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Global bifurcation for nonlinear Dirac problems

Ziyatkhan S. Aliyev✉ [1,2] and **Humay Sh. Rzayeva**[3]

[1]Department of Mathematical Analysis, Baku State University,
Z. Khalilov Str. 23, Baku, AZ-1148, Azerbaijan
[2]Institute of Mathematics and Mechanics NAS of Azerbaijan,
F. Agaev Str. 9, Baku, AZ-1141, Azerbaijan
[3]Department of Mathematical Analysis, Ganja State University,
S. I. Khetayi Ave 187, Ganja, AZ-2000, Azerbaijan

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Abstract. In this paper we consider the nonlinear eigenvalue problems for the one-dimensional Dirac equation. To exploit oscillatory properties of the components of the eigenvector-functions of linear one-dimensional Dirac system an appropriate family of sets is introduced. We show the existence of two families of continua of solutions contained in these sets and bifurcating from the intervals of the line of trivial solutions.

Keywords: nonlinear one-dimensional Dirac system, bifurcation point, eigenvalue, eigenvector-function, oscillation properties of the eigenvector-functions.

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1 Introduction

We consider the following nonlinear Dirac equation

$$\ell w(x) \equiv B w'(x) - P(x) w(x) = \lambda w(x) + h(x, w(x), \lambda), \qquad 0 < x < \pi, \tag{1.1}$$

with the boundary conditions $U(w) = \begin{pmatrix} U_1(w) \\ U_2(w) \end{pmatrix} = 0$ given by

$$U_1(w) := (\sin\alpha,\, \cos\alpha)\, w(0) = v(0)\cos\alpha + u(0)\sin\alpha = 0, \tag{1.2}$$
$$U_2(w) := (\sin\beta,\, \cos\beta)\, w(\pi) = v(\pi)\cos\beta + u(\pi)\sin\beta = 0, \tag{1.3}$$

where

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad P(x) = \begin{pmatrix} p(x) & 0 \\ 0 & r(x) \end{pmatrix}, \qquad w(x) = \begin{pmatrix} u(x) \\ v(x) \end{pmatrix},$$

$\lambda \in \mathbb{R}$ is a spectral parameter, $p(x)$ and $r(x)$ are real valued, continuous functions on the interval $[0, \pi]$, α and β are real constants: moreover $0 \le \alpha, \beta < \pi$. We assume that the

✉Corresponding author. Email: z_aliyev@mail.ru

nonlinear term h has the form $h = f + g$, where $f = \binom{f_1}{f_2}$ and $g = \binom{g_1}{g_2}$ are continuous functions on $C\left([0,\pi] \times \mathbb{R}^2 \times \mathbb{R}\,;\, \mathbb{R}^2\right)$ and satisfy the conditions:

$$|f_1(x,w,\lambda)| \leq K|w|, \qquad |f_2(x,w,\lambda)| \leq M|w|, \qquad x \in [0,\pi],\ 0 < |w| \leq 1,\ \lambda \in \mathbb{R}, \tag{1.4}$$

where K and M are the positive constants;

$$g(x,w,\lambda) = o(|w|) \quad \text{as } |w| \to 0, \tag{1.5}$$

uniformly with respect to $x \in [0,\pi]$ and $\lambda \in \Lambda$, for every compact interval $\Lambda \subset \mathbb{R}$ (here $|\cdot|$ denotes a norm in $\mathbb{R}^2$).

