Electronic Journal of Qualitative Theory of Differential Equations
2016, No. **33**, 1–43; doi: 10.14232/ejqtde.2016.1.33 http://www.math.u-szeged.hu/ejqtde/

Asymptotic integration of functional differential systems with oscillatory decreasing coefficients: a center manifold approach

Pavel Nesterov ✉

Yaroslavl State University, Sovetskaya str. 14, Yaroslavl, 150003, Russia

Received 25 November 2015, appeared 2 June 2016
Communicated by Mihály Pituk

Abstract. In this paper we study the asymptotic integration problem in the neighborhood of infinity for a certain class of linear functional differential systems. We propose a method for the construction of the asymptotics of solutions in the critical case. Using the ideas of the center manifold theory we show the existence of the so called critical manifold that is positively invariant for trajectories of the initial system. We establish that the dynamics of solutions lying on this manifold defines the asymptotics for all solutions. We illustrate the proposed method with an example of the construction of the asymptotics for solutions of a certain scalar delay differential equation.

Keywords: asymptotic integration, functional differential systems, center manifold, method of averaging, Levinson's theorem, oscillatory decreasing coefficients.

2010 Mathematics Subject Classification: 34K06, 34K19, 34K25, 34C29.

1 Introduction

We study the asymptotic integration problem for the functional differential system

$$\dot{x} = B_0 x_t + G(t, x_t) \tag{1.1}$$

as $t \to \infty$. Here $x \in \mathbb{C}^m$, $x_t(\theta) = x(t+\theta)$ $(-h \le \theta \le 0)$ denotes the element of C_h, where $C_h \equiv C\big([-h,0],\mathbb{C}^m\big)$ is the set of all continuous functions defined on $[-h,0]$ and acting to $\mathbb{C}^m$. Further, B_0 is a bounded linear functional acting from C_h to $\mathbb{C}^m$ and $G(t,x_t)$ has the form

$$G(t,x_t) = B(t,x_t) + R(t,x_t). \tag{1.2}$$

We assume that $B(t,\cdot)$ and $R(t,\cdot)$ are linear bounded functionals from C_h to $\mathbb{C}^m$ such that for each $\varphi \in C_h$ functions $B(\cdot,\varphi)$ and $R(\cdot,\varphi)$ are Lebesgue measurable for $t \ge t_0$ and, moreover,

$$|R(t,\varphi)| \le \gamma(t)\|\varphi\|_{C_h}, \quad \gamma(t) \in L_1[t_0,\infty) \quad \Big(\|\varphi\|_{C_h} = \sup_{-h\le\theta\le 0}|\varphi(\theta)|\Big). \tag{1.3}$$

✉ Email: nesterov.pn@gmail.com

The structure of the functional $B(t,\cdot)$ will be defined later. We note only that for each $\varphi \in C_h$ function $B(\cdot,\varphi)$ has, in general, an oscillatory decreasing form as $t \to \infty$.

This paper continues our studies of the asymptotic integration problem for Eq. (1.1) that we began in [25]. In the mentioned paper the case of the zero operator B_0 was discussed. In the case of nonzero operator B_0 the asymptotic integration method, developed in [25], cannot be applied. Since the case of nonzero operator B_0 is the situation of general position, it is of significant importance to provide the method for constructing the asymptotics in this case. We emphasize that the method, we propose below, is not an adaptation or extension of the corresponding method from [25]. However, both methods in the final step use the so-called method of averaging together with the known asymptotic theorems. This will be discussed in details in Section 5 of this paper.

Functional differential systems of the form (1.1) with linear and nonlinear functionals on the right-hand side, considered as perturbations of linear autonomous system

$$\dot{x} = B_0 x_t, \tag{1.4}$$

were studied by many authors (see, e.g., [1–4,9,11,13,14,27–29]). We also remark that the first asymptotic theorems for scalar delay differential equations were proposed by R. Bellman and K. L. Cooke [8] (see also [19, Chapter 9] for a brief survey).

Throughout the paper we study Eq. (1.1) under the condition that the characteristic equation

$$\det \Delta(\lambda) = 0, \qquad \Delta(\lambda) = \lambda I - B_0(e^{\lambda\theta} I), \tag{1.5}$$

has N roots (with account of their multiplicities) $\lambda_1, \ldots, \lambda_N$ with zero real parts and all other roots have negative real parts. This makes possible to use the ideas of the center manifold theory (see, e.g., [2,5,6,10]) for asymptotic integration of Eq. (1.1). The paper is devoted to the adaptation of this technique for the considered asymptotic integration problem. Particularly, a significant role will be also played by the averaging method proposed in [23] for the asymptotic integration of the ordinary differential systems with oscillatory decreasing coefficients.

This paper is organized as follows. In Section 2 we give some notations and facts from the theory of functional differential systems needed for the sequel. In Section 3 we propose an algorithm for an approximate construction of the so called critical manifold in C_h that is positively invariant for sufficiently large t for trajectories $x_t(\theta)$ of Eq. (1.1). It turns out that the dynamics of solutions of Eq. (1.1) lying on this manifold defines the asymptotics for all its solutions. The main theorems describing the properties of critical manifold are established in Section 4. Finally, in Section 5 we study the asymptotic integration problem as $t \to \infty$ for the system on critical manifold. In this section we also use the developed technique to construct the asymptotics as $t \to \infty$ for solutions of the scalar delay differential equation

$$\dot{x} = -\frac{\pi}{2} x(t-1) + \frac{a \sin \omega t}{t^{\rho}} x(t), \tag{1.6}$$

where $a, \omega \in \mathbb{R} \backslash \{0\}$ and $\rho > 0$. The proof of Theorem 3.2 from Section 3, concerning the solvability of certain algebraic problems, is given in Appendix.

2 Preliminaries

The facts and notations given in this section may be found in [19] (see also [20,22]).

We say that function $x(t)$ with values in $\mathbb{C}^m$ satisfies (1.1) for $t \geq T$ if $x(t)$ is continuous on $[T-h,\infty)$, absolutely continuous on $[T,\infty)$ and (1.1) holds almost everywhere on $[T,\infty)$. Under the above conditions, for each $\varphi \in C_h$ and each $T \geq t_0$ there is a unique $x(t)$ satisfying (1.1) for $T \geq t_0$ with $x_T = \varphi$. We will call the function $x(t)$ the solution of Eq. (1.1) with initial value $x_T = \varphi$.

It is known that linear autonomous equation (1.4) generates in C_h for $t \geq 0$ a strongly continuous semigroup $T(t)\colon C_h \to C_h$. The solution operator $T(t)$ of Eq. (1.4) is defined by $T(t)\varphi = x_t^{\varphi}(\theta)$, where $\varphi \in C_h$ and $x_t^{\varphi}(\theta)$ is a unique solution of (1.4) with initial value $x_0^{\varphi}(\theta) = \varphi$. The infinitesimal generator A of this semigroup is defined by $A\varphi = \varphi'(\theta)$ for $\varphi \in D(A)$. The domain of A

$$D(A) = \{\varphi \in C_h \mid \varphi'(\theta) \in C_h,\ \varphi'(0) = B_0\varphi\}$$

is dense in C_h. The following equalities hold:

$$\frac{d}{dt}T(t)\varphi = T(t)A\varphi = AT(t)\varphi, \qquad \varphi \in D(A). \tag{2.1}$$

In the sequel we will use the Riesz representation of B_0:

$$B_0\varphi = \int_{-h}^{0} d\eta(\theta)\varphi(\theta), \tag{2.2}$$

where $\eta(\theta)$ is $(m \times m)$-matrix function of bounded variation on $[-h,0]$. Using (2.2), we obtain the following expressions for matrix $\Delta(\lambda)$ from (1.5) and its derivatives:

$$\Delta(\lambda) = \lambda I - \int_{-h}^{0} d\eta(\theta)e^{\lambda\theta}, \qquad \Delta'(\lambda) = I - \int_{-h}^{0} \theta d\eta(\theta)e^{\lambda\theta},$$
$$\Delta^{(j)}(\lambda) = -\int_{-h}^{0} \theta^j d\eta(\theta)e^{\lambda\theta}, \qquad j \geq 2. \tag{2.3}$$

We can associate with (1.4) the transposed equation

$$\dot{y} = -\int_{-h}^{0} y(t-\theta)d\eta(\theta), \tag{2.4}$$

where $y(t)$ is an m-dimensional complex row vector. The phase space for (2.4) is $C_h' \equiv C([0,h],\mathbb{C}^{m*})$, where $\mathbb{C}^{m*}$ is the space of m-dimensional row vectors. For $\psi \in C_h'$ and $\varphi \in C_h$ we define the bilinear form

$$\big(\psi(\xi),\varphi(\theta)\big) = \psi(0)\varphi(0) - \int_{-h}^{0}\int_{0}^{\theta} \psi(\xi-\theta)d\eta(\theta)\varphi(\xi)d\xi. \tag{2.5}$$

If

$$\Lambda = \{\lambda_i \in \mathbb{C} \mid \det\Delta(\lambda_i) = 0,\ i = 1,\dots,N\}, \tag{2.6}$$

then we can decompose C_h into a direct sum

$$C_h = P_\Lambda \oplus Q_\Lambda. \tag{2.7}$$

Here P_Λ is the generalized eigenspace associated with Λ and Q_Λ is the complementary subspace of C_h such that $T(t)Q_\Lambda \subseteq Q_\Lambda$, $t \geq 0$. Let $\Phi(\theta)$ be the $(m \times N)$-matrix function whose columns are the generalized eigenfunctions $\varphi_1(\theta),\dots,\varphi_N(\theta)$ of A corresponding to the eigenvalues from Λ. Thus, the columns of $\Phi(\theta)$ form the basis of P_Λ. Moreover, let $\Psi(\xi)$ be

$(N \times m)$-matrix whose rows $\psi_1(\xi), \dots, \psi_N(\xi)$ form the basis of the generalized eigenspace P_Λ^T of the transposed equation (2.4) associated with Λ. We can choose matrices $\Phi(\theta)$ and $\Psi(\xi)$ such that

$$\big(\Psi(\xi), \Phi(\theta)\big) = \big\{\big(\psi_i(\xi), \varphi_j(\theta)\big)\big\}_{1 \le i,j \le N} = I. \tag{2.8}$$

Since $\Phi(\theta)$ is the basis of P_Λ and $AP_\Lambda \subseteq P_\Lambda$, there exists $(N \times N)$-matrix D, whose spectrum is Λ, such that $A\Phi(\theta) = \Phi(\theta)D$. From (2.1) and the definition of A, we deduce that

$$\Phi(\theta) = \Phi(0)e^{D\theta}, \qquad T(t)\Phi(\theta) = \Phi(\theta)e^{Dt} = \Phi(0)e^{D(t+\theta)}, \tag{2.9}$$

where $-h \le \theta \le 0$ and $t \ge 0$. Analogously, for matrix $\Psi(\xi)$ we have

$$\Psi(\xi) = e^{-D\xi}\Psi(0), \tag{2.10}$$

where $0 \le \xi \le h$. Matrices $\Phi(0)$ and $\Psi(0)$ are chosen in the following way. Since the columns of matrix $\Phi(\theta)$ are the generalized eigenfunctions of A, they should belong to $D(A)$. This implies that

$$\Phi'(0) = \Phi(0)D = B_0\Phi = \int_{-h}^{0} d\eta(\theta)\Phi(0)e^{D\theta}. \tag{2.11}$$

The same reasoning, using (2.4) and (2.10), yields

$$\Psi'(0) = -D\Psi(0) = -\int_{-h}^{0} e^{D\theta}\Psi(0)d\eta(\theta). \tag{2.12}$$

Finally, the spaces P_Λ and Q_Λ from decomposition (2.7) of C_h may be defined as follows:

$$\begin{aligned} P_\Lambda &= \big\{\varphi \in C_h \mid \varphi(\theta) = \Phi(\theta)a,\ a \in \mathbb{C}^N\big\}, \\ Q_\Lambda &= \big\{\varphi \in C_h \mid (\Psi, \varphi) = 0\big\}. \end{aligned} \tag{2.13}$$

Let $x_t(\theta)$ be the solution of (1.1) for $t \ge t_0$ with initial value $x_{t_0} = \varphi$. The following variation-of-constants formula holds (see [20]):

$$x_t(\theta) = T(t - t_0)\varphi + \int_{t_0}^{t} dK(t,s)G(s, x_s)ds, \qquad t \ge t_0. \tag{2.14}$$

Here the kernel $K(t, \cdot)\colon [t_0, t] \to C_h$ is given by

$$K(t,s)(\theta) = \int_{t_0}^{s} X(t + \theta - \alpha)d\alpha, \tag{2.15}$$

where $X(t)$ is the fundamental matrix of (1.4), i.e., the unique matrix solution of (1.4) with initial condition

$$X_0(\theta) = \begin{cases} I, & \theta = 0, \\ 0, & -h \le \theta < 0. \end{cases} \tag{2.16}$$

We can write (2.14) formally as (see [19])

$$x_t(\theta) = T(t - t_0)\varphi + \int_{t_0}^{t} T(t - s)X_0(\theta)G(s, x_s)ds, \qquad t \ge t_0, \tag{2.17}$$

where $T(t - s)X_0(\theta) = X(t + \theta - s)$.

We decompose now solution $x_t(\theta)$ of Eq. (1.1) with initial value $x_{t_0} = \varphi$ according to (2.7). By (2.14), we have

$$x_t(\theta) = x_t^{P_\Lambda} + x_t^{Q_\Lambda}, \qquad \varphi(\theta) = \varphi^{P_\Lambda} + \varphi^{Q_\Lambda}, \tag{2.18}$$

$$x_t^{P_\Lambda}(\theta) = T(t-t_0)\varphi^{P_\Lambda} + \int_{t_0}^{t} T(t-s)X_0^{P_\Lambda}(\theta)G(s,x_s)ds, \tag{2.19}$$

$$x_t^{Q_\Lambda}(\theta) = T(t-t_0)\varphi^{Q_\Lambda} + \int_{t_0}^{t} d\big[K(t,s)^{Q_\Lambda}\big]G(s,x_s)ds, \tag{2.20}$$

where $t \geq t_0$ and

$$X_0^{P_\Lambda}(\theta) = \Phi(\theta)\Psi(0), \qquad K(t,s)^{Q_\Lambda} = K(t,s) - \Phi(\theta)\big(\Psi, K(t,s)\big). \tag{2.21}$$

If we make decomposition (2.18) in (2.17), we obtain formulas analogous to (2.19), (2.20). The only difference is that (2.20) should be replaced by formula

$$x_t^{Q_\Lambda}(\theta) = T(t-t_0)\varphi^{Q_\Lambda} + \int_{t_0}^{t} T(t-s)X_0^{Q_\Lambda}(\theta)G(s,x_s)ds, \tag{2.22}$$

where $X_0^{Q_\Lambda} = X_0(\theta) - X_0^{P_\Lambda}(\theta)$. It is sometimes more appropriate to use (2.22) instead of (2.20). We should only keep in mind that to attain the necessary mathematical strictness we need to replace integrands of the form $T(t-s)X_0(\theta)(\ldots)ds$ and $T(t-s)X_0^{Q_\Lambda}(\ldots)ds$ in the obtained formulas by integrands $dK(t,s)(\theta)(\ldots)$ and $d\big[K(t,s)^{Q_\Lambda}(\theta)\big](\ldots)$ respectively. Let

$$x_t^{P_\Lambda}(\theta) = \Phi(\theta)u(t), \qquad u(t) \in \mathbb{C}^N, \tag{2.23}$$

then $u(t) = (\Psi, x_t)$ and, moreover, function $u(t)$ is the solution of ordinary differential system

$$\dot{u} = Du + \Psi(0)G(t, x_t), \qquad t \geq t_0 \tag{2.24}$$

with initial condition $u(t_0) = (\Psi, \varphi)$.

Assume that Λ is defined by (2.6) and suppose that it coincides with the set

$$\big\{\lambda \in \mathbb{C} \mid \det\Delta(\lambda) = 0,\ \operatorname{Re}\lambda > \beta\big\} \tag{2.25}$$

for some $\beta \in \mathbb{R}$. Then for any $\varepsilon > 0$ there exists constant $M = M(\varepsilon)$ such that the following inequalities hold:

$$\|T(t)\varphi^{Q_\Lambda}\|_{C_h} \leq Me^{(\beta+\varepsilon)t}\|\varphi^{Q_\Lambda}\|_{C_h}, \qquad t \geq 0,\ \varphi \in C_h, \tag{2.26}$$

$$\|T(t)X_0^{Q_\Lambda}\|_{C_h} \leq Me^{(\beta+\varepsilon)t}, \qquad t \geq 0, \tag{2.27}$$

$$\left\|\int_{t_0}^{t} d\big[K(t,s)^{Q_\Lambda}\big]G(s,x_s)ds\right\|_{C_h} \leq M\int_{t_0}^{t} e^{(\beta+\varepsilon)(t-s)}\big|G(s,x_s)\big|ds, \qquad t \geq t_0. \tag{2.28}$$

Note that the matrix $T(t)X_0^{Q_\Lambda}$ on the left-hand side of inequality (2.27) belongs C_h only for $t \geq h$. Nevertheless, we can use the norm of C_h on the left-hand of (2.27) since $T(t)X_0^{Q_\Lambda}$ is bounded in $\theta \in [-h, 0]$ for $t \in [0, h]$.

3 Approximate construction of the critical manifold

We begin this section by clarifying the form of the operator $B(t,\varphi)$ in (1.2). Suppose that

$$B(t,\varphi)=\sum_{i=1}^{n}v_i(t)B_i(t,\varphi)+\sum_{1\le i_1\le i_2\le n}v_{i_1}(t)v_{i_2}(t)B_{i_1i_2}(t,\varphi)\\ +\cdots+\sum_{1\le i_1\le\cdots\le i_k\le n}v_{i_1}(t)\cdots v_{i_k}(t)B_{i_1\dots i_k}(t,\varphi). \tag{3.1}$$

Here $B_{i_1\dots i_l}(t,\cdot)$ are bounded linear functionals acting from C_h to $\mathbb{C}^m$. We assume that

$$B_{i_1\dots i_l}(t,\varphi)=\sum_{j=1}^{L}\Gamma_j^{(i_1\dots i_l)}(t)\ell_j^{(i_1\dots i_l)}(\varphi),\qquad \varphi\in C_h. \tag{3.2}$$

In the formula above, $\ell_j^{(i_1\dots i_l)}(\varphi)$ are bounded linear functionals acting from C_h to $\mathbb{C}^m$ that do not depend on t and $\Gamma_j^{(i_1\dots i_l)}(t)$ are some matrices whose entries are trigonometric polynomials, i.e.,

$$\Gamma_j^{(i_1\dots i_l)}(t)=\sum_{s=1}^{M}\beta_{sj}^{(i_1\dots i_l)}e^{\mathrm{i}\omega_s t}, \tag{3.3}$$

where $\beta_{sj}^{(i_1\dots i_l)}$ are constant complex $(m\times m)$-matrices and ω_s are real numbers. Finally, $v_1(t),\dots,v_n(t)$ are absolutely continuous functions acting from $[t_0,\infty)$ to $\mathbb{C}$ such that

1^0. $v_1(t)\to 0, v_2(t)\to 0,\dots,v_n(t)\to 0$ as $t\to\infty$;

2^0. $\dot v_1(t),\dot v_2(t),\dots,\dot v_n(t)\in L_1[t_0,\infty)$;

3^0. There exists $k\in\mathbb{N}$ such that $v_{i_1}(t)v_{i_2}(t)\cdots v_{i_{k+1}}(t)\in L_1[t_0,\infty)$ for any sequence $1\le i_1\le i_2\le\cdots\le i_{k+1}\le n$.

We now define the set Λ by formula (2.6). Assume that the following hypotheses hold.

$\mathbf{H_1}$. $\operatorname{Re}\lambda=0$ for all $\lambda\in\Lambda$;

$\mathbf{H_2}$. The set Λ coincides with set (2.25) for some $\beta<0$.

Note that if hypotheses $\mathbf{H_1}$, $\mathbf{H_2}$ do not hold for Eq. (1.1) with operator $G(t,x_t)$ having form (1.2), (1.3), (3.1), then we can make the change of variable $x(t)=y(t)e^{dt}$, where

$$d=\sup\{\operatorname{Re}\lambda\mid\det\Delta(\lambda)=0\}.$$

The transformed system will have the same structure as the initial one (with another functionals having the same properties) and, moreover, hypotheses $\mathbf{H_1}$, $\mathbf{H_2}$ will hold for it. We remark that the verification of hypotheses $\mathbf{H_1}$, $\mathbf{H_2}$ for a certain Eq. (1.1) is not a trivial problem. This problem is typical, say, for bifurcation theory. As a rule, various algebraic methods, methods from complex analysis and the methods to study location of the operator spectrum are used to establish the validity of hypotheses $\mathbf{H_1}$, $\mathbf{H_2}$ or, at least, to find the quantity d (see, e.g., [8, Chapters 12, 13]). Thus, it is a distinct and a serious problem and, therefore, it is not studied here. Finally, we note that hypothesis $\mathbf{H_2}$ ensures that the critical manifold, defining below, possesses the property of global attraction, i.e., it attracts all the solutions of Eq. (1.1).

We decompose now C_h by Λ into direct sum (2.7).

Definition 3.1. A set $\mathcal{W}(t) \subset C_h$ (linear space) is said to be a critical (or center-like) manifold of Eq. (1.1) for $t \geq t_* \geq t_0$ if the following conditions hold.

1. There exists an $(m \times N)$-matrix function $H(t,\theta)$ which is continuous in $t \geq t_*$ and $\theta \in [-h,0]$ with columns belonging to Q_Λ for $t \geq t_*$ such that $\|H(t,\cdot)\|_{C_h} \to 0$ as $t \to \infty$, where
$$\|H(t,\cdot)\|_{C_h} = \sup_{-h\leq\theta\leq 0} |H(t,\theta)|$$
and $|\cdot|$ is some matrix norm;
2. For $t \geq t_*$, the set $\mathcal{W}(t)$ has the form
$$\mathcal{W}(t) = \left\{\varphi(\theta) \in C_h \mid \varphi(\theta) = \Phi(\theta)u + H(t,\theta)u,\ u \in \mathbb{C}^N\right\}, \tag{3.4}$$
where $\Phi(\theta)$ is a basis for a generalized eigenspace P_Λ from (2.7);
3. The set $\mathcal{W}(t)$ is positively invariant for trajectories of Eq. (1.1) for $t \geq t_*$, i.e., if $x_T \in \mathcal{W}(T)$, $T \geq t_*$, then $x_t \in \mathcal{W}(t)$ for $t \geq T$.

Assume that a critical manifold $\mathcal{W}(t)$ of Eq. (1.1) exists for sufficiently large t (the corresponding theorem will be proved in the next section). We propose the method for construction of a certain matrix that is an approximation in some sense for the matrix $H(t,\theta)$ from (3.4). An algorithm we describe below has much in common with an approximation scheme of a center manifold for nonlinear functional differential systems (see, e.g., [2]).

