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Upper and lower solution method for boundary value problems at resonance

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Abstract. We consider two simple boundary value problems at resonance for an ordinary differential equation. Employing a shift argument, a regular fixed point operator is constructed. We employ the monotone method coupled with a method of upper and lower solutions and obtain sufficient conditions for the existence of solutions of boundary value problems at resonance for nonlinear boundary value problems. Three applications are presented in which explicit upper solutions and lower solutions are exhibited for the first boundary value problem. Two applications are presented for the second boundary value problem. Of interest, the upper and lower solutions are easily and explicitly constructed. Of primary interest, the upper and lower solutions are elements of the kernel of the linear problem at resonance.

Keywords: boundary value problems at resonance, upper and lower solutions, differential inequalities, monotone methods.

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1 Introduction

We consider two boundary value problems at resonance for second order ordinary differential equations. Specifically, we shall consider

$$y''(t) = f(t, y(t)), \quad 0 \leq t \leq 1, \tag{1.1}$$

$$y'(0) = 0, \quad y'(1) = 0, \tag{1.2}$$

where $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous and

$$y''(t) = f(t, y(t), y'(t)), \quad 0 \leq t \leq 1, \tag{1.3}$$

$$y(0) = 0, \quad y'(0) = y'(1), \tag{1.4}$$

where $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous. We shall employ the method of upper and lower solutions coupled with monotone methods.

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The boundary value problem (1.1)–(1.2) is said to be at resonance because the homogeneous problem

$$y''(t) = 0, \quad 0 \leq t \leq 1, \quad y'(0) = 0, \quad y'(1) = 0,$$

has nontrivial constant solutions. Similarly, the boundary value problem (1.3)–(1.4) is said to be at resonance because the homogeneous problem

$$y''(t) = 0, \quad 0 \leq t \leq 1, \quad y(0) = 0, \quad y'(0) = y'(1),$$

has nontrivial solutions of the form $y(t) = ct$.

Boundary value problems at resonance have been investigated for many years; coincidence degree theory, credited to Mawhin [24, 25], has been employed by many researchers and we cite, for example, [3, 6–8, 10, 13, 18, 19, 21, 22]. More recently, beginning with interest to obtain sufficient conditions for the existence of solutions in a cone, researchers have been developing a variety of new methods. As examples, the following methods have been developed: (i) a coincidence theorem of Schauder type [30], (ii) a Lyapunov–Schmidt procedure [23], (iii) topological degree [5, 12, 27, 29], (iv) a Leggett–Williams type theorem for coincidences [9, 15, 28], (v) a fixed point index theorem [2, 4, 19, 20], and (vi) fixed point index theory [32]. More in line with the approach employed in this work, Han [14] modified the problem at resonance and considered a regular boundary value problem (a method referred as the shift argument by Infante, Pietramala and Tojo [16]) in order to apply the Krasnosel′skiĭ–Guo fixed point theorem [11].

Szymańska-Dębowska [31] generalized Miranda's theorem [26] and provided applications to boundary value problems at resonance for second order ordinary differential equations. Yang et al. [33] recently extended the work in [31] to nth order ordinary differential equations.

Infante, Pietramala and Tojo [16] provided a thorough study of boundary value problems related to the Neumann boundary conditions, (1.2), using the shift argument. Motivated by [16], Almansour and Eloe [1] applied the shift argument and presented three applications, one using the Krasnosel′skiĭ–Guo fixed point theorem, motivated by Han [14], one using the Schauder fixed point theorem and one using the Leray–Schauder nonlinear alternative.

In this work, we develop the monotone method, coupled with the method of upper and lower solutions, for the shifted boundary value problem. We revisit the applications of Almansour and Eloe [1]. We also present some new applications; in particular, we develop the monotone method, coupled with the method of upper and lower solutions for the more complicated problem, (1.3)–(1.4). The boundary value problem (1.3)–(1.4) is more complicated because nonlinear dependence on velocity is assumed. In each application, explicit upper and lower solutions are exhibited and thus, a numerical algorithm to estimate solutions is implied. However, the primary contribution of this work is that the upper and lower solutions, in each application, are nontrivial solutions of the homogeneous problem at resonance.

For boundary value problems not at resonance, the method of upper and lower solutions provides a stand alone method for studying existence of solutions of boundary value problems [17]. In this case, one employs the upper solution and the lower solution to truncate the problem and then applies the Schauder fixed point theorem to a bounded nonlinearity. We are unsuccessful to employ the method of upper and lower solutions as a stand alone method for boundary value problems at resonance and we shall address this observation in a remark in Section 2.

