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Existence of entire radial solutions to a class of quasilinear elliptic equations and systems

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Abstract. In this paper, by a monotone iterative method and the Arzelà–Ascoli theorem, we obtain the existence of entire positive radial solutions to the following quasilinear elliptic equations

$$\operatorname{div}(\phi_1(|\nabla u|)\nabla u) + a_1(|x|)\phi_1(|\nabla u|)|\nabla u| = b_1(|x|)f(u), \qquad x \in \mathbb{R}^N,$$

and systems

$$\begin{cases} \operatorname{div}(\phi_1(|\nabla u|)\nabla u) + a_1(|x|)\phi_1(|\nabla u|)|\nabla u| = b_1(|x|)f_1(u,v), & x \in \mathbb{R}^N, \\ \operatorname{div}(\phi_2(|\nabla v|)\nabla v) + a_2(|x|)\phi_2(|\nabla v|)|\nabla v| = b_2(|x|)f_2(u,v), & x \in \mathbb{R}^N, \end{cases}$$

under simple conditions on f, f_i, a_i and b_i $(i = 1, 2)$.

Keywords: quasilinear elliptic equations, systems, entire radial solutions, existence.

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1 Introduction

The purpose of this paper is to investigate the existence of entire positive radial solutions to the following quasilinear elliptic equation

$$\operatorname{div}(\phi_1(|\nabla u|)\nabla u) + a_1(|x|)\phi_1(|\nabla u|)|\nabla u| = b_1(|x|)f(u), \qquad x \in \mathbb{R}^N, \tag{1.1}$$

and system

$$\begin{cases} \operatorname{div}(\phi_1(|\nabla u|)\nabla u) + a_1(|x|)\phi_1(|\nabla u|)|\nabla u| = b_1(|x|)f_1(u,v), & x \in \mathbb{R}^N, \\ \operatorname{div}(\phi_2(|\nabla v|)\nabla v) + a_2(|x|)\phi_2(|\nabla v|)|\nabla v| = b_2(|x|)f_2(u,v), & x \in \mathbb{R}^N, \end{cases} \tag{1.2}$$

where a_i, b_i, f, f_i $(i = 1, 2)$ satisfy

$(\mathbf{S_1})$ $a_i, b_i : \mathbb{R}^N \to [0, \infty)$ are continuous;

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$(\mathbf{S_2})$ $f : [0,\infty) \to [0,\infty)$ is continuous and increasing, $f_i : [0,\infty)\times[0,\infty) \to [0,\infty)$ are continuous and increasing (i.e., $f_i(s_2,t_2) \geq f_i(s_1,t_1)$, $\forall s_2 \geq s_1 \geq 0$ and $t_2 \geq t_1 \geq 0$),

and $\phi_i \in C^1((0,\infty),(0,\infty))$ satisfy:

$(\mathbf{S_3})$ $(t\phi_i(t))' > 0,\ \forall t > 0;$

$(\mathbf{S_4})$ there exist $p_i, q_i > 1$ such that

$$p_i \leq \frac{t\Psi_i'(t)}{\Psi_i(t)} \leq q_i, \qquad \forall t > 0,$$

where $\Psi_i(t) = \int_0^t s\phi_i(s)ds,\ t > 0;$

$(\mathbf{S_5})$ there exist $k_i, l_i > 0$ such that

$$k_i \leq \frac{t\Psi_i''(t)}{\Psi_i'(t)} \leq l_i, \qquad \forall t > 0.$$

$\Delta_{\phi_1} u = \operatorname{div}(\phi_1(|\nabla u|)\nabla u)$ is called the ϕ_1-Laplacian operator, which includes special cases appearing in mathematical models in nonlinear elasticity, plasticity, generalized Newtonian fluids, and in quantum physics, see e.g., Benci, Fortunato and Pisani [5], Cencelj, Repovš and Virk [6], Fuchs and Li [9], Fuchs and Osmolovski [10], Fukagai and Narukawa [11] and [12] and the references therein.