Let $x(t)$ be the solution of Eq. (1.1) with initial value at $t = T \geq t_* \geq t_0$. Then for $t+\theta \geq T$ we have the following equalities:
$$\frac{d}{dt}x_t(\theta) = \begin{cases} \dfrac{d}{d\theta}x_t(\theta), & -h \leq \theta < 0, \\ B_0 x_t + G(t,x_t), & \theta = 0. \end{cases} \tag{3.5}$$
Suppose that at the initial moment $t = T$ the vector function $x_T(\theta)$ belongs to $\mathcal{W}(T)$. Due to the positively invariance of $\mathcal{W}(t)$ we obtain that
$$x_t(\theta) = \Phi(\theta)u(t) + H(t,\theta)u(t), \qquad t \geq T, \quad u(t) \in \mathbb{C}^N. \tag{3.6}$$
We remark that formula (3.6) is, actually, decomposition (2.18). Consequently, by (2.24), function $u(t)$ satisfies the ordinary differential system
$$\dot{u} = \Big[D + \Psi(0)G\big(t,\Phi(\theta) + H(t,\theta)\big)\Big]u, \qquad t \geq T. \tag{3.7}$$
This system will be referred to as a projection of Eq. (1.1) on critical manifold $\mathcal{W}(t)$ or, simply, as a *system on critical manifold*. We substitute (3.6) in (3.5). This gives for $t+\theta \geq T$
$$\big(\Phi(\theta) + H(t,\theta)\big)\dot{u}(t) + \frac{\partial H}{\partial t}u = \begin{cases} \left[\dfrac{\partial \Phi}{\partial \theta} + \dfrac{\partial H}{\partial \theta}\right]u, & -h \leq \theta < 0, \\ \big[B_0\Phi + B_0 H + G\big(t,\Phi(\theta) + H(t,\theta)\big)\big]u, & \theta = 0. \end{cases}$$
We then use (3.7) for $\dot{u}$ and also (2.9), (2.11). We conclude that
$$\begin{aligned} \Phi(\theta)\Psi(0)G\big(t,\Phi(\theta) + H(t,\theta)\big) + H(t,\theta)\,\big(D + \Psi(0)G\big(t,\Phi(\theta) + H(t,\theta)\big)\big) + \frac{\partial H}{\partial t} \\ = \begin{cases} \dfrac{\partial H}{\partial \theta}, & -h \leq \theta < 0, \\ B_0 H + G\big(t,\Phi(\theta) + H(t,\theta)\big), & \theta = 0. \end{cases} \end{aligned} \tag{3.8}$$

Therefore, if a critical manifold $\mathcal{W}(t)$ exists for $t \geq t_*$ then for all (t,θ) such that $t+\theta \geq t_*$ matrix $H(t,\theta)$ should satisfy Eq. (3.8). Particularly, since the solution $x(t)$ ($x_{t_*} = \varphi \in \mathcal{W}(t_*)$) is absolutely continuous for $t \geq t_*$ it follows from (3.6) that matrix $H(t,\theta)$ is absolutely continuous in t for $t \geq t_* - \theta$ and in θ for $\theta \geq t_* - t$.

We will try now to satisfy Eq. (3.8) up to terms $\hat{R}(t,\theta)$ such that $\|\hat{R}(t,\cdot)\|_{C_h} \in L_1[t_0,\infty)$. Namely, let

$$\hat{H}(t,\theta) = \sum_{i=1}^{n} v_i(t)H_i(t,\theta) + \sum_{1\leq i_1\leq i_2\leq n} v_{i_1}(t)v_{i_2}(t)H_{i_1i_2}(t,\theta) + \cdots + \sum_{1\leq i_1\leq\cdots\leq i_k\leq n} v_{i_1}(t)\cdots v_{i_k}(t)H_{i_1\ldots i_k}(t,\theta). \tag{3.9}$$

Here the entries of $(m\times N)$-matrices $H_{i_1\ldots i_l}(t,\theta)$ to be found are trigonometric polynomials in t and continuously differentiable in $\theta \in [-h,0]$. The natural number k is defined by property 3^0 of functions $v_1(t),\ldots,v_n(t)$. Moreover, we assume that the columns of matrices $H_{i_1\ldots i_l}(t,\theta)$ belong to Q_Λ for all $t\in\mathbb{R}$. We substitute (3.9) for $H(t,\theta)$ in (3.8) and collect terms corresponding to factors $v_{i_1}(t)\cdots v_{i_l}(t)$ ($l\leq k$). We obtain the following equations for matrices $H_{i_1\ldots i_l}(t,\theta)$:

$$F^{P_\Lambda}_{i_1\ldots i_l}(t,\theta) + H_{i_1\ldots i_l}(t,\theta)D + F^{Q_\Lambda}_{i_1\ldots i_l}(t,\theta) + \frac{\partial H_{i_1\ldots i_l}}{\partial t} = \begin{cases} \dfrac{\partial H_{i_1\ldots i_l}}{\partial\theta}, & -h\leq\theta<0,\\ B_0H_{i_1\ldots i_l} + G_{i_1\ldots i_l}(t), & \theta=0.\end{cases} \tag{3.10}$$

Here we also used formulas (1.2), (3.1). These formulas yield, in particular, that to solve Eq. (3.10) we need to compute matrices $H_{j_1\ldots j_s}(t,\theta)$ with $s<l$. Then $F^{P_\Lambda}_{i_1\ldots i_l}(t,\theta)$ and $F^{Q_\Lambda}_{i_1\ldots i_l}(t,\theta)$ in Eq. (3.10) are some well-defined matrices that include matrices $H_{j_1\ldots j_s}(t,\theta)$ ($s<l$) defined in the earlier steps. By (1.2), (3.1), (3.2) and constraints imposed on matrices $H_{j_1\ldots j_s}(t,\theta)$ ($s<l$) we conclude that the entries of matrices $F^{P_\Lambda}_{i_1\ldots i_l}(t,\theta)$ and $F^{Q_\Lambda}_{i_1\ldots i_l}(t,\theta)$ are trigonometric polynomials in t. Moreover, the columns of matrix $F^{P_\Lambda}_{i_1\ldots i_l}(t,\theta)$ belong to P_Λ and the columns of matrix $F^{Q_\Lambda}_{i_1\ldots i_l}(t,\theta)$ belong to Q_Λ for all $t\in\mathbb{R}$. Further, $G_{i_1\ldots i_l}(t)$ is a certain matrix, whose entries are trigonometric polynomials:

$$G_{i_1\ldots i_l}(t) = \sum_j G_j^{(i_1\ldots i_l)} e^{\mathrm{i}\omega_j t}, \tag{3.11}$$

where $G_j^{(i_1\ldots i_l)}$ are constant $(m\times N)$-matrices and ω_j are real numbers. It follows from (3.8) that matrices $F^{P_\Lambda}_{i_1\ldots i_l}(t,\theta)$ and $F^{Q_\Lambda}_{i_1\ldots i_l}(t,\theta)$ have the following form:

$$F^{P_\Lambda}_{i_1\ldots i_l}(t,\theta) = \Phi(\theta)\Psi(0)G_{i_1\ldots i_l}(t) = \sum_j P_j^{(i_1\ldots i_l)}(\theta)e^{\mathrm{i}\omega_j t}, \tag{3.12}$$

$$P_j^{(i_1\ldots i_l)}(\theta) = \Phi(\theta)\Psi(0)G_j^{(i_1\ldots i_l)}, \tag{3.13}$$

$$F^{Q_\Lambda}_{i_1\ldots i_l}(t,\theta) = \sum_j Q_j^{(i_1\ldots i_l)}(\theta)e^{\mathrm{i}\omega_j t}, \tag{3.14}$$

where $P_j^{(i_1\ldots i_l)}(\theta)$ and $Q_j^{(i_1\ldots i_l)}(\theta)$ are certain matrices continuously differentiable on $-h\leq\theta\leq 0$.

We seek solution of Eq. (3.10) in the form

$$H_{i_1\ldots i_l}(t,\theta) = \sum_j \beta_j^{(i_1\ldots i_l)}(\theta)e^{\mathrm{i}\omega_j t}, \tag{3.15}$$

where $\beta_j^{(i_1\dots i_l)}(\theta)$ are some continuously differentiable on $-h \le \theta \le 0$ matrices to be defined. We substitute (3.11)–(3.15) in (3.10) and match the coefficients of the corresponding exponentials. Omitting for the sake of brevity the dependence of matrices on the indices set $(i_1 \dots i_l)$ and also on the index j, we obtain the following functional boundary value problem for matrix $\beta(\theta) = \beta_j^{(i_1\dots i_l)}(\theta)$:

$$\begin{cases} \dfrac{d\beta}{d\theta} = \beta(\theta)(D + \mathrm{i}\omega_j I) + P(\theta) + Q(\theta), \quad -h \le \theta < 0, \\ \beta(0)(D + \mathrm{i}\omega_j I) + P(0) + Q(0) = B_0\beta + G, \end{cases} \tag{3.16}$$

where $P(\theta) = P_j^{(i_1\dots i_l)}(\theta)$, $Q(\theta) = Q_j^{(i_1\dots i_l)}(\theta)$ and $G = G_j^{(i_1\dots i_l)}$. Note that since solution $\beta(\theta)$ of (3.16) should be continuous on $-h \le \theta \le 0$ we need the solve the first equation of this problem with initial condition $\beta(0)$ that is defined from the second equation. We also should take into account that the columns of $\beta(\theta)$ belong to Q_Λ. Consequently, due to (2.13), we should solve problem (3.16) together with additional condition

$$\big(\Psi(\xi), \beta(\theta)\big) = 0. \tag{3.17}$$

We assume that matrix D, whose spectrum is Λ, has Jordan canonical form

$$D = \mathrm{diag}(D^{(1)}, \dots, D^{(l)}), \qquad D^{(i)} = \begin{pmatrix} \lambda^{(i)} & 1 & 0 & \dots & 0 \\ 0 & \lambda^{(i)} & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & \lambda^{(i)} & 1 \\ 0 & \dots & \dots & 0 & \lambda^{(i)} \end{pmatrix}, \tag{3.18}$$

where $\lambda^{(i)} \in \Lambda$, $D^{(i)}$ is $(N_i \times N_i)$-matrix and $N_1 + \dots + N_l = N$. We write matrices $\beta(\theta)$, $P(\theta)$, $Q(\theta)$ and G in the following form:

$$\begin{aligned} \beta(\theta) &= \big[\beta^{(1)}(\theta), \dots, \beta^{(l)}(\theta)\big], & P(\theta) &= \big[P^{(1)}(\theta), \dots, P^{(l)}(\theta)\big], \\ Q(\theta) &= \big[Q^{(1)}(\theta), \dots, Q^{(l)}(\theta)\big], & G &= \big[G^{(1)}, \dots, G^{(l)}\big]. \end{aligned} \tag{3.19}$$

Here $\beta^{(i)}(\theta)$, $P^{(i)}(\theta)$, $Q^{(i)}(\theta)$, $G^{(i)}$ are $(m \times N_i)$-matrices and square brackets $[\cdot, \dots, \cdot]$ stand for the matrix whose columns are vectors pointed inside the brackets and located in the natural order from left to right. Then we can rewrite (3.16), (3.17) in the form of l independent subsystems

$$\begin{cases} \dfrac{d\beta^{(i)}}{d\theta} = \beta^{(i)}(\theta)(D^{(i)} + \mathrm{i}\omega_j I) + P^{(i)}(\theta) + Q^{(i)}(\theta), \qquad -h \le \theta < 0, \\ \beta^{(i)}(0)(D^{(i)} + \mathrm{i}\omega_j I) + P^{(i)}(0) + Q^{(i)}(0) = B_0\beta^{(i)} + G^{(i)}, \\ \big(\Psi(\xi), \beta^{(i)}(\theta)\big) = 0, \end{cases} \tag{3.20}$$

where I is the identity matrix of the order N_i and $i = 1, \dots, l$. Let

$$\begin{aligned} \beta^{(i)}(\theta) &= \big[z_1^{(i)}(\theta), \dots, z_{N_i}^{(i)}(\theta)\big], & P^{(i)}(\theta) &= \big[p_1^{(i)}(\theta), \dots, p_{N_i}^{(i)}(\theta)\big], \\ Q^{(i)}(\theta) &= \big[q_1^{(i)}(\theta), \dots, q_{N_i}^{(i)}(\theta)\big], & G^{(i)} &= \big[g_1^{(i)}, \dots, g_{N_i}^{(i)}\big], \end{aligned} \tag{3.21}$$

where $z_s^{(i)}(\theta)$, $p_s^{(i)}(\theta)$, $q_s^{(i)}(\theta)$, $g_s^{(i)}$ $(s = 1, \dots, N_i)$ are m-dimensional column vectors. Finally, let

$$\begin{aligned} \tilde{Z}^{(i)}(\theta) &= \mathrm{col}\big(z_1^{(i)}(\theta), \dots, z_{N_i}^{(i)}(\theta)\big), & \tilde{P}^{(i)}(\theta) &= \mathrm{col}\big(p_1^{(i)}(\theta), \dots, p_{N_i}^{(i)}(\theta)\big), \\ \tilde{Q}^{(i)}(\theta) &= \mathrm{col}\big(q_1^{(i)}(\theta), \dots, q_{N_i}^{(i)}(\theta)\big), & \tilde{G}^{(i)} &= \mathrm{col}\big(g_1^{(i)}, \dots, g_{N_i}^{(i)}\big), \\ B_0\tilde{Z}^{(i)} &= \mathrm{col}\big(B_0 z_1^{(i)}, \dots, B_0 z_{N_i}^{(i)}\big) \end{aligned} \tag{3.22}$$

denote mN_i-dimensional column vectors composed from the vectors pointed in $\mathrm{col}(\cdot, \dots, \cdot)$ located from the top downward in the natural order. Using these notations we rewrite (3.20) as follows:

$$\begin{cases} \dfrac{d\tilde{Z}^{(i)}}{d\theta} = \mathbf{A}^{(i)}\tilde{Z}^{(i)}(\theta) + \tilde{P}^{(i)}(\theta) + \tilde{Q}^{(i)}(\theta), \qquad -h \le \theta < 0, \\ \mathbf{A}^{(i)}\tilde{Z}^{(i)}(0) + \tilde{P}^{(i)}(0) + \tilde{Q}^{(i)}(0) = B_0\tilde{Z}^{(i)} + G^{(i)}, \\ \big(\Psi(\xi), z_s^{(i)}(\theta)\big) = 0, \qquad s = 1, \dots, N_i. \end{cases} \tag{3.23}$$

Here the $(mN_i \times mN_i)$-matrix $\mathbf{A}^{(i)}$ is defined by formula

$$\mathbf{A}^{(i)} = \begin{pmatrix} \mu^{(i)} I & 0 & \dots & \dots & 0 \\ I & \mu^{(i)} I & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & I & \mu^{(i)} I & 0 \\ 0 & \dots & 0 & I & \mu^{(i)} I \end{pmatrix}, \qquad \mu^{(i)} = \lambda^{(i)} + \mathrm{i}\omega_j, \tag{3.24}$$

where I is the identity matrix of the order m.

Solving the first equation in (3.23), we obtain

$$\tilde{Z}^{(i)}(\theta) = e^{\mathbf{A}^{(i)}\theta}\tilde{Z}^{(i)}(0) + \int_0^\theta e^{\mathbf{A}^{(i)}(\theta - s)}\big(\tilde{P}^{(i)}(s) + \tilde{Q}^{(i)}(s)\big)ds, \quad -h \le \theta < 0. \tag{3.25}$$

We substitute this expression into the second equation in (3.23). This results in the following linear algebraic equation for vector $\tilde{Z}^{(i)}(0)$:

$$\big(\mathbf{A}^{(i)} - B_0 e^{\mathbf{A}^{(i)}\theta}\big)\tilde{Z}^{(i)}(0) = B_0\left(\int_0^\theta e^{\mathbf{A}^{(i)}(\theta - s)}\left(\tilde{P}^{(i)}(s) + \tilde{Q}^{(i)}(s)\right)ds\right) + G^{(i)} - \tilde{P}^{(i)}(0) - \tilde{Q}^{(i)}(0).$$

Here we apply the functional B_0 to the columns of $(mN_i \times mN_i)$-matrix $e^{\mathbf{A}^{(i)}\theta}$ in the same way as in (3.22) taking into account that

$$e^{\mathbf{A}^{(i)}\theta} = \begin{pmatrix} I & 0 & \dots & \dots & 0 \\ \theta I & I & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \frac{\theta^{N_i - 2}}{(N_i - 2)!} I & \dots & \theta I & I & 0 \\ \frac{\theta^{N_i - 1}}{(N_i - 1)!} I & \frac{\theta^{N_i - 2}}{(N_i - 2)!} I & \dots & \theta I & I \end{pmatrix} e^{\mu^{(i)}\theta}.$$

We recall now formulas (2.2), (2.3), notations (3.22) and the third equation in (3.23). We obtain the following algebraic problems for vectors $z_1^{(i)}(0),\dots,z_{N_i}^{(i)}(0)$ $(i=1,\dots,l)$.

$$\mathbf{P_1}:\quad \begin{cases} \Delta(\mu^{(i)})z_1^{(i)}(0)=B_0\Big(\int_0^\theta e^{\mu^{(i)}(\theta-s)}\big(p_1^{(i)}(s)+q_1^{(i)}(s)\big)ds\Big)+g_1^{(i)}-p_1^{(i)}(0)-q_1^{(i)}(0),\\ \big(\Psi(\xi),e^{\mu^{(i)}\theta}I\big)z_1^{(i)}(0)=-\Big(\Psi(\xi),\int_0^\theta e^{\mu^{(i)}(\theta-s)}\big(p_1^{(i)}(s)+q_1^{(i)}(s)\big)ds\Big); \end{cases} \tag{3.26}$$

$$\mathbf{P_2}:\quad \begin{cases} \Delta'(\mu^{(i)})z_1^{(i)}(0)+\Delta(\mu^{(i)})z_2^{(i)}(0)\\ \quad =B_0\Big(\int_0^\theta\Big((\theta-s)e^{\mu^{(i)}(\theta-s)}\big(p_1^{(i)}(s)+q_1^{(i)}(s)\big)+e^{\mu^{(i)}(\theta-s)}\big(p_2^{(i)}(s)+q_2^{(i)}(s)\big)\Big)ds\Big)\\ \qquad +g_2^{(i)}-p_2^{(i)}(0)-q_2^{(i)}(0),\\ \big(\Psi(\xi),\theta e^{\mu^{(i)}\theta}I\big)z_1^{(i)}(0)+\big(\Psi(\xi),e^{\mu^{(i)}\theta}I\big)z_2^{(i)}(0)\\ \quad =-\Big(\Psi(\xi),\int_0^\theta(\theta-s)e^{\mu^{(i)}(\theta-s)}\big(p_1^{(i)}(s)+q_1^{(i)}(s)\big)ds\Big)\\ \qquad -\Big(\Psi(\xi),\int_0^\theta e^{\mu^{a(i)}(\theta-s)}\big(p_2^{(i)}(s)+q_2^{(i)}(s)\big)ds\Big); \end{cases} \tag{3.27}$$

$$\vdots$$

$$\mathbf{P_{N_i}}:\quad \begin{cases} \dfrac{\Delta^{(N_i-1)}(\mu^{(i)})}{(N_i-1)!}z_1^{(i)}(0)+\cdots+\Delta'(\mu^{(i)})z_{N_i-1}^{(i)}(0)+\Delta(\mu^{(i)})z_{N_i}^{(i)}(0)\\ \quad =B_0\Big(\int_0^\theta\Big(\dfrac{(\theta-s)^{N_i-1}}{(N_i-1)!}e^{\mu^{(i)}(\theta-s)}\big(p_1^{(i)}(s)+q_1^{(i)}(s)\big)\\ \qquad +\cdots+e^{\mu^{(i)}(\theta-s)}\big(p_{N_i}^{(i)}(s)+q_{N_i}^{(i)}(s)\big)\Big)ds\Big)+g_{N_i}^{(i)}-p_{N_i}^{(i)}(0)-q_{N_i}^{(i)}(0),\\ \Big(\Psi(\xi),\dfrac{\theta^{N_i-1}}{(N_i-1)!}e^{\mu^{(i)}\theta}I\Big)z_1^{(i)}(0)+\cdots+\big(\Psi(\xi),\theta e^{\mu^{(i)}\theta}I\big)z_{N_i-1}^{(i)}(0)\\ +\big(\Psi(\xi),e^{\mu^{(i)}\theta}I\big)z_{N_i}^{(i)}(0)\\ \quad =-\Big(\Psi(\xi),\int_0^\theta\dfrac{(\theta-s)^{N_i-1}}{(N_i-1)!}e^{\mu^{(i)}(\theta-s)}\big(p_1^{(i)}(s)+q_1^{(i)}(s)\big)ds\Big)\\ \qquad -\cdots-\Big(\Psi(\xi),\int_0^\theta e^{\mu^{(i)}(\theta-s)}\big(p_{N_i}^{(i)}(s)+q_{N_i}^{(i)}(s)\big)ds\Big). \end{cases} \tag{3.28}$$

We can now formulate the main result of this section.

Theorem 3.2. *System* (3.23) *has a unique solution* $\tilde Z^{(i)}(\theta)$ $(i=1,\dots,l)$ *that is continuously differentiable on* $-h\le\theta\le 0$. *This solution is defined by formula* (3.25), *where the components of the initial vector* $\tilde Z^{(i)}(0)$ *are unique solutions of problems* $\mathbf{P_1},\dots,\mathbf{P_{N_i}}$.

We note that, since $\tilde P^{(i)}(\theta)$ and $\tilde Q^{(i)}(\theta)$ are smooth on $-h\le\theta\le0$, the continuous differentiability of $\tilde Z^{(i)}(\theta)$ follows immediately from (3.25). Moreover, solution $\tilde Z^{(i)}(\theta)$ is infinitely differentiable on $-h\le\theta\le0$ because the entries of matrix $\Phi(\theta)$ (and, therefore, the components of vectors $\tilde P^{(i)}(\theta)$, $\tilde Q^{(i)}(\theta)$ as well) are infinitely differentiable. The proof of Theorem 3.2 is given in the Appendix.

We conclude that due to properties 2^0 and 3^0 of functions $v_1(t),\dots,v_n(t)$ the constructed matrix $\hat{H}(t,\theta)$ of the form (3.9) satisfies the following equation:

$$\Phi(\theta)\Psi(0)G\big(t,\Phi(\theta)+\hat{H}(t,\theta)\big)+\hat{H}(t,\theta)\Big(D+\Psi(0)G\big(t,\Phi(\theta)+\hat{H}(t,\theta)\big)\Big)+\frac{\partial\hat{H}}{\partial t}$$
$$=\begin{cases}\dfrac{\partial\hat{H}}{\partial\theta}+R_1(t,\theta), & -h\le\theta<0,\\ B_0\hat{H}+G\big(t,\Phi(\theta)+\hat{H}(t,\theta)\big)+R_1(t,0)-R_2(t), & \theta=0.\end{cases} \tag{3.29}$$

Here $R_1(\cdot,\theta)$ and $R_2(\cdot)$ are $(m\times N)$-matrix functions which are absolutely integrable on $[t_0,\infty)$, i.e., $\|R_1(t,\cdot)\|_{C_h}$, $R_2(t)\in L_1[t_0,\infty)$. The matrix $R_2(t)$ is composed of the absolutely integrable on $[t_0,\infty)$ part of the matrix $G\big(t,\Phi(\theta)+\hat{H}(t,\theta)\big)$.

At the end of this section we remark that in the case of ordinary differential system (1.1) to construct the approximation for critical manifold we can use the scheme proposed in [26].