2 The monotone method coupled with the method of upper and lower solutions

Since we couple the monotone method with the method of upper and lower solutions, the analysis is simple. Hence, as this is not the primary contribution of this work, we present the method briefly. Throughout, $C[0,1]$ will denote the space of continuous real valued functions defined on $[0,1]$, where for $y \in C[0,1]$ the norm is the usual supremum norm,

$$\|y\|_0 = \max_{0\le x\le 1} |y(x)|;$$

$C^1[0,1]$ will denote the space of continuously differentiable real valued functions defined on $[0,1]$, where for $y \in C^1[0,1]$ the norm is the standard

$$\|y\| = \max\{\|y\|_0, \|y'\|_0\}.$$

First, consider the boundary value problem (1.1)–(1.2). Assume $\beta \in \mathbb{R}$ and define $g(t,y) = f(t,y) + \beta^2 y$. To employ the monotone methods, we shall assume that g is increasing in y. Consider an equivalent boundary value problem,

$$y''(t) + \beta^2 y(t) = f(t,y(t)) + \beta^2 y(t) = g(t,y(t)), \quad 0 \le t \le 1, \tag{2.1}$$

with the boundary conditions (1.2). Assume throughout that $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous and when considering the boundary value problem (1.1)–(1.2), we shall assume

$$\beta \in \left(0, \frac{\pi}{2}\right). \tag{2.2}$$

The Green's function, $G_1(\beta; t, s)$, for the boundary value problem (2.1)–(1.2) exists and has the form

$$G_1(\beta;t,s) = \frac{1}{\beta \sin(\beta)} \begin{cases} \cos(\beta t)\cos\beta(s-1), & 0 \le t \le s \le 1, \\ \cos(\beta s)\cos\beta(t-1), & 0 \le s \le t \le 1; \end{cases} \tag{2.3}$$

in particular, y is a solution of the boundary value problem (2.1)–(1.2) if, and only if, $y \in C[0,1]$ and

$$y(t) = \int_0^1 G_1(\beta;t,s) g(s,y(s))\, ds, \quad 0 \le t \le 1.$$

Define $K_1 : C[0,1] \to C[0,1]$ by

$$K_1 y(t) = \int_0^1 G_1(\beta;t,s) g(s,y(s))\, ds, \quad 0 \le t \le 1. \tag{2.4}$$

Then y is a solution of the boundary value problem (1.1)–(1.2) if, and only if, $y \in C[0,1]$ and $y(t) = K_1 y(t)$, $0 \le t \le 1$.

Note that under the assumption (2.2), it follows that

$$G_1(\beta;t,s) > 0, \quad (t,s) \in (0,1) \times (0,1);$$

so under an additional assumption that $g(t,y) = f(t,y) + \beta^2 y$ is increasing in y, it is the case that K_1 is a monotone operator; that is, if $y_1, y_2 \in C[0,1]$ and $y_1(t) \le y_2(t)$, $t \in [0,1]$, then

$$\int_0^1 G_1(\beta;t,s) g(s,y_1(s))\, ds \le \int_0^1 G_1(\beta;t,s) g(s,y_2(s))\, ds, \quad t \in [0,1].$$

In particular,

$$y_1(t) \leq y_2(t), \quad t \in [0,1] \quad \text{implies} \quad K_1 y_1(t) \leq K_1 y_2(t), \quad t \in [0,1]. \tag{2.5}$$

Theorem 2.1. *Assume $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous. Assume β satisfies (2.2) and assume $g(t,y) = f(t,y) + \beta^2 y$ increasing in y. Assume there exist lower and upper solutions, $w_0, v_0 \in C^2[0,1]$, respectively, of (1.1)–(1.2), such that*

$$\begin{gathered} w_0(t) \leq v_0(t), \quad 0 \leq t \leq 1, \\ w_0''(t) + \beta^2 w_0(t) \leq g(t, w_0(t)), \quad 0 \leq t \leq 1, \quad w_0'(0) = 0, \quad w_0'(1) = 0, \\ v_0''(t) + \beta^2 v_0(t) \geq g(t, v_0(t)), \quad 0 \leq t \leq 1, \quad v_0'(0) = 0, \quad v_0'(1) = 0. \end{gathered}$$

