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On sequences of large homoclinic solutions for a difference equation on the integers involving oscillatory nonlinearities

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Abstract. In this paper, we determine a concrete interval of positive parameters λ, for which we prove the existence of infinitely many homoclinic solutions for a discrete problem

$$-\Delta\left(a(k)\phi_p(\Delta u(k-1))\right)+b(k)\phi_p(u(k))=\lambda f(k,u(k)),\qquad k\in\mathbb{Z},$$

where the nonlinear term $f:\mathbb{Z}\times\mathbb{R}\to\mathbb{R}$ has an appropriate oscillatory behavior at infinity, without any symmetry assumptions. The approach is based on critical point theory.

Keywords: difference equations, discrete p-Laplacian, variational methods, infinitely many solutions.

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1 Introduction

In the present paper we deal with the following nonlinear second-order difference equation:

$$\begin{cases}-\Delta\left(a(k)\phi_p(\Delta u(k-1))\right)+b(k)\phi_p(u(k))=\lambda f(k,u(k)) & \text{for all } k\in\mathbb{Z}\\ u(k)\to 0 & \text{as } |k|\to\infty.\end{cases}\tag{1.1}$$

Here $p>1$ is a real number, λ is a positive real parameter, $\phi_p(t)=|t|^{p-2}t$ for all $t\in\mathbb{R}$, $a,b:\mathbb{Z}\to(0,+\infty)$, while $f:\mathbb{Z}\times\mathbb{R}\to\mathbb{R}$ is a continuous function. Moreover, the forward difference operator is defined as $\Delta u(k-1)=u(k)-u(k-1)$. We say that a solution $u=\{u(k)\}$ of (1.1) is homoclinic if $\lim_{|k|\to\infty}u(k)=0$.

The problem (1.1) is in a class of partial difference equations which usually describe the evolution of certain phenomena over the course of time. The theory of nonlinear discrete dynamical systems has been used to examine discrete models appearing in many fields such as computing, economics, biology and physics.

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Boundary value problems for difference equations can be studied in several ways. It is well known that variational method in such problems is a powerful tool. Many authors have applied different results of critical point theory to prove existence and multiplicity results for the solutions of discrete nonlinear problems. Studying such problems on bounded discrete intervals allows for the search for solutions in a finite-dimensional Banach space (see [1, 2, 5, 6, 14]). The issue of finding solutions on unbounded intervals is more delicate. To study such problems directly by variational methods, [13] and [8] introduced coercive weight functions which allow for preservation of certain compactness properties on l^p-type spaces.

The goal of the present paper is to establish the existence of a sequence of homoclinic solutions for the problem (1.1), which has been studied recently in several papers. Infinitely many solutions were obtained in [20] by employing Nehari manifold methods, in [9] by applying a variant of the fountain theorem (but see Section 5), and in [18] by use of the Ricceri's theorem (see [3, 17]). In this present paper, the result will be achieved by providing the nonlinearity with a suitable oscillatory behavior. For this kind of nonlinearity see [10–12]. We refer to [7, 15, 16, 19] for related results that involve differential operators with variable exponents.

A special case of our contributions reads as follows. For $b : \mathbb{Z} \to \mathbb{R}$ and the continuous mapping $f : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ define the following conditions:

(B) $b(k) \geq b_0 > 0$ for all $k \in \mathbb{Z}$, $b(k) \to +\infty$ as $|k| \to +\infty$;

(F_1) $\displaystyle\lim_{t\to 0} \frac{|f(k,t)|}{|t|^{p-1}} = 0$ uniformly for all $k \in \mathbb{Z}$;

(F_2) there are sequences $\{c_n\}, \{d_n\}$ such that $0 < c_n < d_n < c_{n+1}$, $\lim_{n\to\infty} c_n = +\infty$ and $f(k,t) \leq 0$ for every $k \in \mathbb{Z}$ and $t \in [c_n, d_n], n \in \mathbb{N}$

(F_3) there is $r < 0$ such that $\sup_{t\in[r,d_n]} |F(\cdot .t)| \in l_1$ for all $n \in \mathbb{N}$;

(F_4^+) $\displaystyle\limsup_{(k,t)\to(+\infty,+\infty)} \frac{F(k,t)}{[a(k+1)+a(k)+b(k)]\,t^p} = +\infty;$

(F_4^-) $\displaystyle\limsup_{(k,t)\to(-\infty,+\infty)} \frac{F(k,t)}{[a(k+1)+a(k)+b(k)]\,t^p} = +\infty;$

(F_5) $\displaystyle\sup_{k\in\mathbb{Z}} \left(\limsup_{t\to+\infty} \frac{F(k,t)}{[a(k+1)+a(k)+b(k)]\,t^p} \right) = +\infty,$

where $F(k,t)$ is the primitive function of $f(k,t)$, that is $F(k,t) = \int_0^t f(k,s)\,ds$ for every $t \in \mathbb{R}$ and $k \in \mathbb{Z}$. The solutions are found in the normed space $(X, \|\cdot\|)$, where

$$X = \left\{ u : \mathbb{Z} \to \mathbb{R} \ : \ \sum_{k\in\mathbb{Z}} \left[a(k)\,|\Delta u(k-1)|^p + b(k)|u(k)|^p \right] < \infty \right\}$$

and

$$\|u\| = \left(\sum_{k\in\mathbb{Z}} \left[a(k)\,|\Delta u(k-1)|^p + b(k)|u(k)|^p \right] \right)^{\frac{1}{p}}.$$

