Correct solvability of Volterra integrodifferential equations in Hilbert space

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Abstract. Correct solvability of abstract integrodifferential equations of the Gurtin–Pipkin type is studied. These equations represent abstract wave equations perturbed by terms that include Volterra integral operators.

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1 Introduction

The paper is concerned with integrodifferential equations with unbounded operator coefficients in a Hilbert space. The main part $(d^2u/dt^2 + A^2u)$ of the equation under consideration is an abstract hyperbolic-type equation disturbed by terms involving Volterra operators. These equations can be looked upon as an abstract form of the Gurtin–Pipkin equation describing thermal phenomena and heat transfer in materials with memory or wave propagation in viscoelastic media. A complete analysis and abundant examples of such equations in Banach and Hilbert spaces can be found in [1–3, 8–11, 23].

Consider the following class of second-order abstract models

$$\frac{d^2u}{dt^2} + A^2u + ku - \int_0^t K(t-s)A^{2\theta}u(s)ds = f(t), \qquad t \in \mathbb{R}_+, \tag{1.1}$$

$$u(+0) = \varphi_0, \qquad u^{(1)}(+0) = \varphi_1, \tag{1.2}$$

where A is a positive self-adjoint operator with domain $\text{dom}(A) \subset H$, H is a Hilbert space, and φ_0, φ_1, $f(t)$ will be described later. The variable θ is a real number in $[0, 1]$, k is a non-negative constant and K is the kernel associated with the equation (1.1) (the Gurtin–Pipkin equation). This type of equations appear in various branches of mechanics and physics, for instance, in heat transfer with finite propagation speed [4], theory of viscoelastic media [2], kinetic theory of gases [5], and thermal systems with memory [24].

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In heat theory and theory of viscoelastic media, the kernel K is determined empirically. Properties of heat conduction with memory were studied, for example, in [15] with a smooth function as a kernel. In [6], the solutions of control problems with compactly supported boundary control and distributed control were studied.

In theory of viscoelasticity, a kernel K is also determined empirically. The curves thus obtained are often approximated by a finite sum of exponentials in the form:

$$K(t) \approx \sum_{k=1}^{N} c_k \exp(-\gamma_k t).$$

Equations with structure and properties that are similar to the Gurtin–Pipkin equations also appear in the kinetic theory of gases. In this theory the equations of a solid medium are derived from the laws of pairwise interactions of molecules. A series of momentum equations can be derived from the Boltzmann equation. Here, the momenta represented by coordinates and velocities with respect to velocity variables with certain weights are average of the distribution function for gas molecules in thermal equilibrium. In particular, the ordinary components of the Navier–Stokes equations represented by velocity, pressure, and density can be represented as momenta in a series of momentum equations.

Phenomena like isotropic materials and ionized atmosphere can also be modeled by the Gurtin–Pipkin equation. For example, system (1.1)–(1.2) represents an isotropic viscoelastic model if $\theta = 1/2$, $k = 0$ and $A^2 u = -\mu \Delta u - (\lambda + \mu)\nabla(\operatorname{div} u)$, where μ and λ are the *Lame coefficients*. Similarly, system (1.1)–(1.2) represents a model of ionized atmosphere if $\theta = 0$, $k > 0$ and $A^2 = -\Delta$ (for more details, see [13, 14]). Since the operators A^2 and A are both positive self-adjoint operators, the operator A^2 is used instead of A.

We want to make it clear that we do not study stability of solutions of abstract integrodifferential equation (1.1). We consider important to mention some results obtained in [3, 12–14], because in the papers [16, 17, 19] were obtained some results associated with asymptotic behavior of solutions for system (1.1)–(1.2) when $\theta \in [0, 1]$, $k = 0$. Results related to asymptotic behavior of solutions for systems with memory, for different $\theta \in [0, 1]$, were extensively studied in recent years (see [3, 12–14] and the references given therein). In [12], for instance, Muñoz Rivera and co-authors showed that the solutions for system (1.1)–(1.2) with $\theta \in [0, 1/2)$ and $k = 0$ decay polynomially as $t \to +\infty$, even if the kernel K decays exponentially. Fabrizio and Lazzari in [3], assuming the exponential decay of kernel K, $\theta = 1/2$ and $k = 0$, proved the exponential decay of the solutions for system (1.1)–(1.2). In [13], for the case $k > 0$ and $\theta = 0$, Muñoz Rivera and co-authors showed that for the ionized atmosphere the dissipation produced by the conductivity kernel alone is not enough to produce an exponential decay of the solution of an integrodifferential equation. In [14], for the case $k = 0$ and $\theta = 1/2$, Muñoz Rivera and Maria Naso proved that the solution of model (1.1)–(1.2) decays exponentially to zero if so does the kernel K.

Vlasov and co-authors [18–20] also studied model (1.1)–(1.2) for the case $\theta = 1$ and $k = 0$. They established the correct solvability of initial boundary value problems in weighted Sobolev space on the positive semi-axis and examined spectral properties of the operator-valued function $L(\lambda) = \lambda^2 I + k + A^2 - \widehat{K}(\lambda)A^{2\theta}$, where $\widehat{K}(\lambda)$ is the Laplace transform of $K(t)$. The generalization of aforementioned results was obtained in the recent paper [21]. On the base of spectral analysis of operator function $L(\lambda)$ Vlasov and Rautian [22] obtained the representation of solutions of model (1.1)–(1.2).

In the present work we obtain the correct solvability of system (1.1)–(1.2) in the cases $k = 0$ and $\theta \in [0, 1]$. The correct solvability of initial boundary problems for the specified equations

is established in weighted Sobolev spaces on a positive semi-axis. Spectral properties of the operator-valued function $L(\lambda)$ corresponding to system (1.2) for the cases $\theta \in [0,1]$ and $k = 0$ were obtained in [16,17].

The paper is divided into four sections. Section 1 gives a brief introduction to the subject matter and describes applications of the Gurtin–Pipkin equation. The main results on the correct solvability for the case $\theta \in [0,1]$ and $k = 0$ are formulated in the Section 2. The proof of these results are given in Section 3. Section 4 contains a comment concerning the spectra of operator-valued function $L(\lambda)$.

Throughout the paper, the expression $a \lesssim b$ will mean that $a \leq Cb$, $C > 0$, and $a \approx b$ will be used to write that $a \lesssim b \lesssim a$.

2 Correct solvability

Let H be a separable Hilbert space and let A be a self-adjoint positive operator in H with compact inverse. We associate the domain $\mathrm{dom}(A^\beta)$ of the operator A^β, $\beta > 0$, with a Hilbert space H_β by introducing on $\mathrm{dom}(A^\beta)$ the norm $\|\cdot\|_\beta = \|A^\beta \cdot \|$, which is equivalent to the graph norm of the operator A^β. We denote by $\{e_n\}_{n=1}^\infty$ the orthonormal basis formed by the eigenvectors of A corresponding to its eigenvalues a_n such that $Ae_n = a_n e_n$, $n \in \mathbb{N}$. The eigenvalues a_n are arranged in increasing order and counted according to multiplicity; that is, $0 < a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots$, where $a_n \to \infty$ as $n \to +\infty$.