Then there exists a solution y of the boundary value problem (1.1)–(1.2) such that

$$w_0(t) \leq y(t) \leq v_0(t), \quad 0 \leq t \leq 1. \tag{2.6}$$

Moreover, construct inductively, $\{w_n(t)\}$ and $\{v_n(t)\}$, $0 \leq t \leq 1$, by

$$w_{n+1}(t) = K_1 w_n(t), v_{n+1}(t) = K_1 v_n(t), \quad t \in [0,1]. \tag{2.7}$$

Then if y is a solution of (1.1)–(1.2) satisfying (2.6), then, for each $n = 0, 1, \ldots,$

$$w_n(t) \leq w_{n+1}(t) \leq y(t) \leq v_{n+1}(t) \leq v_n(t), \quad 0 \leq t \leq 1. \tag{2.8}$$

In addition, $\{w_n(t)\}$ converges in $C[0,1]$ to $w(t)$, $\{v_n(t)\}$ converges in $C[0,1]$ to $v(t)$ where

$$w(t) \leq y(t) \leq v(t), \quad 0 \leq t \leq 1, \tag{2.9}$$

and each of w and v are solutions of the boundary value problem (1.1)–(1.2).

Proof. Define the operator K_1 by (2.4). Since $g(t,y) = f(t,y) + \beta^2 y$ is increasing in y, and $G_1(\beta; t, s) > 0$ on $(0,1) \times (0,1)$, then K_1 is monotone as stated in (2.5).

Define sequences by $\{w_n(t)\}$ and $\{v_n(t)\}$ by (2.7). Since K_1 is monotone, and $w_0(t) \leq v_0(t)$, for $0 \leq t \leq 1$, it follows inductively that

$$w_n(t) \leq v_n(t), \quad 0 \leq t \leq 1, \tag{2.10}$$

for each $n = 0, 1, \ldots$.

Moreover, it is the case that

$$w_n(t) \leq w_{n+1}(t), \quad v_{n+1}(t) \leq v_n(t), \quad 0 \leq t \leq 1.$$

To see this, note that w_0 is the solution of

$$y''(t) + \beta^2 y(t) = w_0''(t) + \beta^2 w_0(t), \quad y'(0) = 0, \quad y'(1) = 0.$$

Thus,

$$w_0(t) = \int_0^1 G_1(\beta; t, s)(w_0''(s) + \beta^2 w_0(s))\, ds.$$

Since w_0 is a lower solution and in particular, satisfies the differential inequality

$$w_0''(t) + \beta^2 w_0(t) \leq g(t, w_0), \quad 0 \leq t \leq 1,$$

it follows that

$$\begin{aligned} w_0(t) &= \int_0^1 G_1(\beta;t,s)(w_0''(s)+\beta^2 w_0(s))\,ds \\ &\le \int_0^1 G_1(\beta;t,s)g(s,w_0(s))\,ds = K_1 w_0(t) = w_1(t). \end{aligned}$$

In particular,

$$w_0(t) \le w_1(t), \quad 0 \le t \le 1,$$

and now inductively,

$$w_n(t) \le w_{n+1}(t), \quad 0 \le t \le 1, \tag{2.11}$$

$n = 0, 1, \ldots$, follows by the monotonicity of K_1. Similarly, it is shown that

$$v_{n+1}(t) \le v_n(t), \quad 0 \le t \le 1, \tag{2.12}$$

$n = 0, 1, \ldots$. And so, it follows from (2.10), (2.11) and (2.12) that (2.8) is valid.

From (2.8), it follows that $\{w_n\}$ is monotone increasing and bounded above by v_0. By Dini's theorem, there exists $w \in C[0,1]$ such that $\{w_n\}$ converges uniformly to w. Similarly, $\{v_n\}$ is a monotone decreasing sequence and bounded below by w_0. Thus there exists $v \in C[0,1]$ such that $\{v_n\}$ converges uniformly to v. Thus,

$$w_n(t) \le w_{n+1}(t) \le w(t) \le v(t) \le v_{n+1}(t) \le v_n(t), \quad 0 \le t \le 1,$$

$n = 0, 1, \ldots$.