The equation (1.1) is equivalent to the system of two consistent first-order ordinary differential equations

$$\begin{aligned} v'(x) - p(x)u(x) &= \lambda u(x) + f_1(x,u(x),v(x),\lambda) + g_1(x,u(x),v(x),\lambda), \\ u'(x) + r(x)v(x) &= -\lambda v(x) - f_2(x,u(x),v(x),\lambda) - g_2(x,u(x),v(x),\lambda). \end{aligned} \tag{1.6}$$

In the study of nonlinear eigenvalue problems, an important role is played, when it exists, by the linearization about zero of the problem under consideration, i.e., its Fréchet derivative at the origin (cf. [11]). In this context of linearizability, Rabinowitz [19] gives a nonlinear version of the classical results for linear Sturm–Liouville problems, namely he shows the existence of two families of unbounded continua of nontrivial solutions bifurcating from the points of the line of trivial solutions, corresponding to the eigenvalues of the linear problem, and containing in the classes of functions having usual oscillation properties.

Because of the presence of the term h, problem (1.1)–(1.3) does not in general have a linearization about zero. For this reason, the set of bifurcation points for this problem with respect to the line of trivial solutions need not be discrete (cf. the example of [6, p. 381]). Therefore, to investigate the question of bifurcation for (1.1)–(1.3), one has to consider bifurcation from intervals rather than bifurcation points. We say that bifurcation occurs from an interval if this interval contains at least one bifurcation point [6].

The global results for nonlinearizable Sturm–Liouville problems were obtained by Berestycki [6], Schmitt and Smith [21], Chiappinelli [8], Przybycin [17], Aliyev [1], Rynne [20], Binding, Browne, Watson [7], Dai [9], Aliyev and Mamedova [3]. These papers prove the existence of two families of continua of solutions, $\mathcal{C}_k^+$ and $\mathcal{C}_k^-$ in $\mathbb{R} \times C^1$, corresponding to the usual nodal properties and emanating from bifurcation intervals (in $\mathbb{R} \times \{0\}$, which we identify with $\mathbb{R}$) surrounding the eigenvalues of the linear problem. Similar results for nonlinearizable Sturm–Liouville problems of fourth order were obtained Makhmudov and Aliev [15], Aliyev [2].

In [21] the authors considered the nonlinear problem (1.1)–(1.3) in the case $K + M < 1/2$ and they show that there exists a natural number k_0, such that their bifurcation intervals (which are the same as Berestycki's) do not overlap for every integer k, $|k| \geq k_0$, and corresponding global bifurcation Theorem 2.2 (from [21]) holds for this case. More precisely, for each k, $|k| \geq k_0$, the connected component $\mathcal{D}_k$ of solutions of problem (1.1)–(1.3) emanating from bifurcation interval surrounding the k-th eigenvalue of the linear problem obtained from (1.1)–(1.3) by setting $h \equiv 0$ either is unbounded in $C([0,\pi];\mathbb{R}^2)$, or meet another bifurcation interval.

Thanks to our recent work [4], which is devoted to the study of the oscillations of the linear problem, in this paper we study the structure of bifurcation points and completely investigate the behavior of two families of continua of solutions of problem (1.1)–(1.3) contained in the classes of vector-functions having the oscillation properties of the eigenvector-functions of the corresponding linear problem, and bifurcating from the points and intervals of the line of trivial solutions. Although the problem (1.1)–(1.3) does not have any linearization at the origin, but still can be related to some linear problems. The general idea is to approximate this equation by linearizable ones, for which we apply the global bifurcation results of Rabinowitz [19]. Then, we pass to the limit using a priori bounds which are obtained with the aid of the asymptotic formulas for the eigenvalues of the linear Dirac systems. Note that in our case the bifurcation intervals may overlap, but the use of nodal properties ensures that this does not invalidate the global bifurcation results.

2 Preliminaries

If $h \equiv 0$, then (1.1)–(1.3) is a linear canonical one-dimensional Dirac system [12, Ch. 1, § 10]

$$\begin{aligned} &\ell w(x) = \lambda w(x), \qquad 0 < x < \pi, \\ &U(w) = 0. \end{aligned} \tag{2.1}$$

It is known (see [12, Ch. 1, § 11]) that eigenvalues of the boundary value problem (2.1) are real, algebraically simple and the values range from $-\infty$ to $+\infty$ and can be numerated in increasing order.