We denote by $W_{2,\gamma}^n(\mathbb{R}_+, A^n)$ the Sobolev space consisting of vector-functions on the semi-axis $\mathbb{R}_+ = (0, \infty)$ with values in H; this space will be equipped with the norm

$$\|u\|_{W_{2,\gamma}^n(\mathbb{R}_+, A^n)} \equiv \left(\int_0^\infty e^{-2\gamma t} \left(\|u^{(n)}(t)\|_H^2 + \|A^n u(t)\|_H^2 \right) dt \right)^{1/2}, \qquad \gamma \geq 0.$$

For a complete description of the space $W_{2,\gamma}^n(\mathbb{R}_+, A^n)$ and some of its properties we refer to the monograph [7, Chapter I]. For $\gamma = 0$ we write $W_{2,0}^n(\mathbb{R}_+, A^n) \equiv W_2^n(\mathbb{R}_+, A^n)$. This space is endowed with the norm

$$\|u\|_{W_2^n(\mathbb{R}_+, A^n)} \equiv \left(\int_0^\infty \left(\|u^{(n)}(t)\|_H^2 + \|A^n u(t)\|_H^2 \right) dt \right)^{1/2}.$$

For $n = 0$, set $W_{2,\gamma}^0(\mathbb{R}_+, A^0) \equiv L_{2,\gamma}(\mathbb{R}_+, H)$, where $L_{2,\gamma}(\mathbb{R}_+, H)$ denotes the space of measurable vector functions with values in H, equipped with the norm

$$\|f\|_{L_{2,\gamma}(\mathbb{R}_+, H)} \equiv \left(\int_0^\infty e^{-2\gamma t} \|f(t)\|_H^2 dt \right)^{1/2}, \qquad \gamma \geq 0.$$

Let us consider the following system on the semi-axis $\mathbb{R}_+ = (0, \infty)$:

$$\frac{d^2 u}{dt^2} + A^2 u - \int_0^t K(t-s) A^{2\theta} u(s) ds = f(t), \qquad \theta \in [0,1], \tag{2.1}$$

$$u(+0) = \varphi_0, \qquad u^{(1)}(+0) = \varphi_1. \tag{2.2}$$

It is assumed that the vector-valued function $A^{2-\theta} f(t)$ belongs to $L_{2,\rho_0}(\mathbb{R}_+, H)$ for some $\rho_0 \geq 0$, and the scalar function $K(t)$ admits the representation $K(t) = \sum_{j=1}^\infty c_j e^{-\gamma_j t}$, where $c_j > 0$, $\gamma_{j+1} > \gamma_j > 0$, $j \in \mathbb{N}$, $\gamma_j \to +\infty$ $(j \to +\infty)$. Moreover, we assume that

$$a) \ \sum_{j=1}^\infty \frac{c_j}{\gamma_j} < 1, \qquad b) \ \sum_{j=1}^\infty c_j < +\infty.$$

Note that if a) holds, then $K \in L_1(\mathbb{R}_+)$ and $\|K\|_{L_1(\mathbb{R}_+)} < 1$. If a) and b) are both satisfied, then the kernel K belongs to Sobolev space $W_1^1(\mathbb{R}_+)$.

Definition 2.1. A vector-valued function u is called a *strong solution* of system (2.1)–(2.2) if, for some $\gamma \geq 0$, $u \in W_{2,\gamma}^2(\mathbb{R}_+, A^2)$ satisfies the equation (2.1) almost everywhere on the semi-axis $\mathbb{R}_+$ and u also satisfies the initial condition (2.2).

In [19, Theorem 1] it was shown the existence of a strong solution u and system (2.1)–(2.2) for $\theta = 1$ was proved to be correctly solvable. In the present paper we establish the correct solvability of system (2.1)–(2.2) for $\theta \in [0,1]$. The result obtained here is more general than that of [19, Theorem 1], although both results coincide for $\theta = 1$.

Theorem 2.2. *Suppose that, for all $\theta \in [0,1]$ and for some $\rho_0 \geq 0$, $A^{2-\theta} f(t)$ belongs to $L_{2,\rho_0}(\mathbb{R}_+, H)$.*

1) *If conditions a) and b) both hold, and $\varphi_0 \in H_2$, $\varphi_1 \in H_1$ for all $\theta \in [0,1]$ then there is a $\widetilde{\rho} > \rho_0$ such that for any $\gamma > \widetilde{\rho}$, system (2.1)–(2.2) has a unique solution in the Sobolev space $W_{2,\gamma}^2(\mathbb{R}_+, A^2)$ and this solution satisfies the estimate*

$$\|u\|_{W_{2,\gamma}^2(\mathbb{R}_+, A^2)} \leq d \left(\|A^{2-\theta} f\|_{L_{2,\gamma}(\mathbb{R}_+, H)}^2 + \|A^2 \varphi_0\|_H + \|A\varphi_1\|_H \right), \tag{2.3}$$

where the constant d is independent of the vector-valued function f and the vectors φ_0, φ_1.

2) *If condition a) is satisfied, but condition b) does not hold (i.e., $K(t) \notin W_1^1(\mathbb{R}_+)$) and $\varphi_0 \in H_{2+\theta}$, $\varphi_1 \in H_{1+\theta}$ for all $\theta \in (0,1]$ then there is a $\widetilde{\rho} > \rho_0$ such that for any $\gamma > \widetilde{\rho}$ system (2.1)–(2.2) has a unique solution in Sobolev space $W_{2,\gamma}^2(\mathbb{R}_+, A^2)$ and this solution satisfies the estimate*

$$\|u\|_{W_{2,\gamma}^2(\mathbb{R}_+, A^2)} \leq d \left(\|A^{2-\theta} f\|_{L_{2,\gamma}(\mathbb{R}_+, H)}^2 + \|A^{2+\theta} \varphi_0\|_H + \|A^{1+\theta} \varphi_1\|_H \right), \tag{2.4}$$

where the constant d is independent of the vector-valued function f and the vectors φ_0, φ_1.

From Theorem 2.2 we obtain the correct solvability of system (2.1)–(2.2) in the Sobolev space $W_2^2((0,T), A^2)$, for every $T > 0$, where the space $W_2^2((0,T), A^2)$ is equipped with the norm

$$\|u\|_{W_2^2((0,T), A^2)} \equiv \left(\int_0^T \left(\|u^{(2)}(t)\|_H^2 + \|A^2 u(t)\|_H^2 \right) dt \right)^{1/2}.$$

Corollary 2.3. *Suppose that, for all $\theta \in [0,1]$ and for some $\rho_0 \geq 0$, $A^{2-\theta} f(t)$ belongs to $L_{2,\rho_0}(\mathbb{R}_+, H)$.*

1) *If condition 1) of Theorem 2.2 is satisfied, then for an arbitrary $T > 0$ the following estimate of the solution u is valid*

$$\|u\|_{W_2^2((0,T), A^2)} \leq \widetilde{d} \left(\|A^{2-\theta} f\|_{L_{2,\gamma}(\mathbb{R}_+, H)}^2 + \|A^2 \varphi_0\|_H + \|A\varphi_1\|_H \right), \tag{2.5}$$

where the constant $\widetilde{d}$ is independent of the vector-valued function f and the vectors φ_0, φ_1.

2) *If condition 2) of Theorem 2.2 is satisfied, then for an arbitrary $T > 0$ the following estimate of the solution u is valid*

$$\|u\|_{W_2^2((0,T), A^2)} \leq \widetilde{d} \left(\|A^{2-\theta} f\|_{L_{2,\gamma}(\mathbb{R}_+, H)}^2 + \|A^{2+\theta} \varphi_0\|_H + \|A^{1+\theta} \varphi_1\|_H \right), \tag{2.6}$$

where the constant $\widetilde{d}$ is independent of the vector-valued function f and the vectors φ_0, φ_1.

3 Proof of Theorem 2.2

In the case of homogeneous initial conditions $\varphi_0 = \varphi_1 = 0$ we need to establish the correct solvability of the Cauchy problem for hyperbolic equations through the Laplace transform. We give here some well-known facts that will be used later.

Definition 3.1. The *Hardy space* $H_2(\operatorname{Re}\lambda > \gamma, H)$ is the class of holomorphic (or analytic) functions $\hat{f}(\lambda)$ on the right half-plane $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \gamma \geq 0\}$ values in H. The space $H_2(\operatorname{Re}\lambda > \gamma, H)$ is endowed with the norm

$$\|\hat{f}\|_{H_2(\operatorname{Re}\lambda>\gamma,H)} = \left(\sup_{\operatorname{Re}\lambda>\gamma} \int_{-\infty}^{+\infty} \|\hat{f}(x+iy)\|_H^2 \, dy \right)^{1/2} < +\infty, \qquad (\lambda = x + iy).$$

Let us formulate a well-known Paley–Wiener theorem about the Hardy space $H_2(\operatorname{Re}\lambda > \gamma, H)$.

Theorem 3.2 (Paley–Wiener).

1. *The space $H_2(\operatorname{Re}\lambda > \gamma, H)$ coincides with the set of vector-valued functions (Laplace transforms), which admit the representation*

$$\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} e^{-\lambda t} f(t) \, dt, \tag{3.1}$$

where $f(t) \in L_{2,\gamma}(\mathbb{R}_+, H)$, $\lambda \in \mathbb{C}$, $\operatorname{Re}\lambda > \gamma \geq 0$.