Some basic examples of ϕ_1-Laplacian operators are

$(\mathbf{1})$ when $\phi_1(t) \equiv 2$, $\Psi_1(t) = t^2$, $t > 0$, $\Delta_{\phi_1} u = \Delta u$ is the Laplacian operator. In this case, $p_1 = q_1 = 2$ in $(\mathbf{S_4})$, and $k_1 = l_1 = 1$ in $(\mathbf{S_5})$;

$(\mathbf{2})$ when $\phi_1(t) = pt^{p-2}$, $\Psi_1(t) = t^p$, $t > 0$, $p > 1$, $\Delta_{\phi_1} u = \Delta_p u$ is the p-Laplacian operator. In this case, $p_1 = q_1 = p$ in $(\mathbf{S_4})$, and $k_1 = l_1 = p-1$ in $(\mathbf{S_5})$;

$(\mathbf{3})$ when $\phi_1(t) = pt^{p-2} + qt^{q-2}$, $\Psi_1(t) = t^p + t^q$, $t > 0$, $1 < p < q$, $\Delta_{\phi_1} u = \Delta_p u + \Delta_q u$ is called as the $(p+q)$-Laplacian operator, $p_1 = p$, $q_1 = q$ in $(\mathbf{S_4})$, and $k_1 = p-1$, $l_1 = q-1$ in $(\mathbf{S_5})$;

$(\mathbf{4})$ when $\phi_1(t) = 2p(1+t^2)^{p-1}$, $\Psi_1(t) = (1+t^2)^p - 1$, $t > 0$, $p > 1/2$, $p_1 = \min\{2, 2p\}$, $q_1 = \max\{2, 2p\}$ in $(\mathbf{S_4})$, and $k_1 = \min\{1, 2p-1\}$, $l_1 = \max\{1, 2p-1\}$ in $(\mathbf{S_5})$;

$(\mathbf{5})$ when $\phi_1(t) = \frac{p(\sqrt{1+t^2}-1)^{p-1}}{\sqrt{1+t^2}}$, $\Psi_1(t) = (\sqrt{1+t^2}-1)^p$, $t > 0$, $p > 1$, $p_1 = p$, $q_1 = 2p$ in $(\mathbf{S_4})$, and $k_1 = p-1$, $l_1 = 2p-1$ in $(\mathbf{S_5})$;

$(\mathbf{6})$ when $\phi_1(t) = pt^{p-2}(\ln(1+t))^q + \frac{qt^{p-1}(\ln(1+t))^{q-1}}{1+t}$, $\Psi_1(t) = t^p(\ln(1+t))^q$, $t > 0$, $p > 1$, $q > 0$, $p_1 = p$, $q_1 = p+q$ in $(\mathbf{S_4})$, and $k_1 = p-1$, $l_1 = p+q-1$ in $(\mathbf{S_5})$.

We say that $u \in C^1(\mathbb{R}^N)$ is a solution to equation (1.1) if for each $\psi \in C_0^\infty(\mathbb{R}^N)$, it holds

$$\int_{\mathbb{R}^N} \phi_1(|\nabla u|)\nabla u \nabla\psi dx - \int_{\mathbb{R}^N} a_1(x)(\phi_1(|\nabla u|)\nabla u)\psi dx = -\int_{\mathbb{R}^N} b_1(x) f(u)\psi dx.$$

Moreover, when $\lim_{|x|\to\infty} u(x) = +\infty$, we say that u is a large solution to equation (1.1).

For convenience, for $i = 1, 2$, we denote by

$$h_i^{-1} \quad \text{the inverses of } h_i(t) = t\phi_i(t), \quad t > 0; \tag{1.3}$$

$$I_{i,\rho,g}(\infty) := \lim_{r\to\infty} I_{i,\rho,g}(r), \qquad I_{i,\rho,g}(r) := \int_0^r h_i^{-1}(\Lambda_{\rho,g}(t))dt, \qquad r \geq 0, \tag{1.4}$$

where $\rho, g \in C([0,\infty),[0,\infty))$ and

$$\Lambda_{\rho,g}(t) := \frac{1}{\Phi_g(t)} \int_0^t \Phi_g(s)\rho(s)ds, \qquad t > 0; \tag{1.5}$$

$$\Phi_g(t) := t^{N-1} \exp\left(\int_0^t g(\tau)d\tau\right), \qquad t > 0; \tag{1.6}$$

$$\theta_i(t) := \min\{t^{p_i}, t^{q_i}\}, \qquad \Theta_i(t) := \max\{t^{p_i}, t^{q_i}\}, \qquad t \geq 0; \tag{1.7}$$