We consider a more general problem

$$\begin{aligned} &\tilde{\ell} w(x) \equiv Bw'(x) - \tilde{P}(x)w(x) = \lambda w(x), \qquad 0 < x < \pi, \\ &U(w) = 0, \end{aligned} \tag{2.2}$$

where

$$\tilde{P}(x) = \begin{pmatrix} p(x) & q(x) \\ s(x) & r(x) \end{pmatrix},$$

$q(x)$ and $s(x)$ are real valued, continuous functions on the interval $[0, \pi]$. The problem (2.2) is equivalent to the following eigenvalue problem for the system of two first-order ordinary differential equations

$$\begin{gathered} v'(x) - p(x)u(x) - q(x)v(x) = \lambda u(x), \\ u'(x) + s(x)u(x) + r(x)v(x) = -\lambda v(x), \\ v(0)\cos\alpha + u(0)\sin\alpha = 0, \\ v(\pi)\cos\beta + u(\pi)\sin\beta = 0. \end{gathered} \tag{2.3}$$

Remark 2.1. Without loss of generality we can assume that $s(x) \equiv q(x)$. Indeed, if $s(x) \not\equiv q(x)$, then using the transformations

$$y(x) = u(x)e^{-\frac{1}{2}\int_0^x (q(t)-s(t))dt} \quad \text{and} \quad z(x) = v(x)e^{-\frac{1}{2}\int_0^x (q(t)-s(t))dt},$$

we can rewrite the system (2.3) in the form

$$\begin{gathered} z'(x) - p(x)y(x) - \tilde{q}(x)z(x) = \lambda y(x), \\ y'(x) + \tilde{s}(x)y(x) + r(x)z(x) = -\lambda z(x), \\ z(0)\cos\alpha + y(0)\sin\alpha = 0, \\ z(\pi)\cos\beta + y(\pi)\sin\beta = 0. \end{gathered} \tag{2.4}$$

where

$$\tilde{q}(x) \equiv \tilde{s}(x) \equiv \frac{1}{2}\left(q(x) + s(x)\right).$$

Remark 2.2. If $s(x) \equiv q(x)$, then the substitution

$$w(x) = H(x)\tilde{w}(x)$$

where

$$H(x) = \begin{pmatrix} \cos\omega(x) & -\sin\omega(x) \\ \sin\omega(x) & \cos\omega(x) \end{pmatrix}, \qquad \omega(x) = \frac{1}{2}\arctan\frac{2q(x)}{p(x) - r(x)},$$

transform problem (2.2) into the following problem (which has of the form (2.1)) (see [12, Ch. 1, § 10]),

$$\begin{aligned} B\tilde{w}(x) - \tilde{Q}(x)\tilde{w}(x) &= \lambda\tilde{w}(x), \qquad 0 < x < \pi, \\ \tilde{U}_1(\tilde{w}) := \ (\sin\tilde{\alpha}, \cos\tilde{\alpha})\tilde{w}(0) &= \tilde{v}(0)\cos\tilde{\alpha} + \tilde{u}(0)\sin\tilde{\alpha} = 0, \\ \tilde{U}_2(\tilde{w}) := \ (\sin\tilde{\beta}, \cos\tilde{\beta})\tilde{w}(\pi) &= \tilde{v}(\pi)\cos\tilde{\beta} + \tilde{u}(\pi)\sin\tilde{\beta} = 0, \end{aligned} \tag{2.5}$$

where

$$\begin{gathered} \tilde{Q} = \begin{pmatrix} \omega' - p\cos^2\omega - q\sin 2\omega - r\sin^2\omega & 0 \\ 0 & \omega' - p\sin^2\omega + q\sin 2\omega - r\cos^2\omega \end{pmatrix}, \\ \tilde{w}(x) = \begin{pmatrix} \tilde{u}(x) \\ \tilde{v}(x) \end{pmatrix}, \qquad \tilde{\alpha} = \alpha + \omega(0), \qquad \tilde{\beta} = \beta + \omega(\pi). \end{gathered} \tag{2.6}$$

Thus, the eigenvalues of the boundary value problem (2.2) are real, algebraically simple and the values range from $-\infty$ to $+\infty$ and can be numerated in increasing order.