4 Main theorems

Main theorems concerning the properties of critical manifold will be established in this section. The proof schemes of the corresponding theorems follow the main steps of the similar results in the center manifolds theory for nonlinear differential systems. In our paper we apply the classical scheme from [10].

In the sequel, the following proposition will play the central role (see, e.g., [18, pp. 18–19], [21, Chapter XIII]).

Proposition 4.1. *Suppose that the nonnegative function $p(t)$ is locally integrable on $[t_*,\infty)$ and*

$$\int_t^{t+1}p(s)ds\to 0,\qquad t\to\infty. \tag{4.1}$$

Let

$$f(t)=\int_{t_*}^{\infty}e^{-\alpha|t-s|}p(s)ds,\qquad \alpha>0. \tag{4.2}$$

Then $f(t)\to 0$ as $t\to\infty$. Moreover, if $p(t)\in L_1[t_*,\infty)$ then $f(t)\in L_1[t_*,\infty)$.

Remark 4.2. If function $p(t)$ satisfies (4.1) then the following inequality holds for function $f(t)$ defined by (4.2) (see, e.g., estimates in [24, Lemma 2.1]):

$$f(t)\le\frac{2N(t_*)}{1-e^{-\alpha}},\qquad N(t)=\max_{s\ge t}\int_s^{s+1}p(s)ds,\qquad t\ge t_*. \tag{4.3}$$

The following theorem is valid.

Theorem 4.3. *For sufficiently large t there exists a critical manifold $\mathcal{W}(t)$ of Eq. (1.1).*

Proof. Since columns of matrix $H(t,\theta)$ belong to Q_Λ this matrix, due to (2.22) (see also (2.20)), (3.6), satisfies for $t\ge t_*$ an integral equation

$$H(t,\theta)u(t)=T(t-t_*)H(t_*,\theta)u(t_*)+\int_{t_*}^{t}T(t-s)X_0^{Q_\Lambda}(\theta)G\big(s,\Phi(\theta)+H(t,\theta)\big)u(s)ds. \tag{4.4}$$

In (4.4), function $u(t)\in\mathbb{C}^N$ is the solution of Eq. (3.7) with initial value at $t=t_*$ equal to $u(t_*)$. We will denote by $U_H(t,s)$ $(t,s\ge t_*)$ the Cauchy matrix (principal matrix solution) of Eq. (3.7)

$(U_H(s,s)=I)$. Since $u(t)=U_H(t,t_*)u(t_*)$, by using the properties of the matrix $U_H(t,s)$ we rewrite Eq. (4.4) in the operator form

$$H(t,\theta)=\mathcal{A}H(t,\theta), \tag{4.5}$$

$$\begin{aligned}\mathcal{A}H(t,\theta) &= T(t-t_*)H(t_*,\theta)U_H(t_*,t)\\ &\quad+\int_{t_*}^{t}T(t-s)X_0^{Q_\Lambda}(\theta)G\big(s,\Phi(\theta)+H(t,\theta)\big)U_H(s,t)ds.\end{aligned} \tag{4.6}$$

Operator $\mathcal{A}$ is defined on the Banach space B of $(m\times N)$-matrix functions $H(t,\theta)$ which are continuous in $t\ge t_*$ and $\theta\in[-h,0]$ with a fixed initial value $H(t_*,\theta)$ such that $\|H(t,\cdot)\|_{C_h}\to 0$ as $t\to\infty$. Moreover, we assume that the columns of the initial matrix $H(t_*,\theta)$ belong to Q_Λ. We introduce the norm in B as follows:

$$\|H\|_B=\sup_{t\ge t_*}\|H(t,\cdot)\|_{C_h}. \tag{4.7}$$

We will show that for sufficiently large t_* and sufficiently small $\|H(t_*,\cdot)\|_{C_h}$ operator $\mathcal{A}$ is a contraction of a certain ball $\|H\|_B\le r_0$ in B.

Fix $r_0>0$. First, we prove that operator $\mathcal{A}$ maps the ball $\|H\|_B\le r_0$ into itself. From (4.6) it follows that matrix $(\mathcal{A}H)(t,\theta)$ is continuous in $t\ge t_*$ and $\theta\in[-h,0]$. Moreover, the columns of this matrix belong to Q_Λ for all $t\ge t_*$ since Q_Λ is invariant under the solution operator $T(t)$ and due to the origin of formula (4.6). Therefore, we need to verify that $\|\mathcal{A}H\|_B\le r_0$ and $\|(\mathcal{A}H)(t,\cdot)\|_{C_h}\to 0$ as $t\to\infty$. In the sequel we will require the estimate for the quantity $|U_H(s,t)|$ when $s\le t$. We can obtain the mentioned estimation in the following way. Note that due to (1.2), (1.3) and (3.1) the inequality

$$|G(t,\varphi)|\le p(t)\|\varphi\|_{C_h},\qquad p(t)=w(t)+\gamma(t) \tag{4.8}$$

holds for all $\varphi(\theta)\in C_h$. Here $w(t)\to 0$ as $t\to\infty$, $\gamma(t)\in L_1[t_0,\infty)$, and, therefore, function $p(t)$ possesses property (4.1). Without loss of generality we may assume that $p(t)>0$ for all $t\ge t_0$. It is evident that inequality (4.8) also holds if $\varphi(\theta)$ is $(m\times N)$-matrix (we only should use some matrix norm instead of vector norm). Due to the properties of the Cauchy matrices, we have

$$\big(U_H(s,t)\big)^*=\big(U_H^*(t,s)\big)^{-1}=U_H^c(t,s),\qquad s\le t, \tag{4.9}$$

where symbol * stands for the Hermitian conjugate and $U_H^c(t,s)$ is the Cauchy matrix of the conjugate to (3.7) system

$$\dot u=\Big[-D^*-\Big(G\big(t,\Phi(\theta)+H(t,\theta)\big)\Big)^*\Psi^*(0)\Big]u,\qquad t\ge t_*. \tag{4.10}$$

Hence,

$$U_H^c(t,s)=e^{-D^*(t-s)}-\int_s^t e^{-D^*(t-\tau)}\Big(G\big(\tau,\Phi(\theta)+H(\tau,\theta)\big)\Big)^*\Psi^*(0)U_H^c(\tau,s)d\tau. \tag{4.11}$$

From hypothesis $\mathbf{H_1}$ it follows that all eigenvalues of the matrix D (and, consequently, the matrix $(-D^*)$ as well) have zero real parts. Thus, $\forall\varepsilon>0$ we can choose $M=M(\varepsilon)$ such that inequality

$$|e^{-D^*(t-\tau)}|\le Me^{\varepsilon(t-\tau)}$$

holds for $t_*\le\tau\le t$. Using (4.8), we deduce from (4.11) that

$$|U_H^c(t,s)|\le Me^{\varepsilon(t-s)}+M\int_s^t e^{\varepsilon(t-\tau)}p(\tau)|U_H^c(\tau,s)|d\tau. \tag{4.12}$$

From now on we will denote all the constants by M if their explicit form does not play any role. We multiply both sides of (4.12) by $e^{-\varepsilon t}$ and then use Gronwall's inequality. We obtain

$$|U_H^c(t,s)| \le M \exp\left\{ \varepsilon(t-s) + M \int_s^t p(\tau)d\tau \right\}.$$

Due to (4.1), we can choose t_* sufficiently large such that for all $\varepsilon > 0$ the following inequality holds:

$$|U_H^c(t,s)| \le M e^{\varepsilon(t-s)}, \qquad t_* \le s \le t. \tag{4.13}$$

Note that we can choose constant M one and the same for all $H(t,\theta)$ from the ball $\|H\|_B \le r_0$. We now conclude from (4.9) that inequality (4.13) also holds for the Cauchy matrix $U_H(s,t)$ (probably, with another constant M).

We return now to (4.6). Hypothesis $\mathbf{H_2}$ implies that inequalities (2.26)–(2.28) hold with a certain exponential rate $-\alpha = \beta + \varepsilon < 0$. It follows form (4.6) that

$$\|(\mathcal{A}H)(t,\cdot)\|_{C_h} \le M e^{(-\alpha+\varepsilon)(t-t_*)} \|H(t_*,\cdot)\|_{C_h} + M \int_{t_*}^t e^{(-\alpha+\varepsilon)(t-s)} p(s)ds, \qquad t \ge t_*. \tag{4.14}$$

Choose $\varepsilon > 0$ such that $-\alpha + \varepsilon < 0$. By Proposition 4.1, the right-hand side of (4.14) tends to zero as $t \to \infty$. Moreover, choosing t_* sufficiently large, $\|H(t_*,\cdot)\|_{C_h}$ sufficiently small and taking into account (4.3) we conclude that $\|\mathcal{A}H\|_B \le r_0$. Consequently, operator $\mathcal{A}$ maps the ball $\|H\|_B \le r_0$ into itself.

We will now show that operator $\mathcal{A}$ is a contraction of the ball $\|H\|_B \le r_0$. Suppose that $H_1(t,\theta), H_2(t,\theta) \in B$ and $\|H_1\|_B \le r_0$, $\|H_2\|_B \le r_0$. We derive from (4.6) that

$$\begin{aligned} &\|(\mathcal{A}H_1)(t,\cdot) - (\mathcal{A}H_2)(t,\cdot)\|_{C_h} \\ &\quad \le M e^{-\alpha(t-t_*)} \|H_1(t_*,\cdot)\|_{C_h} |U_{H_1}(t_*,t) - U_{H_2}(t_*,t)| \\ &\qquad + M \int_{t_*}^t e^{-\alpha(t-s)} p(s) |U_{H_1}(s,t) - U_{H_2}(s,t)| ds \\ &\qquad + M \int_{t_*}^t e^{-\alpha(t-s)} p(s) \|H_1(s,\cdot) - H_2(s,\cdot)\|_{C_h} |U_{H_1}(s,t)| ds \\ &\qquad + M \int_{t_*}^t e^{-\alpha(t-s)} p(s) \|H_2(s,\cdot)\|_{C_h} |U_{H_1}(s,t) - U_{H_2}(s,t)| ds. \end{aligned} \tag{4.15}$$

Here we used the equality $H_1(t_*,\theta) = H_2(t_*,\theta)$ and also added and subtracted the quantity

$$\int_{t_*}^t T(t-s) X_0^{Q_\Lambda}(\theta) G\big(s, H_2(s,\theta)\big) U_{H_1}(s,t) ds$$

on the right-hand side of the expression for $\mathcal{A}H_1$–$\mathcal{A}H_2$. Let us estimate the difference $|U_{H_1}(s,t) - U_{H_2}(s,t)|$ ($s \le t$). It follows from the above that we need to obtain estimate for $|U_{H_1}^c(t,s) - U_{H_2}^c(t,s)|$, where $U_{H_1}^c(t,s)$, $U_{H_2}^c(t,s)$ are the Cauchy matrices of (4.10) with $H(t,\theta)$ equal to $H_1(t,\theta)$ and $H_2(t,\theta)$ respectively. From (4.10) we deduce that

$$\begin{aligned} &\frac{\partial}{\partial t}\Big(U_{H_1}^c(t,s) - U_{H_2}^c(t,s) \Big) \\ &\quad = -D^*\big(U_{H_1}^c(t,s) - U_{H_2}^c(t,s)\big) - \Big(G\big(t, \Phi(\theta)\big) \Big)^* \Psi^*(0) \big(U_{H_1}^c(t,s) - U_{H_2}^c(t,s)\big) \\ &\qquad - \Big(G\big(t, H_1(t,\theta)\big) \Big)^* \Psi^*(0) U_{H_1}^c(t,s) + \Big(G\big(t, H_2(t,\theta)\big) \Big)^* \Psi^*(0) U_{H_2}^c(t,s) \end{aligned}$$

$$= -D^*\left(U^c_{H_1}(t,s) - U^c_{H_2}(t,s)\right) - \Big(G\big(t,\Phi(\theta)\big)\Big)^*\Psi^*(0)\left(U^c_{H_1}(t,s) - U^c_{H_2}(t,s)\right)$$
$$- \Big(G\big(t, H_1(t,\theta) - H_2(t,\theta)\big)\Big)^*\Psi^*(0)U^c_{H_1}(t,s)$$
$$- \Big(G\big(t, H_2(t,\theta)\big)\Big)^*\Psi^*(0)\left(U^c_{H_1}(t,s) - U^c_{H_2}(t,s)\right).$$

Thus, the difference $U^c_{H_1}(t,s) - U^c_{H_2}(t,s)$ is the solution (with respect to variable t) of a certain inhomogeneous linear differential equation whose homogeneous part coincides with coefficients matrix of Eq. (4.10), where $H(t,\theta) = H_2(t,\theta)$. Application of the variation-of-constants formula with account of equality $U^c_{H_1}(s,s) - U^c_{H_2}(s,s) = 0$ yields that

$$U^c_{H_1}(t,s) - U^c_{H_2}(t,s) = -\int_s^t U^c_{H_2}(t,\tau)\Big(G\big(\tau, H_1(\tau,\theta) - H_2(\tau,\theta)\big)\Big)^*\Psi^*(0)U^c_{H_1}(\tau,s)d\tau. \quad (4.16)$$

We deduce from (4.16) with account of (4.1), (4.8) and (4.13) the following estimate if $t_* \le s \le t$:

$$|U^c_{H_1}(t,s) - U^c_{H_2}(t,s)| \le Me^{\varepsilon(t-s)}\int_s^t p(\tau)\|H_1(\tau,\cdot) - H_2(\tau,\cdot)\|_{C_h}d\tau$$
$$\le Me^{2\varepsilon(t-s)}\|H_1 - H_2\|_B. \quad (4.17)$$

As it was pointed above, inequality (4.17) is also valid for estimation of the difference $|U_{H_1}(s,t) - U_{H_2}(s,t)|$ ($t_* \le s \le t$). We now use (4.17) to estimate the latter quantity in (4.15). Provided that $-\alpha + 2\varepsilon < 0$ we obtain

$$\|\mathcal{A}H_1 - \mathcal{A}H_2\|_B \quad (4.18)$$
$$\le M\left\{\|H_1(t_*,\cdot)\|_{C_h} + \int_{t_*}^t e^{(-\alpha+2\varepsilon)(t-s)}p(s)ds + \int_{t_*}^t e^{(-\alpha+\varepsilon)(t-s)}p(s)ds\right\}\|H_1 - H_2\|_B.$$

We now choose t_* sufficiently large and $\|H_1(t_*,\cdot)\|_{C_h}$ sufficiently small to state, using (4.3), that operator $\mathcal{A}$ is a contraction of the ball $\|H\|_B \le r_0$ in the space B. This concludes the proof. □

The next step is to justify the method proposed in the previous section for the approximate construction of the critical manifold $\mathcal{W}(t)$.

Theorem 4.4. *Suppose that $\mathcal{W}(t)$ is a critical manifold which according to Theorem 4.3 exists for sufficiently large t. Then there exists a sufficiently large t_* such that for $t \ge t_*$ matrix $H(t,\theta)$ from (3.4) admits the following representation:*

$$H(t,\theta) = \hat{H}(t,\theta) + Z(t,\theta), \qquad t \ge t_* \ge t_0, \quad -h \le \theta \le 0. \quad (4.19)$$

Here matrix $\hat{H}(t,\theta)$ is defined by (3.9) and satisfies Eq. (3.29). Moreover, $Z(t,\theta)$ is a certain $(m \times N)$-matrix function such that $\|Z(t,\cdot)\|_{C_h} \to 0$ as $t \to \infty$ and $\|Z(t,\cdot)\|_{C_h} \in L_1[t_*,\infty)$.

Proof. We write the solution $H(t,\theta)$ of the operator equation (4.5) as sum (4.19), where $\hat{H}(t,\theta)$ is the element of B, constructed in the previous section, and $Z(t,\theta)$ is a certain matrix from B. Then we can regard equation (4.5) as the equation for $Z(t,\theta)$:

$$Z(t,\theta) = \mathcal{S}Z(t,\theta), \quad (4.20)$$
$$\mathcal{S}Z(t,\theta) = \mathcal{A}\big(\hat{H}(t,\theta) + Z(t,\theta)\big) - \hat{H}(t,\theta), \quad (4.21)$$

where operator $\mathcal{A}$ is defined by (4.6). We will assume that the domain of the operator $\mathcal{S}$ is the Banach space BL of all matrix functions $Z(t,\theta)$ continuous in $t \geq t_*$ and $\theta \in [-h,0]$ with a fixed initial value $Z(t_*,\theta)$ such that $\|Z(t,\cdot)\|_{C_h} \to 0$ as $t \to \infty$ and $p(t)\|Z(t,\cdot)\|_{C_h} \in L_1[t_*,\infty)$. Here function $p(t)$ is defined by (4.8) and satisfies condition (4.1). Moreover, the columns of matrix $Z(t_*,\theta)$ belong to Q_Λ. We introduce the norm in BL as follows:

$$\|Z\|_{BL} = \|Z\|_B + \|Z\|_L, \qquad \|Z\|_L = \int_{t_*}^{\infty} p(t)\|Z(t,\cdot)\|_{C_h} dt, \tag{4.22}$$

where the norm $\|\cdot\|_B$ is described by formula (4.7). Our aim is to show that the operator $\mathcal{S}$ is a contraction of a certain ball $\|Z\|_{BL} \leq r_0$ in BL provided that initial condition $\|Z(t_*,\cdot)\|_{C_h}$ is sufficiently small and t_* is sufficiently large.

First, we show that $\mathcal{S}$ acts to BL. The properties of the operator $\mathcal{A}$ and the matrix $\hat{H}(t,\theta)$ imply that the matrix $(\mathcal{S}Z)(t,\theta)$ is continuous in $t \geq t_*$ and $\theta \in [-h,0]$. Moreover, $\|(\mathcal{S}Z)(t,\cdot)\|_{C_h} \to 0$ as $t \to \infty$. The properties of $\mathcal{A}$ and $\hat{H}(t,\theta)$ also yield that the columns of the matrix $(\mathcal{S}Z)(t,\theta)$ belong to Q_Λ for all $t \geq t_*$. Therefore, we need to verify that $p(t)\|(\mathcal{S}Z)(t,\cdot)\|_{C_h} \in L_1[t_*,\infty)$. To achieve this goal we will obtain a more appropriate representation for the operator $\mathcal{S}$.

Let $u(t) \in \mathbb{C}^N$ be the solution of Eq. (3.7), where $H(t,\theta)$ is sum (4.19), with initial value at $t = t_*$ equal to $u(t_*)$. Due to the form of the matrix $\hat{H}(t,\theta)$ (see (3.9)), absolute continuity of functions $v_1(t),\dots v_n(t)$ and the properties of the solution operator $T(t)$, we conclude that function $T(t-s)\hat{H}(s,\theta)u(s)$ is absolutely continuous in s for $t_* \leq s \leq t$. Hence,

$$\hat{H}(t,\theta)u(t) = T(t-t_*)\hat{H}(t_*,\theta)u(t_*) + \int_{t_*}^{t} \frac{d}{ds}\big(T(t-s)\hat{H}(s,\theta)u(s)\big)ds. \tag{4.23}$$

Note that for each continuously differentiable function $\varphi \in C_h$ the following equalities hold (see deduction of formulas (2.1) in [19, p. 167]):

$$\begin{aligned}
&\int_{t_*}^{t} \frac{d}{ds}\big(T(t-s)\varphi(\theta)\big) \\
&\quad = -\int_{t_*}^{t} T(t-s)\begin{cases} \dfrac{d\varphi}{d\theta}, & -h \leq \theta < 0, \\ B_0\varphi(\theta), & \theta = 0. \end{cases} \quad ds \\
&\quad = -\int_{t_*}^{t} T(t-s)\frac{d\varphi}{d\theta}ds + \int_{t_*}^{t} T(t-s)X_0(\theta)\frac{d\varphi}{d\theta}ds - \int_{t_*}^{t} T(t-s)X_0(\theta)B_0\varphi\, ds.
\end{aligned} \tag{4.24}$$

We recall that to provide the mathematical strictness of equality (4.24) we should rewrite it in the form

$$\int_{t_*}^{t} \frac{d}{ds}\big(T(t-s)\varphi(\theta)\big)ds = -\int_{t_*}^{t} T(t-s)\frac{d\varphi}{d\theta}ds + \int_{t_*}^{t} dK(t,s)\frac{d\varphi}{d\theta} - \int_{t_*}^{t} dK(t,s)B_0\varphi,$$

where $K(t,s)(\theta)$ is defined by (2.15). Since matrix $\hat{H}(t,\theta)$ is absolutely continuous in t for

$t \geq t_0$ and continuously differentiable in θ for $\theta \in [-h, 0]$, it is easy to show that

$$\begin{aligned}
&\int_{t_*}^{t} \frac{d}{ds}\big(T(t-s)\hat{H}(s,\theta)u(s)\big)ds \\
&\quad = \int_{t_*}^{t} T(t-s)\left\{-\frac{\partial \hat{H}}{\partial \theta}u(s) + X_0(\theta)\frac{\partial \hat{H}}{\partial \theta}u(s) - X_0(\theta)B_0\hat{H}u(s) + \frac{\partial \hat{H}}{\partial s}u(s) + \hat{H}(s,\theta)\frac{du}{ds}\right\}ds \\
&\quad = \int_{t_*}^{t} T(t-s)\bigg\{-\frac{\partial \hat{H}}{\partial \theta} + X_0(\theta)\frac{\partial \hat{H}}{\partial \theta} - X_0(\theta)B_0\hat{H} + \frac{\partial \hat{H}}{\partial s} \\
&\qquad\qquad + \hat{H}(s,\theta)\Big(D + \Psi(0)G\big(s, \Phi(\theta) + \hat{H}(s,\theta) + Z(s,\theta)\big)\Big)\bigg\}u(s)ds. \qquad (4.25)
\end{aligned}$$

Here we used the fact that function $u(t)$ is a solution of Eq. (3.7). Again, we emphasize that analogous to (4.24) the integrands of the form $T(t-s)X_0(\theta)(\ldots)ds$ on the right-hand side of (4.25) should be replaced by $dK(t,s)(\theta)(\ldots)$. We remind that matrix $\hat{H}(t,\theta)$ satisfies Eq. (3.29). We rewrite the latter as follows:

$$\begin{aligned}
&\Phi(\theta)\Psi(0)G\big(t, \Phi(\theta) + \hat{H}(t,\theta)\big) + \hat{H}(t,\theta)\Big(D + \Psi(0)G\big(t, \Phi(\theta) + \hat{H}(t,\theta)\big)\Big) + \frac{\partial \hat{H}}{\partial t} \qquad (4.26) \\
&\quad = \frac{\partial \hat{H}}{\partial \theta} + R_1(t,\theta) - X_0(\theta)\frac{\partial \hat{H}}{\partial \theta} + X_0(\theta)B_0\hat{H} + X_0(\theta)G\big(t, \Phi(\theta) + \hat{H}(t,\theta)\big) - X_0(\theta)R_2(t).
\end{aligned}$$