$$\theta_i^{-1}(t) := \min\{t^{1/p_i}, t^{1/q_i}\}, \qquad \Theta_i^{-1}(t) := \max\{t^{1/p_i}, t^{1/q_i}\}, \qquad t \geq 0; \tag{1.8}$$

and, for an arbitrary $\alpha > 0$ and $t \geq \alpha$,

$$\Upsilon_{1,\alpha}(\infty) := \lim_{t\to\infty} \Upsilon_{1,\alpha}(t), \qquad \Upsilon_{1,\alpha}(t) := \int_\alpha^t \frac{d\tau}{\Theta_1^{-1}(f(\tau))}; \tag{1.9}$$

$$\Upsilon_{2,\alpha}(\infty) := \lim_{t\to\infty} \Upsilon_{2,\alpha}(t), \qquad \Upsilon_{2,\alpha}(t) := \int_\alpha^t \frac{d\tau}{\Theta_1^{-1}(f_1(\tau,\tau)) + \Theta_2^{-1}(f_2(\tau,\tau))}. \tag{1.10}$$

We see that for $t > \alpha$

$$\Upsilon'_{1,\alpha}(t) = \frac{1}{\Theta_1^{-1}(f(t))} > 0,$$

$$\Upsilon'_{2,\alpha}(t) = \frac{1}{\Theta_1^{-1}(f_1(t,t)) + \Theta_2^{-1}(f_2(t,t))} > 0,$$

and $\Upsilon_{1,\alpha}, \Upsilon_{2,\alpha}$ have the inverse functions $\Upsilon_{1,\alpha}^{-1}$ and $\Upsilon_{2,\alpha}^{-1}$ on $[0, \Upsilon_{1,\alpha}(\infty))$ and $[0, \Upsilon_{2,\alpha}(\infty))$, respectively.

First, let us review the following model

$$\Delta u = b_1(|x|)f(u), \qquad x \in \mathbb{R}^N. \tag{1.11}$$

For $b_1(x) \equiv 1$ on $\mathbb{R}^N$: when f satisfies (S_2), Keller [14] and Osserman [19] first supplied a necessary and sufficient condition

$$\int_1^\infty \frac{dt}{\sqrt{2F(t)}} = \infty, \qquad F(t) = \int_0^t f(s)ds, \tag{1.12}$$

for the existence of entire positive radial large solutions to equation (1.11).

For $N \geq 3$, $f(u) = u^\gamma$, $\gamma \in (0,1]$, and b_1 satisfies (S_1) with $b_1(x) = b_1(|x|)$, Lair and Wood [16] first showed that equation (1.11) has infinitely many entire positive radial large solutions if and only if

$$\int_0^\infty r b_1(r)dr = \infty. \tag{1.13}$$

The above results have been extended by many authors and in many contexts, see, for instance, [1–3, 8, 21–23] and the references therein.

Next let us review the system

$$\begin{cases} \Delta u = b_1(|x|)v^{\gamma_1}, & x \in \mathbb{R}^N, \\ \Delta v = b_2(|x|)u^{\gamma_2}, & x \in \mathbb{R}^N. \end{cases} \tag{1.14}$$

When $N \geq 3$ and $0 < \gamma_1 \leq \gamma_2$, Lair and Wood [17] have considered the existence and nonexistence of entire positive radial solutions to system (1.14).

For the further results, see, for instance, [4,7,13,15,18,24] and the references therein.

Now let us return to equation (1.1). Recently, C. A. Santos, J. Zhou, J. A. Santos [20] considered the existence of entire positive radial and nonradial large solutions to equation

$$\operatorname{div}(\phi_1(|\nabla u|)\nabla u) = b_1(x)f(u), \qquad x \in \mathbb{R}^N.$$

A basic result in [20] is the following.

Lemma 1.1 ([20, Corollary 1.2]). *Let* $(\mathbf{S_3})$–$(\mathbf{S_5})$ *hold,* f *satisfy* $(\mathbf{S_2})$*, and* b_1 *satisfy* $(\mathbf{S_1})$ *with* $b_1(x) = b_1(|x|)$, $x \in \mathbb{R}^N$. *If*

$$I_{1,b_1,0}(\infty) = \infty,$$

then equation (1.1) *admits a sequence of symmetric radial large solutions* $u_m(|x|) \in C^1(\mathbb{R}^N)$ *with* $u_m(0) \to \infty$ as $m \to \infty$ *if and only if* f *satisfies*

$$\int_1^\infty \frac{dt}{\Psi_1^{-1}(F(t))} = \infty,$$

where Ψ_1^{-1} *is the inverse of* Ψ_1 *which is given as in* $(\mathbf{S_4})$*, and* F *is given as in* (1.12).