Theorem 1.1. *Assume that (A), (F_1), (F_2) and (F_3) are satisfied. Moreover, assume that at least one of the conditions (F_4^+), (F_4^-), (F_5) is satisfied. Then, for any $\lambda > 0$, the problem (1.1) admits a sequence of non-negative solutions in X whose norms tend to infinity.*

The plan of the paper is as follows: Section 2 is devoted to our abstract framework, while Section 3 is dedicated to the main result. In Section 4 we give two examples of the independence of conditions (F_4^+) and (F_5). Finally, we compare our result with other known results.

2 Abstract framework

We begin by defining some Banach spaces. For all $1 \le p < +\infty$, we denote ℓ^p the set of all functions $u : \mathbb{Z} \to \mathbb{R}$ such that

$$\|u\|_p^p = \sum_{k\in\mathbb{Z}} |u(k)|^p < +\infty.$$

Moreover, we denote ℓ^∞ the set of all functions $u : \mathbb{Z} \to \mathbb{R}$ such that

$$\|u\|_\infty = \sup_{k\in\mathbb{Z}} |u(k)| < +\infty$$

We set

$$X = \left\{ u : \mathbb{Z} \to \mathbb{R} \ : \ \sum_{k\in\mathbb{Z}} \left[a(k)\,|\Delta u(k-1)|^p + b(k)|u(k)|^p\right] < \infty \right\}$$

and

$$\|u\| = \left(\sum_{k\in\mathbb{Z}} \left[a(k)\,|\Delta u(k-1)|^p + b(k)|u(k)|^p\right] \right)^{\frac{1}{p}}.$$

Clearly we have

$$\|u\|_\infty \le \|u\|_p \le b_0^{-\frac{1}{p}} \|u\| \quad \text{for all } u \in X. \tag{2.1}$$

As is shown in [8, Proposition 3], $(X, \|\cdot\|)$ is a reflexive Banach space and the embedding $X \hookrightarrow l^p$ is compact.

Let

$$\Phi(u) := \frac{1}{p} \sum_{k\in\mathbb{Z}} \left[a(k)\,|\Delta u(k-1)|^p + b(k)\,|u(k)|^p\right] \quad \text{for all } \ u \in X$$

and

$$\Psi(u) := \sum_{k\in\mathbb{Z}} F(k, u(k)) \quad \text{for all } \ u \in l^p$$

where $F(k,s) = \int_0^s f(k,t)dt$ for $s \in \mathbb{R}$ and $k \in \mathbb{Z}$. Let $J : X \to \mathbb{R}$ be the functional associated to problem (1.1) defined by

$$J_\lambda(u) = \Phi(u) - \lambda\Psi(u).$$

Proposition 2.1. *Assume that (A) and (F_1) are satisfied. Then*

(a) $\Phi \in C^1(X)$;

(b) $\Psi \in C^1(l^p)$ *and* $\Psi \in C^1(X)$;

(c) $J_\lambda \in C^1(X)$ *and every critical point* $u \in X$ *of* J_λ *is a homoclinic solution of problem* (1.1);

(d) J_λ *is sequentially weakly lower semicontinuous functional on* X.

This version of the proposition, parts $(a), (b)$ and (c), can be proved essentially by the same way as Propositions 5, 6 and 7 in [8], where $a(k) \equiv 1$ on $\mathbb{Z}$ and the norm on X is slightly different. See also Lemma 2.3 in [9]. The proof of part (d) is standard.

3 Main theorem

Now we will formulate and prove a stronger form of Theorem 1.1. Let

$$B_\pm := \limsup_{(k,t)\to(\pm\infty,+\infty)} \frac{F(k,t)}{[a(k+1)+a(k)+b(k)]\, t^p}$$

and

$$B_0 := \sup_{k\in\mathbb{Z}} \left(\limsup_{t\to+\infty} \frac{F(k,t)}{[a(k+1)+a(k)+b(k)]\, t^p} \right).$$

Set $B = \max\{B_\pm, B_0\}$. For conveniece we put $\frac{1}{+\infty} = 0$.

Theorem 3.1. *Assume that (A), $(F_1), (F_2)$ and (F_3) are satisfied and assume that $B > 0$. Then, for any $\lambda > \frac{1}{Bp}$, the problem* (1.1) *admits a sequence of non-negative solutions in X whose norms tend to infinity.*

Proof. Put $\lambda > \frac{1}{Bp}$ and put Φ, Ψ and J_λ as in the previous section. By Proposition 2.1 we need to find a sequence $\{u_n\}$ of critical points of J_λ with non-negative terms whose norms tend to infinity.