2. *For any $\hat{f}(\lambda) \in H_2(\operatorname{Re}\lambda > \gamma, H)$ there is exactly one representation of the form (3.1), where $f(t) \in L_{2,\gamma}(\mathbb{R}_+, H)$. Moreover, the following inversion formula holds:*

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{f}(\gamma + iy) e^{(\gamma+iy)t} \, dy, \qquad t \in \mathbb{R}_+, \ \gamma \geq 0.$$

3. *For $\hat{f}(\lambda) \in H_2(\operatorname{Re}\lambda > \gamma, H)$ and $f(t) \in L_{2,\gamma}(\mathbb{R}_+, H)$ as in (3.1), the following relation holds:*

$$\|\hat{f}(\lambda)\|_{H_2(\operatorname{Re}\lambda>\gamma,H)}^2 \equiv \sup_{\operatorname{Re}\lambda>\gamma} \int_{-\infty}^{+\infty} \|\hat{f}(x+iy)\|_H^2 dy$$

$$= \int_0^{+\infty} e^{-2\gamma t} \|f(t)\|_H^2 dt \equiv \|f(t)\|_{L_{2,\gamma}(\mathbb{R}_+,H)}^2.$$

Proof. We begin with the proof of Theorem 2.2 in the case of homogeneous initial conditions $\varphi_0 = \varphi_1 = 0$. We note that the Laplace transform $\hat{u}(\lambda)$ of any strong solution of equation (2.1) with the initial condition (2.2) has the form

$$\hat{u}(\lambda) = L^{-1}(\lambda)\hat{f}(\lambda), \tag{3.2}$$

where the operator-valued function $L(\lambda)$ is the symbol of equation (2.1), which can be represented as

$$L(\lambda) = \lambda^2 I + A^2 - \widehat{K}(\lambda)A^{2\theta} := \lambda^2 I + A^2 - \left(\sum_{k=1}^{\infty} \frac{c_k}{\lambda + \gamma_k} \right) A^{2\theta}, \qquad 0 \leq \theta \leq 1. \tag{3.3}$$

Here the operator I is the identity operator in the Hilbert space H.

If we can prove that the vector-valued function of equation (3.2) is such that $A^2\hat{u}(\lambda)$ and $\lambda^2\hat{u}(\lambda)$ both belong to the Hardy space $H_2(\text{Re}\,\lambda > \gamma, H)$ for some $\gamma > \rho_0 \geq 0$, then by the Paley–Wiener theorem we will be able to prove that $A^2u(t)$ and $d^2u(t)/dt^2$ both belong to $L_{2,\gamma}(\mathbb{R}_+, H)$, which would imply that $u(t) \in W^2_{2,\gamma}(\mathbb{R}_+, A^2)$. Then, the solvability of system (2.1)–(2.2) in the Sobolev space $W^2_{2,\gamma}(\mathbb{R}_+, A^2)$ will be established.

With that idea in our mind, let us consider the projection $\hat{u}_n(\lambda)$ of the vector-valued function $\hat{u}(\lambda)$ on the one-dimensional subspace spanned by the vector e_n:

$$\hat{u}_n(\lambda) = \ell_n^{-1}(\lambda)\hat{f}_n(\lambda), \tag{3.4}$$

where $\hat{f}_n(\lambda) = (\hat{f}(\lambda), e_n)$ and

$$\ell_n(\lambda) := (L(\lambda)e_n, e_n) = \lambda^2 + a_n^2 - a_n^{2\theta}\left(\sum_{k=1}^{\infty} \frac{c_k}{\lambda + \gamma_k}\right).$$

The restriction of $A^2\hat{u}(\lambda)$ to the one-dimensional space spanned by e_n has the form

$$\left(A^2\hat{u}(\lambda), e_n\right) = \frac{a_n^\theta \hat{g}_n(\lambda)}{\ell_n(\lambda)}, \qquad 0 \leq \theta \leq 1, \tag{3.5}$$

where $\hat{g}_n(\lambda)$ is the nth coordinate of the vector function $\hat{g}(\lambda) = A^{2-\theta}\hat{f}(\lambda)$. According to the hypotheses of Theorem 2.2, the vector-valued function $g(t) = A^{2-\theta}f(t)$ belongs to the space $L_{2,\rho_0}(\mathbb{R}_+, H)$. Consequently, the Laplace transform $\hat{g}(\lambda)$ of the function $g(t)$ belongs to Hardy space $H_2(\text{Re}\,\lambda > \rho_0, H)$.

In order to prove that $A^2\hat{u}(\lambda)$ belongs to $H_2(\text{Re}\,\lambda > \gamma, H)$, it is enough to establish the estimate

$$\sup_{\substack{\text{Re}\,\lambda > \gamma \\ n\in\mathbb{N}}} \left|\frac{a_n^\theta}{\ell_n(\lambda)}\right| \leq \text{const}, \qquad \text{for all } \theta \in [0,1], \tag{3.6}$$

which is uniform with respect to $\lambda(\text{Re}\,\lambda > \gamma)$ and $n \in \mathbb{N}$.

For that purpose, we consider the function $\mathfrak{m}_n(\lambda) = \frac{\ell_n(\lambda)}{a_n^2}$. We estimate this function from below by means of its real and imaginary parts:

$$\text{Re}\,\mathfrak{m}_n(\lambda) = \frac{x^2 - y^2}{a_n^2} + 1 - \frac{1}{a_n^{2(1-\theta)}}\left(\sum_{k=1}^{\infty} \frac{c_k(x + \gamma_k)}{(x + \gamma_k)^2 + y^2}\right), \qquad \lambda = x + iy,$$

$$\text{Im}\,\mathfrak{m}_n(\lambda) = \frac{2xy}{a_n^2} + \frac{y}{a_n^{2(1-\theta)}}\left(\sum_{k=1}^{\infty} \frac{c_k}{(x + \gamma_k)^2 + y^2}\right).$$

First, we seek a lower bound for $|\text{Im}\,\mathfrak{m}_n(\lambda)|$ with $|y| > x$, where $x > \gamma \geq \rho_1 \geq 0$:

$$|\text{Im}\,\mathfrak{m}_n(\lambda)| > \frac{2x|y|}{a_n^2} + \frac{1}{|y|a_n^{2(1-\theta)}}\left(\sum_{k=1}^{\infty} \frac{c_k}{\left(1 + \frac{\gamma_k}{|y|}\right)^2 + 1}\right) > \frac{2\gamma y^2 + k_0(\gamma)a_n^{2\theta}}{|y|a_n^2},$$

where $k_0(\gamma) = \frac{c_1}{\left(1 + \frac{\gamma_1}{\gamma}\right)^2 + 1}$. Hence, for $|y| > x$ with $x > \gamma \geq \rho_1 \geq 0$ we have

$$\frac{1}{|\ell_n(\lambda)|} \leq \frac{1}{a_n^2|\text{Im}\,\mathfrak{m}_n(\lambda)|} < \frac{|y|}{2\gamma y^2 + k_0(\gamma)a_n^{2\theta}} < \frac{1}{a_n^\theta \sqrt{2\gamma \cdot k_0(\gamma)}}. \tag{3.7}$$

Second, we estimate $|\operatorname{Re} \mathfrak{m}_n(\lambda)|$ from below with $|y| < x$, where $x > \gamma > \rho_1 \geq 0$. For fixed ϵ it is possible to find a $N = N(\epsilon)$ such that

$$\sum_{k=N+1}^{\infty} \frac{c_k}{x + \gamma_k} < \sum_{k=N+1}^{\infty} \frac{c_k}{\gamma_k} < \frac{\epsilon}{2}, \qquad x > 0. \tag{3.8}$$

In turn for the finite sum $\sum_{k=1}^{N} \frac{c_k}{x+\gamma_k}$ it is possible to find a $\rho^* > 0$ such that for $x > \rho^* > 0$,

$$\sum_{k=1}^{N} \frac{c_k}{x + \gamma_k} < \frac{\epsilon}{2}. \tag{3.9}$$

Hence from (3.8) and (3.9) we obtain the following inequality

$$\sum_{k=1}^{\infty} \frac{c_k}{x + \gamma_k} < \epsilon, \qquad x > \rho^* > 0. \tag{3.10}$$