Recently, when $a_i \equiv 0$ in $\mathbb{R}^N$, $f_1(u,v) = f(v)$, $f_2(u,v) = g(u)$, and g satisfies $(\mathbf{S_2})$, Zhang [25] showed existence of entire positive radial solutions to (1.1) and system (1.2).

In this paper, we extend the results of [25] and show existence of entire positive radial solutions to (1.1) and (1.2) for more general a_i and f_i.

Our main results for equation (1.1) are as follows.

Theorem 1.2. *Let the hypotheses* $(\mathbf{S_1})$–$(\mathbf{S_5})$ *hold. If*

$(\mathbf{S_6})$ $\Upsilon_{1,\alpha}(\infty) = \infty$,

then equation (1.1) *has one entire positive radial solution* $u \in C^1(\mathbb{R}^N)$*. Moreover, when* $I_{1,a_1,b_1}(\infty) < \infty$, u *is bounded, and* $\lim_{r\to\infty} u(r) = \infty$ *provided* $I_{1,a_1,b_1}(\infty) = \infty$*, where* I_{1,a_1,b_1} *is given as in* (1.4).

Theorem 1.3. *Under the hypotheses* $(\mathbf{S_1})$–$(\mathbf{S_5})$ *and*

$(\mathbf{S_7})$ $I_{1,a_1,b_1}(\infty) < \Upsilon_{1,\alpha}(\infty) < \infty$,

equation (1.1) *has one entire positive radial bounded solution* $u \in C^1(\mathbb{R}^N)$ *satisfying*

$$\alpha + \theta_1^{-1}(f(\alpha))I_{1,a_1,b_1}(r) \leq u(r) \leq \Upsilon_{1,\alpha}^{-1}\left(I_{1,a_1,b_1}(r)\right), \qquad \forall r \geq 0,$$

where θ_1^{-1} *is given as in* (1.8).

Remark 1.4. When $\int_0^1 \frac{d\tau}{\Theta_1^{-1}(f(\tau))} = \infty$, one can see that there is $\alpha > 0$ sufficiently small such that $(\mathbf{S_7})$ holds provided $I_{1,a_1,b_1}(\infty) < \infty$ and $\Upsilon_{1,\alpha}(\infty) < \infty$.

Remark 1.5. For $f(s) = s^{\gamma_1}$, $s \geq 0$, $\gamma_1 > 0$, since $\Theta_1^{-1}(t) = t^{1/p_1}$, $t \geq 1$, one can see that when $\gamma_1 > p_1$, $\Upsilon_{1,\alpha}(\infty) < \infty$, and $\Upsilon_{1,\alpha}(\infty) = \infty$ provided $\gamma_1 \leq p_1$, where p_1 is given as in $(\mathbf{S_4})$.

Remark 1.6. For $f(s) = (1+s)^{\gamma_1}(\ln(1+s))^{\mu_1}$, $s \geq 0$, $\mu_1, \gamma_1 > 0$, one can see that when $\gamma_1 > p_1$ or $\gamma_1 = p_1$ and $\mu_1 > p_1$, $\Upsilon_{1,\alpha}(\infty) < \infty$, and $\Upsilon_{1,\alpha}(\infty) = \infty$ provided $\gamma_1 < p_1$ or $\gamma_1 = p_1$ and $\mu_1 \leq p_1$.

Remark 1.7. For $f(s) = \exp(c_1 s)$, $s \geq 0$, $c_1 > 0$, one can see that $\Upsilon_{1,\alpha}(\infty) < \infty$.

Our main results for system (1.2) are as follows.

