v \ge 0,\ \ i = 1,2,$$

where

$$M_5 = \max_{i=1,2}\left\{ \max_{t\in[0,1],\, u,v\ge 0,\, u+v\le M_4} \beta_i(t,u,v) \right\}.$$

We define now

$$\begin{aligned} R_4 = \ \max\Big\{ & 2R_3, 4\lambda\sigma_1 \max\{M_5,1\} \int_0^1 h_1(s)(\alpha_1(s)+p_1(s))\,ds, \\ & 4\lambda\sigma_2 \max\{M_5,1\} \int_0^1 h_2(s)(\alpha_2(s)+p_2(s))\,ds, \\ & 4\lambda\sigma_3 \max\{M_5,1\} \int_0^1 h_2(s)(\alpha_2(s)+p_2(s))\,ds, \\ & 4\lambda\sigma_4 \max\{M_5,1\} \int_0^1 h_1(s)(\alpha_1(s)+p_1(s))\,ds \Big\}, \end{aligned}$$

and let $\Omega_4 = \{(x,y) \in P,\ \|(x,y)\|_Y < R_4\}$.

For any $(x,y) \in P \cap \partial\Omega_4$, we have

$$\begin{aligned} T_1(x,y)(t) \le\ & \lambda \int_0^1 \sigma_1 h_1(s)[\alpha_1(s)\beta_1(s,[x(s)-q_1(s)]^*,[y(s)-q_2(s)]^*) + p_1(s)]ds \\ & + \lambda \int_0^1 \sigma_2 h_2(s)[\alpha_2(s)\beta_2(s,[x(s)-q_1(s)]^*,[y(s)-q_2(s)]^*) + p_2(s)]ds \\ \le\ & \lambda\sigma_1 \int_0^1 h_1(s)[\alpha_1(s)(M_5 + \varepsilon([x(s)-q_1(s)]^* + [y(s)-q_2(s)]^*)) + p_1(s)]\,ds \\ & + \lambda\sigma_2 \int_0^1 h_2(s)[\alpha_2(s)(M_5 + \varepsilon([x(s)-q_1(s)]^* + [y(s)-q_2(s)]^*)) + p_2(s)]\,ds \\ \le\ & \lambda\sigma_1 \max\{M_5,1\} \int_0^1 h_1(s)(\alpha_1(s)+p_1(s))\,ds + \lambda\sigma_1\varepsilon R_4 \int_0^1 h_1(s)\alpha_1(s)\,ds \\ & + \lambda\sigma_2 \max\{M_5,1\} \int_0^1 h_2(s)(\alpha_2(s)+p_2(s))\,ds + \lambda\sigma_2\varepsilon R_4 \int_0^1 h_2(s)\alpha_2(s)\,ds \\ \le\ & \frac{R_4}{4} + \frac{R_4}{4} + \frac{R_4}{4} + \frac{R_4}{4} = R_4 = \|(x,y)\|_Y, \qquad \forall\, t \in [0,1], \end{aligned}$$

and so $\|T_1(x,y)\| \le \|(x,y)\|_Y$ for all $(x,y) \in P \cap \partial\Omega_4$.

In a similar manner we obtain

$$
\begin{aligned}
T_2(x,y)(t) &\le \lambda \int_0^1 \sigma_3 h_2(s)[\alpha_2(s)\beta_2(s,[x(s)-q_1(s)]^*,[y(s)-q_2(s)]^*)+p_2(s)]ds \\
&\quad + \lambda \int_0^1 \sigma_4 h_1(s)[\alpha_1(s)\beta_1(s,[x(s)-q_1(s)]^*,[y(s)-q_2(s)]^*)+p_1(s)]ds \\
&\le \lambda\sigma_3 \int_0^1 h_2(s)[\alpha_2(s)(M_5+\varepsilon([x(s)-q_1(s)]^*+[y(s)-q_2(s)]^*))+p_2(s)]\,ds \\
&\quad + \lambda\sigma_4 \int_0^1 h_1(s)[\alpha_1(s)(M_5+\varepsilon([x(s)-q_1(s)]^*+[y(s)-q_2(s)]^*))+p_1(s)]\,ds \\
&\le \lambda\sigma_3 \max\{M_5,1\} \int_0^1 h_2(s)(\alpha_2(s)+p_2(s))\,ds + \lambda\sigma_3\varepsilon R_4 \int_0^1 h_2(s)\alpha_2(s)\,ds \\
&\quad + \lambda\sigma_4 \max\{M_5,1\} \int_0^1 h_1(s)(\alpha_1(s)+p_1(s))\,ds + \lambda\sigma_4\varepsilon R_4 \int_0^1 h_1(s)\alpha_1(s)\,ds \\
&\le \frac{R_4}{4}+\frac{R_4}{4}+\frac{R_4}{4}+\frac{R_4}{4} = R_4 = \|(x,y)\|_Y, \qquad \forall\, t\in[0,1],
\end{aligned}
$$