Let $\{c_n\}, \{d_n\}$ be sequences and $r < 0$ a number satisfying conditions (F_2) and (F_3). For every $n \in \mathbb{N}$ define the set

$$W_n = \{u \in X : r \le u(k) \le d_n \text{ for every } k \in \mathbb{Z}\}.$$

Claim 3.2. *For every $n \in \mathbb{N}$, the functional J_λ is bounded from below on W_n and its infimum on W_n is attained.*

Clearly, the set W_n is weakly closed in X. By condition (F_3) we have

$$\begin{aligned} J(u) &= \frac{1}{p} \sum_{k\in\mathbb{Z}} \left[a(k)\, |\Delta u(k-1)|^p + b(k)\, |u(k)|^p \right] - \lambda \sum_{k\in\mathbb{Z}} F(k, u(k)) \\ &\ge -\lambda \sum_{k\in\mathbb{Z}} \max_{t\in[r,d_n]} F(k,t) > -\infty \end{aligned}$$

for $u \in W_n$. Thus, J_λ is bounded from below on W_n. Let $\eta_n = \inf_{W_n} J_\lambda$ and $\{\tilde{u}_l\}$ be sequence in X such that $\eta_n \le J_\lambda(\tilde{u}_l) \le \eta_n + \frac{1}{l}$ for all $l \in \mathbb{N}$. Then

$$\begin{aligned} \frac{1}{p} \|\tilde{u}_l\|^p &= \frac{1}{p} \sum_{k\in\mathbb{Z}} \left[a(k)\, |\Delta \tilde{u}_l(k-1)|^p + b(k)\, |\tilde{u}_l(k)|^p \right] = J(\tilde{u}_l) + \lambda \sum_{k\in\mathbb{Z}} F(k, \tilde{u}_l(k)) \\ &\le \eta_n + 1 + \lambda \sum_{k\in\mathbb{Z}} \max_{t\in[r,d_n]} F(k,t) \end{aligned}$$

for all $l \in \mathbb{N}$, i.e. $\{\tilde{u}_l\}$ is bounded in X. So, up to subsequence, $\{\tilde{u}_l\}$ weakly converges in X to some $u_n \in W_n$. By the sequentially weakly lower semicontinuity of J_λ we conclude that $J_\lambda(u_n) = \eta_n = \inf_{W_n} J_\lambda$. This proves Claim 3.2.

Claim 3.3. *For every $n \in \mathbb{N}$, let $u_n \in W_n$ be such that $J_\lambda(u_n) = \inf_{W_n} J_\lambda$. Then, $0 \leq u_n(k) \leq c_n$ for all $k \in \mathbb{Z}$.*

Let $K = \{k \in \mathbb{Z} : u_n(k) \notin [0, c_n]\}$ and suppose that $K \neq \emptyset$. We then introduce the sets

$$K_- = \{k \in K : \ u_n(k) < 0\} \quad \text{and} \quad K_+ = \{k \in K : \ u_n(k) > c_n\}.$$

Thus, $K = K_- \cup K_+$.

Define the truncation function $\gamma : \mathbb{R} \to \mathbb{R}$ by $\gamma(s) = \min(s_+, c_n)$, where $s_+ = \max(s, 0)$. Now, set $w_n = \gamma \circ u_n$. Clearly $w_n \in X$. Moreover, $w_n(k) \in [0, c_n]$ for every $k \in \mathbb{Z}$; thus $w_n \in W_n$.

We also have that $w_n(k) = u_n(k)$ for all $k \in \mathbb{Z} \setminus K$, $w_n(k) = 0$ for all $k \in K_-$, and $w_n(k) = c_n$ for all $k \in K_+$. Furthermore, we have

$$\begin{aligned} J_\lambda(w_n) - J_\lambda(u_n) &= \frac{1}{p} \sum_{k\in\mathbb{Z}} a(k) \left(|\Delta w_n(k-1)|^p - |\Delta u_n(k-1)|^p\right) + \\ &+ \frac{1}{p} \sum_{k\in\mathbb{Z}} b(k) \left(|w_n(k)|^p - |u_n(k)|^p\right) - \lambda \sum_{k\in\mathbb{Z}} [F(k, w_n(k)) - F(k, u_n(k))] \\ &=: \frac{1}{p} I_1 + \frac{1}{p} I_2 - \lambda I_3. \end{aligned} \tag{3.1}$$

Since γ is a Lipschitz function with Lipschitz-constant 1, and $w = \gamma \circ \tilde{u}$, we have

$$\begin{aligned} I_1 &= \sum_{k\in\mathbb{Z}} a(k) \left(|\Delta w_n(k-1)|^p - |\Delta u_n(k-1)|^p\right) \\ &= \sum_{k\in\mathbb{Z}} a(k) \left(|w_n(k) - w_n(k-1)|^p - |u_n(k) - u_n(k-1)|^p\right) \\ &\leq 0. \end{aligned} \tag{3.2}$$

Moreover, we have

$$\begin{aligned} I_2 &= \sum_{k\in\mathbb{Z}} b(k) \left(|w_n(k)|^p - |u_n(k)|^p\right) = \sum_{k\in K} b(k) \left(|w_n(k)|^p - (u_n(k))^p\right) \\ &= \sum_{k\in K_-} -b(k) \, |u_n(k)|^p + \sum_{k\in K_+} b(k)[c_n^p - |u_n(k)|^p] \\ &\leq 0. \end{aligned} \tag{3.3}$$