We use (4.26) in (4.25) and represent $u(t)$ in the form $u(t) = U_{\hat{H}+Z}(t,t_*)u(t_*)$. Here, $U_{\hat{H}+Z}(t,s)$ $(t,s \geq t_*)$ is the Cauchy matrix of (3.7) $(U_{\hat{H}+Z}(s,s) = I)$, where $H(t,\theta)$ is sum (4.19). Recalling the notation $X_0^{Q_\Lambda} = X_0(\theta) - \Phi(\theta)\Psi(0)$, we deduce from (4.23) that

$$\begin{aligned}
\hat{H}(t,\theta) &= T(t-t_*)\hat{H}(t_*,\theta)U_{\hat{H}+Z}(t_*,t) \\
&\quad + \int_{t_*}^{t} T(t-s)\Big\{R_1(s,\theta) + X_0^{Q_\Lambda}G\big(s, \Phi(\theta) + \hat{H}(s,\theta)\big) \\
&\qquad\qquad - X_0(\theta)R_2(s) + \hat{H}(s,\theta)\Psi(0)G\big(s, Z(s,\theta)\big)\Big\}U_{\hat{H}+Z}(s,t)ds.
\end{aligned} \qquad (4.27)$$

Due to (4.6), (4.19) and (4.27), we obtain the following representation for the operator $\mathcal{S}$ from (4.21):

$$\begin{aligned}
\mathcal{S}Z(t,\theta) &= T(t-t_*)Z(t_*,\theta)U_{\hat{H}+Z}(t_*,t) \\
&\quad + \int_{t_*}^{t} T(t-s)\Big\{X_0^{Q_\Lambda}G\big(s, Z(s,\theta)\big) - R_1(s,\theta) \\
&\qquad\qquad + X_0(\theta)R_2(s) - \hat{H}(s,\theta)\Psi(0)G\big(s, Z(s,\theta)\big)\Big\}U_{\hat{H}+Z}(s,t)ds.
\end{aligned} \qquad (4.28)$$

Since matrix $R_1(t,\theta)$ is continuous in θ for $\theta \in [-h, 0]$, by (2.7) we can decompose it into direct sum

$$R_1(t,\theta) = R_1^{P_\Lambda}(t,\theta) + R_1^{Q_\Lambda}(t,\theta),$$

where the columns of $(m \times N)$-matrices $R_1^{P_\Lambda}(t,\theta)$ and $R_1^{Q_\Lambda}(t,\theta)$ belong to P_Λ and Q_Λ respectively for $t \geq t_0$. We also note that functions $\|R_1^{P_\Lambda}(t,\cdot)\|_{C_h}$ and $\|R_1^{Q_\Lambda}(t,\cdot)\|_{C_h}$ belong to $L_1[t_0,\infty)$. Since the columns of all matrices $H_{i_1\ldots i_l}(t,\theta)$ in (3.9) belong to Q_Λ for all $t \in \mathbb{R}$, it follows that the columns of matrices $\hat{H}(t,\theta)$ and $\frac{\partial \hat{H}}{\partial t}$ also belong to Q_Λ for $t \geq t_0$. Hence, we derive from

(3.29) that $R_1^{P_\Lambda}(t,\theta) = \Phi(\theta)\Psi(0)R_2(t)$ since matrix $R_2(t)$ is an absolutely integrable on $[t_0,\infty)$ part of the matrix $G\big(t,\Phi(\theta)+\hat{H}(t,\theta)\big)$. Due to the latter, equation (4.28) takes the form

$$\begin{aligned}\mathcal{S}Z(t,\theta) &= T(t-t_*)Z(t_*,\theta)U_{\hat{H}+Z}(t_*,t)\\ &\quad+\int_{t_*}^{t} T(t-s)\Big\{X_0^{Q_\Lambda}G\big(s,Z(s,\theta)\big)+X_0^{Q_\Lambda}R_2(s)\\ &\qquad\qquad - R_1^{Q_\Lambda}(s,\theta)-\hat{H}(s,\theta)\Psi(0)G\big(s,Z(s,\theta)\big)\Big\}U_{\hat{H}+Z}(s,t)ds.\end{aligned} \tag{4.29}$$

Here we need to replace integrands of the form $T(t-s)X_0^{Q_\Lambda}(\ldots)ds$ by the integrands $dK(t,s)^{Q_\Lambda}(\theta)(\ldots)$, where $K(t,s)^{Q_\Lambda}$ is defined by (2.21).

We now apply inequalities (2.26)-(2.28) and also estimate (4.13) for $U_{\hat{H}+Z}(s,t)$ to obtain

$$\begin{aligned}\|(\mathcal{S}Z)(t,\cdot)\|_{C_h} &\le Me^{(-\alpha+\varepsilon)(t-t_*)}\|Z(t_*,\cdot)\|_{C_h}+M\int_{t_*}^{t} e^{(-\alpha+\varepsilon)(t-s)}p(s)\|Z(s,\cdot)\|_{C_h}ds\\ &\quad+M\int_{t_*}^{t} e^{(-\alpha+\varepsilon)(t-s)}\big(|R_2(s)|+\|R_1^{Q_\Lambda}(s,\cdot)\|_{C_h}\big)ds, \qquad t\ge t_*.\end{aligned} \tag{4.30}$$

Changing the order of integration yields that

$$\begin{aligned}\int_{t_*}^{\infty} p(t)\|(\mathcal{S}Z)(t,\cdot)\|_{C_h}dt &\le M\|Z(t_*,\cdot)\|_{C_h}\int_{t_*}^{\infty} e^{(-\alpha+\varepsilon)(t-t_*)}p(t)dt\\ &\quad+M\int_{t_*}^{\infty} p(s)\|Z(s,\cdot)\|_{C_h}\int_{t_*}^{\infty} e^{(-\alpha+\varepsilon)|t-s|}p(t)dtds\\ &\quad+M\int_{t_*}^{\infty}\big(|R_2(s)|+\|R_1^{Q_\Lambda}(s,\cdot)\|_{C_h}\big)\int_{t_*}^{\infty} e^{(-\alpha+\varepsilon)|t-s|}p(t)dtds.\end{aligned} \tag{4.31}$$

Finally, we use Proposition 4.1, inequality (4.3) and also the fact that functions $\|R_1^{Q_\Lambda}(t,\cdot)\|_{C_h}$, $|R_2(t)|$ and $p(t)\|Z(t,\cdot)\|_{C_h}$ $(Z(t,\theta)\in BL)$ belong to $L_1[t_*,\infty)$. Consequently, all the integrals on the right-hand side of (4.31) exist, and, therefore, function $p(t)\|(\mathcal{S}Z)(t,\cdot)\|_{C_h}$ belongs to $L_1[t_*,\infty)$. Thus, we have established that operator $\mathcal{S}$ maps BL into itself.

Fix $r_0>0$. We remark that, due to (4.11), (4.12), constant M in (4.31) may be chosen one and the same for all matrices $Z(t,\theta)$ from the ball $\|Z\|_{BL}\le r_0$. Then, by (4.3), we deduce from (4.31) that $\|\mathcal{S}Z\|_L \le MN(t_*)$ for all $\|Z\|_{BL}\le r_0$. Further, due to (4.14), (4.21), for all $\|Z\|_{BL}\le r_0$ we have

$$\begin{aligned}\|\mathcal{S}Z\|_{BL} &= \|\mathcal{A}\big(\hat{H}(t,\theta)+Z(t,\theta)\big)-\hat{H}(t,\theta)\|_B+\|\mathcal{S}Z\|_L\\ &\le M\big(\|\hat{H}\|_B+\|Z(t_*,\cdot)\|_{C_h}+N(t_*)\big).\end{aligned} \tag{4.32}$$

Since $\|\hat{H}(t,\cdot)\|_{C_h}\to 0$ and $N(t)\to 0$ as $t\to\infty$, by choosing t_* sufficiently large and $\|Z(t_*,\cdot)\|_{C_h}$ sufficiently small we get the inequality $\|\mathcal{S}Z\|_{BL}\le r_0$. Thus, the operator $\mathcal{S}$ maps the ball $\|Z\|_{BL}\le r_0$ into itself. What is left is to show that $\mathcal{S}$ is a contraction of this ball for sufficiently large t_*. From (4.18), (4.21) it follows that for any $Z_1(t,\theta)$, $Z_2(t,\theta)$ from the ball $\|Z\|_{BL}\le r_0$ the following inequality holds:

$$\begin{aligned}\|\mathcal{S}Z_1-\mathcal{S}Z_2\|_{BL} &= \|\mathcal{A}(\hat{H}+Z_1)-\mathcal{A}(\hat{H}+Z_2)\|_B+\|\mathcal{S}Z_1-\mathcal{S}Z_2\|_L\\ &\le q\|Z_1-Z_2\|_B+\|\mathcal{S}Z_1-\mathcal{S}Z_2\|_L, \qquad 0\le q<1,\end{aligned} \tag{4.33}$$

if the quantity $\|Z_1(t_*,\cdot)\|_{C_h} = \|Z_2(t_*,\cdot)\|_{C_h}$ is sufficiently small and t_* is sufficiently large. We derive from (4.29) that

$$\begin{aligned}
&\|\mathcal{S}Z_1 - \mathcal{S}Z_2\|_L \\
&\quad\le M\|Z_1(t_*,\cdot)\|_{C_h}\int_{t_*}^{\infty} e^{-\alpha(t-t_*)}p(t)|U_{H_1}(t_*,t) - U_{H_2}(t_*,t)|dt \\
&\qquad + M\int_{t_*}^{\infty} p(t)\int_{t_*}^{t} e^{(-\alpha+\varepsilon)(t-s)}p(s)\|Z_1(s,\cdot) - Z_2(s,\cdot)\|_{C_h}dsdt \\
&\qquad + M\int_{t_*}^{\infty} p(t)\int_{t_*}^{t} e^{-\alpha(t-s)}p(s)\|Z_2(s,\cdot)\|_{C_h}|U_{H_1}(s,t) - U_{H_2}(s,t)|dsdt \\
&\qquad + M\int_{t_*}^{\infty} p(t)\int_{t_*}^{t} e^{-\alpha(t-s)}\left(|R_2(s)| + \|R_1^{Q_\Lambda}(s,\cdot)\|_{C_h}\right)|U_{H_1}(s,t) - U_{H_2}(s,t)|dsdt \\
&\qquad + M\int_{t_*}^{\infty} p(t)\int_{t_*}^{t} e^{(-\alpha+\varepsilon)(t-s)}p(s)\|\hat{H}(s,\cdot)\|_{C_h}\|Z_1(s,\cdot) - Z_2(s,\cdot)\|_{C_h}dsdt \\
&\qquad + M\int_{t_*}^{\infty} p(t)\int_{t_*}^{t} e^{-\alpha(t-s)}p(s)\|\hat{H}(s,\cdot)\|_{C_h}\|Z_2(s,\cdot)\|_{C_h}|U_{H_1}(s,t) - U_{H_2}(s,t)|dsdt, \qquad (4.34)
\end{aligned}$$

where $H_1 = \hat{H} + Z_1$ and $H_2 = \hat{H} + Z_2$. To obtain (4.34) we used the equality $Z_1(t_*,\theta) = Z_2(t_*,\theta)$, inequality (4.13) and also added and subtracted the difference

$$\int_{t_*}^{t} T(t-s)X_0^{Q_\Lambda}(\theta)G\big(s, Z_2(t,\theta)\big)U_{H_1}(s,t)ds - \int_{t_*}^{t} T(t-s)\hat{H}(s,\theta)\Psi(0)G\big(s, Z_2(s,\theta)\big)U_{H_1}(s,t)ds$$

on the right-hand side of the expression for the difference $\mathcal{S}Z_1 - \mathcal{S}Z_2$. From the left inequality in (4.17) for any $Z_1(t,\theta)$, $Z_2(t,\theta)$ such that $\|Z\|_{BL} \le r_0$ the following estimate follows for $t_* \le s \le t$:

$$|U_{H_1}(s,t) - U_{H_2}(s,t)| \le Me^{\varepsilon(t-s)}\|Z_1 - Z_2\|_L, \qquad (4.35)$$

where $H_1 = \hat{H} + Z_1$ and $H_2 = \hat{H} + Z_2$. Changing the order of integration in (4.34) and taking into account (4.3), (4.35) and inequalities $\|Z_i\|_{BL} \le r_0$ $(i = 1,2)$, we conclude that

$$\|\mathcal{S}Z_1 - \mathcal{S}Z_2\|_L \le MN(t_*)\|Z_1 - Z_2\|_L \le q\|Z_1 - Z_2\|_L, \qquad 0 \le q < 1, \qquad (4.36)$$

if t_* is sufficiently large. Applying (4.36) in (4.33) yields that the operator $\mathcal{S}$ is a contraction of the ball $\|Z\|_{BL} \le r_0$.

Assume now that $Z(t,\theta)$ is the solution of Eq. (4.20) that belongs to BL. Since functions $p(t)\|Z(t,\cdot)\|_{C_h}$, $\|R_1^{Q_\Lambda}(t,\cdot)\|_{C_h}$ and $|R_2(t)|$ belong to $L_1[t_*,\infty)$, it follows from Proposition 4.1 that the right-hand side of inequality (4.30) also belongs to $L_1[t_*,\infty)$. Thus, $\|Z(t,\cdot)\|_{C_h} \in L_1[t_*,\infty)$ and the proof is complete. □

To construct the asymptotics for solutions of Eq. (3.7) sometimes we need a detailed information concerning the decay rate of function $\|Z(t,\cdot)\|_{C_h}$ as $t \to \infty$. In this connection the following corollary of Theorem 4.4 is useful.

Corollary 4.5. *Let the following inequality hold:*

$$\|R_1^{Q_\Lambda}(t,\cdot)\|_{C_h} + |R_2(t)| \le \varphi(t), \qquad t \ge t_0, \qquad (4.37)$$

where $\varphi(t) > 0$ for $t \ge t_0$. Moreover, suppose that there exists $\beta \in (0,\alpha)$ such that

$$\varphi(t_1)e^{\beta t_1} \le \varphi(t_2)e^{\beta t_2}, \qquad t_0 \le t_1 \le t_2. \qquad (4.38)$$

Then the solution of Eq. (4.20) *for* $t \geq t_* \geq t_0$ *satisfies inequality*

$$\|Z(t,\cdot)\|_{C_h} \leq K\varphi(t) \tag{4.39}$$

with a certain constant K.

The proof of Corollary 4.5 may be handled in much the same way as in [26, Theorem 5].

Remark 4.6. From (1.2), (1.3), (3.9), (3.29) and conditions 1^0–3^0 imposed on functions $v_1(t), \ldots, v_n(t)$, it follows that function $\|R_1^{Q_\Lambda}(t,\cdot)\|_{C_h} + |R_2(t)|$ has the following asymptotic estimate as $t \to \infty$:

$$O\left(\sum_{1 \leq i_1 \leq \cdots \leq i_{k+1} \leq n} |v_{i_1}(t) \cdots v_{i_{k+1}}(t)|\right) + O\left(\sum_{i=1}^{n} |\dot{v}_i(t)|\right) + O\big(\gamma(t)\big).$$

We establish now the property of global attraction for the manifold $\mathcal{W}(t)$.

Theorem 4.7. *Suppose that* $x(t)$ *is a solution of Eq.* (1.1) *defined for* $t \geq T \geq t_0$. *Then there exists a sufficiently large* $t_* \geq T$ *such that the following asymptotic formula holds for* $t \geq t_*$:

$$x_t(\theta) = \Phi(\theta)u_H(t) + H(t,\theta)u_H(t) + O\big(e^{(-\alpha+\varepsilon)t}\big), \qquad t \to \infty. \tag{4.40}$$

Here $\alpha > 0$ *is chosen in the way that inequalities* (2.26)–(2.28) *hold with exponential rate equal to* $(-\alpha)$, $\varepsilon \in (0,\alpha)$ *is an arbitrary real number and* $u_H(t)$ $(t \geq t_*)$ *is a certain solution of system on the critical manifold* (3.7).

Proof. By (2.18), (2.23), we have

$$x_t(\theta) = \Phi(\theta)u(t) + x_t^{Q_\Lambda}(\theta), \qquad t \geq t_*, \tag{4.41}$$

where $x_t^{Q_\Lambda}(\theta)$ is defined by (2.20) (see also (2.22)) with $t_0 = t_*$ and fucntion $u(t)$ is the solution of Eq. (2.24) with initial value $u(t_*) = \big(\Psi(\xi), x_{t_*}(\theta)\big)$. Let $\mathcal{W}(t)$ be a critical manifold for Eq. (1.1) that exists for sufficiently large $t \geq t_*$ according to Theorem 4.3. We recall that this manifold is described by formula (3.4). Choose $u_H(t_*) = u(t_*)$, then

$$\tilde{x}_t(\theta) = \Phi(\theta)u_H(t) + H(t,\theta)u_H(t) \tag{4.42}$$

is a certain solution of Eq. (1.1) lying on a critical manifold $\mathcal{W}(t)$ for $t \geq t_*$. Our aim is to show that $x_t(\theta) = \tilde{x}_t(\theta) + O(e^{(-\alpha+\varepsilon)t})$. By setting $z(t,\theta) = x_t^{Q_\Lambda}(\theta) - H(t,\theta)u_H(t)$, $r(t) = u(t) - u_H(t)$ and subtracting (4.42) from (4.41), we obtain

$$x_t(\theta) - \tilde{x}_t(\theta) = \Phi(\theta)r(t) + z(t,\theta), \qquad t \geq t_*. \tag{4.43}$$

Note that function $H(t,\theta)u_H(t)$ is the solution of integral equation (4.4). Subtracting (4.4) from (2.22) (where $t_0 = t_*$) and taking into account (4.41), we get the following equation for $z(t,\theta)$:

$$z(t,\theta) = T(t-t_*)z(t_*,\theta) + \int_{t_*}^{t} T(t-s)X_0^{Q_\Lambda}G\big(s, \Phi(\theta)r(s) + z(s,\theta)\big)ds, \qquad t \geq t_*. \tag{4.44}$$

We emphasize that the quantity $\|z(t_*,\cdot)\|_{C_h}$ can be assumed to be as small as we wish. Indeed, due to the linearity of Eq. (1.1) we can replace solution $x_t(\theta)$ by $\delta x_t(\theta)/\|x_{t_*}(\theta)\|_{C_h}$ with initial value at $t = t_*$ equal to $\varphi(\theta)$ such that $\|\varphi(\theta)\|_{C_h} = \delta$ for any prescribed $\delta > 0$.

We subtract (3.7) (where $u(t) = u_H(t)$) from (2.24) an use (4.41). We conclude that function $r(t)$ is the solution of the Cauchy problem

$$\dot{r} = Dr(t) + \Psi(0)G\big(t, \Phi(\theta)r(t) + z(t,\theta)\big), \qquad r(t_*) = 0. \tag{4.45}$$

We write matrix D having the form (3.18) as follows

$$D = D_1 + D_2, \tag{4.46}$$

where $D_1 = \operatorname{diag} D$ and D_2 is a nilpotent matrix. We can regard that $|D_2| < \delta$ for any prescribed $\delta > 0$. Actually, in Eq. (4.45) we can make the change of variable $r = C_\delta \tilde{r}$, where matrix C_δ brings matrix $\delta^{-1}D$ to Jordan form. Then this change of the variable does not effect the matrix D_1 and the only nonzero entries of matrix D_2 are, possibly, $d_{i,i+1} = \delta$. We write (4.45) as an integral equation

$$r(t) = \int_{t_*}^{t} e^{D_1(t-s)}\big[D_2 r(s) + \Psi(0)G\big(s, \Phi(\theta)r(s) + z(s,\theta)\big)\big]ds. \tag{4.47}$$

We introduce the space B_1 whose elements are the pairs $\big(z(t,\theta), r(t)\big)$. Here function $z(t,\theta)$ is continuous in $\theta \in [-h, 0]$ and $t \geq t_*$, and function $r(t)$ is continuous for $t \geq t_*$. We also assume that function $z(t_*,\theta)$ is fixed and belongs to Q_Λ. Moreover, suppose that the following inequalities hold:

$$\|z(t,\cdot)\|_{C_h} \leq Ke^{(-\alpha+\varepsilon)(t-t_*)}, \qquad |r(t)| \leq Ke^{(-\alpha+\varepsilon)(t-t_*)}, \qquad t \geq t_*, \tag{4.48}$$

where $K > 0$ is a certain constant and $\varepsilon \in (0, \alpha)$ is an arbitrarily chosen real number. The space B_1 becomes the Banach space with the norm

$$\big\|\big(z(t,\theta), r(t)\big)\big\|_{B_1} = \sup_{t \geq t_*}\big(e^{(\alpha-\varepsilon)(t-t_*)}(\|z(t,\cdot)\|_{C_h} + |r(t)|)\big).$$

If the system of equations (4.44), (4.47) has solution $\big(z(t,\theta), r(t)\big) \in B_1$, then we can rewrite (4.47) in the following equivalent form. Letting $t \to \infty$ and taking into account the right inequality in (4.48), we get

$$\int_{t_*}^{\infty} e^{-D_1 s}\big[D_2 r(s) + \Psi(0)G\big(s, \Phi(\theta)r(s) + z(s,\theta)\big)\big]ds = 0.$$

We also use the fact that, due to hypothesis $\mathbf{H_1}$, all eigenvalues of matrix D_1 have zero real parts. Finally, we use the above equality in (4.47) to write the latter as follows:

$$r(t) = -\int_{t}^{\infty} e^{D_1(t-s)}\big[D_2 r(s) + \Psi(0)G\big(s, \Phi(\theta)r(s) + z(s,\theta)\big)\big]ds, \qquad t \geq t_*. \tag{4.49}$$

We write system (4.44), (4.49) as an operator equation in the space B_1:

$$\big(z(t,\theta), r(t)\big) = \mathcal{L}\big(z(t,\theta), r(t)\big) = \big(\mathcal{L}_1\big(z(t,\theta), r(t)\big), \mathcal{L}_2\big(z(t,\theta), r(t)\big)\big), \tag{4.50}$$

where operators $\mathcal{L}_1$, $\mathcal{L}_2$ are defined by the right-hand sides of equations (4.44) and (4.49) respectively. We want to show that the operator $\mathcal{L}$ is contracting in B_1 provided that $\|z(t_*,\cdot)\|$ is sufficiently small and t_* is sufficiently large.

First, we show that the operator $\mathcal{L}$ maps B_1 to itself. Assume that $(z,r) \in B_1$, then, by using (4.3), we derive from (4.44) that

$$\begin{aligned} \|(\mathcal{L}_1(z,r))(t,\cdot)\|_{C_h} &\le Me^{-\alpha(t-t_*)}\|z(t_*,\cdot)\|_{C_h} + MKe^{(-\alpha+\varepsilon)(t-t_*)}\int_{t_*}^{t} e^{-\varepsilon(t-s)}p(s)ds \\ &\le M\left(\|z(t_*,\cdot)\|_{C_h} + \frac{2KN(t_*)}{1-e^{-\varepsilon}}\right)e^{(-\alpha+\varepsilon)(t-t_*)}, \qquad t \ge t_*. \end{aligned}$$

If we now choose $\|z(t_*,\cdot)\|_{C_h}$ sufficiently small and t_* sufficiently large we get that function $\mathcal{L}_1(z,r)$ satisfies left inequality in (4.48). Note that $|e^{D_1(t-s)}| \le C$ for a certain constant C (the latter depends only on the used matrix norm) and $|D_2| < \delta$. Hence, it follows from (4.49) that

$$\begin{aligned} |(\mathcal{L}_2(z,r))(t)| &\le \frac{CK\delta}{\alpha-\varepsilon}e^{(-\alpha+\varepsilon)(t-t_*)} + MK\int_t^{\infty} e^{(-\alpha+\varepsilon)(s-t_*)}p(s)ds \\ &\le K\left(\frac{C\delta}{\alpha-\varepsilon} + M\hat{\varepsilon} + M\int_t^{\infty}\gamma(s)ds\right)e^{(-\alpha+\varepsilon)(t-t_*)}, \qquad t \ge t_*, \end{aligned}$$

where M depends, actually, on δ. Here we applied formula (4.8) that describes function $p(t)$ and used the fact that $w(t) < \hat{\varepsilon}$ for any prescribed $\hat{\varepsilon} > 0$ provided that t_* is sufficiently large. Since $\gamma(t) \in L_1[t_0,\infty)$ and δ, $\hat{\varepsilon}$ are arbitrary real constants, by choosing t_* sufficiently large we establish that the right inequality in (4.48) holds for function $\mathcal{L}_2(z,r)$.