From the continuity of g and K_1 (not shown here), and from the uniform convergence of $w_{n+1}(t) = K_1 w_n(t)$ and $v_{n+1}(t) = K_1 v_n(t)$, it follows that $w(t) = K_1 w(t)$ or $v(t) = K_2 v(t)$ and the proof is complete. □

Remark 2.2. It is interesting to note that we are unable to develop a stand alone method of upper and lower solutions for the boundary value problem at resonance (1.1)–(1.2). For the regular boundary value problem (1.1) with Dirichlet boundary conditions,

$$y(0) = 0, \quad y(1) = 0,$$

the corresponding Green's function for this boundary value problem is negative on $(0,1) \times (0,1)$ and in the definition of upper solution, one assumes,

$$v_0''(t) \le f(t, v_0(t)), \quad 0 \le t \le 1.$$

One then shows that the solution y of the truncated problem (obtained as an application of the Schauder fixed point theorem) satisfies

$$y(t) \le v_0(t), \quad 0 < t < 1,$$

by showing that sign of the differential inequality contradicts the second derivative test for local maximum values. For the problem considered in Theorem 2.1, the Green's function, G_1, is positive on $(0,1) \times (0,1)$. This implies that in the definition of upper solution, the differential inequality is reversed; in particular,

$$v_0''(t) \ge f(t, v_0(t)), \quad 0 < t < 1.$$

There is no contradiction to the second derivative test.

Second, consider the boundary value problem (1.3)–(1.4). Replace the assumption (2.2) by the assumption $\beta > 0$ and define $g(t, y_1, y_2) = f(t, y_1, y_2) + \beta y_2$. To employ monotone methods, we shall assume that g is increasing in each of y_1 and y_2. Consider an equivalent boundary value problem,

$$y''(t) + \beta y'(t) = f(t, y(t), y'(t)) + \beta y'(t) = g(t, y(t), y'(t)), \quad 0 \leq t \leq 1, \tag{2.13}$$

with the boundary conditions (1.4). Assume throughout that $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous.

The Green's function for the boundary value problem (2.13)–(1.4) has the form

$$G_2(\beta; t, s) = \begin{cases} \frac{e^{-\beta(1-s)} - e^{-\beta} e^{-\beta(t-s)}}{\beta(1-e^{-\beta})}, & 0 \leq t \leq s \leq 1, \\ \\ \frac{e^{-\beta(1-s)} - e^{-\beta} e^{-\beta(t-s)}}{\beta(1-e^{-\beta})} + \frac{1-e^{-\beta(t-s)}}{\beta}, & 0 \leq s \leq t \leq 1; \end{cases} \tag{2.14}$$

in particular, y is a solution of the boundary value problem (2.13)–(1.4) if, and only if, $y \in C^1[0,1]$ and

$$y(t) = \int_0^1 G_2(\beta; t, s) g(s, y(s), y'(s))\, ds, \quad 0 \leq t \leq 1.$$

Define $K_2 : C^1[0,1] \to C^1[0,1]$ by

$$K_2 y(t) = \int_0^1 G_2(\beta; t, s) g(s, y(s), y'(s))\, ds, \quad 0 \leq t \leq 1. \tag{2.15}$$

Then y is a solution of the boundary value problem (1.3)–(1.4) if, and only if, $y \in C^1[0,1]$ and $y(t) = K_2 y(t)$, $0 \leq t \leq 1$.

Note that

$$\frac{\partial}{\partial t} G_2(\beta; t, s) = \begin{cases} \frac{e^{-\beta} e^{-\beta(t-s)}}{(1-e^{-\beta})}, & 0 \leq t \leq s \leq 1, \\ \\ \frac{e^{-\beta} e^{-\beta(t-s)}}{(1-e^{-\beta})} + e^{-\beta(t-s)}, & 0 \leq s \leq t \leq 1; \end{cases}$$

It is the case that

$$G_2(\beta; t, s) > 0, \quad 0 < t < 1, \quad 0 < s < 1,$$

and

$$\frac{\partial}{\partial t} G_2(\beta; t, s) > 0, \quad 0 < t < 1, \quad 0 < s < 1.$$

Then, under an additional hypothesis that $g(t, y_1, y_2)$ is increasing in each of y_1 and y_2, it follows that $K_2 : C^1[0,1] \to C^1[0,1]$ is a monotone map in the following sense. If $y_1, y_2 \in C^1[0,1]$,

$$y_1^{(i)}(t) \leq y_2^{(i)}(t), \quad 0 \leq t \leq 1, \quad i = 0, 1,$$

then

$$(K_2 y_1)^{(i)}(t) \leq (K_2 y_2)^{(i)}(t), \quad 0 \leq t \leq 1, \quad i = 0, 1.$$

We state the following application of the method of upper and lower solutions, coupled with monotone methods, without proof.