Next, we estimate I_3. First, $F(k, s) = 0$ for $s \leq 0$, $k \in \mathbb{Z}$, and consequently $\sum_{k\in K_-} [F(k, w_n(k)) - F(k, u_n(k))] = 0$. By the mean value theorem, for every $k \in K_+$, there exists $\xi_k \in [c_n, u_n(k)] \subset [c_n, d_n]$ such that $F(k, w_n(k)) - F(k, u_n(k)) = F(k, c_n) - F(k, u_n(k)) = f(k, \xi_k)(c_n - u_n(k))$. Taking into account hypothesis (F_2), we have that $F(k, w_n(k)) - F(k, u_n(k)) \geq 0$ for every $k \in K_+$. Consequently,

$$\begin{aligned} I_3 &= \sum_{k\in\mathbb{Z}} [F(k, w_n(k)) - F(k, u_n(k))] = \sum_{k\in K} [F(k, w_n(k)) - F(k, u_n(k))] \\ &= \sum_{k\in K_+} [F(k, w_n(k)) - F(k, u_n(k))] \geq 0. \end{aligned} \tag{3.4}$$

Combining relations (3.2)–(3.4) with (3.1), we have that

$$J_\lambda(w_n) - J_\lambda(u_n) \le 0.$$

But $J_\lambda(w_n) \ge J_\lambda(u_n) = \inf_{W_n} J_\lambda$ since $w_n \in W_n$. So, every term in $J_\lambda(w_n) - J_\lambda(u_n)$ should be zero. In particular, from I_2, we have

$$\sum_{k\in K_-} |u_n(k)|^p = \sum_{k\in K_+} [c_n^p - |u_n(k)|^p] = 0,$$

which imply that $u_n(k) = 0$ for every $k \in K_-$ and $u_n(k) = c_n$ for every $k \in K_+$. By definition of the sets K_- and K_+, we must have $K_- = K_+ = \emptyset$, which contradicts $K_- \cup K_+ = K \neq \emptyset$; therefore $K = \emptyset$. This proves Claim 3.3.

Claim 3.4. *For every $n \in \mathbb{N}$, let $u_n \in W_n$ be such that $J_\lambda(u_n) = \inf_{W_n} J_\lambda$. Then, u_n is a critical point of J_λ.*

It is sufficient to show that u_n is local minimum point of J_λ in X. Assuming the contrary, consider a sequence $\{v_i\} \subset X$ which converges to u_n and $J_\lambda(v_i) < J_\lambda(u_n) = \inf_{W_n} J_\lambda$ for all $i \in \mathbb{N}$. From this inequality it follows that $v_i \notin W_n$ for any $i \in \mathbb{N}$. Since $v_i \to u_n$ in X, then due to (2.1), $v_i \to u_n$ in l_∞ as well. Choose a positive δ such that $\delta < \frac{1}{2}\min\{-r, d_n - c_n\}$. Then, there exists $i_\delta \in \mathbb{N}$ such that $\|v_i - u_n\|_\infty < \delta$ for every $i \ge i_\delta$. By using Claim 3.3 and taking into account the choice of the number δ, we conclude that $r < v_i(k) < d_n$ for all $k \in \mathbb{Z}$ and $i \ge i_\delta$, which contradicts the fact $v_i \notin W_n$. This proves Claim 3.4.

Claim 3.5. *For every $n \in \mathbb{N}$, let $\eta_n = \inf_{W_n} J_\lambda$. Then $\lim_{n\to+\infty} \eta_n = -\infty$.*

Firstly, we assume that $B = B_\pm$. Without loss of generality we can assume that $B = B_+$. We begin with $B = +\infty$. Then there exists a number $\sigma > \frac{1}{\lambda p}$, a sequence of positive integers $\{k_n\}$ and a sequence of real numbers $\{t_n\}$ which tends to $+\infty$, such that

$$F(k_n, t_n) > \sigma(a(k_n+1) + a(k_n) + b(k_n))t_n^p$$

for all $n \in \mathbb{N}$. Up to extracting a subsequence, we may assume that $d_n \ge t_n \ge 1$ for all $n \in \mathbb{N}$. Define in X a sequence $\{w_n\}$ such that, for every $n \in \mathbb{N}$, $w_n(k_n) = t_n$ and $w_n(k) = 0$ for every $k \in \mathbb{Z}\backslash\{k_n\}$. It is clear that $w_n \in W_n$. One then has

$$\begin{aligned} J_\lambda(w_n) &= \frac{1}{p}\sum_{k\in\mathbb{Z}} \left(a(k)\,|\Delta w_n(k-1)|^p + b(k)\,|w_n(k)|^p\right) - \lambda \sum_{k\in\mathbb{Z}} F(k, w_n(k)) \\ &< \frac{1}{p}\left(a(k_n+1) + a(k_n)\right) t_n^p + \frac{1}{p} b(k_n) t_n^p - \lambda\sigma(a(k_n+1) + a(k_n) + b(k_n))t_n^p \\ &= \left(\frac{1}{p} - \lambda\sigma\right)(a(k_n+1) + a(k_n) + b(k_n))t_n^p \end{aligned}$$