Theorem 1.8. *Let the hypotheses* $(\mathbf{S_1})$–$(\mathbf{S_5})$ *hold. If*

$(\mathbf{S_8})$ $\Upsilon_{2,\alpha}(\infty) = \infty$,

then system (1.2) *has one entire positive radial solution* (u, v) *in* $C^1(\mathbb{R}^N) \times C^1(\mathbb{R}^N)$. *Moreover, when* $I_{1,a_1,b_1}(\infty) + I_{2,a_2,b_2}(\infty) < \infty$, u *and* v *are bounded; when* $I_{1,a_1,b_1}(\infty) = I_{2,a_2,b_2}(\infty) = \infty$, $\lim_{r\to\infty} u(r) = \lim_{r\to\infty} v(r) = \infty$.

Theorem 1.9. *Under the hypotheses* $(\mathbf{S_1})$–$(\mathbf{S_5})$ *and*

$$(\mathbf{S_9}) \qquad I_{1,a_1,b_1}(\infty) + I_{2,a_2,b_2}(\infty) < \Upsilon_{2,\alpha}(\infty) < \infty,$$

system (1.2) *has one entire positive radial bounded solution* (u, v) *in* $C^1(\mathbb{R}^N) \times C^1(\mathbb{R}^N)$ *satisfying*

$$\alpha/2 + \theta_1^{-1}(f_1(\alpha/2, \alpha/2)) I_{1,a_1,b_1}(r) \leq u(r) \leq \Upsilon_{2,\alpha}^{-1}(I_{1,a_1,b_1}(r) + I_{2,a_2,b_2}(r)), \qquad \forall r \geq 0;$$

$$\alpha/2 + \theta_2^{-1}(f_2(\alpha/2, \alpha/2)) I_{2,a_2,b_2}(r) \leq v(r) \leq \Upsilon_{2,\alpha}^{-1}(I_{1,a_1,b_1}(r) + I_{2,a_2,b_2}(r)), \qquad \forall r \geq 0.$$

Remark 1.10. For $f_1(s, s) = s^{\gamma_1}$, $f_2(s, s) = s^{\gamma_2}$, $s \geq 0$, $\gamma_1, \gamma_2 > 0$, when $\gamma_1 > p_1$ or $\gamma_2 > p_2$, $\Upsilon_{2,\alpha}(\infty) < \infty$, and $\Upsilon_{2,\alpha}(\infty) = \infty$ provided $\gamma_1 \leq p_1$ and $\gamma_2 \leq p_2$, where p_1 and p_2 are given as in $(\mathbf{S}_4)$.

Remark 1.11. For $f_1(s, s) = (1 + s)^{\gamma_1}(\ln(1 + s))^{\mu_1}$, $f_2(s, s) = (1 + s)^{\gamma_2}(\ln(1 + s))^{\mu_2}$, $s \geq 0$, $\gamma_i, \mu_i > 0$ $(i = 1, 2)$, when $\gamma_1 > p_1$ or $\gamma_2 > p_2$; or $\gamma_1 = p_1$ and $\mu_1 > p_1$; or $\gamma_2 = p_2$ and $\mu_2 > p_2$, $\Upsilon_{2,\alpha}(\infty) < \infty$, and $\Upsilon_{2,\alpha}(\infty) = \infty$ provided $\gamma_1 < p_1$ and $\gamma_2 < p_2$; or $\gamma_1 = p_1$, $\mu_1 \leq p_1$ and $\gamma_2 = p_2$, $\mu_2 \leq p_2$.

Remark 1.12. For $f_1(s, s) = \exp(c_1 s)$ or $f_2(s, s) = \exp(c_2 s)$, $s \geq 0$, $c_1, c_2 > 0$, one can see that $\Upsilon_{2,\alpha}(\infty) < \infty$.

Remark 1.13. We note that the paper [26] by X. Zhang et al. studied the nonexistence and existence of positive radial large solutions to system (1.2). But, since their basic assumption is that $\phi_i \in C^1((0, \infty), [0, \infty))$ $(i = 1, 2)$ are nondecreasing and for any $c \in (0, 1)$, there exist constants $\sigma_i \in (0, 1)$ such that

$$\phi_i(cs) \leq c^{\sigma_i} \phi_i(s), \qquad \forall s > 0, \tag{1.15}$$

it is $c^{\sigma_i} < 1$, hence (1.15) can not be set up when $\phi_i \equiv 1$ on $(0, \infty)$ (in this case, $\Delta_{\phi_1} u = \Delta u$ is the Laplacian operator).

2 Proof of Theorems 1.2 and 1.3

In this section we prove Theorems 1.2 and 1.3.