We show now that the operator $\mathcal{L}$ is a contraction of B_1. Suppose that (z_1,r_1) and (z_2,r_2) belong to B_1. Then

$$\begin{aligned} &\|\mathcal{L}(z_1,r_1) - \mathcal{L}(z_2,r_2)\|_{B_1} \\ &\quad= \sup_{t\ge t_*}\{e^{(\alpha-\varepsilon)(t-t_*)}(\|(\mathcal{L}_1(z_1,r_1)-\mathcal{L}_1(z_2,r_2))(t,\cdot)\|_{C_h} + |(\mathcal{L}_2(z_1,r_1)-\mathcal{L}_2(z_2,r_2))(t)|)\} \\ &\quad\le M\sup_{t\ge t_*}\int_{t_*}^{t} e^{-\varepsilon(t-s)}p(s)e^{(\alpha-\varepsilon)(s-t_*)}\{|r_1(s)-r_2(s)| + \|z_1(s,\cdot)-z_2(s,\cdot)\|_{C_h}\}ds \\ &\qquad+ \sup_{t\ge t_*}\Big\{C\delta e^{(\alpha-\varepsilon)t}\int_t^{\infty} e^{(-\alpha+\varepsilon)s}e^{(\alpha-\varepsilon)(s-t_*)}|r_1(s)-r_2(s)|ds \\ &\qquad+ Me^{(\alpha-\varepsilon)t}\int_t^{\infty} e^{(-\alpha+\varepsilon)s}p(s)e^{(\alpha-\varepsilon)(s-t_*)}(|r_1(s)-r_2(s)| + \|z_1(s,\cdot)-z_2(s,\cdot)\|_{C_h})ds\Big\} \\ &\quad\le \left\{\frac{2MN(t_*)}{1-e^{-\varepsilon}} + \frac{C\delta}{\alpha-\varepsilon} + M\hat{\varepsilon} + M\int_{t_*}^{\infty}\gamma(s)ds\right\}\|(z_1,r_1)-(z_2,r_2)\|_{B_1}. \end{aligned}$$

Again we choose t_* sufficiently large and $\delta, \hat{\varepsilon}$ sufficiently small to state that $\mathcal{L}$ is a contraction of B_1. Consequently, equation (4.50) has a unique solution in B_1.

By recalling (4.43), we complete the proof of the theorem. □

Suppose that $u^{(1)}(t), \ldots, u^{(N)}(t)$ are fundamental solutions of system on critical manifold (3.7) and $x(t)$ is an arbitrary solution of Eq. (1.1) defined for $t \ge T$. By Theorem 4.7, this solution has the following asymptotic representation as $t \to \infty$:

$$x(t) = x_t(0) = (\Phi(0) + H(t,0))\sum_{i=1}^{N} c_i u^{(i)}(t) + O(e^{-\beta t}), \tag{4.51}$$

where $c_1, \ldots, c_N$ are arbitrary complex constants and $\beta > 0$ is a certain real number.

5 Asymptotic integration of system on critical manifold. Example

System (3.7), describing the dynamics of the initial Eq. (1.1) on critical manifold $\mathcal{W}(t)$, belongs to the class of the dynamical systems with oscillatory decreasing coefficients. The method for asymptotic integration of this kind of systems was proposed in [23]. Due to (3.9), (4.19), system (3.7) has the form

$$\dot{u} = \Big[D + \sum_{i=1}^{n} v_i(t)A_i(t) + \sum_{1\le i_1\le i_2\le n} v_{i_1}(t)v_{i_2}(t)A_{i_1i_2}(t) \\ + \cdots + \sum_{1\le i_1\le\cdots\le i_k\le n} v_{i_1}(t)\cdots v_{i_k}(t)A_{i_1\dots i_k}(t) + W(t)\Big]u, \qquad u\in\mathbb{C}^N. \quad (5.1)$$

Here matrix D is defined by (3.18), $A_{i_1\dots i_l}(t)$ are $(N\times N)$-matrices whose entries are trigonometric polynomials (i.e., matrices having the form (3.3)) and $W(t)$ is a certain matrix from $L_1[t_*,\infty)$. Without loss of generality we may assume that matrix D in Eq. (5.1) is, actually, matrix D_2, whose only nonzero entries are, possibly, $d_{i,i+1} = 1$ for certain $i\in\mathbb{N}$. Indeed, we write matrix D as sum (4.46), where, by hypothesis $\mathbf{H}_1$, all the eigenvalues of the diagonal matrix D_1 have zero real parts. In (5.1) we can make the change of variable $u = e^{D_1t}\hat{u}$ that does not change the properties of matrices $A_{i_1\dots i_l}(t)$ and $W(t)$. In the transformed system matrix D will be replaced by D_2.

From now on we will assume that the only eigenvalue of the Jordan matrix D in (5.1) is zero. The main difficulty in the asymptotic integration of system (5.1) as $t\to\infty$ is that its coefficients have an oscillatory behaviour. Thus, on the first step we utilize in (5.1) the averaging change of variable that makes it possible to exclude the oscillating coefficients from the main part of the system. The following theorem holds (see [23]).

Theorem 5.1. *For sufficiently large t, system (5.1) by the change of variable*

$$u = \Big[I + \sum_{i=1}^{n} Y_i(t)v_i(t) + \sum_{1\le i_1\le i_2\le n} Y_{i_1i_2}(t)v_{i_1}(t)v_{i_2}(t) \\ + \cdots + \sum_{1\le i_1\le\cdots\le i_k\le n} Y_{i_1\dots i_k}(t)v_{i_1}(t)\cdots v_{i_k}(t)\Big]u_1 \quad (5.2)$$

can be reduced to its averaged form

$$\dot{u}_1 = \Big[D + \sum_{i=1}^{n} A_iv_i(t) + \sum_{1\le i_1\le i_2\le n} A_{i_1i_2}v_{i_1}(t)v_{i_2}(t) \\ + \ldots + \sum_{1\le i_1\le\cdots\le i_k\le n} A_{i_1\dots i_k}v_{i_1}(t)\cdots v_{i_k}(t) + W_1(t)\Big]u_1(t) \quad (5.3)$$

with constant matrices $A_{i_1\dots i_l}$ and with matrix $W_1(t)$ from $L_1[t_*,\infty)$. In (5.2), I is the identity matrix and the entries of matrices $Y_{i_1\dots i_l}(t)$ are trigonometric polynomials having zero mean value.

As a rule, to construct the asymptotics for solutions of Eq. (5.3) we need to compute only a few constant matrices. Hence, we give the explicit formulas only for matrices A_i and A_{ij}. We have

$$A_i = \mathrm{M}\big[A_i(t)\big], \qquad i=1,\dots,n, \qquad \Big(\mathrm{M}\big[F(t)\big] = \lim_{T\to\infty}\frac{1}{T}\int_0^T F(s)ds\Big). \quad (5.4)$$

Further,

$$A_{ij} = \mathrm{M}\big[A_{ij}(t) + A_i(t)Y_j(t) + A_j(t)Y_i(t)\big], \quad 1 \le i < j \le n$$

and

$$A_{ii} = \mathrm{M}\big[A_{ii}(t) + A_i(t)Y_i(t)\big], \quad i = 1, \dots, n. \tag{5.5}$$

Here matrices $Y_i(t)$ with zero mean value are the solutions of the equations

$$\dot{Y}_i - DY_i + Y_iD = A_i(t) - A_i, \qquad i = 1, \dots, n. \tag{5.6}$$

The subsequent transformation of the averaged system (5.3) depends significantly on the structure of matrix D. We consider in details only the case when matrix D in (5.1) and (5.3) is zero matrix. Suppose that there exists a leading term in system (5.3) and this term is the matrix $A_{i_1\dots i_s}v_{i_1}(t)\cdots v_{i_s}(t)$. This means that we can rewrite system (5.3) in the following form:

$$\dot{u}_1 = \big[A_{i_1\dots i_s} + V(t)\big]v_{i_1}(t)\cdots v_{i_s}(t)u_1(t) + W_1(t)u_1(t), \tag{5.7}$$

where the matrix $V(t) \to 0$ as $t \to \infty$ and $\dot{V}(t) \in L_1[t_*, \infty)$. The following lemma holds (see, for instance, [7,15,18]).

Lemma 5.2 (Diagonalization of variable matrices)**.** *Suppose that all eigenvalues of the matrix $A_{i_1\dots i_s}$ are distinct. Moreover, suppose that the matrix $V(t) \to 0$ as $t \to \infty$ and $\dot{V}(t) \in L_1[t_*, \infty)$. Then for sufficiently large t there exists a nonsingular matrix $C(t)$ such that*

- *(i) the columns of this matrix are the eigenvectors of the matrix $A_{i_1\dots i_s} + V(t)$ and $C(t) \to C_0$ as $t \to \infty$. The columns of the constant matrix C_0 are the eigenvectors of the matrix $A_{i_1\dots i_s}$;*
- *(ii) the derivative $\dot{C}(t) \in L_1[t_*, \infty)$;*
- *(iii) it brings the matrix $A_{i_1\dots i_s} + V(t)$ to diagonal form, i.e.,*

$$C^{-1}(t)\big[A_{i_1\dots i_s} + V(t)\big]C(t) = \hat{\Lambda}(t),$$

where $\hat{\Lambda}(t) = \mathrm{diag}\big(\tilde{\lambda}_1(t), \dots, \tilde{\lambda}_N(t)\big)$ and $\tilde{\lambda}_j(t)$ $(j = 1, \dots, N)$ are the eigenvalues of the matrix $A_{i_1\dots i_s} + V(t)$.

In (5.3), we make the change of variable

$$u_1(t) = C(t)u_2(t), \tag{5.8}$$

where $C(t)$ is the matrix from Lemma 5.2. This change of variable brings system (5.3) to *L-diagonal* form:

$$\dot{u}_2 = \big[\Lambda(t) + W_2(t)\big]u_2, \tag{5.9}$$

where $\Lambda(t) = \mathrm{diag}\big(\lambda_1(t), \dots, \lambda_N(t)\big)$, $\lambda_j(t) = \tilde{\lambda}_j(t)v_{i_1}(t)\cdots v_{i_s}(t)$ $(j = 1, \dots, N)$ and

$$W_2(t) = -C^{-1}(t)\dot{C}(t) + C^{-1}(t)W_1(t)C(t).$$

The properties (i) and (ii) of the matrix $C(t)$ imply that matrix $W_2(t)$ belongs to $L_1[t_*, \infty)$.

To construct the asymptotics for solutions of the L-diagonal system (5.9) as $t \to \infty$ the well-known Theorem of Levinson can be used. Suppose that the following dichotomy condition holds for the entries of the matrix $\Lambda(t)$: either the inequality

$$\int_{t_1}^{t_2} \mathrm{Re}\big(\lambda_i(s) - \lambda_j(s)\big)ds \le K_1, \qquad t_2 \ge t_1 \ge t_*, \tag{5.10}$$

or the inequality

$$\int_{t_1}^{t_2} \mathrm{Re}\big(\lambda_i(s) - \lambda_j(s)\big)ds \geq K_2, \qquad t_2 \geq t_1 \geq t_*, \tag{5.11}$$

is valid for each pair of indices (i,j), where K_1, K_2 are some constants. The following is Levinson's fundamental theorem (see, e.g., [15, 18]).

Theorem 5.3 (Levinson)**.** *Let the dichotomy condition* (5.10)*,* (5.11) *be satisfied. Then the fundamental matrix $U(t)$ of system* (5.9) *has the following asymptotics as $t \to \infty$:*

$$U(t) = \big(I + o(1)\big) \exp\left\{\int_{t^*}^{t} \Lambda(s)ds\right\}. \tag{5.12}$$

If matrix D in (5.3) is a nonzero Jordan matrix or matrix $A_{i_1 \ldots i_s}$ in (5.7) is a Jordan matrix there is no general method to construct the asymptotics in this case. The structure of the matrix $W_1(t)$ from $L_1[t_*, \infty)$ may play a significant role in this situation. This question will not be discussed here. We only give some references dealing with the problem (see, e.g., [12, 16–18]).

Example.

We start with a remark concerning the following equation, studied in [25]:

$$\ddot{x} + x + \frac{a \sin \lambda t}{t^\rho} x(t-h) = 0, \tag{5.13}$$

where $a, \lambda \in \mathbb{R}$, $\rho > 0$ and $h > 0$. We wish to emphasize that this equation differs significantly from Eq. (1.6). Namely, we do not need to construct the critical manifold to get the asymptotics for solutions of Eq. (5.13). On the contrary, the method from [25] cannot be used to construct the asymptotic formulas for solutions of Eq. (1.6), since the corresponding functional $B_0\varphi = -\frac{\pi}{2}\varphi(-1)$ is nonzero and, consequently, a critical manifold should be constructed for this equation.

So, consider now the asymptotic integration problem for Eq. (1.6) as $t \to \infty$. It is known (see, e.g., [19, 20]) that the characteristic quasipolynomial for the corresponding unperturbed equation (1.4)

$$p(\lambda) = \lambda + \frac{\pi}{2} e^{-\lambda}$$

has purely imaginary roots $\lambda_{1,2} = \pm \mathrm{i}\pi/2$ and all the other roots have negative real parts. Therefore, the hypotheses $\mathbf{H_1}$, $\mathbf{H_2}$ hold and we can apply the asymptotic integration method developed in the paper. Easy computations show that the (1×2)-matrix $\Phi(\theta)$ and the (2×1)-matrix $\Psi(\xi)$, satisfying the normalization condition (2.8), may be defined as follows:

$$\Phi(\theta) = \big(e^{\mathrm{i}\frac{\pi}{2}\theta}, e^{-\mathrm{i}\frac{\pi}{2}\theta}\big), \qquad \Psi(\xi) = \frac{4}{4+\pi^2}\begin{pmatrix} (1 - \mathrm{i}\frac{\pi}{2})e^{-\mathrm{i}\frac{\pi}{2}\xi} \\ (1 + \mathrm{i}\frac{\pi}{2})e^{\mathrm{i}\frac{\pi}{2}\xi} \end{pmatrix},$$

where $\theta \in [-1, 0]$ and $\xi \in [0, 1]$. Since $G(t, \varphi) = at^{-\rho} \sin \omega t \varphi(0)$, the system on the critical manifold (3.7) has the form

$$\dot{u} = \Big[D + t^{-\rho} B_1(t) + W(t)\Big] u, \qquad t \geq t_*, \tag{5.14}$$

where

$$D = \frac{\pi}{2}\begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}, \qquad B_1(t) = -\frac{2a\mathrm{i}}{4+\pi^2}\big(e^{\mathrm{i}\omega t} - e^{-\mathrm{i}\omega t}\big)\begin{pmatrix} 1 - \mathrm{i}\frac{\pi}{2} & 1 - \mathrm{i}\frac{\pi}{2} \\ 1 + \mathrm{i}\frac{\pi}{2} & 1 + \mathrm{i}\frac{\pi}{2} \end{pmatrix}, \tag{5.15}$$

and $W(t) = \Psi(0)G\big(t, H(t,\theta)\big)$. Due to the form of operator $G(t,\varphi)$ and the form of matrix (3.9), it follows from Theorem 4.4, Corollary 4.5 and Remark 4.6 that $W(t) = O\big(t^{-2\rho}\big)$. We should consider several cases.

If

$$\rho > 1,$$

then system (5.14) has the L-diagonal form (5.9). According to Theorem 5.3, the fundamental solutions of Eq. (5.14) have the following asymptotics as $t \to \infty$:

$$u^{(j)}(t) = \big(e_j + o(1)\big)e^{\pm i\frac{\pi}{2}t}, \qquad j = 1, 2. \tag{5.16}$$

Here e_j are the standard basis vectors in $\mathbb{R}^2$. Due to (4.51) and the properties of matrix $H(t,\theta)$, all solutions of Eq. (1.6) have the following asymptotic representation as $t \to \infty$:

$$x(t) = c_1\big(1 + o(1)\big)e^{i\frac{\pi}{2}t} + c_2\big(1 + o(1)\big)e^{-i\frac{\pi}{2}t} + O\big(e^{-\beta t}\big), \tag{5.17}$$

where c_1, c_2 are arbitrary complex constants and $\beta > 0$ is a certain real number.

Let

$$\frac{1}{2} < \rho \le 1.$$

We make the change of variable $u = \operatorname{diag}\big(e^{i\frac{\pi}{2}t}, e^{-i\frac{\pi}{2}t}\big)u_1$ to reduce (5.14) to the form

$$\dot{u}_1 = \big[t^{-\rho}A_1(t) + W_1(t)\big]u_1, \tag{5.18}$$

where

$$A_1(t) = -\frac{2ai}{4+\pi^2}\begin{pmatrix} (1 - i\frac{\pi}{2})(e^{i\omega t} - e^{-i\omega t}) & (1 - i\frac{\pi}{2})(e^{i\omega t} - e^{-i\omega t})e^{-i\pi t} \\ (1 + i\frac{\pi}{2})(e^{i\omega t} - e^{-i\omega t})e^{i\pi t} & (1 + i\frac{\pi}{2})(e^{i\omega t} - e^{-i\omega t}) \end{pmatrix}, \tag{5.19}$$

and $W_1(t) \in L_1[t_*, \infty)$. According to Theorem 5.1, we make in (5.18) the change of variable $u_1 = \big(I + t^{-\rho}Y_1(t)\big)u_2$ to obtain the averaged system

$$\dot{u}_2 = \big[t^{-\rho}A_1 + W_2(t)\big]u_2. \tag{5.20}$$

Here $A_1 = \mathrm{M}\big[A_1(t)\big]$ and $W_2(t)$ is a certain matrix from $L_1[t_*, \infty)$. The form of the matrix A_1 will be different in the following cases.

Assume that

$$\omega \neq \pm\pi. \tag{5.21}$$

Matrix A_1 is a zero matrix and, hence, system (5.20) has L-diagonal form. By the Levinson Theorem, fundamental solutions of this system have the following asymptotics as $t \to \infty$:

$$u_2^{(j)}(t) = e_j + o(1), \qquad j = 1, 2.$$

If we return to system (5.14) we get the asymptotic formulas (5.16) for its fundamental solutions as $t \to \infty$. Consequently, all solutions of Eq. (1.6) have asymptotics (5.17) as $t \to \infty$.

Suppose now that

$$\omega = \pi. \tag{5.22}$$

The case $\omega = -\pi$ is reduced to the case $\omega = \pi$ if we replace a with $(-a)$ in (1.6). We have

$$A_1 = -\frac{2ai}{4+\pi^2}\begin{pmatrix} 0 & 1 - i\frac{\pi}{2} \\ -1 - i\frac{\pi}{2} & 0 \end{pmatrix}. \tag{5.23}$$

The eigenvalues of matrix A_1 are $\mu_{1,2} = \pm\frac{a}{\sqrt{4+\pi^2}}$. By the change of variable $u_2 = Cu_3$, where matrix $C = [f_1, f_2]$ brings A_1 to diagonal form, system (5.20) is reduced to L-diagonal form. Fundamental solutions of system (5.20) have the following asymptotics as $t \to \infty$:

$$u_2^{(j)}(t) = \big(f_j + o(1)\big)\exp\Big\{\mu_j \int t^{-\rho}dt\Big\}, \qquad j = 1, 2,$$

where

$$f_1 = \begin{pmatrix} 1 \\ \delta \end{pmatrix}, \qquad f_2 = \begin{pmatrix} 1 \\ -\delta \end{pmatrix}, \qquad \delta = \frac{-\pi + 2\mathrm{i}}{\sqrt{4+\pi^2}}.$$

Therefore, all solutions of Eq. (1.6) have the following asymptotic representation as $t \to \infty$:

$$\begin{aligned} x(t) = c_1\Big(e^{\mathrm{i}\frac{\pi}{2}t}\big(1+o(1)\big) + e^{-\mathrm{i}\frac{\pi}{2}t}\big(\delta + o(1)\big)\Big)\exp\Big\{\mu_1\int t^{-\rho}dt\Big\} \\ + c_2\Big(e^{\mathrm{i}\frac{\pi}{2}t}\big(1+o(1)\big) + e^{-\mathrm{i}\frac{\pi}{2}t}\big(-\delta + o(1)\big)\Big)\exp\Big\{\mu_2\int t^{-\rho}dt\Big\} + O\big(e^{-\beta t}\big), \end{aligned} \tag{5.24}$$

where c_1, c_2 are arbitrary real constants and $\beta > 0$ is a certain real number.