which gives $\lim_{n\to+\infty} J(w_n) = -\infty$. Next, assume that $B < +\infty$. Since $\lambda > \frac{1}{Bp}$, we can fix $\varepsilon < B - \frac{1}{\lambda p}$. Therefore, also taking $\{k_n\}$ a sequence of positive integers and $\{t_n\}$ a sequence of real numbers with $\lim_{n\to+\infty} t_n = +\infty$ and $d_n \ge t_n \ge 1$ for all $n \in \mathbb{N}$ such that

$$F(k_n, t_n) > (B - \varepsilon)(a(k_n+1) + a(k_n) + b(k_n))t_n^p$$

for all $n \in \mathbb{N}$, choosing $\{w_n\}$ in W_n as above, one has

$$J_\lambda(w_n) < \left(\frac{1}{p} - \lambda(B-\varepsilon)\right)(a(k_n+1) + a(k_n) + b(k_n))t_n^p.$$

So, also in this case, $\lim_{n\to+\infty} J(w_n) = -\infty$.

Now, assume that $B = B_0$. We begin with $B = +\infty$. Then there exists a number $\sigma > \frac{1}{\lambda p}$ and an index $k_0 \in \mathbb{Z}$ such that

$$\limsup_{t\to+\infty} \frac{F(k_0, t)}{(a(k_0+1) + a(k_0) + b(k_0))\,|t|^p} > \sigma.$$

Then, there exists a sequence of real numbers $\{t_n\}$ such that $\lim_{n\to+\infty} t_n = +\infty$ and

$$F(k_0, t_n) > \sigma(a(k_0+1) + a(k_0) + b(k_0))t_n^p$$

for all $n \in \mathbb{N}$. Up to considering a subsequence, we may assume that $d_n \geq t_n \geq 1$ for all $n \in \mathbb{N}$. Thus, take in X a sequence $\{w_n\}$ such that, for every $n \in \mathbb{N}$, $w_n(k_0) = t_n$ and $w_n(k) = 0$ for every $k \in \mathbb{Z}\backslash\{k_0\}$. Then, one has $w_n \in W_n$ and

$$\begin{aligned}
J_\lambda(w_n) &= \frac{1}{p}\sum_{k\in\mathbb{Z}} \left(a(k)\,|\Delta w_n(k-1)|^p + b(k)\,|w_n(k)|^p\right) - \lambda \sum_{k\in\mathbb{Z}} F(k, w_n(k)) \\
&< \frac{1}{p}\left(a(k_0+1) + a(k_0)\right) t_n^p + \frac{1}{p} b(k_0) t_n^p - \lambda\sigma(a(k_0+1) + a(k_0) + b(k_0))t_n^p \\
&= \left(\frac{1}{p} - \lambda\sigma\right)(a(k_0+1) + a(k_0) + b(k_0))t_n^p,
\end{aligned}$$

which gives $\lim_{n\to+\infty} J(w_n) = -\infty$. Next, assume that $B < +\infty$. Since $\lambda > \frac{1}{Bp}$, we can fix $\varepsilon > 0$ such that $\varepsilon < B - \frac{1}{\lambda p}$. Therefore, there exists an index $k_0 \in \mathbb{Z}$ such that

$$\limsup_{t\to+\infty} \frac{F(k_0, t)}{(a(k_0+1) + a(k_0) + b(k_0))t^p} > B - \varepsilon.$$

and taking $\{t_n\}$ a sequence of real numbers with $\lim_{n\to+\infty} t_n = +\infty$ and $d_n \geq t_n \geq 1$ for all $n \in \mathbb{N}$ and

$$F(k_0, t_n) > (B - \varepsilon)\,(a(k_0+1) + a(k_0) + b(k_0))t_n^p$$

for all $n \in \mathbb{N}$, choosing $\{w_n\}$ in W_n as above, one has

$$J_\lambda(w_n) < \left(\frac{1}{p} - \lambda(B - \varepsilon)\right)(a(k_0+1) + a(k_0) + b(k_0))t_n^p.$$

So, also in this case, $\lim_{n\to+\infty} J_\lambda(w_n) = -\infty$. This proves Claim 3.5.

Now we are ready to end the proof of Theorem 3.1. With Proposition 2.1, Claims 3.3–3.5, up to a subsequence, we have infinitely many pairwise distinct non-negative homoclinic solutions u_n of (1.1) with $u_n \in W_n$. To finish the proof, we will prove that $\|u_n\| \to +\infty$ as $n \to +\infty$. Let us assume the contrary. Therefore, there is a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ which is bounded in X. Thus, it is also bounded in l_∞. Consequently, we can find $m_0 \in \mathbb{N}$ such that $u_{n_i} \in W_{m_0}$ for all $i \in \mathbb{N}$. Then, for every $n_i \geq m_0$ one has

$$\eta_{m_0} = \inf_{W_{m_0}} J \leq J(u_{n_i}) = \inf_{W_{n_i}} J = \eta_{n_i} \leq \eta_{m_0},$$

which proves that $\eta_{n_i} = \eta_{m_0}$ for all $n_i \geq m_0$, contradicting Claim 3.5. This concludes our proof. □

Remark 3.6. Theorem 1.1 follows now from Theorem 3.1.