Theorem 2.3. *Assume $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous. Let $\beta > 0$ and assume $g(t, y_1, y_2) = f(t, y_1, y_2) + \beta y_2$ is increasing in each of y_1 and y_2. Assume there exist lower and upper solutions, $w_0, v_0 \in C^2[0,1]$, respectively, of (1.3)–(1.4), such that*

$$\begin{aligned} w_0^{(i)}(t) &\le v_0^{(i)}(t), \quad 0 \le t \le 1, \quad i = 0, 1, \\ w_0''(t) + \beta w_0(t) &\le g(t, w_0(t), w_0'(t)), \quad 0 \le t \le 1, \quad w_0(0) = 0, \quad w_0'(0) = w_0'(1), \\ v_0''(t) + \beta v_0(t) &\ge g(t, v_0(t), v_0'(t)), \quad 0 \le t \le 1, \quad v_0(0) = 0, \quad v_0'(0) = v_0'(1). \end{aligned}$$

Then there exists a solution y of the boundary value problem (1.3)–(1.4) such that

$$w_0^{(i)}(t) \le y^{(i)}(t) \le v_0^{(i)}(t), \quad 0 \le t \le 1, \quad i = 0, 1. \tag{2.16}$$

Moreover, construct inductively, $\{w_n(t)\}$ and $\{v_n(t)\}$, $0 \le t \le 1$, by

$$w_{n+1}(t) = K_2 w_n(t), \quad v_{n+1}(t) = K_2 v_n(t), \quad t \in [0,1].$$

If y is a solution of (1.3)–(1.4) satisfying (2.16), then, for each $n = 0, 1, \ldots,$

$$w_n^{(i)}(t) \le w_{n+1}^{(i)}(t) \le y^{(i)}(t) \le v_{n+1}^{(i)}(t) \le v_n^{(i)}(t), \quad 0 \le t \le 1, \quad i = 0, 1.$$

In addition, $\{w_n(t)\}$ converges in $C^1[0,1]$ to $w(t)$, $\{v_n(t)\}$ converges in $C^1[0,1]$ to $v(t)$ where

$$w^{(i)}(t) \le y^{(i)}(t) \le v^{(i)}(t), \quad 0 \le t \le 1, \quad i = 0, 1,$$

and each of w and v are solutions of the boundary value problem (1.3)–(1.4).

3 Construction of upper and lower solutions

The method of upper and lower solutions is of value in the case when explicit upper and lower solutions can be constructed. In this section we exhibit explicit upper and lower solutions for five applications. Each application can be obtained using standard fixed point theorems (following the shift argument). In each application, the explicit upper and lower solutions are nontrivial solutions of the original linear problem at resonance. The first three applications illustrate the usage of Theorem 2.1. The fourth and fifth applications will illustrate the usage of Theorem 2.3.

Theorem 3.1. *Assume $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous. Assume there exists $\beta \in (0, \frac{\pi}{2})$ such that*

$$g(t, y) = f(t, y) + \beta^2 y$$

is bounded on $[0,1] \times \mathbb{R}$ and g is increasing in y. Then there exists a solution of the boundary value problem (1.1)–(1.2).

Remark 3.2. Remove the hypothesis that g is increasing in y and the Schauder fixed point theorem implies the existence of a solution of the shifted boundary value problem (1.1)–(1.2) in the case that g is bounded.

Proof. Since g is bounded, assume $M > 0$ such that

$$|g(t, y)| \le M, \quad (t, y) \in [0,1] \times \mathbb{R}.$$

Construct constant upper and lower solutions,

$$v_0 = \frac{M}{\beta^2}, \quad w_0 = \frac{-M}{\beta^2}$$

which implies

$$\begin{aligned} w_0(t) &\leq v_0(t), \quad 0 \leq t \leq 1, \\ w_0''(t) + \beta^2 w_0(t) = -M \leq g(t, w_0(t)), &\quad 0 \leq t \leq 1, \quad w_0'(0) = 0, \quad w_0'(1) = 0, \\ v_0''(t) + \beta^2 v_0(t) = M \geq g(t, v_0(t)), &\quad 0 \leq t \leq 1, \quad v_0'(0) = 0, \quad v_0'(1) = 0. \end{aligned}$$