Consider now the case

$$\frac{1}{3} < \rho \leq \frac{1}{2}. \tag{5.25}$$

The matrix $W(t) = \Psi(0)G\big(t, H(t,\theta)\big)$ in (5.14) does not belong to $L_1[t_*, \infty)$ any more. Thus, we need to construct an approximation for the (1×2)-matrix $H(t,\theta)$ that describes the critical manifold $\mathcal{W}(t)$. By (3.9), (4.19), we have

$$H(t,\theta) = t^{-\rho}H_1(t,\theta) + t^{-2\rho}H_2(t,\theta) + Z(t,\theta), \tag{5.26}$$

where $\|Z(t,\cdot)\|_{C_h} \in L_1[t_*, \infty)$. We note that only the situation when inequality (5.21) holds is of interest. Actually, if equality (5.22) holds then we can, firstly, represent the averaged system on the critical manifold in the form (5.7), where $A_{i_1\ldots i_s} = A_1$, and matrix A_1 is described by formula (5.23). And, secondly, we apply Lemma 5.2 to reduce this system to the L-diagonal form (5.9) and, then, use Levinson's Theorem. It is easy to verify that the asymptotic formula of the form (5.24) as $t \to \infty$ holds for solutions of Eq. (1.6) provided that equality (5.22) is valid and $\rho \leq 1/2$. The only difference is that we should replace the quantity $\int t^{-\rho}dt$ in (5.24) by

$$\frac{t^{1-\rho}}{1-\rho}\big(1+o(1)\big). \tag{5.27}$$

We assume now that inequality (5.21) holds. Let us calculate the matrix $H_1(t,\theta)$ using the approximation scheme described in Section 3. We substitute (5.26) in (3.8) and collect terms with $t^{-\rho}$. We obtain the following equation for the (1×2)-matrix $H_1(t,\theta)$:

$$\begin{aligned} \frac{a}{2\mathrm{i}}\big(e^{\mathrm{i}\omega t} - e^{-\mathrm{i}\omega t}\big)\Phi(\theta)\Psi(0)\Phi(0) + H_1(t,\theta)D + \frac{\partial H_1}{\partial t} \\ = \begin{cases} \dfrac{\partial H_1}{\partial\theta}, & -1 \leq \theta < 0, \\ -\dfrac{\pi}{2}H_1(t,-1) + \dfrac{a}{2\mathrm{i}}\big(e^{\mathrm{i}\omega t} - e^{-\mathrm{i}\omega t}\big)\Phi(0), & \theta = 0. \end{cases} \end{aligned} \tag{5.28}$$

By setting $H_1(t,\theta) = \big(h_1(t,\theta), h_2(t,\theta)\big)$ we obtain the following equations for $h_j(t,\theta)$:

$$-\frac{2a\mathrm{i}}{4+\pi^2}\big(e^{\mathrm{i}\omega t} - e^{-\mathrm{i}\omega t}\big)\left(\Big(1-\mathrm{i}\frac{\pi}{2}\Big)e^{\mathrm{i}\frac{\pi}{2}\theta} + \Big(1+\mathrm{i}\frac{\pi}{2}\Big)e^{-\mathrm{i}\frac{\pi}{2}\theta}\right) + (-1)^{j-1}\mathrm{i}\frac{\pi}{2}h_j(t,\theta) + \frac{\partial h_j}{\partial t}$$
$$= \begin{cases} \dfrac{\partial h_j}{\partial \theta}, & -1 \le \theta < 0, \\ -\dfrac{\pi}{2}h_j(t,-1) + \dfrac{a}{2\mathrm{i}}\big(e^{\mathrm{i}\omega t} - e^{-\mathrm{i}\omega t}\big), & \theta = 0, \end{cases} \qquad j = 1,2. \tag{5.29}$$

Each of these equations is uniquely solvable due to Theorem 3.2. From the latter and the form of equations (5.29) we can derive that $h_2(t,\theta) = \overline{h_1(t,\theta)}$, where notation $\bar{z}$ means the complex conjugate for $z \in \mathbb{C}$. We seek solution $h_1(t,\theta)$ as

$$h_1(t,\theta) = g_1(\theta)e^{\mathrm{i}\omega t} + g_2(\theta)e^{-\mathrm{i}\omega t}. \tag{5.30}$$

We substitute (5.30) in (5.29) and collect the terms with the same exponentials. We get

$$(-1)^j\frac{2a\mathrm{i}}{4+\pi^2}\left(\Big(1-\mathrm{i}\frac{\pi}{2}\Big)e^{\mathrm{i}\frac{\pi}{2}\theta} + \Big(1+\mathrm{i}\frac{\pi}{2}\Big)e^{-\mathrm{i}\frac{\pi}{2}\theta}\right) + \mathrm{i}\frac{\pi}{2}g_j(\theta) + (-1)^{j-1}\mathrm{i}\omega g_j(\theta)$$
$$= \begin{cases} \dfrac{dg_j}{d\theta}, & -1 \le \theta < 0, \\ -\dfrac{\pi}{2}g_j(-1) + (-1)^j\dfrac{a\mathrm{i}}{2}, & \theta = 0, \end{cases} \qquad j = 1,2. \tag{5.31}$$

Some easy computations show that

$$g_{1,2}(\theta) = K_{1,2}e^{\mathrm{i}(\frac{\pi}{2}\pm\omega)\theta} + \frac{2a}{\omega(4+\pi^2)}\Big(1-\mathrm{i}\frac{\pi}{2}\Big)e^{\mathrm{i}\frac{\pi}{2}\theta} + \frac{2a}{(\omega\pm\pi)(4+\pi^2)}\Big(1+\mathrm{i}\frac{\pi}{2}\Big)e^{-\mathrm{i}\frac{\pi}{2}\theta}, \tag{5.32}$$
$$K_{1,2} = \mp\frac{a}{2\big(\frac{\pi}{2} \pm \omega - \frac{\pi}{2}\exp\{\mp\mathrm{i}\omega\}\big)}.$$

Here the upper sign and constant K_1 stand for function $g_1(\theta)$ and the lower sign with constant K_2 stand for $g_2(\theta)$. Thus, matrix $H_1(t,\theta)$ has the form

$$H_1(t,\theta) = \big(h_1(t,\theta), \overline{h_1(t,\theta)}\big), \tag{5.33}$$

where $h_1(t,\theta)$ is described by formulas (5.30), (5.32).

We return to system (5.14). Taking into account (5.26), (5.33), we obtain

$$\dot{u} = \Big[D + t^{-\rho}B_1(t) + t^{-2\rho}B_2(t) + R(t)\Big]u, \qquad t \ge t_*, \tag{5.34}$$

where matrices D and $B_1(t)$ are defined in (5.15), matrix $R(t)$ belongs to $L_1[t_*,\infty)$ and

$$B_2(t) = -\frac{a\mathrm{i}}{2}(e^{\mathrm{i}\omega t} - e^{-\mathrm{i}\omega t})\Psi(0)H_1(t,0)$$
$$= -\frac{2a\mathrm{i}}{4+\pi^2}\big(e^{\mathrm{i}\omega t} - e^{-\mathrm{i}\omega t}\big)\begin{pmatrix} (1-\mathrm{i}\frac{\pi}{2})h_1(t,0) & (1-\mathrm{i}\frac{\pi}{2})\overline{h_1(t,0)} \\ (1+\mathrm{i}\frac{\pi}{2})h_1(t,0) & (1+\mathrm{i}\frac{\pi}{2})\overline{h_1(t,0)} \end{pmatrix}. \tag{5.35}$$

In (5.34) we make the change of variable $u = \mathrm{diag}\big(e^{\mathrm{i}\frac{\pi}{2}t}, e^{-\mathrm{i}\frac{\pi}{2}t}\big)u_1$ to get

$$\dot{u}_1 = \big[t^{-\rho}A_1(t) + t^{-2\rho}A_2(t) + R_1(t)\big]u_1. \tag{5.36}$$

Here $R_1(t) \in L_1[t_*,\infty)$, matrix $A_1(t)$ is defined by (5.19) and

$$A_2(t) = -\frac{2a\mathrm{i}}{4+\pi^2}\begin{pmatrix} (1-\mathrm{i}\frac{\pi}{2})(e^{\mathrm{i}\omega t}-e^{-\mathrm{i}\omega t})h_1(t,0) & (1-\mathrm{i}\frac{\pi}{2})(e^{\mathrm{i}\omega t}-e^{-\mathrm{i}\omega t})e^{-\mathrm{i}\pi t}\overline{h_1(t,0)} \\ (1+\mathrm{i}\frac{\pi}{2})(e^{\mathrm{i}\omega t}-e^{-\mathrm{i}\omega t})e^{\mathrm{i}\pi t}h_1(t,0) & (1+\mathrm{i}\frac{\pi}{2})(e^{\mathrm{i}\omega t}-e^{-\mathrm{i}\omega t})\overline{h_1(t,0)} \end{pmatrix}. \tag{5.37}$$

In (5.36) we apply the averaging change of variable $u_1 = \left[I + t^{-\rho}Y_1(t) + t^{-2\rho}Y_2(t)\right]u_2$ according to Theorem 5.1. This yields that

$$\dot{u}_2 = \left[t^{-2\rho}A_2 + R_2(t)\right]u_2. \tag{5.38}$$

We used the fact that, due to (5.21), $A_1 = \mathrm{M}\left[A_1(t)\right]$ is a zero matrix. Further,

$$A_2 = \mathrm{M}\left[A_2(t) + A_1(t)Y_1(t)\right], \qquad \dot{Y}_1 = A_1(t) - A_1 = A_1(t) \tag{5.39}$$

and $R_2(t) \in L_1[t_*,\infty)$.

We assume first that

$$\omega = \frac{\pi}{2}, \qquad \left(\omega = -\frac{\pi}{2}\right). \tag{5.40}$$

After some easy but tedious computations one gets that

$$A_2 = \frac{a^2}{4+\pi^2}\begin{pmatrix} -\frac{2}{5}+\frac{8}{5\pi}+\mathrm{i}\left(-\frac{4}{5}-\frac{4}{5\pi}\right) & -\frac{2}{\pi}+\mathrm{i} \\ -\frac{2}{\pi}-\mathrm{i} & -\frac{2}{5}+\frac{8}{5\pi}-\mathrm{i}\left(-\frac{4}{5}-\frac{4}{5\pi}\right) \end{pmatrix}.$$

The eigenvalues of matrix A_2 are

$$\mu_{1,2} = \frac{a^2}{5\pi(4+\pi^2)}\left(8 - 2\pi \pm \sqrt{9\pi^2 - 32\pi + 84}\right). \tag{5.41}$$

System (5.38) by the change of variable $u_2 = Cu_3$, where C is a certain constant matrix, may be reduced to L-diagonal form. Hence, the linearly independent solutions of Eq. (5.38) have the following asymptotics as $t \to \infty$:

$$u_2^{(j)}(t) = \left(f_j + o(1)\right)\exp\left\{\mu_j \int t^{-2\rho}dt\right\}, \quad j = 1,2,$$

where $f_1 = \left(\delta_1, \delta_2\right)^T$ and f_2 are the eigenvectors of A_2 corresponding to the eigenvalues μ_1 and μ_2 respectively. Solutions of initial Eq. (1.6) are described by the following asymptotic formula as $t \to \infty$:

$$x(t) = c_1\left(e^{\mathrm{i}\frac{\pi}{2}t}\left(\delta_1 + o(1)\right) + e^{-\mathrm{i}\frac{\pi}{2}t}\left(\delta_2 + o(1)\right)\right)\exp\left\{\mu_1 \int t^{-2\rho}dt\right\} + O\left(\exp\left\{\mu_2 \int t^{-2\rho}dt\right\}\right), \tag{5.42}$$

where c_1 is an arbitrary complex constant.

We suppose now that

$$\omega \neq \pm\frac{\pi}{2}, \pm\pi \tag{5.43}$$

and inequality (5.25) holds. Matrix A_2, that is defined in (5.39), has the form

$$A_2 = \begin{pmatrix} \nu & 0 \\ 0 & \bar{\nu} \end{pmatrix}, \qquad \nu = -\frac{a^2\mathrm{i}}{4+\pi^2}\left(1 - \mathrm{i}\frac{\pi}{2}\right)\left(\frac{1}{\frac{\pi}{2} - \omega - \frac{\pi}{2}e^{\mathrm{i}\omega}} + \frac{1}{\frac{\pi}{2} + \omega - \frac{\pi}{2}e^{-\mathrm{i}\omega}}\right). \tag{5.44}$$

Thus, the fundamental solutions of (5.38) have asymptotics

$$u_2^{(1,2)}(t) = \left(e_{1,2} + o(1)\right)\exp\left\{\left(\mathrm{Re}\,\nu \pm \mathrm{i}\,\mathrm{Im}\,\nu\right)\int t^{-2\rho}dt\right\}, \qquad t \to \infty,$$

where e_j are the standard basis vectors in $\mathbb{R}^2$. Consequently, all solutions of Eq. (1.6) have the following asymptotics as $t \to \infty$:

$$x(t) = c_1(1+o(1)) \exp\left\{ \mathrm{i}\left(\frac{\pi}{2}t + \operatorname{Im}\nu \int t^{-2\rho} dt\right)\right\} \exp\left\{\operatorname{Re}\nu \int t^{-2\rho} dt\right\}$$
$$+ c_2(1+o(1)) \exp\left\{ -\mathrm{i}\left(\frac{\pi}{2}t + \operatorname{Im}\nu \int t^{-2\rho} dt\right)\right\} \exp\left\{\operatorname{Re}\nu \int t^{-2\rho} dt\right\} + O(e^{-\beta t}), \quad (5.45)$$

where c_1, c_2 are arbitrary complex constants and $\beta > 0$ is a certain real number. Evidently, the qualitative behaviour of solutions of Eq. (1.6) will be defined in this situation by the sign of the real part of ν. It can be shown (e.g., by using the packages for symbolic calculations) that this quantity has the following expression:

$$\operatorname{Re}\nu = \frac{2\pi^2 a^2 \sin^2\left(\frac{\omega}{2}\right)\left(2\omega^2 + 4\omega\sin\omega + \pi^2\cos\omega - \pi^2\right)}{(4+\pi^2)\left(4\omega^4 + \pi^4 + \pi^2\left(4\omega^2 - 2\pi^2\right)\cos\omega + \pi^2\left(\pi^2 - 4\omega^2\right)\cos^2\omega\right)}. \quad (5.46)$$

The graph of $\operatorname{Re}\nu$ as a function of ω is given in Fig. 5.1.

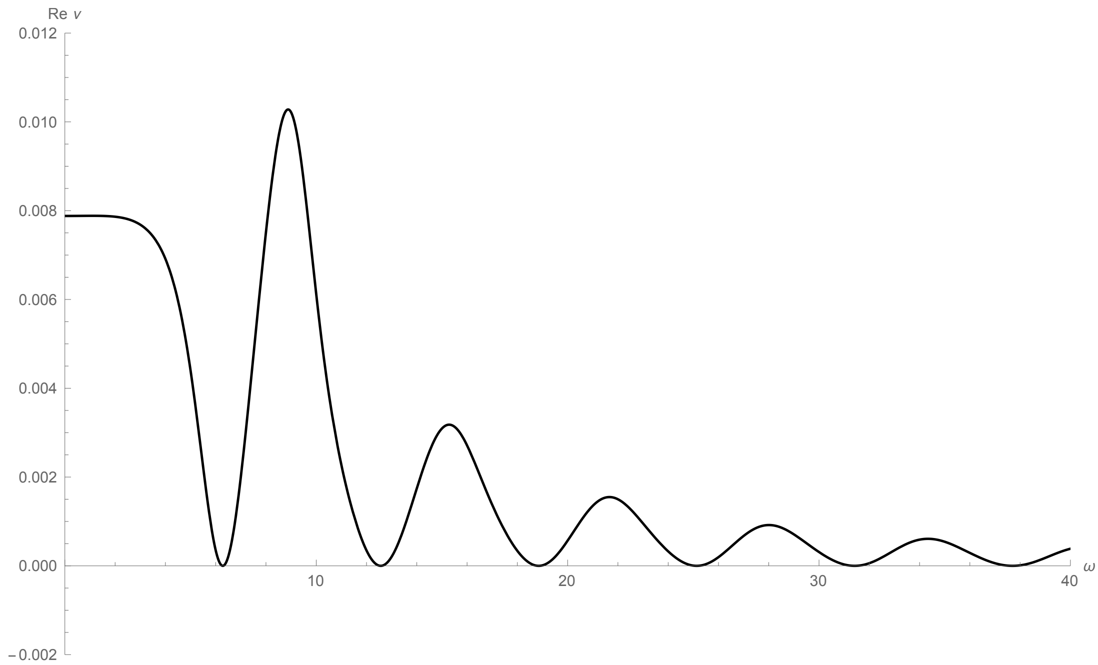

Figure 5.1: Graph of quantity (5.46), when $a = 1$.

By using (5.46), it is not difficult to check that $\operatorname{Re}\nu > 0$ for all $\omega \neq 2\pi n$, $n \in \mathbb{N}$ $(\omega > 0)$. Moreover,

$$\lim_{\omega\to 0} \operatorname{Re}\nu = \frac{\pi^2 a^2 (12 - \pi^2)}{(4+\pi^2)^3}, \qquad \lim_{\omega\to \pi} \operatorname{Re}\nu = \frac{a^2(\pi^2 - 4)}{4(4+\pi^2)^2} \quad (5.47)$$

and

$$\operatorname{Re}\nu = \frac{\pi^2 a^2 \sin^2(\frac{\omega}{2})}{(4+\pi^2)\omega^2}\left(1 + O(\omega^{-1})\right), \qquad \omega \to +\infty. \quad (5.48)$$

Finally, consider the case

$$\rho \le \frac{1}{3}. \quad (5.49)$$

Due to (3.9), (4.19), we have

$$H(t,\theta) = t^{-\rho}H_1(t,\theta) + \cdots + t^{-k\rho}H_k(t,\theta) + Z(t,\theta). \tag{5.50}$$

Here $k \in \mathbb{N}$ is chosen in the way that $k\rho \le 1 < (k+1)\rho$ and $Z(t,\theta)$ is a (1×2)-matrix function such that $\|Z(t,\cdot)\|_{C_h} \in L_1[t_*,\infty)$. The system on the critical manifold has the following form:

$$\dot{u} = \Big[D + t^{-\rho}B_1(t) + t^{-2\rho}B_2(t) + \cdots + t^{-k\rho}B_k(t) + R(t)\Big]u, \qquad t \ge t_*, \tag{5.51}$$

where matrices D, $B_1(t)$ are defined by formulas (5.15), matrix $B_2(t)$ is described by (5.35) and matrix $R(t)$ is a certain matrix from $L_1[t_*,\infty)$. In (5.51), we utilize the change of variable $u = \operatorname{diag}\big(e^{\mathrm{i}\frac{\pi}{2}t}, e^{-\mathrm{i}\frac{\pi}{2}t}\big)u_1$ to obtain system

$$\dot{u}_1 = \big[t^{-\rho}A_1(t) + t^{-2\rho}A_2(t) + \cdots + t^{-k\rho}A_k(t) + R_1(t)\big]u_1. \tag{5.52}$$

Then, according to Theorem 5.1, in (5.52) we make the averaging change of variable $u_1 = \big[I + t^{-\rho}Y_1(t) + t^{-2\rho}Y_2(t) + \cdots + t^{-k\rho}Y_k(t)\big]u_2$ to get

$$\dot{u}_2 = \big[A_1t^{-\rho} + t^{-2\rho}A_2 + \cdots + t^{-k\rho}A_k + R_2(t)\big]u_2. \tag{5.53}$$

We write (5.53) in the form (5.7) and then use Lemma 5.2 to reduce this system to the L-diagonal form (5.9). If $\omega = \pm\pi$ then the eigenvalues of matrix A_1 are distinct. Thus, in this case the asymptotic representation (5.24) holds for solutions of Eq. (1.6), where the integral $\int t^{-\rho}dt$ should be replaced by expression (5.27). Further, if $\omega = \pm\frac{\pi}{2}$ then $A_1 = 0$ and the eigenvalues of A_2 are distinct and have the form (5.41). Therefore, the asymptotics of all solutions of Eq. (1.6) is defined by formula (5.42), where $\int t^{-2\rho}dt$ should be replaced by

$$\frac{t^{1-2\rho}}{1-2\rho}\big(1+o(1)\big). \tag{5.54}$$

Finally, if inequalities (5.43) hold then $A_1 = 0$ and matrix A_2 is defined by (5.44). The eigenvalues of A_2 are distinct provided that $\operatorname{Im}\nu \ne 0$. Suppose that $\operatorname{Re}\nu \ne 0$, then the asymptotics for all solutions of Eq. (1.6) has the form (5.45), where $\int t^{-2\rho}dt$ should be replaced by (5.54).

6 Appendix. Proof of Theorem 3.2

Before we proceed to the proof of Theorem 3.2 we need to introduce some notation and formulate two auxiliary propositions. Analogously to (3.19), (3.21), we write the $(m\times N)$-matrix function $\Phi(\theta)$, whose columns form the basis of the generalized eigenspace P_Λ, as follows:

$$\Phi(\theta) = \big[\Phi^{(1)}(\theta),\dots,\Phi^{(l)}(\theta)\big], \qquad \Phi^{(p)}(\theta) = \big[\varphi_1^{(p)}(\theta),\dots,\varphi_{N_p}^{(p)}(\theta)\big], \tag{6.1}$$

where $\Phi^{(p)}(\theta)$ are $(m\times N_p)$-matrices and $\varphi_s^{(p)}(\theta)$ are m-dimensional column vectors. Taking into account (3.18) and also (2.9), (2.11), we obtain

$$\Phi^{(p)}(\theta) = \Phi^{(p)}(0)e^{D^{(p)}\theta}, \quad \Phi^{(p)}(0)D^{(p)} - \int_{-h}^{0} d\eta(\theta)\Phi^{(p)}(0)e^{D^{(p)}\theta} = 0, \quad -h \le \theta \le 0. \tag{6.2}$$

Using (2.2), (2.3), we conclude that

$$\begin{aligned}
\varphi_1^{(p)}(\theta) &= \varphi_1^{(p)}(0)e^{\lambda^{(p)}\theta}, \\
\varphi_2^{(p)}(\theta) &= \big(\varphi_1^{(p)}(0)\theta + \varphi_2^{(p)}(0)\big)e^{\lambda^{(p)}\theta}, \\
&\vdots \\
\varphi_{N_p}^{(p)}(\theta) &= \Big(\varphi_1^{(p)}(0)\frac{\theta^{N_p-1}}{(N_p-1)!} + \cdots + \varphi_{N_p-1}^{(p)}(0)\theta + \varphi_{N_p}^{(p)}(0)\Big)e^{\lambda^{(p)}\theta},
\end{aligned} \tag{6.3}$$

and vectors $\varphi_1^{(p)}(0), \dots, \varphi_{N_p}^{(p)}(0)$ are defined from the equations

$$\begin{aligned}
\Delta(\lambda^{(p)})\varphi_1^{(p)}(0) &= 0, \\
\Delta'(\lambda^{(p)})\varphi_1^{(p)}(0) + \Delta(\lambda^{(p)})\varphi_2^{(p)}(0) &= 0, \\
&\vdots \\
\frac{\Delta^{(N_p-1)}(\lambda^{(p)})}{(N_p-1)!}\varphi_1^{(p)}(0) + \cdots + \Delta'(\lambda^{(p)})\varphi_{N_p-1}^{(p)}(0) + \Delta(\lambda^{(p)})\varphi_{N_p}^{(p)}(0) &= 0.
\end{aligned} \tag{6.4}$$

In much the same way we handle the $(N \times m)$-matrix function $\Psi(\xi)$, whose rows form the basis of the generalized eigenspace P_Λ^T of the transposed equation (2.4). We have

$$\Psi(\xi) = \operatorname{col}(\Psi^{(1)}(\xi), \dots, \Psi^{(l)}(\xi)), \qquad \Psi^{(p)}(\xi) = \operatorname{col}\big(\psi_1^{(p)}(\xi), \dots, \psi_{N_p}^{(p)}(\xi)\big), \tag{6.5}$$

where $\Psi^{(p)}(\xi)$ are $(N_p \times m)$-matrices and $\psi_s^{(p)}(\xi)$ are m-dimensional row vectors. We then use (3.18) and also (2.10), (2.12) to derive that

$$\Psi^{(p)}(\xi) = e^{-D^{(p)}\xi}\Psi^{(p)}(0), \quad D^{(p)}\Psi^{(p)}(0) - \int_{-h}^{0} e^{D^{(p)}\theta}\Psi^{(p)}(0)d\eta(\theta) = 0, \quad 0 \le \xi \le h. \tag{6.6}$$

We recall (2.2), (2.3) to deduce that

$$\begin{aligned}
\psi_{N_p}^{(p)}(\xi) &= \psi_{N_p}^{(p)}(0)e^{-\lambda^{(p)}\xi}, \\
\psi_{N_p-1}^{(p)}(\xi) &= \big(\psi_{N_p}^{(p)}(0)(-\xi) + \psi_{N_p-1}^{(p)}(0)\big)e^{-\lambda^{(p)}\xi}, \\
&\vdots \\
\psi_1^{(p)}(\xi) &= \Big(\psi_{N_p}^{(p)}(0)\frac{(-\xi)^{N_p-1}}{(N_p-1)!} + \cdots + \psi_2^{(p)}(0)(-\xi) + \psi_1^{(p)}(0)\Big)e^{-\lambda^{(p)}\xi},
\end{aligned} \tag{6.7}$$

and vectors $\psi_1^{(p)}(0), \dots, \psi_{N_p}^{(p)}(0)$ are defined from the equations

$$\begin{aligned}
\psi_{N_p}^{(p)}(0)\Delta(\lambda^{(p)}) &= 0, \\
\psi_{N_p}^{(p)}(0)\Delta'(\lambda^{(p)}) + \psi_{N_p-1}^{(p)}(0)\Delta(\lambda^{(p)}) &= 0, \\
&\vdots \\
\frac{\psi_{N_p}^{(p)}(0)\Delta^{(N_p-1)}(\lambda^{(p)})}{(N_p-1)!} + \cdots + \psi_2^{(p)}(0)\Delta'(\lambda^{(p)}) + \psi_1^{(p)}(0)\Delta(\lambda^{(p)}) &= 0.
\end{aligned} \tag{6.8}$$

The following propositions hold.