We know that the sequence $\{a_n\}_{n=1}^{\infty}$ is such that $a_n \to \infty$ when $n \to \infty$ and the eigenvalues a_n are arranged in increasing order. Hence we can choose a sufficiently small $\epsilon > 0$ such that $\epsilon < a_1^{2(1-\theta)}/2$. Consequently for $x > \rho^* > 0$ the following inequality holds

$$\frac{1}{a_n^{2(1-\theta)}} \sum_{k=1}^{\infty} \frac{c_k(x + \gamma_k)}{(x + \gamma_k)^2 + y^2} < \frac{\sum_{k=1}^{\infty} \frac{c_k}{x+\gamma_k}}{a_n^{2(1-\theta)}} < \frac{1}{a_1^{2(1-\theta)}} \sum_{k=1}^{\infty} \frac{c_k}{x + \gamma_k} < \frac{1}{2}. \tag{3.11}$$

It follows that

$$|\operatorname{Re} \mathfrak{m}_n(\lambda)| > \left| 1 - \frac{1}{a_n^{2(1-\theta)}} \sum_{k=1}^{\infty} \frac{c_k}{x + \gamma_k} \right| > \left| 1 - \frac{1}{a_1^{2(1-\theta)}} \sum_{k=1}^{\infty} \frac{c_k}{x + \gamma_k} \right| > \frac{1}{2}.$$

Therefore, for $|y| < x$, with $x > \widetilde{\rho} = \max(\rho_0, \rho_1, \rho^*) > 0$, we have

$$\frac{1}{|\ell_n(\lambda)|} \leq \frac{1}{a_n^2 |\operatorname{Re} \mathfrak{m}_n(\lambda)|} < \frac{1}{a_n^2 \left| 1 - \frac{1}{a_1^{2(1-\theta)}} \sum_{k=1}^{\infty} \frac{c_k}{x+\gamma_k} \right|} < \frac{2}{a_n^2}. \tag{3.12}$$

From estimates (3.7) and (3.12) we obtain

$$\left| \frac{a_n^{\theta}}{\ell_n(\lambda)} \right| < \frac{1}{\sqrt{2\gamma \cdot k_0(\gamma)}}, \qquad |y| > x > \gamma > \rho_1 \geq 0,$$

$$\left| \frac{a_n^{\theta}}{\ell_n(\lambda)} \right| < \frac{2}{a_n^{2-\theta}} < \frac{2}{a_1^{2-\theta}}, \qquad |y| < x, \quad x > \widetilde{\rho} > 0.$$

Therefore, for $\gamma > \max\{\rho_1, \widetilde{\rho}\}$ we have

$$\sup_{\substack{\operatorname{Re} \lambda > \gamma \\ n \in \mathbb{N}}} \left| \frac{a_n^{\theta}}{\ell_n(\lambda)} \right| < \frac{2}{\min\left(\sqrt{2\gamma \cdot k_0(\gamma)}, a_1^{2-\theta} \right)}, \qquad \text{for all } \theta \in [0, 1]. \tag{3.13}$$

Remark 3.3. Estimate (3.13) implies that

$$\sup_{\operatorname{Re} \lambda > \gamma} \left\| A^{\theta} L^{-1}(\lambda) \right\| \leq \text{const.} \tag{3.14}$$

The Hardy space $H_2(\operatorname{Re}\lambda > \gamma, H)$ is invariant with respect to multiplication of functions of the form $\frac{a_n^\theta}{\ell_n(\lambda)}$, since they are analytic and bounded in view of (3.13). Consequently, $\hat{g}(\lambda) = A^{2-\theta}\hat{f}(\lambda) \in H_2(\operatorname{Re}\lambda > \gamma, H)$ implies that $A^2\hat{u}(\lambda)$ belongs to $H_2(\operatorname{Re}\lambda > \gamma, H)$.

Let us establish the estimate for the norm of the vector-valued function $A^2u(t) \in L_{2,\gamma}(\mathbb{R}_+, H)$. From (3.2) it follows that

$$A^2\hat{u}(\lambda) = A^2 L^{-1}(\lambda)\hat{f}(\lambda) = A^\theta L^{-1}(\lambda) A^{2-\theta}\hat{f}(\lambda). \tag{3.15}$$

This function can be represented in the form

$$A^2\hat{u}(\lambda) = \sum_{k=1}^{\infty} \frac{a_k^\theta}{\ell_k(\lambda)} \cdot a_k^{2-\theta}\hat{f}_k(\lambda)e_k.$$

According to the hypothesis of Theorem 2.2, the vector-valued function $A^{2-\theta}f(t)$ belongs to $L_{2,\rho_0}(\mathbb{R}_+, H)$. Therefore, by the Paley–Wiener theorem, $A^{2-\theta}\hat{f}(\lambda) \in H_2(\operatorname{Re}\lambda > \rho_0, H)$ and

$$\|A^{2-\theta}f\|_{L_{2,\rho_0}(\mathbb{R}_+,H)} = \|A^{2-\theta}\hat{f}\|_{H_2(\operatorname{Re}\lambda>\rho_0,H)}.$$

By (3.13) and Paley–Wiener theorem, the following relations hold for $\gamma > \tilde{\rho}$:

$$\|A^2u(t)\|_{L_{2,\gamma}(\mathbb{R}_+,H)}^2 = \|A^2\hat{u}(\lambda)\|_{H_2(\operatorname{Re}\lambda>\gamma,H)}^2 = \|A^\theta L^{-1}(\lambda) A^{2-\theta}\hat{f}(\lambda)\|_{H_2(\operatorname{Re}\lambda>\gamma,H)}^2.$$

But

$$\begin{aligned}
\|A^\theta L^{-1} A^{2-\theta}\hat{f}\|_{H_2(\operatorname{Re}\lambda>\gamma,H)}^2 &= \sup_{\operatorname{Re}\lambda>\gamma} \int_{-\infty}^{+\infty} \left(\sum_{k=1}^{\infty} \left| \frac{a_k^\theta}{\ell_k(\lambda)} \cdot a_k^{2-\theta}\hat{f}_k(\lambda) \right|^2 \right) dy \\
&\leq \sup_{\substack{\operatorname{Re}\lambda>\gamma \\ k\in\mathbb{N}}} \left| \frac{a_k^\theta}{\ell_k(\lambda)} \right|^2 \cdot \sup_{\operatorname{Re}\lambda>\gamma} \int_{-\infty}^{+\infty} \left(\sum_{k=1}^{\infty} \left| a_k^{2-\theta}\hat{f}_k(\lambda) \right|^2 \right) dy \qquad (3.16) \\
&\leq \sup_{\substack{\operatorname{Re}\lambda>\gamma \\ k\in\mathbb{N}}} \left| \frac{a_k^\theta}{\ell_k(\lambda)} \right|^2 \|A^{2-\theta}\hat{f}(\lambda)\|_{H_2(\operatorname{Re}\lambda>\gamma,H)}^2 \\
&\leq d_1^2 \|A^{2-\theta}f\|_{L_{2,\gamma}(\mathbb{R}_+,H)}^2,
\end{aligned}$$

where

$$d_1 = \sup_{\substack{\operatorname{Re}\lambda>\gamma \\ k\in\mathbb{N}}} \left| \frac{a_k^\theta}{\ell_k(\lambda)} \right|.$$

This shows that $A^2u(t)$ belongs to $L_{2,\gamma}(\mathbb{R}_+, H)$, for $\gamma > \tilde{\rho}$, and the following inequality

$$\|A^2u\|_{L_{2,\gamma}(\mathbb{R}_+,H)} \leq d_1 \left(\|A^{2-\theta}f\|_{L_{2,\gamma}(\mathbb{R}_+,H)} \right) \tag{3.17}$$

holds.