4 Examples

Now, we will show the example of a function for which we can apply Theorem 1.1. First we give an example of a function f for which (F_4^+) arise, but (F_5) is not satisfied.

Example 4.1. Let $\{a(k)\}, \{b(k)\}$ be two sequences of positive numbers such that $\lim_{k\to+\infty} b(k) = +\infty$. Let $\{c_n\}, \{d_n\}$ be sequences such that $0 < c_n < d_n < c_{n+1}$ and $\lim_{n\to\infty} c_n = +\infty$. Let $\{h_n\}$ be a sequence such that

$$h_n > n \ \left(a(n+1) + a(n) + b(n)\right) c_{n+1}^p$$

for every $n \in \mathbb{N}$. For every nonpositive integer k let $f(k,\cdot) : \mathbb{R} \to \mathbb{R}$ be identically zero function. For every positive integer k let $f(k,\cdot) : \mathbb{R} \to \mathbb{R}$ be any nonnegative continuous function such that $f(k,t) = 0$ for $t \in \mathbb{R}\setminus(d_k, c_{k+1})$ and $\int_{d_k}^{c_{k+1}} f(k,t)dt = h_k$. The conditions (F_1) and (F_2) are now obviously satisfied.

Set $F(k,t) := \int_0^t f(k,s)ds$ for every $t \in \mathbb{R}$ and $k \in \mathbb{Z}$. Since for every $n \in \mathbb{N}$ and all $r < 0$ only finitely many $\max_{t\in[r,d_n]} F(k,t)$ is nonzero, (F_3) is satisfied. By our choosing of the sequence $\{h_n\}$ we have

$$\begin{aligned}\limsup_{(k,t)\to(+\infty,+\infty)} \frac{F(k,t)}{(a(k+1)a(k)+b(k))\,|t|^p} &\geq \lim_{n\to+\infty} \frac{F(n,c_{n+1})}{(a(n+1)+a(n)+b(n))c_{n+1}^p} \\ &= \lim_{n\to+\infty} \frac{h_n}{(a(n+1)+a(n)+b(n))c_{n+1}^p} = +\infty\end{aligned}$$

and

$$\sup_{k\in\mathbb{Z}} \left(\limsup_{t\to+\infty} \frac{F(k,t)}{(a(k+1)+a(k)+b(k))\,|t|^p} \right) = 0.$$

Now we give an example of a function f for which (F_5) arises, but (F_4^+) is not satisfied.

Example 4.2. Let $\{a(k)\}, \{b(k)\}$ be two sequences of positive numbers such that $\lim_{k\to+\infty} b(k) = +\infty$. Let $\{c_n\}, \{d_n\}$ be sequences such that $0 < c_n < d_n < c_{n+1}$ and $\lim_{n\to\infty} c_n = +\infty$. Let $\{h_n\}$ be a sequence of nonnegative numbers satisfying

$$\frac{\sum_{k=1}^n h_k}{(a(1)+a(0)+b(0))c_{n+1}^p} > n$$

for every $n \in \mathbb{N}$. Let $\tilde{f} : \mathbb{R} \to \mathbb{R}$ be the continuous nonnegative function given by

$$\tilde{f}(s) := \sum_{n\in\mathbb{N}} 2h_n \left(c_{n+1} - d_n - 2\left| s - \frac{1}{2}(d_n + c_{n+1}) \right| \right) \cdot \mathbf{1}_{[d_n,c_{n+1}]}$$

where $\mathbf{1}_{[d,c]}$ is the indicator of the interval $[d,c]$. We check at once that, for every $n \in \mathbb{N}$,

$$\int_{d_n}^{c_{n+1}} \tilde{f}(s)\, ds = h_n.$$

Set $f(0,s) := \tilde{f}(s)$ for $s \in \mathbb{R}$ and $f(k,s) = 0$ for $k \in \mathbb{Z}\setminus\{0\}$ and $s \in \mathbb{R}$. Set $F(k,t) := \int_0^t f(k,s)ds$ for every $t \in \mathbb{R}$ and $k \in \mathbb{Z}$. Then $F(0,c_{n+1}) = \sum_{k=1}^n h_k$. The conditions $(F_1), (F_2)$ and (F_3) are satisfied and

$$\sup_{k\in\mathbb{Z}}\left(\limsup_{t\to+\infty}\frac{F(k,t)}{(a(k+1)+a(k)+b(k))\,|t|^p}\right)=\limsup_{t\to+\infty}\frac{F(0,t)}{(a(1)+a(0)+b(0))\,|t|^p}$$
$$\geq\lim_{n\to+\infty}\frac{F(0,c_{n+1})}{(a(1)+a(0)+b(0))c_{n+1}^p}$$
$$=\lim_{n\to+\infty}\frac{\sum_{k=1}^n h_k}{(a(1)+a(0)+b(0))c_{n+1}^p}=+\infty.$$

Moreover,

$$\limsup_{(k,t)\to(+\infty,+\infty)}\frac{F(k,t)}{(a(k+1)+a(k)+b(k))t^p}=0.$$

5 Comparison with other known results

In the paper [9], the following theorem is presented.