and so $\|T_2(x,y)\| \le \|(x,y)\|_Y$ for all $(x,y)\in P\cap\partial\Omega_4$.

Therefore, we deduce

$$
\|\mathcal{T}(x,y)\|_Y \le \|(x,y)\|_Y, \qquad \forall\,(x,y)\in P\cap\partial\Omega_4. \tag{3.12}
$$

By Theorem 2.11, (3.11) and (3.12), we conclude that $\mathcal{T}$ has a fixed point $(x_1,y_1)\in P\cap(\overline{\Omega_4}\setminus\Omega_3)$, so $R_3\le\|(x_1,y_1)\|_Y\le R_4$. Therefore, we deduce that for all $t\in[0,1]$

$$
\begin{aligned}
x_1(t)-q_1(t) &= x_1(t) - \lambda\int_0^1 G_1(t,s)p_1(s)\,ds - \lambda\int_0^1 G_2(t,s)p_2(s)\,ds \\
&\ge x_1(t) - \lambda\delta_1\int_0^1 t^{\alpha-1}p_1(s)\,ds - \lambda\delta_2\int_0^1 t^{\alpha-1}p_2(s)\,ds \\
&\ge \gamma t^{\alpha-1}\|(x_1,y_1)\|_Y - \lambda t^{\alpha-1}\int_0^1(\delta_1p_1(s)+\delta_2p_2(s))\,ds \\
&\ge \gamma t^{\alpha-1}R_3 - \lambda t^{\alpha-1}\int_0^1(\delta_1p_1(s)+\delta_2p_2(s))\,ds \\
&\ge 2\lambda t^{\alpha-1}\int_0^1(\delta_1p_1(s)+\delta_2p_2(s))\,ds - \lambda t^{\alpha-1}\int_0^1(\delta_1p_1(s)+\delta_2p_2(s))\,ds \\
&= \lambda t^{\alpha-1}\int_0^1(\delta_1p_1(s)+\delta_2p_2(s))\,ds \ge \lambda_* t^{\alpha-1}\int_0^1(\delta_1p_1(s)+\delta_2p_2(s))\,ds \ge \frac{M_3}{c^{\alpha-1}}t^{\alpha-1},
\end{aligned}
$$

and

$$
\begin{aligned}
y_1(t)-q_2(t) &= y_1(t) - \lambda\int_0^1 G_3(t,s)p_2(s)\,ds - \lambda\int_0^1 G_4(t,s)p_1(s)\,ds \\
&\ge y_1(t) - \lambda\delta_3\int_0^1 t^{\beta-1}p_2(s)\,ds - \lambda\delta_4\int_0^1 t^{\beta-1}p_1(s)\,ds \\
&\ge \gamma t^{\beta-1}\|(x_1,y_1)\|_Y - \lambda t^{\beta-1}\int_0^1(\delta_3p_2(s)+\delta_4p_1(s))\,ds \\
&\ge \gamma t^{\beta-1}R_3 - \lambda t^{\beta-1}\int_0^1(\delta_3p_2(s)+\delta_4p_1(s))\,ds \\
&\ge 2\lambda t^{\beta-1}\int_0^1(\delta_3p_2(s)+\delta_4p_1(s))\,ds - \lambda t^{\beta-1}\int_0^1(\delta_3p_2(s)+\delta_4p_1(s))\,ds \\
&= \lambda t^{\beta-1}\int_0^1(\delta_3p_2(s)+\delta_4p_1(s))\,ds \ge \lambda_* t^{\beta-1}\int_0^1(\delta_3p_2(s)+\delta_4p_1(s))\,ds \ge \frac{M_3}{c^{\beta-1}}t^{\beta-1}.
\end{aligned}
$$