Now let us prove that $\lambda^2\hat{u}$ also belongs to $H_2(\operatorname{Re}\lambda > \gamma, H)$. We set

$$\widehat{K}(\lambda) = \sum_{k=1}^{\infty} \frac{c_k}{\lambda + \gamma_k}.$$

Note that for $\operatorname{Re}\lambda > \gamma$ we can write the identity operator I as

$$I = \lambda^2 L^{-1}(\lambda) + \left(I - \widehat{K}(\lambda) A^{-2(1-\theta)} \right) A^2 L^{-1}(\lambda).$$

Hence, for $\operatorname{Re}\lambda > \gamma$, we obtain

$$\hat{f}(\lambda) = \lambda^2 L^{-1}(\lambda)\hat{f}(\lambda) + \left(I - \widehat{K}(\lambda) A^{-2(1-\theta)} \right) A^2 L^{-1}(\lambda)\hat{f}(\lambda)$$
$$= \lambda^2 \hat{u}(\lambda) + \left(I - \widehat{K}(\lambda) A^{-2(1-\theta)} \right) A^\theta L^{-1}(\lambda) A^{2-\theta}\hat{f}(\lambda). \tag{3.18}$$

Here, $A^{2-\theta}\hat{f}(\lambda) \in H_2(\operatorname{Re}\lambda > \gamma, H)$ and $\sup_{\operatorname{Re}\lambda > \gamma}\left\| A^\theta L^{-1}(\lambda) \right\| \leq$ const. From (3.18) it follows that

$$\lambda^2 \hat{u}(\lambda) = \hat{f}(\lambda) - \left(I - \widehat{K}(\lambda) A^{-2(1-\theta)} \right) A^\theta L^{-1}(\lambda) A^{2-\theta}\hat{f}(\lambda). \tag{3.19}$$

Therefore, the function $\lambda^2 \hat{u}_n(\lambda)$ can be represented in the form

$$\lambda^2 \hat{u}_n(\lambda) = \hat{f}_n(\lambda) - \left(1 - \frac{\widehat{K}(\lambda)}{a_n^{2(1-\theta)}} \right) \frac{a_n^\theta}{\ell_n(\lambda)} \hat{g}_n(\lambda). \tag{3.20}$$

From the assumptions imposed on the sequences $\{c_k\}_{k=1}^\infty$ and $\{\gamma_k\}_{k=1}^\infty$, we have that the function $\widehat{K}(\lambda)$ is analytic and bounded on the right half-plane $\operatorname{Re}\lambda > 0$. From the inequality (3.11) for $x > \tilde{\rho}$ and for all $n \in \mathbb{N}$ we have

$$\left| 1 - \frac{\widehat{K}(\lambda)}{a_n^{2(1-\theta)}} \right| < \frac{3}{2}. \tag{3.21}$$

Since the vector-function $A^{2-\theta}f(t) \in L_{2,\gamma}(\mathbb{R}_+, H)$, it follows that its Laplace transform $A^{2-\theta}\hat{f}(\lambda)$ belongs to Hardy space $H_2(\operatorname{Re}\lambda > \gamma, H)$. It implies that

$$\|A^{2-\theta}\hat{f}\|^2_{H_2(\operatorname{Re}\lambda > \gamma, H)} = \sup_{\operatorname{Re}\lambda > \gamma} \int_{-\infty}^{+\infty} \|A^{2-\theta}\hat{f}(x+iy)\|^2_H dy < +\infty, \qquad \lambda = x + iy.$$

Hence we obtain the following relation

$$\|A^{2-\theta}\hat{f}\|^2_{H_2(\operatorname{Re}\lambda > \gamma, H)} = \sup_{\substack{\operatorname{Re}\lambda > \gamma \\ n \in \mathbb{N}}} \int_{-\infty}^{+\infty} \left(\sum_{n=1}^\infty |a_n^{2-\theta}\hat{f}_n(x+iy)|^2 \right) dy$$
$$> \sup_{\substack{\operatorname{Re}\lambda > \gamma \\ n \in \mathbb{N}}} a_1^{2(2-\theta)} \int_{-\infty}^{+\infty} \left(\sum_{n=1}^\infty |\hat{f}_n(x+iy)|^2 \right) dy$$
$$= a_1^{2(2-\theta)} \sup_{\operatorname{Re}\lambda > \gamma} \int_{-\infty}^{+\infty} \|\hat{f}(x+iy)\|^2_H dy$$
$$= a_1^{2(2-\theta)} \|\hat{f}(\lambda)\|^2_{H_2(\operatorname{Re}\lambda > \gamma, H)}. \tag{3.22}$$

For that reason, $\hat{f}(\lambda) \in H_2(\operatorname{Re}\lambda > \gamma, H)$ and $f(t) \in L_{2,\gamma}(\mathbb{R}_+, H)$. Now, taking into account (3.13) and (3.22) we obtain the following estimate:

$$\sup_{\substack{\operatorname{Re}\lambda > \gamma \\ n \in \mathbb{N}}} \int_{-\infty}^{+\infty} |\lambda^2 \hat{u}_n(\lambda)|^2 dy < \sup_{\substack{\operatorname{Re}\lambda > \gamma \\ n \in \mathbb{N}}} \left(\frac{1}{a_1^{2(2-\theta)}} + 4 \left| \frac{a_n^\theta}{\ell_n(\lambda)} \right|^2 \right) \cdot \sup_{\substack{\operatorname{Re}\lambda > \gamma \\ n \in \mathbb{N}}} \int_{-\infty}^{+\infty} |\hat{g}_n(\lambda)|^2 dy < +\infty.$$

Therefore $\lambda^2 \hat{u}_n(\lambda) \in H_2(\operatorname{Re} \lambda > \gamma, \mathbb{C})$ and $\frac{d^2}{dt^2} u_n(t) \in L_{2,\gamma}(\mathbb{R}_+, \mathbb{C})$.

By (3.13) and Paley–Wiener theorem we have

$$
\begin{aligned}
\left\| \frac{d^2}{dt^2} u(t) \right\|_{L_{2,\gamma}(\mathbb{R}_+,H)}^2 &= \|\lambda^2 \hat{u}(\lambda)\|_{\mathcal{H}_2}^2 \\
&= \left\| \hat{f}(\lambda) - \left(1 - \widehat{K}(\lambda) A^{-2(1-\theta)}\right) A^\theta L^{-1}(\lambda) A^{2-\theta} \hat{f}(\lambda) \right\|_{\mathcal{H}_2}^2 \\
&< \frac{1}{a_1^{2(2-\theta)}} \|A^{2-\theta} \hat{f}(\lambda)\|_{\mathcal{H}_2}^2 + 4\|A^\theta L^{-1}(\lambda) A^{2-\theta} \hat{f}(\lambda)\|_{\mathcal{H}_2}^2 \\
&\le \frac{1}{a_1^{2(2-\theta)}} \|A^{2-\theta} \hat{f}(\lambda)\|_{\mathcal{H}_2}^2 + 4 \sup_{\substack{\operatorname{Re}\lambda>\gamma \\ k\in\mathbb{N}}} \left| \frac{a_k^\theta}{\ell_k(\lambda)} \right|^2 \|A^{2-\theta} \hat{f}(\lambda)\|_{\mathcal{H}_2}^2 \\
&< \left(\frac{1}{a_1^{2-\theta}} + 2 \sup_{\substack{\operatorname{Re}\lambda>\gamma \\ k\in\mathbb{N}}} \left| \frac{a_k^\theta}{\ell_k(\lambda)} \right| \right)^2 \|A^{2-\theta} \hat{f}(\lambda)\|_{\mathcal{H}_2}^2 \\
&\le d_2^2 \|A^{2-\theta} f\|_{L_{2,\gamma}(\mathbb{R}_+,H)}^2,
\end{aligned}
\tag{3.23}
$$

where $\mathcal{H}_2 := H_2(\operatorname{Re} \lambda > \gamma, H)$ and $d_2 = \left(\frac{1}{a_1^{2-\theta}} + 2d_1 \right)$. Hence, $\frac{d^2}{dt^2} u(t) \in L_{2,\gamma}(\mathbb{R}_+, H)$ and the inequality

$$
\left\| \frac{d^2}{dt^2} u(t) \right\|_{L_{2,\gamma}(\mathbb{R}_+,H)} \le d_2 \left(\|A^{2-\theta} f\|_{L_{2,\gamma}(\mathbb{R}_+,H)} \right)
\tag{3.24}
$$

holds. Accordingly, combining estimates (3.17) and (3.24), we come to the desired inequality

$$
\|u(t)\|_{W_{2,\gamma}^2(\mathbb{R}_+,A^2)} \le d \left(\|A^{2-\theta} f\|_{L_{2,\gamma}(\mathbb{R}_+,H)} \right),
\tag{3.25}
$$

where the constant d is independent of f. Consequently, this implies that the equation (2.1) has a solution $u(t)$, which belongs to Sobolev space $W_{2,\gamma}^2(\mathbb{R}_+, A^2)$.