The hypotheses of Theorem 2.1 are satisfied. □

Theorem 3.3. *Assume $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous. Assume there exists $\beta \in (0, \frac{\pi}{2})$ such that*

$$g(t,y) = f(t,y) + \beta^2 y$$

is increasing in y. Assume, moreover, that

$$f(t,y) \geq -\beta^2 y$$

holds. Assume f satisfies the asymptotic properties

(1) $\limsup_{y \to +\infty} \max_{t \in [0,1]} \frac{f(t,y)}{y} = -\beta^2.$

(2) $\liminf_{y \to 0^+} \min_{t \in [0,1]} \frac{f(t,y)}{y} = +\infty.$

Then there is at least one positive solution for the boundary value problem (1.1)–(1.2).

Remark 3.4. Remove the hypothesis that g is increasing in y and the compression contraction fixed point theorem often credited to Krasnosel′skiĭ-Guo [11] implies the existence of a positive solution of the shifted boundary value problem (1.1)–(1.2) in the case that $f(t,y) \geq -\beta^2 y$ and f satisfies (1) and (2).

Proof. Since

$$\limsup_{y \to +\infty} \max_{t \in [0,1]} \frac{f(t,y)}{y} = -\beta^2$$

then

$$\limsup_{y \to +\infty} \max_{t \in [0,1]} \frac{g(t,y)}{y} = 0.$$

Let $\epsilon = \beta^2$. Find $M > 0$ such that if $y \geq M$ then,

$$\frac{g(t,y)}{y} \leq \epsilon = \beta^2,$$

or

$$g(t,y) \leq \beta^2 y.$$

Choose

$$v_0 = M.$$

Then

$$v_0''(t) + \beta^2 v_0 = \beta^2 v_0 \geq g(t, v_0).$$

So, the constant $v_0 = M$ serves as an appropriate upper solution.

Now construct w_0, a positive constant, such that

$$w_0''(t) + \beta^2 w_0 \leqslant g(t, w_0(t)).$$

Since

$$\liminf_{y\to 0^+} \min_{t\in[0,1]} \frac{f(t,y)}{y} = +\infty$$

then

$$\liminf_{y\to 0^+} \min_{t\in[0,1]} \frac{g(t,y)}{y} = +\infty.$$

Let $\epsilon = \beta^2$, and find $M > \delta > 0$ such that if $0 < y < \delta$ then,

$$\frac{g(t,y)}{y} \geq \epsilon,$$

or

$$g(t,y) \geq \epsilon y \geq \beta^2 y.$$

Choose

$$w_0 = \delta.$$

Then w is lower solution such that,

$$w_0''(t) + \beta^2 w_0 = \beta^2 w_0 \leq\ g(t, w_0).$$

Thus,

$$\begin{gathered} w_0(t) \leq v_0(t), \quad 0 \leq t \leq 1, \\ w_0''(t) + \beta^2 w_0(t) = \beta^2 w_0(t) \leq g(t, w_0(t)), \quad 0 \leq t \leq 1, \quad w_0'(0) = 0, \quad w_0'(1) = 0, \\ v_0''(t) + \beta^2 v_0(t) = \beta^2 v_0(t) \geq g(t, v_0(t)), \quad 0 \leq t \leq 1, \quad v_0'(0) = 0, \quad v_0'(1) = 0. \end{gathered}$$

Again, the hypotheses of Theorem 2.1 are satisfied. □

Theorem 3.5. *Assume $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous. Assume there exists $\beta \in (0, \frac{\pi}{2})$ such that*

$$g(t,y) = f(t,y) + \beta^2 y$$

is increasing in y. Assume there exist $\sigma \in C[0,1]$ and a nondecreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$|g(t,y)| \leqslant \sigma(t)\psi(|y|), \quad (t,y) \in [0,1] \times \mathbb{R}.$$

Moreover, assume there exists $M > 0$ such that

$$\frac{\beta^2 M}{\|\sigma\|_0 \psi(M)} > 1.$$

Then the boundary value problem (1.1)–(1.2) has a solution.

Remark 3.6. Remove the hypothesis that g is increasing in y and the Leray–Schauder alternative theorem implies the existence of a solution of the shifted boundary value problem (1.1)–(1.2).