Let $u_1(t) = x_1(t) - q_1(t)$ and $v_1(t) = y_1(t) - q_2(t)$ for all $t \in [0,1]$. Then $u_1(t) \geq \widetilde{\Lambda}_1 t^{\alpha-1}$ and $v_1(t) \geq \widetilde{\Lambda}_2 t^{\beta-1}$ for all $t \in [0,1]$, where $\widetilde{\Lambda}_1 = \frac{M_3}{c^{\alpha-1}}$, $\widetilde{\Lambda}_2 = \frac{M_3}{c^{\beta-1}}$. Hence we deduce that (u_1, v_1) is a positive solution of (S)–(BC), which completes the proof of Theorem 3.5. □

In a similar manner as we proved Theorem 3.5, we obtain the following theorems.

Theorem 3.6. *Assume that $\lambda = \mu$, and $(H1)$, $(H4)$ and $(H6)$ hold. In addition if*

$(\widetilde{H7})$ *there exists $c \in (0, 1/2)$ such that*

$$f^i_\infty = \liminf_{\substack{u+v\to\infty\\ u,v\geq 0}} \min_{t\in[c,1-c]} f(t,u,v) > \widetilde{L}_0 \quad \text{or} \quad g^i_\infty = \liminf_{\substack{u+v\to\infty\\ u,v\geq 0}} \min_{t\in[c,1-c]} g(t,u,v) > \widetilde{L}_0,$$

where

$$\begin{aligned}\widetilde{L}_0 = \ &\max\left\{\frac{2}{\gamma}\int_0^1 (\delta_1 p_1(s) + \delta_2 p_2(s))ds,\ \frac{2}{\gamma}\int_0^1 (\delta_3 p_2(s) + \delta_4 p_1(s))ds\right\}\\ &\times\left(\min\left\{c^{\beta-1}\varrho_4 \int_c^{1-c} h_1(s)\,ds,\ c^{\beta-1}\varrho_3 \int_c^{1-c} h_2(s)\,ds\right\}\right)^{-1},\end{aligned}$$

then there exists $\widetilde{\lambda}_* > 0$ such that for any $\lambda \geq \widetilde{\lambda}_*$ problem (S)–(BC) (with $\lambda = \mu$) has at least one positive solution.

Theorem 3.7. *Assume that $\lambda = \mu$, and $(H1)$, $(H4)$ and $(H6)$ hold. In addition if*

$(H7')$ *there exists $c \in (0, 1/2)$ such that*

$$f^i_\infty = \liminf_{\substack{u+v\to\infty\\ u,v\geq 0}} \min_{t\in[c,1-c]} f(t,u,v) > L'_0 \quad \text{or} \quad g^i_\infty = \liminf_{\substack{u+v\to\infty\\ u,v\geq 0}} \min_{t\in[c,1-c]} g(t,u,v) > L'_0,$$

where

$$\begin{aligned}L'_0 = \ &\max\left\{\frac{2}{\gamma}\int_0^1 (\delta_1 p_1(s) + \delta_2 p_2(s))ds, \frac{2}{\gamma}\int_0^1 (\delta_3 p_2(s) + \delta_4 p_1(s))ds\right\}\\ &\times\left(\min\left\{c^{\alpha-1}\varrho_1 \int_c^{1-c} h_1(s)\,ds,\ c^{\beta-1}\varrho_3 \int_c^{1-c} h_2(s)\,ds\right\}\right)^{-1},\end{aligned}$$

then there exists $\lambda'_* > 0$ such that for any $\lambda \geq \lambda'_*$ problem (S)–(BC) (with $\lambda = \mu$) has at least one positive solution.