Now, let us prove that the solution $u(t)$ satisfies the initial conditions $u(+0) = 0$ and $u^{(1)}(+0) = 0$.

Remark 3.4. If $\varphi(\lambda) \in H_2(\operatorname{Re} \lambda > \gamma, \mathbb{C})$, then for any $\eta > \gamma$ there is a sequence $\{\eta_k\}_{k=1}^\infty$ such that $\lim_{k\to\infty} \eta_k = +\infty$ and

$$
\lim_{k\to\infty} \int_\gamma^\eta |\varphi(x \pm i\eta_k)| dx = 0.
$$

Proof. Indeed, for any $\eta > \gamma$ and $\eta_k > 0$, we have

$$
\int_{-\eta_k}^{\eta_k} \left(\int_\gamma^\eta |\varphi(x \pm i\eta_k)|^2 dx \right) dy \le \int_\gamma^\eta \left(\int_{-\infty}^{+\infty} |\varphi(x \pm i\eta_k)|^2 dy \right) dx < +\infty.
$$

Therefore, for any $\eta > 0$ there is a sequence $\{\eta_k\}_{k=1}^\infty$ such that its limit tends to $+\infty$ as $k \to \infty$ and

$$
\lim_{k\to\infty} \int_\gamma^\eta |\varphi(x \pm i\eta_k)|^2 dx = 0.
$$

Now it remains to use the Cauchy inequality. $\square$

The above argument shows that $u(t) \in L_{2,\gamma}(\mathbb{R}_+, H)$, which implies that $\hat{u}(\lambda)$ belongs to space $H_2(\operatorname{Re}\lambda > \gamma, H)$ and $\hat{u}_n(\lambda)$ belongs to $H_2(\operatorname{Re}\lambda > \gamma, \mathbb{C})$. Let us prove, moreover, that $\lambda \hat{u}_n(\lambda) \in H_2(\operatorname{Re}\lambda > \gamma, \mathbb{C})$. Indeed, since $\lambda^2 \hat{u}_n(\lambda)$ belongs to Hardy space $H_2(\operatorname{Re}\lambda > \gamma, \mathbb{C})$, we have

$$\sup_{\substack{\operatorname{Re}\lambda>\gamma \\ n\in\mathbb{N}}} \int_{-\infty}^{+\infty} |(\operatorname{Re}\lambda + iy)\hat{u}_n(\operatorname{Re}\lambda + iy)|^2 dy = \sup_{\substack{\operatorname{Re}\lambda>\gamma \\ n\in\mathbb{N}}} \int_{-\infty}^{+\infty} \frac{|(\operatorname{Re}\lambda + iy)^2 \hat{u}_n(\operatorname{Re}\lambda + iy)|^2}{(\operatorname{Re}\lambda)^2 + y^2} dy$$

$$< \frac{1}{\gamma} \sup_{\substack{\operatorname{Re}\lambda>\gamma \\ n\in\mathbb{N}}} \int_{-\infty}^{+\infty} |\lambda^2 \hat{u}_n(\lambda)|^2 dy < +\infty.$$

By the Paley–Wiener theorem,

$$\hat{u}_n(+0) = \frac{1}{\sqrt{2\pi}} \lim_{\eta_k \to \infty} \int_{-\eta_k}^{\eta_k} \hat{u}_n(x+iy)dy = \frac{1}{\sqrt{2\pi i}} \lim_{\eta_k \to \infty} \int_{\gamma-i\eta_k}^{\gamma+i\eta_k} \hat{u}_n(\lambda)d\lambda,$$

$$\hat{u}_n^{(1)}(+0) = \frac{1}{\sqrt{2\pi}} \lim_{\eta_k \to \infty} \int_{-\eta_k}^{\eta_k} (x+iy)\hat{u}_n(x+iy)dy = \frac{1}{\sqrt{2\pi i}} \lim_{\eta_k \to \infty} \int_{\gamma-i\eta_k}^{\gamma+i\eta_k} \lambda\hat{u}_n(\lambda)d\lambda.$$

Since the functions $\hat{u}_n(+0)$ and $\hat{u}_n^{(1)}(+0)$ are analytic functions on the right half-plane $\operatorname{Re}\lambda > \gamma \geq 0$, it follows from Cauchy's theorem that, for any $\eta > \gamma$,

$$\int_{\gamma-i\eta_k}^{\gamma+i\eta_k} \hat{u}_n(\lambda)d\lambda = \left(\int_{\gamma-i\eta_k}^{\eta-i\eta_k} - \int_{\gamma+i\eta_k}^{\eta+i\eta_k} + \int_{\eta-i\eta_k}^{\eta+i\eta_k} \right) \hat{u}_n(\lambda)d\lambda$$

$$= \int_{\gamma}^{\eta} \hat{u}_n(x-i\eta_k)dx - \int_{\gamma}^{\eta} \hat{u}_n(x+i\eta_k)dx + i\int_{-\eta_k}^{\eta_k} \hat{u}_n(\eta+iy)dy,$$

and

$$\int_{\gamma-i\eta_k}^{\gamma+i\eta_k} \lambda\hat{u}_n(\lambda)d\lambda = \left(\int_{\gamma-i\eta_k}^{\eta-i\eta_k} - \int_{\gamma+i\eta_k}^{\eta+i\eta_k} + \int_{\eta-i\eta_k}^{\eta+i\eta_k} \right) \lambda\hat{u}_n(\lambda)d\lambda$$

$$= \int_{\gamma}^{\eta} (x-i\eta_k)\hat{u}_n(x-i\eta_k)dx - \int_{\gamma}^{\eta} (x+i\eta_k)\hat{u}_n(x+i\eta_k)dx$$

$$+ i\int_{-\eta_k}^{\eta_k} (\eta+iy)\hat{u}_n(\eta+iy)dy.$$

According to Remark 3.4 and the fact that $\lambda^2 \hat{u}_n(\lambda) \in H_2(\operatorname{Re}\lambda > \gamma, \mathbb{C})$ we have

$$\lim_{k\to\infty} \int_{\gamma}^{\eta} |\hat{u}_n(x\pm i\eta_k)|^2 dx = 0,$$

$$\lim_{k\to\infty} \int_{\gamma}^{\eta} |(x\pm i\eta_k)\hat{u}_n(x\pm i\eta_k)|^2 dx = 0.$$

Thus, for $\eta > \gamma$,

$$|\hat{u}_n(+0)| \leq C_1 \lim_{\eta_k\to\infty} \int_{-\eta_k}^{\eta_k} |\hat{u}_n(\eta+iy)|dy = C_1 \int_{-\infty}^{+\infty} \left| \frac{(\eta+iy)^2 \hat{u}_n(\eta+iy)}{(\eta+iy)^2} \right| dy$$

$$\leq C_1 \left(\int_{-\infty}^{+\infty} |(\eta+iy)^2 \hat{u}_n(\eta+iy)|^2 dy \right)^{1/2} \left(\int_{-\infty}^{+\infty} \frac{dy}{(\eta^2+y^2)^2} \right)^{1/2}$$

$$\lesssim \frac{1}{\eta^{3/2}}, \tag{3.26}$$

$$|\hat{u}_n^{(1)}(+0)| \leq C_1 \lim_{\eta_k \to \infty} \int_{-\eta_k}^{\eta_k} |(\eta + iy)\hat{u}_n(\eta + iy)|dy = C_1 \int_{-\infty}^{+\infty} \left| \frac{(\eta + iy)^2 \hat{u}_n(\eta + iy)}{\eta + iy} \right| dy$$

$$\leq C_1 \left(\int_{-\infty}^{+\infty} |(\eta + iy)^2 \hat{u}_n(\eta + iy)|^2 dy \right)^{1/2} \left(\int_{-\infty}^{+\infty} \frac{dy}{\eta^2 + y^2} \right)^{1/2}$$

$$\lesssim \frac{1}{\eta^{1/2}}. \tag{3.27}$$

Here, $C_1 := \frac{1}{\sqrt{2\pi}}$. From inequalities (3.26) and (3.27) it follows that $u(+0) = 0$ and $u^{(1)}(+0) = 0$ as $\eta \to +\infty$.