Theorem 5.1. *Assume that a function $b:\mathbb{Z}\to\mathbb{R}$ and a continuous function $f:\mathbb{Z}\times\mathbb{R}\to\mathbb{R}$ satisfy conditions:*

(B) $b(k)\geq b_0>0$ *for all* $k\in\mathbb{Z}$, $b(k)\to+\infty$ *as* $|k|\to+\infty$;

(H_1) $\sup_{|t|\leq T}|F(\cdot.t)|\in l_1$ *for all* $T>0$;

(H_2) $f(k,-t)=-f(k,t)$ *for all* $k\in\mathbb{Z}$ *and* $t\in\mathbb{R}$;

(H_3) *there exist* $d>0$ *and* $q>p$ *such that* $|F(k,t)|\leq d\,|t|^q$ *for all* $k\in\mathbb{Z}$ *and* $t\in\mathbb{R}$;

(H_4) $\lim_{|t|\to+\infty}\frac{f(k,t)t}{|t|^p}=+\infty$ *uniformly for all* $k\in\mathbb{Z}$;

(H_5) *there exists* $\sigma\geq1$ *such that* $\sigma\mathcal{F}(k,t)\geq\mathcal{F}(k,st)$ *for* $k\in\mathbb{Z},t\in\mathbb{R}$, *and* $s\in[0,1]$,

where $F(k,t)$ is the primitive function of $f(k,t)$, that is $F(k,t)=\int_0^t f(k,s)ds$ for every $t\in\mathbb{R}$ and $k\in\mathbb{Z}$, and $\mathcal{F}(k,t)=tf(k,t)-pF(k,t)$. Then, for any $\lambda>0$, problem (1.1) has a sequence $\{u_n(k)\}$ of nontrivial solutions such that $J_\lambda(u_n)\to+\infty$ as $n\to+\infty$.

As an example of function, which satisfied conditions (H_1)–(H_5) is given the function

$$f(k,t)=\frac{1}{k^\mu}\,|t|^{p-2}\,t\ln\left(1+|t|^\nu\right),\qquad (k,t)\in\mathbb{Z}\times\mathbb{R}$$

with $\mu>1$ and $\nu\geq1$. But the theorem cannot be applied to this function, because it does not satisfy the condition (H_4). Moreover, the conditions (H_1) and (H_4) are contradictory. Indeed, since $p>1$ the hypothesis (H_4) does give us $T_1>0$ such that $|f(k,t)|\geq1$ for all $|t|\geq T_1$ and $k\in\mathbb{Z}$. Put $\alpha_k=F(k,T_1)$ for all $k\in\mathbb{Z}$. Then $\{\alpha_k\}\in l_1$, by (H_1). As f is continuous we have for $T>T_1$ and $k\in\mathbb{Z}$

$$|F(k,T)|=\left|\int_0^T f(k,t)dt\right|=\left|\int_0^{T_1} f(k,t)dt+\int_{T_1}^T f(k,t)dt\right|=\left|\alpha_k+\int_{T_1}^T f(k,t)dt\right|$$
$$\geq\left|\int_{T_1}^T f(k,t)dt\right|-|\alpha_k|=\int_{T_1}^T|f(k,t)|\,dt-|\alpha_k|\geq(T-T_1)-|\alpha_k|,$$

and so $|F(\cdot,T)| \notin l_1$, contrary to (H_1).

In the paper [20], the problem (1.1) with $a(k) \equiv 1$ and $\lambda = 1$ was considered. The authors obtained infinitely many pairs of homoclinic solutions assuming, among other things, that $f(k,t)$ is odd in t for each $k \in \mathbb{Z}$, i.e. (H_2). Our Theorem 3.1 has no symmetry assumptions and, for instance, the function in our Example 1 is not odd. On the other hand, Example 7 in [20] shows the function $f : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ satisfying assumptions of the main theorem in [20] with $f(k,t) > 0$ for all $t > 1$ and $k \in \mathbb{Z}$. Such a function does not satisfy (F_2) and Theorem 3.1 does not apply to it.

In the paper [18], the problem (1.1) with $a(k) \equiv 1$ was considered and the following theorem was obtained.

Theorem 5.2. *Assume that a function $b : \mathbb{Z} \to \mathbb{R}$ and a continuous function $f : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ satisfy conditions:*

(B) *$b(k) \geq b_0 > 0$ for all $k \in \mathbb{Z}$, $b(k) \to +\infty$ as $|k| \to +\infty$;*

(F_1) *$\lim_{t\to 0} \frac{|f(k,t)|}{|t|^{p-1}} = 0$ uniformly for all $k \in \mathbb{Z}$.*

Put

$$A := \liminf_{t\to+\infty} \frac{\sum_{k\in\mathbb{Z}} \max_{|\xi|\leq t} F(k,\xi)}{t^p},$$

$$B_{\pm,\pm} := \limsup_{(k,t)\to(\pm\infty,\pm\infty)} \frac{F(k,t)}{(2+b(k))\,|t|^p},$$

$$B_{\pm} := \sup_{k\in\mathbb{Z}} \left(\limsup_{t\to\pm\infty} \frac{F(k,t)}{(2+b(k))\,|t|^p} \right)$$

and $B := \max\{B_{\pm,\pm}, B_{\pm}\}$, where $F(k,t)$ is the primitive function of $f(k,t)$. If $A < b_0 \cdot B$, then for each $\lambda \in I := \left(\frac{1}{Bp}, \frac{b_0}{Ap}\right)$ problem (1.1) admits a sequence of solutions.