Theorem 3.8. *Assume that $\lambda = \mu$, and $(H1)$, $(H4)$ and $(H6)$ hold. In addition if*

$(H7'')$ *there exists $c \in (0, 1/2)$ such that*

$$f^i_\infty = \liminf_{\substack{u+v\to\infty\\ u,v\geq 0}} \min_{t\in[c,1-c]} f(t,u,v) > L''_0 \quad \text{or} \quad g^i_\infty = \liminf_{\substack{u+v\to\infty\\ u,v\geq 0}} \min_{t\in[c,1-c]} g(t,u,v) > L''_0,$$

where

$$\begin{aligned}L''_0 = \ &\max\left\{\frac{2}{\gamma}\int_0^1 (\delta_1 p_1(s) + \delta_2 p_2(s))ds,\ \frac{2}{\gamma}\int_0^1 (\delta_3 p_2(s) + \delta_4 p_1(s))ds\right\}\\ &\times\left(\min\left\{c^{\beta-1}\varrho_4 \int_c^{1-c} h_1(s)\,ds,\ c^{\alpha-1}\varrho_2 \int_c^{1-c} h_2(s)\,ds\right\}\right)^{-1},\end{aligned}$$

then there exists $\lambda''_* > 0$ such that for any $\lambda \geq \lambda''_*$ problem (S)–(BC) (with $\lambda = \mu$) has at least one positive solution.

Theorem 3.9. *Assume that $\lambda = \mu$, and $(H1)$, $(H4)$ and $(H6)$ hold. In addition if*

$(H8)$ *there exists $c \in (0, 1/2)$ such that*

$$\hat{f}_\infty = \lim_{\substack{u+v\to\infty \\ u,v\geq 0}} \min_{t\in[c,1-c]} f(t,u,v) = \infty \quad or \quad \hat{g}_\infty = \lim_{\substack{u+v\to\infty \\ u,v\geq 0}} \min_{t\in[c,1-c]} g(t,u,v) = \infty,$$

then there exists $\hat{\lambda}_* > 0$ such that for any $\lambda \geq \hat{\lambda}_*$ problem (S)–(BC) (with $\lambda = \mu$) has at least one positive solution.

4 Examples

Let $\alpha = 7/3$ $(n = 3)$, $\beta = 5/2$ $(m = 3)$, $H(t) = t^3$, $K(t) = t^4$. Then $\int_0^1 u(s)dK(s) = 4\int_0^1 s^3u(s)\,ds$ and $\int_0^1 v(s)dH(s) = 3\int_0^1 s^2v(s)\,ds$.

We consider the system of fractional differential equations

$$\begin{cases} D_{0+}^{7/3}u(t) + \lambda f(t,u(t),v(t)) = 0, & t \in (0,1), \\ D_{0+}^{5/2}v(t) + \mu g(t,u(t),v(t)) = 0, & t \in (0,1), \end{cases} \tag{S_0}$$

with the boundary conditions

$$\begin{cases} u(0) = u'(0) = 0, \ \ u'(1) = 3\displaystyle\int_0^1 s^2v(s)\,ds, \\ v(0) = v'(0) = 0, \ \ v'(1) = 4\displaystyle\int_0^1 s^3u(s)\,ds. \end{cases} \tag{BC_0}$$

Then we obtain that $\Delta = (\alpha-1)(\beta-1) - \left(\int_0^1 s^{\alpha-1}dK(s)\right)\left(\int_0^1 s^{\beta-1}dH(s)\right) = \frac{3}{2} > 0$, and $\int_0^1 \tau^{\alpha-1}\,dK(\tau) = \frac{3}{4} > 0$, $\int_0^1 \tau^{\beta-1}dH(\tau) = \frac{2}{3} > 0$. The functions H and K are nondecreasing, and so assumption $(H1)$ is satisfied. Besides, we deduce

$$g_1(t,s) = \frac{1}{\Gamma(7/3)} \begin{cases} t^{4/3}(1-s)^{1/3} - (t-s)^{4/3}, & 0 \leq s \leq t \leq 1, \\ t^{4/3}(1-s)^{1/3}, & 0 \leq t \leq s \leq 1, \end{cases}$$