Proposition 6.1. *Let $\mu \neq \lambda^{(p)}$ for some $p = 1, \dots, l$. Then, for every $f(\theta) \in C_h$ and $\nu \in \mathbb{N}$ the following formulas are valid:*

$$\begin{aligned}&\left(\Psi^{(p)}(\xi), \int_0^\theta e^{\mu(\theta-s)} f(s)ds\right)\\ &\quad = -(\mu I - D^{(p)})^{-1}\left[\Psi^{(p)}(0)B_0\left(\int_0^\theta e^{\mu(\theta-s)}f(s)ds\right) + \left(\Psi^{(p)}(\xi), f(\theta)\right) - \Psi^{(p)}(0)f(0)\right],\end{aligned} \tag{6.9}$$

$$\begin{aligned}&\left(\Psi^{(p)}(\xi), \int_0^\theta \frac{(\theta-s)^\nu}{\nu!} e^{\mu(\theta-s)} f(s)ds\right)\\ &\quad = -(\mu I - D^{(p)})^{-1}\Psi^{(p)}(0)B_0\left(\int_0^\theta \frac{(\theta-s)^\nu}{\nu!}e^{\mu(\theta-s)}f(s)ds\right)\\ &\qquad + (\mu I - D^{(p)})^{-2}\Psi^{(p)}(0)B_0\left(\int_0^\theta \frac{(\theta-s)^{\nu-1}}{(\nu-1)!}e^{\mu(\theta-s)}f(s)ds\right)\\ &\qquad + \cdots + (-1)^{\nu+1}(\mu I - D^{(p)})^{-(\nu+1)}\left[\Psi^{(p)}(0)B_0\left(\int_0^\theta e^{\mu(\theta-s)}f(s)ds\right)\right.\\ &\qquad\qquad \left. + \left(\Psi^{(p)}(\xi), f(\theta)\right) - \Psi^{(p)}(0)f(0)\right].\end{aligned} \tag{6.10}$$

Corollary 6.2. *Let $\mu \neq \lambda^{(p)}$ for some $p = 1, \dots, l$. Then for every $\nu \in \mathbb{N}$ the following formulas hold:*

$$\left(\Psi^{(p)}(\xi), e^{\mu\theta} I\right) = (\mu I - D^{(p)})^{-1}\Psi^{(p)}(0)\Delta(\mu), \tag{6.11}$$

$$\begin{aligned}\left(\Psi^{(p)}(\xi), \frac{\theta^\nu}{\nu!}e^{\mu\theta}\right) &= (\mu I - D^{(p)})^{-1}\Psi^{(p)}(0)\frac{\Delta^{(\nu)}(\mu)}{\nu!}\\ &\quad - (\mu I - D^{(p)})^{-2}\Psi^{(p)}(0)\frac{\Delta^{(\nu-1)}(\mu)}{(\nu-1)!}\\ &\quad + \cdots + (-1)^\nu(\mu I - D^{(p)})^{-(\nu+1)}\Psi^{(p)}(0)\Delta(\mu).\end{aligned} \tag{6.12}$$

Proposition 6.3. *Let $\mu = \lambda^{(p)}$ for some $p = 1, \dots, l$. Then for every $f(\theta) \in C_h$ and $\nu \in \mathbb{N}$ the following formula holds:*

$$\begin{aligned}&\left(\Psi^{(p)}(\xi), \int_0^\theta \frac{(\theta-s)^{\nu-1}}{(\nu-1)!}e^{\mu(\theta-s)}f(s)ds\right)\\ &= -\int_{-h}^0\int_0^\theta \begin{pmatrix} \frac{(\theta-\xi)^\nu}{\nu!} & \frac{(\theta-\xi)^{\nu+1}}{(\nu+1)!} & \cdots & \cdots & \frac{(\theta-\xi)^{\nu+N_p-1}}{(\nu+N_p-1)!} \\ 0 & \frac{(\theta-\xi)^\nu}{\nu!} & \frac{(\theta-\xi)^{\nu+1}}{(\nu+1)!} & \cdots & \frac{(\theta-\xi)^{\nu+N_p-2}}{(\nu+N_p-2)!} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \frac{(\theta-\xi)^\nu}{\nu!} & \frac{(\theta-\xi)^{\nu+1}}{(\nu+1)!} \\ 0 & \cdots & \cdots & 0 & \frac{(\theta-\xi)^\nu}{\nu!} \end{pmatrix} \Psi^{(p)}(0)d\eta(\theta)e^{\mu(\theta-\xi)}f(\xi)d\xi.\end{aligned} \tag{6.13}$$

Corollary 6.4. *Let $\mu = \lambda^{(p)}$ for some $p = 1, \dots, l$. Then for every $\nu \in \mathbb{N}$ the following formulas hold:*

$$\big(\Psi^{(p)}(\xi), e^{\mu\theta} I\big) = \Psi^{(p)}(0) - \int_{-h}^{0} \begin{pmatrix} \theta & \frac{\theta^2}{2!} & \dots & \dots & \frac{\theta^{N_p}}{N_p!} \\ 0 & \theta & \frac{\theta^2}{2!} & \dots & \frac{\theta^{N_p-1}}{(N_p-1)!} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \theta & \frac{\theta^2}{2!} \\ 0 & \dots & \dots & 0 & \theta \end{pmatrix} \Psi^{(p)}(0) d\eta(\theta) e^{\mu\theta}, \tag{6.14}$$

$$\Big(\Psi^{(p)}(\xi), \frac{\theta^\nu}{\nu!} e^{\mu\theta}\Big) = -\int_{-h}^{0} \begin{pmatrix} \frac{\theta^{\nu+1}}{(\nu+1)!} & \frac{\theta^{\nu+2}}{(\nu+2)!} & \dots & \dots & \frac{\theta^{\nu+N_p}}{(\nu+N_p)!} \\ 0 & \frac{\theta^{\nu+1}}{(\nu+1)!} & \frac{\theta^{\nu+2}}{(\nu+2)!} & \dots & \frac{\theta^{\nu+N_p-1}}{(\nu+N_p-1)!} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \frac{\theta^{\nu+1}}{(\nu+1)!} & \frac{\theta^{\nu+2}}{(\nu+2)!} \\ 0 & \dots & \dots & 0 & \frac{\theta^{\nu+1}}{(\nu+1)!} \end{pmatrix} \Psi^{(p)}(0) d\eta(\theta) e^{\mu\theta}. \tag{6.15}$$

We emphasize that the proofs of Propositions 6.1, 6.3 and the corresponding Corollaries 6.2, 6.4 are, actually, technical exercises. We omit them here in order to focus on the main result. The corresponding proofs may be provided by the author on request.

We are now in a position to prove Theorem 3.2.

Proof of Theorem 3.2. For the sake of brevity in the sequel we will omit the upper index $^{(i)}$ in the notation of the coefficients and variables in problems $\mathbf{P_1}, \dots, \mathbf{P_{N_i}}$, setting $\mu = \mu^{(i)}$, $g_r = g_r^{(i)}$, $p_r(\theta) = p_r^{(i)}(\theta)$, $q_r(\theta) = q_r^{(i)}(\theta)$ and $z_r(0) = z_r^{(i)}(0)$ $(r = 1, \dots N_i)$. We need to consider two cases.

Case 1. Number $\mu \in \mathbb{C}$ is not a root of characteristic equation (1.5).

Consider the problem $\mathbf{P_1}$. Since matrix $\Delta(\mu)$ is nonsingular, vector $z_1(0)$ is uniquely defined from the first equation of system (3.26)

$$z_1(0) = \Delta^{-1}(\mu) \Big[B_0 \Big(\int_0^\theta e^{\mu(\theta - s)} \big(p_1(s) + q_1(s)\big) ds \Big) + g_1 - p_1(0) - q_1(0) \Big]. \tag{6.16}$$

We substitute this expression into the left-hand side of the second equation of system (3.26) and set $\Psi(\xi) = \Psi^{(p)}(\xi)$, where p is an arbitrary natural number from 1 to l. Using formula (6.11) from Corollary 6.2, we get

$$\begin{aligned} \big(\Psi^{(p)}(\xi), e^{\mu\theta} I\big) z_1(0) &= (\mu I - D^{(p)})^{-1} \Psi^{(p)}(0) \\ &\times \Big[B_0 \Big(\int_0^\theta e^{\mu(\theta - s)} \big(p_1(s) + q_1(s)\big) ds \Big) + g_1 - p_1(0) - q_1(0) \Big]. \end{aligned} \tag{6.17}$$

Now, consider the right-hand side of the second equation in system (3.26). It follows from (2.13) that

$$\big(\Psi^{(p)}(\xi), q_r(\theta)\big) = 0, \qquad r = 1, \dots, N_i, \tag{6.18}$$

since $q_r(\theta) \in Q_\Lambda$ due to (3.14), (3.19) and (3.21). Moreover, we derive from (2.8), (3.12), (3.13), (3.19), (3.21), that

$$\big(\Psi^{(p)}(\xi), p_r(\theta)\big) = \Psi^{(p)}(0) g_r, \qquad r = 1, \dots, N_i. \tag{6.19}$$

We apply now formula (6.9) from Proposition 6.1 with $f(\theta) = p_1(\theta) + q_1(\theta)$ and take into account (6.18), (6.19). The right-hand side of the second equation in (3.26) takes the form

$$\begin{aligned} &-\Big(\Psi^{(p)}(\xi), \int_0^\theta e^{\mu(\theta-s)}(p_1(s)+q_1(s))ds\Big) \\ &\quad = (\mu I - D^{(p)})^{-1} \\ &\qquad \times \Big[\Psi^{(p)}(0)B_0\Big(\int_0^\theta e^{\mu(\theta-s)}(p_1(s)+q_1(s))ds\Big) \\ &\qquad\qquad + \Psi^{(p)}(0)g_1 - \Psi^{(p)}(0)(p_1(0)+q_1(0))\Big]. \end{aligned} \tag{6.20}$$

Comparing the right-hand sides of (6.17) and (6.20) and varying p from 1 to l we state the unique solvability of the problem $\mathbf{P_1}$.

Let us turn to the problem $\mathbf{P_2}$. We have

$$\begin{aligned} z_2(0) = \Delta^{-1}(\mu)\Big[&B_0\Big(\int_0^\theta \Big((\theta-s)e^{\mu(\theta-s)}(p_1(s)+q_1(s)) + e^{\mu(\theta-s)}(p_2(s)+q_2(s))\Big)ds\Big) \\ &+ g_2 - p_2(0) - q_2(0) - \Delta'(\mu)z_1(0)\Big]. \end{aligned}$$

We substitute the above expression into the left-hand side of the second equation of system (3.27) and set $\Psi(\xi) = \Psi^{(p)}(\xi)$, where p is an arbitrary natural number from 1 to l. Using formulas (6.11), (6.12) from Corollary 6.2 and also (6.16) yields

$$\begin{aligned} &\big(\Psi^{(p)}(\xi), \theta e^{\mu\theta} I\big)z_1(0) + \big(\Psi(\xi), e^{\mu\theta} I\big)z_2(0) \\ &\quad = -(\mu I - D^{(p)})^{-2}\Psi^{(p)}(0)\Big[B_0\Big(\int_0^\theta e^{\mu(\theta-s)}(p_1(s)+q_1(s))ds\Big) + g_1 - p_1(0) - q_1(0)\Big] \\ &\qquad + (\mu I - D^{(p)})^{-1}\Psi^{(p)}(0) \\ &\qquad \times \Big[B_0\Big(\int_0^\theta \Big((\theta-s)e^{\mu(\theta-s)}(p_1(s)+q_1(s)) + e^{\mu(\theta-s)}(p_2(s)+q_2(s))\Big)ds\Big) \\ &\qquad\qquad + g_2 - p_2(0) - q_2(0)\Big]. \end{aligned} \tag{6.21}$$

We apply (6.9), (6.10) to modify the right-hand side of the second equation in (3.27). By (6.18), (6.19), we conclude that the obtained expression coincides with (6.21). Varying p from 1 to l we state the unique solvability of the problem $\mathbf{P_2}$.

In much the same manner we can prove the unique solvability of the problems $\mathbf{P_r}$ ($r = 3, \dots, N_i$). All we need is to substitute the obtained expressions for $z_1(0), \dots, z_r(0)$ into the left-hand side of the second equation of the corresponding system for the problem $\mathbf{P_r}$ and, then, use formulas (6.9), (6.10), (6.11), (6.12), (6.18), (6.19).

Case 2. Number $\mu \in \mathbb{C}$ is a root of characteristic equation (1.5).

In this case matrix $\Delta(\mu)$ is singular. The first equations in the problems $\mathbf{P_r}$ ($r = 1, \dots, N_i$) may be written in the form

$$\Delta(\mu)z_r(0) = f_r, \tag{6.22}$$

where $f_r \in \mathbb{C}^m$ is the corresponding right-hand side of the first equation in $\mathbf{P_r}$. It is well-known that Eq. (6.22) has solution iff $y^* f_r = 0$ for all fundamental solutions of adjoint system

$$\Delta^*(\mu)y = 0. \tag{6.23}$$

Here $(^*)$ denotes the Hermitian conjugate. Conjugating both sides of (6.23) and using (6.8), we get the following solvability condition for Eq. (6.22):

$$\psi_{N_p}^{(p)}(0) f_r = 0. \tag{6.24}$$

This condition should be fulfilled for numbers p such that $\mu = \lambda^{(p)}$. We note that, due to (6.4), the fundamental solutions of the corresponding homogeneous equation (6.22) are the vectors $\varphi_1^{(p)}(0)$ with numbers p ($p = 1, \dots, l$) such that $\mu = \lambda^{(p)}$. Consequently, if Eq. (6.22) is solvable, its general solution has the form

$$z_r(0) = \sum_{p:\ \lambda^{(p)}=\mu} c_p \varphi_1^{(p)}(0) + \tilde{z}_r, \tag{6.25}$$

where $c_p \in \mathbb{C}$ are arbitrary constants and $\tilde{z}_r$ is a certain partial solution of Eq. (6.22).

As in Case 1 we will study problems $\mathbf{P_1}, \dots, \mathbf{P_{N_i}}$ successively. First we consider the problem $\mathbf{P_1}$. Let us establish the solvability of the first equation in (3.26). We have

$$\begin{aligned}
&\psi_{N_p}^{(p)}(0)\left[B_0\left(\int_0^\theta e^{\mu(\theta-s)}(p_1(s)+q_1(s))ds\right) + g_1 - p_1(0) - q_1(0)\right] \\
&\quad = \int_{-h}^0 \int_0^\theta \psi_{N_p}^{(p)}(0) e^{\mu(\theta-s)} d\eta(\theta)(p_1(s)+q_1(s))ds + \psi_{N_p}^{(p)}(0)(g_1 - p_1(0) - q_1(0)) \\
&\quad = -\left(\psi_{N_p}^{(p)}(\xi), p_1(\theta)+q_1(\theta)\right) + \psi_{N_p}^{(p)}(0)(p_1(0)+q_1(0)) + \psi_{N_p}^{(p)}(0)(g_1 - p_1(0) - q_1(0)) \\
&\quad = 0.
\end{aligned}$$

Here we used equality $\mu = \lambda^{(p)}$ and formulas (6.7), (6.18), (6.19). Thus, the general solution of the first equation in (3.26) may be written in form (6.25), where $r = 1$. We substitute this expression for $z_1(0)$ in the left-hand side of the second equation in (3.26) and use (2.8), (6.3). We get

$$\begin{aligned}
\left(\Psi(\xi), e^{\mu\theta} I\right) z_1(0) &= \sum_{p:\ \lambda^{(p)}=\mu} c_p\left(\Psi(\xi), \varphi_1^{(p)}(\theta)\right) + \left(\Psi(\xi), e^{\mu\theta}\tilde{z}_1\right) \\
&= \sum_{p:\ \lambda^{(p)}=\mu} c_p e^{(p)} + \left(\Psi(\xi), e^{\mu\theta}\tilde{z}_1\right).
\end{aligned} \tag{6.26}$$

Here, $e^{(p)}$ is the N-dimensional column vector whose only nonzero element is the number 1 in position $1 + N_1 + \cdots + N_{p-1}$ ($N_0 = 0$). Taking into account the right-hand side of the second equation in (3.26), we uniquely define constants c_p:

$$\begin{aligned}
c_p &= -(e^{(p)})^*\left(\Psi(\xi), \int_0^\theta e^{\mu(\theta-s)}(p_1(s)+q_1(s))ds\right) - (e^{(p)})^*\left(\Psi(\xi), e^{\mu\theta}\tilde{z}_1\right) \\
&= -\left(\psi_1^{(p)}(\xi), \int_0^\theta e^{\mu(\theta-s)}(p_1(s)+q_1(s))ds\right) - \left(\psi_1^{(p)}(\xi), e^{\mu\theta}\tilde{z}_1\right).
\end{aligned} \tag{6.27}$$

Therefore, if solution $z_1(0)$ of (3.26) exists, it has form (6.25), where $r = 1$, and constants c_p are described by (6.27). Moreover, if this solution exists it is uniquely defined. Namely, if we consider the corresponding homogeneous system (3.26), by (6.25), (6.27) we conclude that it has only zero solution.

We will now show that the constructed solution $z_1(0)$ of the first equation in (3.26) satisfies the second equation, where $\Psi(\xi) = \Psi^{(p)}(\xi)$ and p varies from 1 to l. We remark that if number

p is such that $\lambda^{(p)} \neq \mu$ this is done in the same way as in Case 1. Actually, equality (6.17) holds for the left-hand side of the second equation in (3.26) and equality (6.20) holds for the right-hand side. Consequently, for each number p such that $\lambda^{(p)} = \mu$ we should verify the validity of the vector equality

$$\big(\Psi^{(p)}(\xi), e^{\mu\theta} I\big) z_1(0) = -\Big(\Psi^{(p)}(\xi), \int_0^\theta e^{\mu(\theta-s)}\big(p_1(s)+q_1(s)\big)ds\Big), \tag{6.28}$$

consisting of N_p scalar equalities. It follows from (6.14), with account of (2.3) and (6.8), that

$$\begin{aligned}
\big(\Psi^{(p)}(\xi), e^{\mu\theta} I\big) z_1(0) &= \begin{pmatrix} \big(\psi_1^{(p)}(\xi), e^{\mu\theta} I\big) \\ \psi_2^{(p)}(0)\Delta'(\mu)+\cdots+\psi_{N_p}^{(p)}(0)\frac{\Delta^{(N_p-1)}(\mu)}{(N_p-1)!} \\ \psi_3^{(p)}(0)\Delta'(\mu)+\cdots+\psi_{N_p}^{(p)}(0)\frac{\Delta^{(N_p-2)}(\mu)}{(N_p-2)!} \\ \vdots \\ \psi_{N_p}^{(p)}(0)\Delta'(\mu) \end{pmatrix} z_1(0) \\
&= \begin{pmatrix} \big(\psi_1^{(p)}(\xi), e^{\mu\theta} I\big) \\ -\psi_1^{(p)}(0)\Delta(\mu) \\ -\psi_2^{(p)}(0)\Delta(\mu) \\ \vdots \\ -\psi_{N_p-1}^{(p)}(0)\Delta(\mu) \end{pmatrix} z_1(0) \\
&= \begin{pmatrix} \big(\psi_1^{(p)}(\xi), e^{\mu\theta} I\big) z_1(0) \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ \psi_1^{(p)}(0) \\ \psi_2^{(p)}(0) \\ \vdots \\ \psi_{N_p-1}(0) \end{pmatrix} \Delta(\mu) z_1(0).
\end{aligned} \tag{6.29}$$

Since the above constructed vector $z_1(0)$ satisfies the first equation in (3.26), we obtain

$$\begin{aligned}
&\big(\Psi^{(p)}(\xi), e^{\mu\theta} I\big) z_1(0) \\
&\quad = \begin{pmatrix} \big(\psi_1^{(p)}(\xi), e^{\mu\theta} I\big) z_1(0) \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \\
&\qquad - \begin{pmatrix} 0 \\ \psi_1^{(p)}(0) \\ \psi_2^{(p)}(0) \\ \vdots \\ \psi_{N_p-1}(0) \end{pmatrix} \Big[B_0\Big(\int_0^\theta e^{\mu(\theta-s)}\big(p_1(s)+q_1(s)\big)ds\Big) + g_1 - p_1(0) - q_1(0)\Big].
\end{aligned} \tag{6.30}$$

Consider the right-hand side of (6.28). We apply formula (6.13) from Proposition 6.3 and use

(6.7). We get

$$
\begin{aligned}
&-\left(\Psi^{(p)}(\xi), \int_0^\theta e^{\mu(\theta-s)} f(s)ds\right) \\
&= -\begin{pmatrix} \left(\psi_1^{(p)}(\xi), \int_0^\theta e^{\mu(\theta-s)} f(s)ds\right) \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \\
&\quad + \int_{-h}^0 \int_0^\theta \begin{pmatrix} 0 \\ \psi_2^{(p)}(0)(\theta-\xi)+\psi_3^{(p)}(0)\frac{(\theta-\xi)^2}{2!}+\cdots+\psi_{N_p}^{(p)}(0)\frac{(\theta-\xi)^{N_p-1}}{(N_p-1)!} \\ \vdots \\ \psi_{N_p-1}^{(p)}(0)(\theta-\xi)+\psi_{N_p}^{(p)}(0)\frac{(\theta-\xi)^2}{2!} \\ \psi_{N_p}^{(p)}(0)(\theta-\xi) \end{pmatrix} e^{\mu(\theta-\xi)} d\eta(\theta) f(\xi) d\xi \\
&= -\begin{pmatrix} \left(\psi_1^{(p)}(\xi), \int_0^\theta e^{\mu(\theta-s)} f(s)ds\right) \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \int_{-h}^0 \int_0^\theta \begin{pmatrix} 0 \\ \psi_1^{(p)}(\xi-\theta)-\psi_1^{(p)}(0)e^{-\mu(\xi-\theta)} \\ \vdots \\ \psi_{N_p-2}^{(p)}(\xi-\theta)-\psi_{N_p-2}^{(p)}(0)e^{-\mu(\xi-\theta)} \\ \psi_{N_p-1}^{(p)}(\xi-\theta)-\psi_{N_p-1}^{(p)}(0)e^{-\mu(\xi-\theta)} \end{pmatrix} d\eta(\theta) f(\xi) d\xi.
\end{aligned}
$$

Hence, by (2.2) and (2.5),

$$
\begin{aligned}
-\left(\Psi^{(p)}(\xi), \int_0^\theta e^{\mu(\theta-s)} f(s)ds\right) &= -\begin{pmatrix} \left(\psi_1^{(p)}(\xi), \int_0^\theta e^{\mu(\theta-s)} f(s)ds\right) \\ \left(\psi_1^{(p)}(\xi), f(\theta)\right) \\ \vdots \\ \left(\psi_{N_p-2}^{(p)}(\xi), f(\theta)\right) \\ \left(\psi_{N_p-1}^{(p)}(\xi), f(\theta)\right) \end{pmatrix} \\
&\quad + \begin{pmatrix} 0 \\ \psi_1^{(p)}(0)f(0) \\ \vdots \\ \psi_{N_p-2}^{(p)}(0)f(0) \\ \psi_{N_p-1}^{(p)}(0)f(0) \end{pmatrix} - \begin{pmatrix} 0 \\ \psi_1^{(p)}(0) \\ \vdots \\ \psi_{N_p-2}^{(p)}(0) \\ \psi_{N_p-1}^{(p)}(0) \end{pmatrix} B_0\left(\int_0^\theta e^{\mu(\theta-s)} f(s)ds\right). \quad (6.31)
\end{aligned}
$$

Setting $f(\theta) = p_1(\theta) + q_1(\theta)$ and recalling (6.18), (6.19), we conclude that the right-hand side of (6.31) coincides with the right-hand side of (6.30) with the possible exception of the first components of the corresponding vectors. But the first components of these vectors coincide due to (6.26) and the choice of c_p according to (6.27). Therefore, all N_p equalities in (6.28) hold.