Lemma 2.1 ([20, Lemma 2.2])**.** *Let* $(\mathbf{S_3})$–$(\mathbf{S_5})$ *hold,* θ_i, Θ_i *and* $\theta_i^{-1}, \Theta_i^{-1}$ $(i = 1, 2)$ *be given as in* (1.7) *and* (1.8)*. We have*

(i) θ_i, Θ_i, θ_i^{-1} *and* Θ_i^{-1} *are strictly increasing on* $(0, \infty)$*;*

(ii) $\theta_i^{-1}(\beta) h_i^{-1}(t) \leq h_i^{-1}(\beta t) \leq \Theta_i^{-1}(\beta) h_i^{-1}(t)$, $\forall \beta, t > 0$.

Let us consider the following initial value problem

$$(\Phi_{a_1}(r)\phi_1(u'(r))u'(r))' = b_1(r)\Phi_{a_1}(r)f(u), \qquad r>0, \quad u(0)=\alpha, \quad u'(0)=0, \tag{2.1}$$

where $\Phi_{a_1}(r)$ is given as in (1.6).

By a simple calculation,

$$u'(r) = h_1^{-1}\left(\frac{1}{\Phi_{a_1}(r)}\int_0^r b_1(s)\Phi_{a_1}(s)f(u(s))ds\right), \qquad r>0, \quad u(0)=\alpha, \tag{2.2}$$

and thus

$$u(r) = \alpha + \int_0^r h_1^{-1}\left(\frac{1}{\Phi_{a_1}(t)}\int_0^t b_1(s)\Phi_{a_1}(s)f(u(s))ds\right)dt, \qquad r\geq 0. \tag{2.3}$$

Note that solutions in $C[0,\infty)$ to problem (2.3) are solutions in $C^1[0,\infty)$ to problem (2.1).

Let $\{u_m\}_{m\geq 1}$ be the sequence of positive continuous functions defined on $[0,\infty)$ by

$$\begin{cases} u_0(r)=\alpha, \\ u_m(r) = \alpha + \displaystyle\int_0^r h_1^{-1}\left(\frac{1}{\Phi_{a_1}(t)}\int_0^t b_1(s)\Phi_{a_1}(s)f(u_{m-1}(s))ds\right)dt, \quad r\geq 0. \end{cases} \tag{2.4}$$

Obviously,

$$u_m'(r) = h_1^{-1}\left(\frac{1}{\Phi_{a_1}(r)}\int_0^r b_1(s)\Phi_{a_1}(s)f(u_{m-1}(s))ds\right), \qquad r>0, \tag{2.5}$$

and, for all $r\geq 0$ and $m\in\mathbb{N}$, $u_m(r)\geq\alpha$, and $u_0\leq u_1$. Then $(\mathbf{S_1})$–$(\mathbf{S_3})$ and Lemma 2.1 yield $u_1(r)\leq u_2(r)$, $\forall r\geq 0$. Continuing this line of reasoning, we obtain that the sequence $\{u_m\}$ is non-decreasing on $[0,\infty)$. Moreover, we obtain by $(\mathbf{S_1})$–$(\mathbf{S_3})$ and Lemma 2.1 that for each $r>0$

$$\begin{aligned} u_m'(r) &= h_1^{-1}\left(\frac{1}{\Phi_{a_1}(r)}\int_0^r b_1(s)\Phi_{a_1}(s)f(u_{m-1}(s))ds\right) \\ &\leq h_1^{-1}\left(f(u_m(r))\frac{1}{\Phi_{a_1}(r)}\int_0^r b_1(s)\Phi_{a_1}(s)ds\right) \\ &\leq \Theta_1^{-1}(f(u_m(r)))h_1^{-1}\left(\frac{1}{\Phi_{a_1}(r)}\int_0^r b_1(s)\Phi_{a_1}(s)ds\right), \end{aligned}$$

and

$$\int_a^{u_m(r)}\frac{d\tau}{\Theta_1^{-1}(f(\tau))}\leq I_{1,a_1,b_1}(r).$$

Consequently, for an arbitrary $R>0$,

$$\Upsilon_{1\alpha}(u_m(r))\leq I_{1,a_1,b_1}(r)\leq I_{1,a_1,b_1}(R), \qquad \forall r\in[0,R]. \tag{2.6}$$