$$g_2(t,s) = \frac{4}{3\sqrt{\pi}} \begin{cases} t^{3/2}(1-s)^{1/2} - (t-s)^{3/2}, & 0 \leq s \leq t \leq 1, \\ t^{3/2}(1-s)^{1/2}, & 0 \leq t \leq s \leq 1, \end{cases}$$

$$G_1(t,s) = g_1(t,s) + \frac{16}{9}t^{4/3}\int_0^1 \tau^3 g_1(\tau,s)\,ds, \qquad G_2(t,s) = 3t^{4/3}\int_0^1 \tau^2 g_2(\tau,s)\,d\tau,$$

$$G_3(t,s) = g_2(t,s) + \frac{3}{2}t^{3/2}\int_0^1 \tau^2 g_2(\tau,s)\,d\tau, \qquad G_4(t,s) = \frac{32}{9}t^{3/2}\int_0^1 \tau^3 g_1(\tau,s)\,d\tau.$$

We also obtain $h_1(s) = \frac{1}{\Gamma(7/3)}s(1-s)^{1/3}$ and $h_2(s) = \frac{4}{3\sqrt{\pi}}s(1-s)^{1/2}$ for all $s \in [0,1]$.

In addition, we have $\sigma_1 = \frac{13}{9}$, $\delta_1 = \frac{4}{3\Gamma(7/3)}$, $\varrho_1 = \frac{4}{3}$, $\sigma_2 = 1$, $\delta_2 = \frac{8}{9\sqrt{\pi}}$, $\varrho_2 = \frac{2}{3}$, $\sigma_3 = \frac{3}{2}$, $\delta_3 = \frac{16}{9\sqrt{\pi}}$, $\varrho_3 = \frac{4}{3}$, $\sigma_4 = \frac{8}{9}$, $\delta_4 = \frac{1}{2\Gamma(4/3)}$, $\varrho_4 = \frac{2}{3}$, $\gamma = \frac{4}{9}$.

Example 4.1. We consider the functions

$$f(t,u,v) = \frac{(u+v)^2}{\sqrt{t(1-t)}} + \ln t, \quad g(t,u,v) = \frac{2+\sin(u+v)}{\sqrt{t(1-t)}} + \ln(1-t), \quad t \in (0,1), \ u,v \geq 0.$$

We have $p_1(t) = -\ln t$, $p_2(t) = -\ln(1-t)$, $\alpha_1(t) = \alpha_2(t) = \frac{1}{\sqrt{t(1-t)}}$ for all $t \in (0,1)$, $\beta_1(t,u,v) = (u+v)^2$, $\beta_2(t,u,v) = 2+\sin(u+v)$ for all $t \in [0,1]$, $u, v \geq 0$, $\int_0^1 p_1(t)\,dt = 1$, $\int_0^1 p_2(t)\,dt = 1$, $\int_0^1 \alpha_i(t)\,dt = \pi$, $i = 1,2$. Therefore, assumption $(H4)$ is satisfied. In addition, for $c \in (0,1/2)$ fixed, assumption $(H5)$ is also satisfied ($f_\infty = \infty$).

After some computations, we deduce $\int_0^1(\delta_1 p_1(s) + \delta_2 p_2(s)\,ds \approx 1.62134837$, $\int_0^1(\delta_3 p_2(s) + \delta_4 p_1(s))\,ds \approx 1.56292696$, $\int_0^1 h_1(s)(\alpha_1(s)+p_1(s))\,ds \approx 0.87405192$, $\int_0^1 h_2(s)(\alpha_2(s)+p_2(s))\,ds \approx 0.71547597$. We choose $R_1 = 4$ which satisfies the condition from the beginning of the proof of Theorem 3.3. Then $M_1 = 16$, $M_2 = 3$, $\lambda^* \approx 0.0990084$ and $\mu^* \approx 0.6211871$. By Theorem 3.3, we conclude that (S_0)–(BC_0) has at least one positive solution for any $\lambda \in (0,\lambda^*]$ and $\mu \in (0,\mu^*]$.