One can readily show that there exists a unique solution $w(x,\lambda) = \left(\begin{smallmatrix} u(x,\lambda) \\ v(x,\lambda) \end{smallmatrix}\right)$ of Dirac equation

$$\tilde{\ell}w(x) = \lambda w(x), \qquad 0 < x < \pi,$$

satisfying the initial condition

$$u(0,\lambda) = \cos\alpha, \qquad v(0,\lambda) = -\sin\alpha\,; \tag{2.7}$$

moreover, for each fixed $x \in [0, \pi]$ the functions $u(x,\lambda)$ and $v(x,\lambda)$ are entire functions of the argument λ. The proof of this assertion reproduces that of Theorem 1.1 from [12, Ch. 1, § 1] with obvious modifications.

We recall the Prüfer angular variable $\theta(x,\lambda) = \tan^{-1}(v(x,\lambda)/u(x,\lambda))$ (see [5, Ch. 8, § 3]), or more precisely,

$$\theta(x,\lambda) = \arg\{u(x,\lambda) + iv(x,\lambda)\}. \tag{2.8}$$

We recall that u, v have fixed initial values for $x = 0$, and all λ, given by (2.7). We define initially

$$\theta(0,\lambda) = -\alpha, \tag{2.9}$$

in view (2.7). For other x and λ, $\theta(x, \lambda)$ is given by (2.8) except for an arbitrary multiple of 2π, since u and v cannot vanish simultaneously. This multiple of 2π is to be fixed so that $\theta(x, \lambda)$ satisfies (2.9) and is continuous in x and λ. Since the (x,λ)-region, namely, $0 \le x \le \pi$, $-\infty < \lambda < +\infty$, is simply-connected, this defines $\theta(x, \lambda)$ uniquely.

Remark 2.3. From (2.8) it is obvious that the zeros of the functions $u(x,\lambda)$ and $v(x,\lambda)$ are the same as the occasions on which $\theta(x,\,\lambda)$ is an odd or even multiple of $\pi/2$, respectively.

Theorem 2.4 ([4, Theorem 2.1]). *The following properties of the angular function $\theta(x,\,\lambda)$ are true:*

(i) $\theta(x,\,\lambda)$ satisfies the differential equation, with respect to x,

$$\theta' = \lambda + p\cos^2\theta + r\sin^2\theta + q(x)\sin 2\theta\,; \tag{2.10}$$

(ii) if $\lambda + p(x) > 0$, $\lambda + r(x) > 0$ for $x \in [0,\,\pi]$, then as x increases, θ cannot tend to a multiple of $\pi/2$ from above, and as x decreases, θ cannot tend to a multiple of $\pi/2$ from below; if $\lambda + p(x) < 0$, $\lambda + r(x) < 0$ for $x \in [0,\,\pi]$, then as x increases, θ cannot tend to a multiple of $\pi/2$ from below, and as x decreases, θ cannot tend to a multiple of $\pi/2$ from above;

(iii) as λ increases, for fixed x, θ is increasing; in particular, $\theta(\pi,\lambda)$ is a strictly increasing function of λ.

We have the following oscillation theorem.