(i) When $(\mathbf{S_6})$ holds, we see that

$$\Upsilon_{1,\alpha}^{-1}(\infty)=\infty \quad\text{and}\quad u_m(r)\leq\Upsilon_{1,\alpha}^{-1}\left(I_{1,a_1,b_1}(r)\right)\leq\Upsilon_{1,\alpha}^{-1}\left(I_{1,a_1,b_1}(R)\right), \qquad \forall r\in[0,R], \tag{2.7}$$

i.e., the sequence $\{u_m\}$ is bounded on $[0,R]$ for an arbitrary $R>0$.

It follows by (2.5) that $\{u_m'\}$ is bounded on $[0,R]$. By the Arzelà–Ascoli theorem, $\{u_m\}$ has a subsequence converging uniformly to u on $[0,R]$. Since $\{u_m\}$ is non-decreasing on $[0,\infty)$, we see that $\{u_m\}$ itself converges uniformly to u on $[0,R]$. By the arbitrariness of R, we see that u is an entire positive radial solution to equation (1.1). Moreover, when $I_{1,a_1,b_1}(\infty) < \infty$, we see by (2.7) that

$$u(r) \leq \Upsilon_{1,\alpha}^{-1}\left(I_{1,a_1,b_1}(\infty)\right), \qquad \forall r \geq 0.$$

Moreover, when $I_{1,a_1,b_1}(\infty) = \infty$, we see by (S$_2$) and Lemma 2.1 that

$$u(r) \geq \alpha + \theta_1^{-1}(f(\alpha)) I_{1,a_1,b_1}(r), \qquad \forall r \geq 0.$$

Thus $\lim_{r\to\infty} u(r) = \infty$.

(ii) When $(\mathbf{S_7})$ holds, we see by (2.6) that

$$\Upsilon_{1,\alpha}(u_m(r)) \leq I_{1,a_1,b_1}(\infty) < \Upsilon_{1,\alpha}(\infty) < \infty. \tag{2.8}$$

Since $\Upsilon_{1,\alpha}^{-1}$ is strictly increasing on $[0, \Upsilon_{1,\alpha}(\infty))$, we have

$$u_m(r) \leq \Upsilon_{1,\alpha}^{-1}\left(I_{1,a_1,b_1}(\infty)\right) < \infty, \ \forall r \geq 0. \tag{2.9}$$

The rest part of the proof follows from (i). The proof is finished.

3 Proof of Theorems 1.8 and 1.9

In this section we prove Theorems 1.8 and 1.9.

Let us consider the following initial value problem

$$\begin{cases} \left(\Phi_{a_1}(r)\phi_1(u'(r))u'(r)\right)' = b_1(r)\Phi_{a_1}(r) f_1(u,v), & r > 0, \\ \left(\Phi_{a_2}(r)\phi_2(v'(r))v'(r)\right)' = b_2(r)\Phi_{a_2}(r) f_2(u,v), & r > 0, \\ u(0) = v(0) = \alpha/2, \quad u'(0) = v'(0) = 0, \end{cases}$$

which is equivalent to

$$\begin{cases} u(r) = \alpha/2 + \displaystyle\int_0^r h_1^{-1}\left(\frac{1}{\Phi_{a_1}(t)} \int_0^t b_1(s)\Phi_{a_1}(s) f_1(u(s), v(s))ds\right)dt, & r \geq 0, \\ v(r) = \alpha/2 + \displaystyle\int_0^r h_2^{-1}\left(\frac{1}{\Phi_{a_2}(t)} \int_0^t b_2(s)\Phi_{a_2}(s) f_2(u(s), v(s))ds\right)dt, & r \geq 0. \end{cases}$$

Let $\{u_m\}_{m\geq 1}$ and $\{v_m\}_{m\geq 0}$ be the sequences of positive continuous functions defined on $[0, \infty)$ by

$$\begin{cases} u_0(r) = v_0(r) = \alpha/2, \\ u_m(r) = \alpha/2 + \displaystyle\int_0^r h_1^{-1}\left(\frac{1}{\Phi_{a_1}(t)} \int_0^t b_1(s)\Phi_{a_1}(s) f_1(u_{m-1}(s), v_{m-1}(s))ds\right)dt, & r \geq 0, \\ v_m(r) = \alpha/2 + \displaystyle\int_0^r h_2^{-1}\left(\frac{1}{\Phi_{a_2}(t)} \int_0^t b_2(s)\Phi_{a_2}(s) f_2(u_{m-1}(s), v_{m-1}(s))ds\right)dt, & r \geq 0. \end{cases}$$