Example 4.2. We consider the functions

$$f(t,u,v) = (u+v)^3 + \cos u, \qquad g(t,u,v) = (u+v)^{1/3} + \cos v, \qquad t \in [0,1],\ u,\ v \geq 0.$$

We have $p_1(t) = p_2(t) = 1$ for all $t \in [0,1]$, and then assumption $(H2)$ is satisfied. Besides, assumption $(H3)$ is also satisfied, because $f(t,0,0) = 1$ and $g(t,0,0) = 1$ for all $t \in [0,1]$.

Let $\delta = \frac{1}{2} < 1$ and $R_0 = 1$. Then

$$f(t,u,v) \geq \delta f(t,0,0) = \frac{1}{2}, \qquad g(t,u,v) \geq \delta g(t,0,0) = \frac{1}{2}, \qquad \forall\, t \in [0,1],\ u,\ v \in [0,1].$$

In addition,

$$\bar{f}(R_0) = \bar{f}(1) = \max_{t\in[0,1],\ u,v\in[0,1]} \{f(t,u,v) + p_1(t)\} \approx 9.5403023,$$
$$\bar{g}(R_0) = \bar{g}(1) = \max_{t\in[0,1],\ u,v\in[0,1]} \{g(t,u,v) + p_2(t)\} \approx 3.04684095.$$

We also obtain $c_1 \approx 0.38994655$, $c_2 \approx 0.20060074$, $c_3 \approx 0.30090111$, $c_4 \approx 0.23996711$, and then $\lambda_0 = \max\left\{\frac{R_0}{4c_1\bar{f}(R_0)}, \frac{R_0}{4c_4\bar{f}(R_0)}\right\} \approx 0.10920088$ and $\mu_0 = \max\left\{\frac{R_0}{4c_2\bar{g}(R_0)}, \frac{R_0}{4c_3\bar{g}(R_0)}\right\} \approx 0.40903238$.

By Theorem 3.2, for any $\lambda \in (0,\lambda_0]$ and $\mu \in (0,\mu_0]$, we deduce that problem (S_0)–(BC_0) has at least one positive solution.

Because assumption $(H4')$ is satisfied ($\alpha_1(t) = \alpha_2(t) = 1$, $\beta_1(t,u,v) = (u+v)^3+1$, $\beta_2(t,u,v) = (u+v)^{1/3}+1$ for all $t \in [0,1]$, $u,v \geq 0$) and assumption $(H5)$ is also satisfied ($f_\infty = \infty$), by Theorem 3.4 we conclude that problem (S_0)–(BC_0) has at least two positive solutions for λ and μ sufficiently small.

Example 4.3. We consider $\lambda = \mu$ and the functions

$$f(t,u,v) = \frac{(u+v)^a}{\sqrt[3]{t^2(1-t)}} - \frac{1}{\sqrt{t}}, \qquad g(t,u,v) = \frac{\ln(1+u+v)}{\sqrt[3]{t(1-t)^2}} - \frac{1}{\sqrt{1-t}}, \qquad t \in (0,1),\ u,v \geq 0,$$

where $a \in (0,1)$.

Here we have $p_1(t) = \frac{1}{\sqrt{t}}$, $p_2(t) = \frac{1}{\sqrt{1-t}}$, $\alpha_1(t) = \frac{1}{\sqrt[3]{t^2(1-t)}}$, $\alpha_2(t) = \frac{1}{\sqrt[3]{t(1-t)^2}}$ for all $t \in (0,1)$, $\beta_1(t,u,v) = (u+v)^a$, $\beta_2(t,u,v) = \ln(1+u+v)$ for all $t \in [0,1]$, $u,v \geq 0$. For $c \in (0,1/2)$ fixed, the assumptions $(H4)$, $(H6)$ and $(H8)$ are satisfied ($\beta_{i\infty} = 0$ for $i = 1,2$ and $\hat{f}_\infty = \infty$).

Then by Theorem 3.9, we deduce that there exists $\hat{\lambda}_* > 0$ such that for any $\lambda \geq \hat{\lambda}_*$ our problem (S_0)–(BC_0) (with $\lambda = \mu$) has at least one positive solution.

Acknowledgements

The authors thank the referee for his/her valuable comments and suggestions. The work of R. Luca was supported by the CNCS grant PN-II-ID-PCE-2011-3-0557, Romania.