Finally, let us prove that the solution $u(t)$ satisfies the equation (2.1). By Paley–Wiener theorem,

$$u(t) = C_1 \lim_{\eta \to +\infty} \int_{-\eta}^{\eta} L^{-1}(\gamma + iy)\hat{f}(\gamma + iy)e^{(\gamma+iy)t}dy = C_i \lim_{\eta \to +\infty} \int_{\gamma-i\eta}^{\gamma+i\eta} L^{-1}(\lambda)\hat{f}(\lambda)e^{\lambda t}d\lambda,$$

where $C_i := \frac{1}{\sqrt{2\pi i}}$. Consequently, we get

$$\frac{d^2}{dt^2}u(t) = \frac{1}{\sqrt{2\pi i}} \lim_{\eta \to +\infty} \int_{\gamma-i\eta}^{\gamma+i\eta} \lambda^2 L^{-1}(\lambda)\hat{f}(\lambda)e^{\lambda t}d\lambda, \tag{3.28}$$

$$A^2 u(t) = \frac{1}{\sqrt{2\pi i}} \lim_{\eta \to +\infty} \int_{\gamma-i\eta}^{\gamma+i\eta} A^2 L^{-1}(\lambda)\hat{f}(\lambda)e^{\lambda t}d\lambda, \tag{3.29}$$

$$\int_0^t K(t-s)A^{2\theta}u(s)ds = C_i \lim_{\eta \to +\infty} \int_{\gamma-i\eta}^{\gamma+i\eta} \left(\sum_{k=1}^{\infty} \frac{c_k}{\lambda + \gamma_k} \right) A^{2\theta} L^{-1}(\lambda)\hat{f}(\lambda)e^{\lambda t}d\lambda. \tag{3.30}$$

From (3.28)–(3.30) it follows that $u(t)$ satisfies equation (2.1).

Let us turn to the proof of Theorem 2.2 in the case of nonhomogeneous initial conditions. For system (2.1)–(2.2) we define $u(t)$ by means of

$$u(t) := \cos(At)\varphi_0 + A^{-1}\sin(At)\varphi_1 + \omega(t).$$

Then for the function $\omega(t)$ we obtain the following system

$$\frac{d^2}{dt^2}\omega(t) + A^2\omega(t) - \int_0^t K(t-s)A^{2\theta}\omega(s)ds = f_1(t), \qquad t \in \mathbb{R}_+,$$

$$\omega(+0) = \omega^{(1)}(+0) = 0,$$

where $f_1(t) = f(t) - h(t)$ and

$$h(t) = \int_0^t K(t-s)A^{2\theta} \left(\cos(As)\varphi_0 + A^{-1}\sin(As)\varphi_1 \right) ds, \tag{3.31}$$

$$A^{2-\theta}h(t) = \int_0^t K(t-s) \left(A^{2+\theta}\cos(As)\varphi_0 + A^{1+\theta}\sin(As)\varphi_1 \right) ds. \tag{3.32}$$

Let us prove that the vector-valued function $f_1(t)$ satisfies the hypotheses of Theorem 2.2 with homogeneous initial conditions. Indeed,

$$\|A^{2-\theta}f_1(t)\|_{L_{2,\gamma}(\mathbb{R}_+,H)} \leq \|A^{2-\theta}f(t)\|_{L_{2,\gamma}(\mathbb{R}_+,H)} + \|A^{2-\theta}h(t)\|_{L_{2,\gamma}(\mathbb{R}_+,H)}.$$

Now, it remains to estimate the norm $\|A^{2-\theta}h(t)\|_{L_{2,\gamma}(\mathbb{R}_+,H)}$.

1. Suppose that condition $b)$ holds. A direct integration shows that

$$\int_0^t e^{-\gamma_k(t-s)}\cos(As)ds = \left(A^2+\gamma_k^2I\right)^{-1}\left\{\gamma_k\{\cos(At)-e^{-\gamma_k t}I\}+A\sin(At)\right\}, \qquad (3.33)$$

$$\int_0^t e^{-\gamma_k(t-s)}\sin(As)ds = \left(A^2+\gamma_k^2I\right)^{-1}\left\{A\{e^{-\gamma_k t}I-\cos(At)\}+\gamma_k\sin(At)\right\}. \qquad (3.34)$$

In what follows we will use the following argument.

Remark 3.5. The following inequality holds

$$\left\|\left(A^2+\gamma_k^2I\right)^{-1}\right\|_H^2 \lesssim \gamma_k^{-2}\|A^{-1}\|_H^2. \qquad (3.35)$$

Proof. Indeed, note that for any vector $v \in H$, $\|v\|_H = 1$, the following relations are valid:

$$\left\|\left(A^2+\gamma_k^2I\right)^{-1}v\right\|_H^2 = \sum_{n=1}^\infty \left(a_n^2+\gamma_k^2\right)^{-2}|v_n|^2 \lesssim \sum_{n=1}^\infty (a_n\gamma_k)^{-2}|v_n|^2 = \gamma_k^{-2}\|A^{-1}v\|_H^2,$$

where $v_n = (v, e_n)$. The above inequality implies (3.35), because the operator A is self-adjoint and γ_k is positive. $\qquad\square$

Using (3.31), (3.33), (3.34), (3.35) and inequalities $\|\cos(At)\| \le 1$, $\|\sin(At)\| \le 1$, we readily obtain the following result:

$$\|A^{2-\theta}h(t)\|_{\mathcal{L}_{2,\gamma}} = \left\|\sum_{k=1}^\infty c_k \int_0^t e^{-\gamma_k(t-s)}\left\{A^{2+\theta}\cos(At)\varphi_0+A^{1+\theta}\sin(At)\varphi_1\right\}ds\right\|_{\mathcal{L}_{2,\gamma}}$$

$$\le \left\|R_v A^{2+\theta}\left\{\gamma_k\{\cos(At)-e^{-\gamma_k t}I\}+A\sin(At)\right\}\varphi_0\right\|_{\mathcal{L}_{2,\gamma}}$$

$$+\left\|R_v A^{1+\theta}\left\{A\{e^{-\gamma_k t}I-\cos(At)\}+\gamma_k\sin(At)\right\}\varphi_1\right\|_{\mathcal{L}_{2,\gamma}}$$

$$= \left\|\cos(At)R_v\gamma_k A^{2+\theta}\varphi_0\right\|_{\mathcal{L}_{2,\gamma}} + \left\|e^{-\gamma_k t}R_v\gamma_k A^{2+\theta}\varphi_0\right\|_{\mathcal{L}_{2,\gamma}}$$

$$+\left\|\sin(At)R_v A^{3+\theta}\varphi_0\right\|_{\mathcal{L}_{2,\gamma}} + \left\|e^{-\gamma_k t}R_v A^{2+\theta}\varphi_1\right\|_{\mathcal{L}_{2,\gamma}}$$

$$+\left\|\cos(At)R_v A^{2+\theta}\varphi_1\right\|_{\mathcal{L}_{2,\gamma}} + \left\|\sin(At)R_v\gamma_k A^{1+\theta}\varphi_1\right\|_{\mathcal{L}_{2,\gamma}}$$

$$\lesssim \left\|\sum_{k=1}^\infty c_k\gamma_k\gamma_k^{-1}A^{-1}A^{2+\theta}\varphi_0\right\|_H + \left\|\sum_{k=1}^\infty c_k\left(A^2+\gamma_k^2I\right)^{-1}A^{3+\theta}\varphi_0\right\|_H$$

$$+\left\|\sum_{k=1}^\infty c_k\left(A^2+\gamma_k^2I\right)^{-1}A^{2+\theta}\varphi_1\right\|_H + \left\|\sum_{k=1}^\infty c_k\gamma_k\gamma_k^{-1}A^{-1}A^{1+\theta}\varphi_1\right\|_H$$

$$\lesssim 2\sum_{k=1}^\infty c_k\left(\|A^{1+\theta}\varphi_0\|_H+\|A^\theta\varphi_1\|_H\right), \qquad (3.36)$$

where $R_v := \sum_{k=1}^\infty c_k\left(A^2+\gamma_k^2I\right)^{-1}$ and $\mathcal{L}_{2,\gamma} := L_{2,\gamma}(\mathbb{R}_+,H)$.