As the example 3 in [18] shows, for any two strictly positive real numbers α, β there is a continuous function $f : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ such that $A = \alpha$ and $B = \dot{\beta}$. So, if we choose $\alpha, \beta > 0$ with $\alpha \geq b_0 \cdot \beta$, we will not be able to apply the above theorem. Since this example is similar to our Example 1, the function f satisfies the condition (F_2) and (F_3), and we can apply Theorem 3.1 to obtain a sequence of solutions. On the other hand, as f in example 3 in [18] is non-negative, it is easy to see, that we can modify it in the way, that for some (or even infinitely many) k we have $f(k,t) > 0$ for all $t \geq 1$ and the interval I differ by as little as we wish. Therefore, such an f does not satisfy (F_2) and cannot be used in Theorem 3.1.

References

[1] R. P. Agarwal, K. Perera, D. O'Regan, Multiple positive solutions of singular and nonsingular discrete problems via variational methods, *Nonlinear Anal.* **58**(2004), 69–73. MR2070806

[2] G. Bonanno, P. Candito, Infinitely many solutions for a class of discrete non-linear boundary value problems, *Appl. Anal.* **88**(2009), 605–616. MR2541143

[3] G. Bonanno, G. Molica Bisci, Infinitely many solutions for a boundary value problem with discontinuous nonlinearities, *Bound. Value Probl.* **2009**, Art. ID 670675, 20 pp. MR2487254

[4] G. Bonanno, G. Molica Bisci, Infinitely many solutions for a Dirichlet problem involving the p-Laplacian, *Proc. Roy. Soc. Edinburgh Sect. A* **140**(2010), 737–752. MR2672068

[5] A. Cabada, A. Iannizzotto, S. Tersian, Multiple solutions for discrete boundary value problems, *J. Math. Anal. Appl.* **356**(2009), 418–428. MR2524278

[6] P. Candito, G. Molica Bisci, Existence of two solutions for a second-order discrete boundary value problem, *Adv. Nonlinear Stud.* **11**(2011), 443–453. MR2810141

[7] A. Iannizzotto, V. Rădulescu, Positive homoclinic solutions for the discrete p-Laplacian with a coercive potential, *Differential Integral Equations* **27**(2014), 35–44. MR3161594

[8] A. Iannizzotto, S. Tersian, Multiple homoclinic solutions for the discrete p-Laplacian via critical point theory, *J. Math. Anal. Appl.* **403**(2013), 173–182. MR3035082

[9] L. Kong, Homoclinic solutions for a second order difference equation with p-Laplacian, *Appl. Math. Comput.* **247**(2014), 1113–1121. MR3270909

[10] A. Kristály, M. Mihăilescu, V. Rădulescu, Discrete boundary value problems involving oscillatory nonlinearities: small and large solutions, *J. Difference Equ. Appl.* **17**(2011), 1431–1440. MR2836872

[11] A. Kristály, G. Moroșanu, S. Tersian, Quasilinear elliptic problems in $\mathbb{R}^n$ involving oscillatory nonlinearities, *J. Differential Equations* **235**(2007), 366–375. MR2317487

[12] A. Kristály, V. Rădulescu, Cs. Varga, *Variational principles in mathematical physics, geometry, and economics. Qualitative analysis of nonlinear equations and unilateral problems*, Encyclopedia of Mathematics and its Applications, Vol. 136, Cambridge University Press, Cambridge, 2010. MR2683404

[13] M. Ma, Z. Guo, Homoclinic orbits for second order self-adjont difference equations, *J. Math. Anal. Appl.* **323**(2006), 513–521. MR2262222

[14] G. Molica Bisci, D. Repovš, Existence of solutions for p-Laplacian discrete equations, *Appl. Math. Comput.* **242**(2014), 454–461. MR3239674

[15] V. Rădulescu, Nonlinear elliptic equations with variable exponent: old and new, *Nonlinear Anal.* **121**(2015), 336–369. MR3348928

[16] V. Rădulescu, D. Repovš, *Partial differential equations with variable exponents. Variational methods and qualitative analysis*, CRC Press, Boca Raton FL, 2015. MR3379920

[17] B. Ricceri, A general variational principle and some of its applications, *J. Comput. Appl. Math.* **133**(2000), 401–410. MR1735837

[18] R. Stegliński, On sequences of large homoclinic solutions for a difference equations on the integers, *Adv. Difference Equ.* **2016**, 2016:38, 11 pp. MR3456503

[19] S. Stević, Solvable subclasses of a class of nonlinear second-order difference equations, *Adv. Nonlinear Anal.*, in press. url

[20] G. Sun, A. Mai, Infinitely many homoclinic solutions for second order nonlinear difference equations with p-Laplacian, *Scientific World J.* **2014**, Art. ID 276372, 6 pp. url