References

[1] R. P. Agarwal, M. Meehan, D. O'Regan, *Fixed point theory and applications*, Cambridge University Press, Cambridge, 2001. MR1825411

[2] J. Caballero, I. Cabrera, K. Sadarangani, Positive solutions of nonlinear fractional differential equations with integral boundary value conditions, *Abstr. Appl. Anal.* **2012**, Art. ID 303545, 1–11. MR2984028

[3] S. Das, *Functional fractional calculus for system identification and controls*, Springer, Berlin, 2008. MR2414740

[4] J. R. Graef, L. Kong, Q. Kong, M. Wang, Uniqueness of positive solutions of fractional boundary value problems with non-homogeneous integral boundary conditions, *Fract. Calc. Appl. Anal.* **15**(2012), No. 3, 509–528. MR2944114

[5] D. Guo, V. Lakshmikantham, *Nonlinear problems in abstract cones*, Academic Press, Inc., Boston, MA, 1988. MR0959889

[6] J. Henderson, R. Luca, Positive solutions for a system of nonlocal fractional boundary value problems, *Fract. Calc. Appl. Anal.* **16**(2013), No. 4, 985–1008. MR3124348

[7] J. Henderson, R. Luca, Existence and multiplicity of positive solutions for a system of fractional boundary value problems, *Bound. Value Prob.* **2014**, 2014:60, 1–17. MR3352633

[8] J. Henderson, R. Luca, Positive solutions for a system of fractional differential equations with coupled integral boundary conditions, *Appl. Math. Comput.* **249**(2014), 182–197. MR3279412

[9] J. Henderson, R. Luca, Nonexistence of positive solutions for a system of coupled fractional boundary value problems, *Bound. Value Prob.* **2015**, 2015:138, 12 pp. MR3382857

[10] J. Henderson, R. Luca, *Boundary value problems for systems of differential, difference and fractional equations. Positive solutions*, Elsevier, Amsterdam, 2016. MR3430818

[11] J. Henderson, R. Luca, A. Tudorache, Positive solutions for a fractional boundary value problem, *Nonlinear Stud.* **22**(2015), No. 1, 139–151 MR3309286

[12] J. Henderson, R. Luca, A. Tudorache, On a system of fractional differential equations with coupled integral boundary conditions, *Fract. Calc. Appl. Anal.* **18**(2015), No. 2, 361–386. MR3323907

[13] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies, Vol. 204, Elsevier Science B.V., Amsterdam, 2006. MR2218073

[14] R. Luca, A. Tudorache, Positive solutions to a system of semipositone fractional boundary value problems, *Adv. Difference Equ.* **2014**, 2014:179, 11 pp. MR3357337

[15] I. Podlubny, *Fractional differential equations. An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, Mathematics in Science and Engineering, Vol. 198, Academic Press, Inc., San Diego, CA, 1999. MR1658022

[16] J. Sabatier, O. P. Agrawal, J. A. T. Machado (Eds.), *Advances in fractional calculus: theoretical developments and applications in physics and engineering*, Springer, Dordrecht, 2007. MR2432163

[17] S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional integrals and derivatives. Theory and applications*, Gordon and Breach Science Publishers, Yverdon, 1993. MR1347689

[18] C. Yuan, Multiple positive solutions for $(n-1,1)$-type semipositone conjugate boundary value problems of nonlinear fractional differential equations, *Electron. J. Qual. Theory Differ. Equ.* **2010**, No. 36, 1–12. MR2652066

[19] C. Yuan, Two positive solutions for $(n-1,1)$-type semipositone integral boundary value problems for coupled systems of nonlinear fractional differential equations, *Commun. Nonlinear Sci. Numer. Simulat.* **17**(2012), No. 2, 930–942. MR2834462

[20] C. Yuan, D. Jiang, D. O'Regan, R. P. Agarwal, Multiple positive solutions to systems of nonlinear semipositone fractional differential equations with coupled boundary conditions, *Electron. J. Qual. Theory Differ. Equ.* **2012**, No. 13, 1–17. MR2889755