Consider now the problem $\mathbf{P_r}$ with $2 \le r \le N_i$, assuming that the unique solvability of all the previous problems is already proven. First, we show that the first equation of the corresponding system is solvable. Writing this equation in the form (6.22) and using the solvability condition (6.24) yields

$$
\begin{aligned}
&\psi_{N_p}^{(p)}(0)\Big[B_0\Big(\int_0^\theta \Big(\frac{(\theta-s)^{r-1}}{(r-1)!} e^{\mu(\theta-s)}\big(p_1(s)+q_1(s)\big) + \cdots + e^{\mu(\theta-s)}\big(p_r(s)+q_r(s)\big)\Big)ds\Big) \\
&\qquad + g_r - p_r(0) - q_r(0) - \Delta'(\mu) z_{r-1}(0) - \cdots - \frac{\Delta^{(r-1)}(\mu)}{(r-1)!} z_1(0)\Big] \\
&= -\left(\psi_{N_p}^{(p)}(\xi), \int_0^\theta \frac{(\theta-s)^{r-2}}{(r-2)!} e^{\mu(\theta-s)}\big(p_1(s)+q_1(s)\big)ds\right)
\end{aligned}
$$

$$
\begin{aligned}
&- \cdots - \left(\psi_{N_p}^{(p)}(\xi), \int_0^\theta e^{\mu(\theta-s)} \big(p_{r-1}(s) + q_{r-1}(s)\big) ds \right) \\
&- \left(\psi_{N_p}^{(p)}(\xi), p_r(\theta) + q_r(\theta) \right) + \psi_{N_p}^{(p)}(0)\big(p_r(0) + q_r(0)\big) \\
&+ \psi_{N_p}^{(p)}(0) \left(g_r - p_r(0) - q_r(0) - \Delta'(\mu) z_{r-1}(0) - \cdots - \frac{\Delta^{(r-1)}(\mu)}{(r-1)!} z_1(0) \right). \qquad (6.32)
\end{aligned}
$$

Here we also used formula (6.13) (to be precise, we used the equality for the last components of the corresponding vectors on the left-hand side and on the right-hand side of this formula). Moreover, we took into account the form of the function $\psi_{N_p}^{(p)}(\xi)$ that is defined by (6.7), the Riesz representation (2.2) of B_0 and the definition of the bilinear form (2.5). Let f_r denote the vector in the square brackets on the left-hand side of (6.32). We use the second equation in the corresponding system of the problem $\mathbf{P_{r-1}}$ and apply (6.18), (6.19). Then it follows from (6.32) that

$$
\begin{aligned}
\psi_{N_p}^{(p)}(0) f_r = &\left(\psi_{N_p}^{(p)}(\xi), \frac{\theta^{r-2}}{(r-2)!} e^{\mu\theta} I \right) z_1(0) + \cdots + \left(\psi_{N_p}^{(p)}(\xi), e^{\mu\theta} I \right) z_{r-1}(0) \\
&- \psi_{N_p}^{(p)}(0) \left(\Delta'(\mu) z_{r-1}(0) + \cdots + \frac{\Delta^{(r-1)}(\mu)}{(r-1)!} z_1(0) \right). \qquad (6.33)
\end{aligned}
$$

What is left is to use formulas (6.14), (6.15) (namely, we should use the equality for the last components of the corresponding vectors on the left-hand side and on the right-hand side of these formulas) to modify the right-hand side of (6.33) with account of (2.3). Finally, some easy computations imply (6.24). Consequently, the first equation in the problem $\mathbf{P_r}$ is solvable and its general solution is described by (6.25). Exactly in the same manner as for the problem $\mathbf{P_1}$ we define constants c_p by substitution of (6.25) into the second equation in $\mathbf{P_r}$:

$$
\begin{aligned}
&\left(\Psi^{(p)}(\xi), \frac{\theta^{r-1}}{(r-1)!} e^{\mu\theta} I \right) z_1(0) + \cdots + \big(\Psi^{(p)}(\xi), \theta e^{\mu\theta} I \big) z_{r-1}(0) + \left(\Psi^{(p)}(\xi), e^{\mu\theta} I \right) z_r(0) \\
&\quad = - \left(\Psi^{(p)}(\xi), \int_0^\theta \frac{(\theta-s)^{r-1}}{(r-1)!} e^{\mu(\theta-s)} \big(p_1(s) + q_1(s)\big) ds \right) \\
&\qquad - \cdots - \left(\Psi^{(p)}(\xi), \int_0^\theta e^{\mu(\theta-s)} \big(p_r(s) + q_r(s)\big) ds \right). \qquad (6.34)
\end{aligned}
$$

Hence

$$
\begin{aligned}
c_p = &- \left(\psi_1^{(p)}(\xi), \int_0^\theta \frac{(\theta-s)^{r-1}}{(r-1)!} e^{\mu(\theta-s)} \big(p_1(s) + q_1(s)\big) ds \right) \\
&- \cdots - \left(\psi_1^{(p)}(\xi), \int_0^\theta e^{\mu(\theta-s)} \big(p_r(s) + q_r(s)\big) ds \right) - \left(\psi_1^{(p)}(\xi), \frac{\theta^{r-1}}{(r-1)!} e^{\mu\theta} I \right) z_1(0) \\
&- \cdots - \big(\psi_1^{(p)}(\xi), \theta e^{\mu\theta} I \big) z_{r-1}(0) - \big(\psi_1^{(p)}(\xi), e^{\mu\theta} \tilde{z}_r \big). \qquad (6.35)
\end{aligned}
$$

We should now verify for each p the validity of N_p scalar equalities in (6.34). If number p is such that $\lambda^{(p)} \neq \mu$, the validity of these equalities is verified in the same way as for the Case 1. We should only note that we need not to have the expression of the form $z_s(0) = \Delta^{-1}(\mu) f_s$ for $z_s(0)$ $(s = 1, \ldots, r)$. It is sufficient to know that there exists $z_s(0)$ that satisfies equation $\Delta(\mu) z_s(0) = f_s$.

In what follows we will analyze equalities in (6.34) only for numbers p such that $\lambda^{(p)} = \mu$. Note that the first equality in (6.34) holds due to the representation (6.25) of $z_r(0)$ and the choice of c_p according to (6.35). Let us verify the validity of the $(\nu+1)$-th equality in (6.34), where $1 \le \nu \le N_p - 1$. The right-hand side of the $(\nu+1)$-th equality in (6.34) with account of (6.13) takes the form

$$\begin{aligned}
&\int_{-h}^{0}\int_{0}^{\theta}\Big(\psi_{\nu+1}^{(p)}(0)\frac{(\theta-\xi)^{r}}{r!}+\psi_{\nu+2}^{(p)}(0)\frac{(\theta-\xi)^{r+1}}{(r+1)!}+\cdots+\psi_{N_p}^{(p)}(0)\frac{(\theta-\xi)^{N_p+r-1-\nu}}{(N_p+r-1-\nu)!}\Big)\\
&\qquad\times e^{\mu(\theta-\xi)}d\eta(\theta)\big(p_1(\xi)+q_1(\xi)\big)d\xi\\
&+\int_{-h}^{0}\int_{0}^{\theta}\Big(\psi_{\nu+1}^{(p)}(0)\frac{(\theta-\xi)^{r-1}}{(r-1)!}+\cdots+\psi_{N_p}^{(p)}(0)\frac{(\theta-\xi)^{N_p+r-2-\nu}}{(N_p+r-2-\nu)!}\Big)\\
&\qquad\times e^{\mu(\theta-\xi)}d\eta(\theta)\big(p_2(\xi)+q_2(\xi)\big)d\xi\\
&+\cdots+\int_{-h}^{0}\int_{0}^{\theta}\Big(\psi_{\nu+1}^{(p)}(0)\frac{(\theta-\xi)^{2}}{2!}+\cdots+\psi_{N_p}^{(p)}(0)\frac{(\theta-\xi)^{N_p+1-\nu}}{(N_p+1-\nu)!}\Big)\\
&\qquad\times e^{\mu(\theta-\xi)}d\eta(\theta)\big(p_{r-1}(\xi)+q_{r-1}(\xi)\big)d\xi\\
&+\int_{-h}^{0}\int_{0}^{\theta}\Big(\psi_{\nu+1}^{(p)}(0)(\theta-\xi)+\cdots+\psi_{N_p}^{(p)}(0)\frac{(\theta-\xi)^{N_p-\nu}}{(N_p-\nu)!}\Big)e^{\mu(\theta-\xi)}d\eta(\theta)\big(p_r(\xi)+q_r(\xi)\big)d\xi.
\end{aligned}\tag{6.36}$$

The ν-th equality in the second equation of the problem $\mathbf{P_{r-1}}$ with account of (6.13) has the form

$$\begin{aligned}
&\Big(\psi_{\nu}^{(p)}(\xi),\frac{\theta^{r-2}}{(r-2)!}e^{\mu\theta}I\Big)z_1(0)+\cdots+\big(\psi_{\nu}^{(p)}(\xi),\theta e^{\mu\theta}I\big)z_{r-2}(0)+\big(\psi_{\nu}^{(p)}(\xi),e^{\mu\theta}I\big)z_{r-1}(0)\\
&=\int_{-h}^{0}\int_{0}^{\theta}\Big(\psi_{\nu}^{(p)}(0)\frac{(\theta-\xi)^{r-1}}{(r-1)!}+\psi_{\nu+1}^{(p)}(0)\frac{(\theta-\xi)^{r}}{r!}+\cdots+\psi_{N_p}^{(p)}(0)\frac{(\theta-\xi)^{N_p+r-1-\nu}}{(N_p+r-1-\nu)!}\Big)\\
&\qquad\times e^{\mu(\theta-\xi)}d\eta(\theta)\big(p_1(\xi)+q_1(\xi)\big)d\xi\\
&\quad+\int_{-h}^{0}\int_{0}^{\theta}\Big(\psi_{\nu}^{(p)}(0)\frac{(\theta-\xi)^{r-2}}{(r-2)!}+\cdots+\psi_{N_p}^{(p)}(0)\frac{(\theta-\xi)^{N_p+r-2-\nu}}{(N_p+r-2-\nu)!}\Big)\\
&\qquad\times e^{\mu(\theta-\xi)}d\eta(\theta)\big(p_2(\xi)+q_2(\xi)\big)d\xi\\
&\quad+\cdots+\int_{-h}^{0}\int_{0}^{\theta}\Big(\psi_{\nu}^{(p)}(0)(\theta-\xi)+\psi_{\nu+1}^{(p)}(0)\frac{(\theta-\xi)^{2}}{2!}+\cdots+\psi_{N_p}^{(p)}(0)\frac{(\theta-\xi)^{N_p+1-\nu}}{(N_p+1-\nu)!}\Big)\\
&\qquad\times e^{\mu(\theta-\xi)}d\eta(\theta)\big(p_{r-1}(\xi)+q_{r-1}(\xi)\big)d\xi.
\end{aligned}\tag{6.37}$$

We express the common part of (6.36) and (6.37) from relation (6.37) and substitute it into (6.36). The right-hand side of the $(\nu+1)$-th equality in (6.34), therefore, takes the form

$$\begin{aligned}
&\int_{-h}^{0}\int_{0}^{\theta}\Big(\psi_{\nu+1}^{(p)}(0)(\theta-\xi)+\cdots+\psi_{N_p}^{(p)}(0)\frac{(\theta-\xi)^{N_p-\nu}}{(N_p-\nu)!}\Big)e^{\mu(\theta-\xi)}d\eta(\theta)\big(p_r(\xi)+q_r(\xi)\big)d\xi\\
&+\Big(\psi_{\nu}^{(p)}(\xi),\frac{\theta^{r-2}}{(r-2)!}e^{\mu\theta}I\Big)z_1(0)+\cdots+\big(\psi_{\nu}^{(p)}(\xi),\theta e^{\mu\theta}I\big)z_{r-2}(0)+\big(\psi_{\nu}^{(p)}(\xi),e^{\mu\theta}I\big)z_{r-1}(0)\\
&-\int_{-h}^{0}\int_{0}^{\theta}\psi_{\nu}^{(p)}(0)\frac{(\theta-\xi)^{r-1}}{(r-1)!}e^{\mu(\theta-\xi)}d\eta(\theta)\big(p_1(\xi)+q_1(\xi)\big)d\xi\\
&-\cdots-\int_{-h}^{0}\int_{0}^{\theta}\psi_{\nu}^{(p)}(0)(\theta-\xi)e^{\mu(\theta-\xi)}d\eta(\theta)\big(p_{r-1}(\xi)+q_{r-1}(\xi)\big)d\xi.
\end{aligned}\tag{6.38}$$

In (6.38), we add and subtract the quantity

$$\int_{-h}^{0}\int_{0}^{\theta}\psi_\nu^{(p)}(0)e^{\mu(\theta-\xi)}d\eta(\theta)\big(p_r(\xi)+q_r(\xi)\big)d\xi=\psi_\nu^{(p)}(0)B_0\bigg(\int_0^\theta e^{\mu(\theta-s)}\big(p_r(s)+q_r(s)\big)ds\bigg).$$

Then, by using (2.2), we deduce from the first equation of the problem $\mathbf{P_r}$ the following representation for (6.38):

$$\begin{aligned}&\int_{-h}^{0}\int_{0}^{\theta}\bigg(\psi_\nu^{(p)}(0)+\psi_{\nu+1}^{(p)}(0)(\theta-\xi)+\cdots+\psi_{N_p}^{(p)}(0)\frac{(\theta-\xi)^{N_p-\nu}}{(N_p-\nu)!}\bigg)\\&\qquad\times e^{\mu(\theta-\xi)}d\eta(\theta)\big(p_r(\xi)+q_r(\xi)\big)d\xi\\&\quad+\bigg(\psi_\nu^{(p)}(\xi),\frac{\theta^{r-2}}{(r-2)!}e^{\mu\theta}I\bigg)z_1(0)+\cdots+\big(\psi_\nu^{(p)}(\xi),\theta e^{\mu\theta}I\big)z_{r-2}(0)+\big(\psi_\nu^{(p)}(\xi),e^{\mu\theta}I\big)z_{r-1}(0)\\&\quad-\psi_\nu^{(p)}(0)\bigg(\frac{\Delta^{(r-1)}(\mu)}{(r-1)!}z_1(0)+\cdots+\Delta'(\mu)z_{r-1}(0)+\Delta(\mu)z_r(0)-g_r+p_r(0)+q_r(0)\bigg).\end{aligned}\tag{6.39}$$

We now use the expression (6.7) for function $\psi_\nu(\xi)$ and apply (2.5) to write the integral term in (6.39) in the form

$$-\big(\psi_\nu^{(p)}(\xi),p_r(\theta)+q_r(\theta)\big)+\psi_\nu^{(p)}(0)\big(p_r(0)+q_r(0)\big).$$

Due to (6.18), (6.19), we can rewrite (6.39) as follows:

$$\begin{aligned}&\bigg(\psi_\nu^{(p)}(\xi),\frac{\theta^{r-2}}{(r-2)!}e^{\mu\theta}I\bigg)z_1(0)+\cdots+\big(\psi_\nu^{(p)}(\xi),\theta e^{\mu\theta}I\big)z_{r-2}(0)+\big(\psi_\nu^{(p)}(\xi),e^{\mu\theta}I\big)z_{r-1}(0)\\&\qquad-\psi_\nu^{(p)}(0)\bigg(\frac{\Delta^{(r-1)}(\mu)}{(r-1)!}z_1(0)+\cdots+\Delta'(\mu)z_{r-1}(0)+\Delta(\mu)z_r(0)\bigg).\end{aligned}\tag{6.40}$$

By applying (2.3), (6.14), (6.15) and collecting terms containing $z_s(0)$, we get the following representation for (6.40):

$$\begin{aligned}&\bigg(\psi_{\nu+1}^{(p)}(0)\frac{\Delta^{(r)}(\mu)}{r!}+\cdots+\psi_{N_p}^{(p)}(0)\frac{\Delta^{(N_p+r-1-\nu)}}{(N_p+r-1-\nu)!}\bigg)z_1(0)\\&\quad+\cdots+\bigg(\psi_{\nu+1}^{(p)}(0)\frac{\Delta''(\mu)}{2!}+\cdots+\psi_{N_p}^{(p)}(0)\frac{\Delta^{(N_p+1-\nu)}}{(N_p+1-\nu)!}\bigg)z_{r-1}(0)-\psi_\nu^{(p)}(0)\Delta(\mu)z_r(0)\\&\quad=\bigg(\psi_{\nu+1}^{(p)}(\xi),\frac{\theta^{r-1}}{(r-1)!}e^{\mu\theta}I\bigg)z_1(0)+\cdots+\big(\psi_{\nu+1}^{(p)}(\xi),\theta e^{\mu\theta}I\big)z_{r-1}(0)\\&\qquad+\big(\psi_{\nu+1}^{(p)}(\xi),e^{\mu\theta}I\big)z_r(0).\end{aligned}\tag{6.41}$$

Here, to obtain the last term on the right-hand side of the above expression we also used formula (6.29) that is evidently valid if we replace $z_1(0)$ by $z_r(0)$. Finally, we note that the right-hand side of (6.41) coincides with the left-hand side of the $(\nu+1)$-th equality in (6.34). Thus, we have established the validity of the $(\nu+1)$-th equality in (6.34).

The proof is complete. □

Acknowledgements

The author wishes to express his thanks to professor Vladimir Burd for his active interest in the publication of this paper. The author is also indebted to the referee for his comments and suggestions.

This research was supported by the grant of the President of the Russian Federation No. MK-4625.2016.1 and by Russian Ministry of Education and Science within project No. 2014/258-1875.

References

[1] S. Ai, Asymptotic integration of delay differential systems, *J. Math. Anal. Appl.* **165**(1992), 71–101. MR1151062; url

[2] M. Ait Babram, M. L. Hbid, O. Arino, Approximation scheme of a center manifold for functional differential equations, *J. Math. Anal. Appl.* **213**(1997), 554–572. MR1470870; url

[3] O. Arino, I. Győri, Asymptotic integration of delay differential systems, *J. Math. Anal. Appl.* **138**(1989), 311–327. MR991026; url

[4] O. Arino, I. Győri, M. Pituk, Asymptotically diagonal delay differential systems, *J. Math. Anal. Appl.* **204**(1996), 701–728. MR1422768; url

[5] O. Arino, M. L. Hbid, E. Ait Dads (eds.), *Delay differential equations and applications*, Springer, Dordrecht, 2006. MR2334175; url

[6] B. Balachandran, T. Kalmár-Nagy, D. E. Gilsinn (eds.), *Delay differential equations: recent advances and new directions*, Springer, New York, 2009. MR2722942; url

[7] R. Bellman, *Stability theory of differential equations*, McGraw-Hill, New York, 1953. MR61235

[8] R. Bellman, K. L. Cooke, *Differential-difference equations*, Academic Press, New York, 1963. MR147745

[9] I. Győri, M. Pituk, L^2-Perturbation of a linear delay differential equation, *J. Math. Anal. Appl.* **195**(1995), 415–427. MR1354552; url

[10] J. Carr, *Applications of centre manifold theory*, Springer-Verlag, New York, 1981. MR635782; url

[11] J. S. Cassel, Z. Hou, L^p-Perturbation of linear functional differential equations, *Monatsh. Math.* **128**(1999), 211–226. MR1719352; url

[12] S. Castillo, M. Pinto, Asymptotic integration of ordinary different systems, *J. Math. Anal. Appl.* **218**(1998), 1–12. MR1601828; url

[13] S. Castillo, M. Pinto, Levinson theorem for functional differential systems, *Nonlinear Anal.* **47**(2001), 3963–3975. MR1972339; url

[14] S. Castillo, M. Pinto, An asymptotic theory for nonlinear functional differential equations, *Comput. Math. Appl.* **44**(2002), 763–775. MR1925819; url

[15] E. A. Coddington, N. Levinson, *Theory of ordinary differential equations*, McGraw-Hill, New York, 1955. MR69338

[16] W. A. Coppel, *Stability and asymptotic behavior of differential equations*, D. C. Heath and Co., Boston, 1965. MR190463

[17] A. Devinatz, The asymptotic nature of the solutions of certain linear systems of differential equations, *Pacific J. Math.* **15**(1965), 75–83. MR178195; url

[18] M. S. P. Eastham, *The asymptotic solution of linear differential systems*, Clarendon Press, Oxford, 1989. MR1006434

[19] J. K. Hale, *Theory of functional differential equations*, Springer-Verlag, New York, 1977. MR0508721

[20] J. K. Hale, S. M. Verduyn Lunel, *Introduction to functional differential equations*, Appl. Math. Sciences, Vol. 99, Springer-Verlag, New York, 1993. MR1243878; url

[21] P. Hartman, *Ordinary differential equations*, John Wiley & Sons, Inc., New York, 1964. MR171038

[22] V. Kolmanovskii, A. Myshkis, *Introduction to the theory and applications of functional differential equations*, Kluwer Academic Publishers, Dordrecht, 1999. MR1680144; url

[23] P. N. Nesterov, Averaging method in the asymptotic integration problem for systems with oscillatory-decreasing coefficients, *Differ. Equ.* **43**(2007), 745–756. MR2383823; url

[24] P. Nesterov, Method of averaging for systems with main part vanishing at infinity, *Math. Nachr.* **284**(2011), 1496–1514. MR2832661; url

[25] P. Nesterov, Asymptotic integration of functional differential systems with oscillatory decreasing coefficients, *Monatsh. Math.* **171**(2013), 217–240. MR3077933; url

[26] P. N. Nesterov, On asymptotics for critical solutions of systems of differential equations with oscillatory decreasing coefficients, *Automatic Control and Computer Sciences* **47**(2013), 500–515. url

[27] M. Pituk, The Hartman–Wintner theorem for functional differential equations, *J. Differential Equations* **155**(1999), 1–16. MR1693226; url

[28] M. Pituk, A Perron type theorem for functional differential equations, *J. Math. Anal. Appl.* **316**(2006), 24–41. MR2201747; url

[29] M. Pituk, Asymptotic behavior and oscillation of functional differential equations, *J. Math. Anal. Appl.* **322**(2006), 1140–1158. MR2250641; url