Proof. To exhibit v_0, an upper solution, set

$$v_0 = M.$$

Then

$$v_0''(t) + \beta^2 v_0 = \beta^2 M \geq \|\sigma\|_0 \psi(M) \geq \sigma(t)\psi(|v_0|) \geq g(t, v_0).$$

To exhibit w_0 a lower solution, set

$$w_0 = -M.$$

Then,

$$w_0''(t) + \beta^2 w_0 = -\beta^2 M \leq -\|\sigma\|_0 \psi(M) \leq -\sigma(t)\psi(|w_0|) \leq g(t, w_0).$$

In particular,

$$\begin{aligned} w_0(t) \leq v_0(t), &\quad 0 \leq t \leq 1, \\ w_0''(t) + \beta^2 w_0(t) = \beta^2 w_0(t) \leq g(t, w_0(t)), &\quad 0 \leq t \leq 1, \quad w_0'(0) = 0, \quad w_0'(1) = 0, \\ v_0''(t) + \beta^2 v_0(t) = \beta^2 v_0(t) \geq g(t, v_0(t)), &\quad 0 \leq t \leq 1, \quad v_0'(0) = 0, \quad v_0'(1) = 0, \end{aligned}$$

and the hypotheses of Theorem 2.1 are satisfied. □

Corollary 3.7. *Assume $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous, there exists $\beta \in (0, \frac{\pi}{2})$ such that*

$$g(t, y) = f(t, y) + \beta^2 y$$

is increasing in y, and there exist $\sigma \in C[0,1]$ and some $0 < \alpha < 1$ such that

$$|g(t, y)| \leq \sigma(t)|y|^\alpha, \quad (t, y) \in [0,1] \times \mathbb{R}.$$

Then the boundary value problem (1.1)–(1.2) has a solution.

The final two applications in this section employ Theorem 2.3.

Theorem 3.8. *Assume $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous. Assume there exists $\beta > 0$ such that*

$$g(t, y_1, y_2) = f(t, y_1, y_2) + \beta y_2$$

is bounded on $[0,1] \times \mathbb{R}^2$. Moreover, assume that g is increasing in each of y_1 and y_2. Then there exists a solution of the boundary value problem (1.3)–(1.4).

Remark 3.9. Analogous to Remark 3.2, remove the hypotheses that g is increasing in each of y_1 and y_2 and the Schauder fixed point theorem implies the existence of solutions in the case that g is bounded.

Proof. Assume there exists $M > 0$ such that $|g| \leq M$ on $[0,1] \times \mathbb{R}^2$. Set $v_0(t) = \frac{M}{\beta} t$ and set $w_0(t) = -v_0(t)$. Then

$$\begin{aligned} w_0^{(i)}(t) \leq v_0^{(i)}(t), &\quad 0 \leq t \leq 1, \quad i = 0, 1, \\ w_0''(t) + \beta w_0'(t) = -M \leq g(t, w_0(t), w_0'(t)), &\quad 0 \leq t \leq 1, \quad w_0(0) = 0, \quad w_0'(0) = w_0'(1), \\ v_0''(t) + \beta v_0'(t) = M \geq g(t, v_0(t), v_0'(t)), &\quad 0 \leq t \leq 1, \quad v_0(0) = 0, \quad v_0'(0) = v_0'(1). \end{aligned}$$

The hypotheses of Theorem 2.3 are satisfied. □

Motivated by the application in Theorem 3.5, we shall provide a second application of Theorem 2.3.

Theorem 3.10. *Assume $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous. Assume there exists $\beta > 0$ such that*

$$g(t, y_1, y_2) = f(t, y_1, y_2) + \beta y_2$$

is increasing in each of y_1 and y_2. Assume there exist $\sigma \in C[0,1]$ and a nondecreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that if

$$|g(t, y_1, y_2)| \leqslant \sigma(t)\psi(|y_2|), \quad (t, y_1, y_2) \in [0,1] \times \mathbb{R}^2.$$

Moreover, assume there exists $M > 0$ such that

$$\frac{\beta M}{\|\sigma\|_0 \psi(M)} > 1.$$

Then the boundary value problem (1.3)–(1.4) has a solution.

Proof. For this application, set $v_0(t) = Mt$. To verify that v_0 satisfies the differential inequality for the upper solution, note that

$$v_0''(t) + \beta v_0'(t) = \beta M \geq \|\sigma\|_0 \psi(M) \geq \sigma(t)\psi(|v_0'(t)|) \geq g(t, v_0(t), v_0'(t)).$$

Set $w_0(t) = -v_0(t)$ and the remainder of the verification is clear. □

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