Thus, from (3.25) and the last estimate (3.36), we get

$$\|\omega\|_{W_{2,\gamma}^2(\mathbb{R}_+,H)} \le d\|A^{2-\theta}f_1\|_{L_{2,\gamma}(\mathbb{R}_+,H)}$$

$$\le d\left(\|A^{2-\theta}f(t)\|_{L_{2,\gamma}(\mathbb{R}_+,H)}+\|A^{1+\theta}\varphi_0\|_H+\|A^\theta\varphi_1\|_H\right). \qquad (3.37)$$

The estimate of

$$\|v(t)\|_{W_{2,\gamma}^2(\mathbb{R}_+,A^2)} := \|\cos(At)\varphi_0\|_{W_{2,\gamma}^2(\mathbb{R}_+,A^2)} + \|A^{-1}\sin(At)\varphi_1\|_{W_{2,\gamma}^2(\mathbb{R}_+,A^2)}$$

is immediate. Indeed,

$$\|\cos(At)\varphi_0\|_{W_{2,\gamma}^n(\mathbb{R}_+,A^2)}^2 = \int_0^\infty e^{-2\gamma t}\left(\|A^2\cos(At)\varphi_0\|_H^2 + \|A^2\cos(At)\varphi_0\|_H^2\right)dt$$

$$\leq 2\int_0^\infty e^{-2\gamma t}\left(\|A^2\varphi_0\|_H^2\right)dt < \frac{1}{\gamma}\|A^2\varphi_0\|_H^2, \tag{3.38}$$

$$\|A^{-1}\sin(At)\varphi_1\|_{W_{2,\gamma}^n(\mathbb{R}_+,A^2)}^2 = \int_0^\infty e^{-2\gamma t}\left(\|A\sin(At)\varphi_1\|_H^2 + \|A\sin(At)\varphi_1\|_H^2\right)dt$$

$$\leq 2\int_0^\infty e^{-2\gamma t}\left(\|A\varphi_1\|_H^2\right)dt < \frac{1}{\gamma}\|A\varphi_1\|_H^2. \tag{3.39}$$

This means that $\varphi_0 \in \mathrm{dom}(A^2) = H_2$ and $\varphi_1 \in \mathrm{dom}(A) = H_1$. Now the first estimate of Theorem 2.2 follows from (3.37) and (3.38)–(3.39),

$$\|u(t)\|_{W_{2,\gamma}^2(\mathbb{R}_+,H)} \leq d\left(\|A^{2-\theta}f(t)\|_{L_{2,\gamma}(\mathbb{R}_+,H)} + \|A^2\varphi_0\|_H + \|A\varphi_1\|_H\right),$$

where the constant d is independent of the vector-valued function f and the vectors φ_0, φ_1.

2. Suppose now that condition $b)$ does not hold. Then $a)$ and (3.31) imply that

$$\|A^{2-\theta}h(t)\|_{\mathcal{L}_{2,\gamma}} = \left\|\int_0^t \sum_{k=1}^\infty c_k e^{-\gamma_k(t-s)}\left\{A^{2+\theta}\cos(As)\varphi_0 + A^{1+\theta}\sin(As)\varphi_1\right\}ds\right\|_{\mathcal{L}_{2,\gamma}}$$

$$\leq \frac{1}{\sqrt{2\gamma}}\sum_{k=1}^\infty \frac{c_k}{\gamma_k}\left(\|A^{2+\theta}\varphi_0\|_H + \|A^{1+\theta}\varphi_1\|_H\right),$$

where $\mathcal{L}_{2,\gamma} := L_{2,\gamma}(\mathbb{R}_+,H)$.

Therefore, for all $\theta \in [0,1]$ we have

$$\|\omega\|_{W_{2,\gamma}^2(\mathbb{R}_+,H)} \leq d\|A^{2-\theta}f_1\|_{L_{2,\gamma}(\mathbb{R}_+,H)}$$

$$\leq d\left(\|A^{2-\theta}f(t)\|_{L_{2,\gamma}(\mathbb{R}_+,H)} + \|A^{2+\theta}\varphi_0\|_H + \|A^{1+\theta}\varphi_1\|_H\right).$$

Consequently, the second estimate of Theorem 2.2 is as follows,

$$\|u(t)\|_{W_{2,\gamma}^2(\mathbb{R}_+,H)} \leq d\left(\|A^{2-\theta}f(t)\|_{L_{2,\gamma}(\mathbb{R}_+,H)} + \|A^{2+\theta}\varphi_0\|_H + \|A^{1+\theta}\varphi_1\|_H\right),$$

where the constant d is independent of the vector-valued function f and the vectors φ_0, φ_1. $\square$

4   Comments

In the papers [16, 17] the structure of spectra of operator-valued function $L(\lambda)$ was studied. By spectrum $\sigma(L) = \mathbb{C} \setminus \rho(L)$ of operator-valued function $L(\lambda)$, where $\rho(L)$ is the resolvent

set of $L(\lambda)$, we understand as the closure of the union of the sets of zeroes of the function $\ell_n(\lambda) = \lambda^2 + a_n^{2\theta}\big(a_n^{2(1-\theta)} - \sum_{n=1}^{\infty} \frac{c_k}{\lambda + \gamma_k}\big)$, i.e.,

$$\sigma(L) = \overline{(\cup_{n,k=1}^{\infty} \lambda_{n,k}) \cup (\cup_{n=1}^{\infty} \lambda_n^{\pm})},$$

where $\{\lambda_{n,k}\}_{n,k=1}^{\infty}$ and $\{\lambda_n^{\pm}\}_{n=1}^{\infty}$ are the real zeroes and complex-conjugate zeroes, respectively, of function $\ell_n(\lambda)$. It is important to underline that the spectrum $\sigma(L)$ of operator-valued function $L(\lambda)$ is contained in the left half-plane, if both conditions $a)$ and $a_1 \geq 1$ are satisfied. Indeed, the function $\varphi(\lambda) = \lambda^2$, where $\lambda = x + iy$, maps the upper right quadrant $\phi_{\pi/2} := \{\lambda \in \mathbb{C} : 0 < \arg(\lambda) < \pi/2\}$ onto the upper half-plane $\operatorname{Im} \lambda > 0$. In turn, the function $\psi(\lambda) = a_n^{2\theta}\big(-a_n^{2(1-\theta)} + \sum_{n=1}^{\infty} \frac{c_k}{\lambda + \gamma_k}\big)$ maps the angle $\phi_{\pi/2}$ into lower half-plane $\operatorname{Im} \lambda < 0$. Thus the equation $\varphi(\lambda) = \psi(\lambda)$, which is equivalent to the equation $\ell_n(\lambda) = 0$, has no solutions within the angle $\phi_{\pi/2}$. Since the function $\ell_n(\lambda)$ has real coefficients, the non-real zeroes are complex-conjugate zeroes of $\ell_n(\lambda)$. Therefore, the equation $\ell_n(\lambda) = 0$ has no zeroes within the lower right quadrant $\phi_{-\pi/2} := \{\lambda \in \mathbb{C} : -\pi/2 < \arg(\lambda) < 0\}$. It is also clear that the equation $\varphi(x) = \psi(x)$ has no solutions lying on the semi-axis $(0, +\infty)$, under the assumption that $a_1 \geq 1$. Consequently, the equation $\varphi(\lambda) = \psi(\lambda)$ has no solutions for such λ on which $\operatorname{Re} \lambda > 0$. We note also that on the imaginary axis and at the origin $(0,0)$ of the complex plane $\mathbb{C}$ there are no points of the spectrum $\sigma(L)$ of operator-valued function $L(\lambda)$. In fact, for $y \neq 0$, $\operatorname{Im} \ell_n(iy) = y\big(\sum_{n=1}^{\infty} \frac{c_k}{y^2 + \gamma_k^2}\big)a_n^{2\theta} \neq 0$. For $x = y = 0$, we have $\operatorname{Re} \ell_n(0) = a_n^{2\theta}\big(a_n^{2(1-\theta)} - \sum_{n=1}^{\infty} \frac{c_k}{\gamma_k}\big) > 0$ under the assumption that $a_1 \geq 1$.

As a result of the aforementioned, the analysis about the stability of solutions for system (2.1)–(2.2) becomes an important subject. But in this paper we have focused only on correct solvability of system (2.1)–(2.2).

Acknowledgements

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