Theorem 2.5 ([4, Theorem 3.1]). *The eigenvalues $\lambda_k, k \in \mathbb{Z}$, of the problem (2.2) can be numbered in ascending order on the real axis*

$$\cdots < \lambda_{-k} < \cdots < \lambda_{-1} < \lambda_0 < \lambda_1 < \cdots < \lambda_k < \cdots,$$

so that the corresponding angular function $\theta(x,\lambda_k)$ at $x=\pi$ satisfy the condition

$$\theta(\pi,\lambda_k) = -\beta + k\pi. \tag{2.11}$$

The eigenvector-functions $w_k(x) = w(x,\lambda_k) = \begin{pmatrix} u(x,\lambda_k) \\ v(x,\lambda_k)\end{pmatrix} = \begin{pmatrix} u_k(x) \\ v_k(x)\end{pmatrix}$ have, with a suitable interpretation, the following oscillation properties: if $k > 0$ and $k = 0$, $\alpha \geq \beta$ (except the cases $\alpha = \beta = 0$ and $\alpha = \beta = \pi/2$), then

$$\begin{pmatrix} s(u_k) \\ s(v_k)\end{pmatrix} = \begin{pmatrix} k - 1 + \chi(\alpha - \pi/2) + \chi(\pi/2 - \beta) \\ k - 1 + \operatorname{sgn}\alpha\end{pmatrix}, \tag{2.12}$$

and if $k < 0$ and $k = 0$, $\alpha < \beta$, then

$$\begin{pmatrix} s(u_k) \\ s(v_k)\end{pmatrix} = \begin{pmatrix} |k| - 1 + \chi(\pi/2 - \alpha) + \chi(\beta - \pi/2) \\ |k| - 1 + \operatorname{sgn}\beta\end{pmatrix}, \tag{2.13}$$

where $s(g)$ the number of zeros of the function $g \in C([0,\pi]\,;\mathbb{R})$ in the interval $(0,\pi)$ and

$$\chi(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ 1, & \text{if } x > 0.\end{cases}$$

Remark 2.6. It is know [4, formula (3.26)] (see also [12, Ch. 1, formulas (11.17) and (11.18)]) that the eigenvalues μ_k of problem (2.1) satisfy the asymptotic formula

$$\mu_k = k + \frac{\alpha - \beta - (1/2)\int_0^\pi \{p(t) + r(t)\}\,dt}{\pi} + O\left(\frac{1}{k}\right). \tag{2.14}$$

Then by Remarks 2.1, 2.2 and by (2.5), (2.6) it follows from (2.14) that for the eigenvalues λ_k of problem (2.2) the following asymptotic formula

$$\lambda_k = k + \frac{\alpha - \beta - (1/2)\int_0^\pi \{p(t) + r(t)\}\,dt}{\pi} + O\left(\frac{1}{k}\right). \tag{2.15}$$

is true.

We define E to be the Banach space $C\left([0,\pi];\mathbb{R}^2\right)\cap\{w : U(w)=0\}$ with the usual norm $\|w\|=\max_{x\in[0,\pi]}|u(x)|+\max_{x\in[0,\pi]}|v(x)|$. Let S be the subset of E given by

$$S=\{w\in E : |u(x)+|v(x)|\,>0,\ \forall x\in[0,\pi]\}$$

with metric inherited from E.

For each $w=\left(\begin{smallmatrix}u\\ v\end{smallmatrix}\right)\in S$ we define $\theta(w,\cdot)$ to be continuous function on $[0,\pi]$ satisfying

$$\theta(w,x)=\arctan\frac{v(x)}{u(x)},\qquad \theta(w,0)=-\alpha$$

(see, e.g. [2, 6]). It is apparent that $\theta : S\times[0,\pi]\to\mathbb{R}$ is continuous. From (2.11) we have

$$\theta(w_k,0)=-\alpha,\qquad \theta(w_k,\pi)=-\beta+k\pi,\qquad k\in\mathbb{Z}, \tag{2.16}$$

where $w_k(x)$ is an eigenvector-function corresponding to the eigenvalue λ_k of problem (2.2).