Obviously, for all $r \geq 0$ and $m \in \mathbb{N}$, $u_m(r) \geq \alpha/2$, $v_m(r) \geq \alpha/2$ and $u_0 \leq u_1$, $v_0 \leq v_1$. $(\mathbf{S_1})$–$(\mathbf{S_3})$ and Lemma 2.1 yield $u_1(r) \leq u_2(r)$ and $v_1(r) \leq v_2(r)$ on $[0, \infty)$. Continuing this line of reasoning, we obtain that the sequences $\{u_m\}$ and $\{v_m\}$ are increasing on $[0,\infty)$. Moreover, we obtain by $(\mathbf{S_1})$–$(\mathbf{S_3})$ and Lemma 2.1 that for each $r > 0$

$$\begin{aligned} u_m'(r) &= h_1^{-1}\left(\frac{1}{\Phi_{a_1}(r)}\int_0^r b_1(s)\Phi_{a_1}(s)f_1(u_{m-1}(s),v_{m-1}(s))ds\right) \\ &\le h_1^{-1}\left(f_1(u_{m-1}(r),v_{m-1}(r))\frac{1}{\Phi_{a_1}(t)}\int_0^t b_1(s)\Phi_{a_1}(s)ds\right) \\ &\le \Theta_1^{-1}(f_1(u_m(r),v_m(r)))h_1^{-1}\left(\frac{1}{\Phi_{a_1}(r)}\int_0^r b_1(s)\Phi_{a_1}(s)ds\right) \\ &\le \Theta_1^{-1}(f_1(u_m(r)+v_m(r),u_m(r)+v_m(r)))\big(h_1^{-1}(\Lambda_{b_1,a_1}(r))+h_2^{-1}(\Lambda_{b_2,a_2}(r))\big), \end{aligned}$$

where $\Lambda_{b_1,a_1}(r)$ and $\Lambda_{b_2,a_2}(r)$ are given as in (1.5).

In a similar way, we can show that

$$\begin{aligned} v_m'(r) &= h_2^{-1}\left(\frac{1}{\Phi_{a_2}(t)}\int_0^t b_2(s)\Phi_{a_2}(s)f_2(u_{m-1}(s),v_{m-1}(s))ds\right)dt \\ &\le \Theta_2^{-1}(f_2(u_m(r),v_m(r)))h_2^{-1}\left(\frac{1}{\Phi_{a_2}(t)}\int_0^t b_2(s)\Phi_{a_2}(s)ds\right) \\ &\le \Theta_2^{-1}(f_2(u_m(r)+v_m(r),u_m(r)+v_m(r)))\big(h_1^{-1}(\Lambda_{b_1,a_1}(r))+h_2^{-1}(\Lambda_{b_2,a_2}(r))\big). \end{aligned}$$

Consequently,

$$\begin{aligned} u_m'(r)+v_m'(r) \le \Big(&\Theta_1^{-1}(f_1(v_m(r)+u_m(r),v_m(r)+u_m(r))) \\ &+\Theta_2^{-1}(f_2(v_m(r)+u_m(r),v_m(r)+u_m(r)))\Big) \\ &\times\big(h_1^{-1}(\Lambda_{b_1,a_1}(r))+h_2^{-1}(\Lambda_{b_2,a_2}(r))\big), \qquad r>0, \end{aligned}$$

and

$$\int_a^{u_m(r)+v_m(r)} \frac{d\tau}{\Theta_1^{-1}(f_1(\tau,\tau))+\Theta_2^{-1}(f_2(\tau,\tau))} \le I_{1,b_1,a_1}(r)+I_{2,b_2,a_2}(r), \qquad r>0,$$
$$\Upsilon_{2,\alpha}(u_m(r)+v_m(r)) \le I_{1,b_1,a_1}(r)+I_{2,b_2,a_2}(r), \qquad \forall r\ge 0.$$

The remaining proofs are similar to that for Theorems 1.2 and 1.3. Here we omit their proof.

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