Let S_k^+ be set of $w\in S$ which satisfy the conditions:

(i) $\theta(w,\pi)=-\beta+k\pi$;

(ii) the function $u(x)$ is positive in a deleted neighborhood of $x=0$;

(iii) if $k>0$ or $k=0$, $\alpha\geq\beta$ (except the cases $\alpha=\beta=0$ and $\alpha=\beta=\pi/2$), then for fixed w, as x increases from 0 to π, the function θ cannot tend to a multiple of $\pi/2$ from above, and as x decreases, the function θ cannot tend to a multiple of $\pi/2$ from below; if $k<0$ or $k=0,\alpha<\beta$, then for fixed w, as x increases, the function θ cannot tend to a multiple of $\pi/2$ from below, and as x decreases, the function θ cannot tend to a multiple of $\pi/2$ from above.

Let $S_k^-=-S_k^+$ and $S_k=S_k^-\cup S_k^+$. It follows by (2.16), Remark 2.3 and Theorems 2.4, 2.5 that $w_k\in S_k$, $k\in\mathbb{Z}$, i.e. the sets S_k^-, S_k^+ and S_k are nonempty. Moreover, if $w(x)=\left(\begin{smallmatrix}u(x)\\ v(x)\end{smallmatrix}\right)\in S_k$, $k\in\mathbb{Z}$, then the number of zeros of functions $u(x)$ and $v(x)$ are determined by (2.12)–(2.13) and there functions have only nodal zeros in $(0,\pi)$.

From now on ν will denote an element of $\{+,-\}$ that is, either $\nu=+$ or $\nu=-$.

Remark 2.7. From the definition of the sets S_k^ν, it follows directly that, they are disjoint and open in E. Furthermore, if $w\in\partial S_k^\nu$, then there exists a point $\tau\in[0,\pi]$ such that $|w(\tau)|=0$, i.e. $u(\tau)=v(\tau)=0$.

Lemma 2.8. *If $(\lambda,w)\in\mathbb{R}\times E$ is a solution of problem (1.1)–(1.3) and $w\in\partial S_k^\nu$, then $w\equiv0$.*

Proof. Let (λ,w) is a solution of problem (1.1)–(1.3) and $w\in\partial S_k^\nu$. Then, by Remark 2.7, there exists $\zeta\in(0,\pi)$ such that $u(\zeta)=v(\zeta)=0$. Taking into account conditions (1.4) and (1.5) from (1.1) we obtain that in some neighborhood of ζ the following inequality holds:

$$|w'(x)|\ \leq\ c_0\,|w(x)|, \tag{2.17}$$

where c_0 is a positive constant. Integrating both sides of the inequality (2.17) from ζ to x, we obtain

$$\left|\int_\zeta^x|w'(t)|\,dt\right|\leq c_0\left|\int_\zeta^x|w(t)|\,dt\right|.$$

Consequently, by virtue of this inequality and equality $|w(\zeta)| = 0$, we have

$$|w(x)| = \left| \int_\zeta^x w'(t)\,dt \right| \le c_0 \left| \int_\zeta^x |w(t)|\,dt \right|. \tag{2.18}$$

Using Gronwall's inequality, we conclude from (2.18) that $|w(x)| = 0$ in a neighborhood of ζ. This shows that the functions $u(x)$ and $v(x)$ is equal to zero in a neighborhood of ζ. Continuing the specified process, we obtain $w(x) \equiv 0$ on $[0, \pi]$. □

Assume that $\lambda = 0$ is not an eigenvalue of (2.1). Then the problem (1.1)–(1.3) can be converted to the equivalent integral equation

$$w(x) = \lambda \int_0^\pi K(x,t)w(t)\,dt + \int_0^\pi K(x,t)h(t, w(t), \lambda)\,dt, \tag{2.19}$$

where $K(x,t) = K(x,t,0)$ is the appropriate Green's matrix (see [12, Ch. 1, formula (13.8)]).

Define $L : E \to E$ by

$$Lw(x) = \int_0^\pi K(x,t)w(t)\,dt, \tag{2.20}$$

$F : \mathbb{R} \times E \to E$ by

$$F(\lambda, w(x)) = \int_0^\pi K(x,t)f(t, w(t), \lambda)\,dt, \tag{2.21}$$

$G : \mathbb{R} \times E \to E$ by

$$G(\lambda, w(x)) = \int_0^\pi K(x,t)g(t, w(t), \lambda)\,dt. \tag{2.22}$$

The Green matrix $K(x,t)$ is continuous in $[0, \pi; 0, \pi]$ everywhere except on the diagonal $x = t$, where it has a jump $K(x, x+0) - K(x, x-0) = B$. Then L is completely continuous in E. The operators F and G can be represented as a compositions of a operator L and the superposition operators $\mathbf{f}(\lambda, w(x)) = f(x, w(x), \lambda)$ and $\mathbf{g}(\lambda, w(x)) = g(x, w(x), \lambda)$, respectively. Since $f(x, w, \lambda) \in C\left([0,\pi] \times \mathbb{R}^2 \times \mathbb{R}\,;\, \mathbb{R}^2\right)$ and $g(x, w, \lambda) \in C\left([0,\pi] \times \mathbb{R}^2 \times \mathbb{R}\,;\, \mathbb{R}^2\right)$, then the operators $\mathbf{f}$ and $\mathbf{g}$ maps $\mathbb{R} \times E$ to $C\left([0,\pi]; \mathbb{R}^2\right)$. Hence the operators F and G are completely continuous. Furthermore, by virtue of (1.5) we have

$$G(\lambda, w) = o(\|w\|) \quad \text{as } \|w\| \to 0, \tag{2.23}$$

uniformly with respect to $\lambda \in \Lambda$.

On the base (2.19)–(2.22) problem (1.1)–(1.3) can be written in the following equivalent form

$$w = \lambda L w + F(\lambda, w) + G(\lambda, w), \tag{2.24}$$

and therefore, it is enough to investigate the structure of the set of solutions of (1.1)–(1.3) in $\mathbb{R} \times E$.

3 Bifurcation for a class of linearizable problems

We suppose that

$$f \equiv 0 \tag{3.1}$$

(in effect, we suppose that the nonlinearity h itself satisfies (1.5)). Then, by (2.24), problem (1.1)–(1.3) is equivalent to the following problem

$$w = \lambda L w + G(\lambda, w). \tag{3.2}$$

Note that problem (3.2) is of the form (0.1) of [19]. The linearization of this problem at $w = 0$ is the spectral problem

$$w = \lambda L w. \tag{3.3}$$

Obviously, the problem (3.3) is equivalent to the spectral problem (2.1).

We denote by Y the closure in $\mathbb{R} \times E$ of the set of nontrivial solutions of (2.24) (i.e. of (1.1)–(1.3)).

In the following, we will denote by $w_k^+(x) = \binom{u_k^+(x)}{v_k^+(x)}$, $k \in \mathbb{Z}$, the unique eigenvector-function of linear problem (2.1) associated to eigenvalue λ_k such that $\lim_{x \to 0+} \operatorname{sgn} u_k^+(x) = 1$ and $\|w_k^+(x)\| = 1$.

The linear existence theory for the problem (2.1) (or problem (3.3)) can be stated as: for each integer k and each ν, there exists a half line of solutions of problem (3.3) in $\mathbb{R} \times S_k^\nu$ of the form $(\mu_k, \gamma w_k^+)$, $\gamma \in \mathbb{R}^\nu$. This half line joins $(\mu_k, 0)$ to infinity in E. (Here $\mathbb{R}^\nu = \{\varsigma \in \mathbb{R} : 0 \leq \varsigma\nu \leq +\infty\}$).

An analogous result holds for problem (3.2).

Theorem 3.1. *Suppose that (3.1) holds. Then for each integer k and each ν, there exists a continuum of solutions C_k^ν of problem (1.1)–(1.3) (or problem (3.2)) in $\left(\mathbb{R} \times S_k^\nu\right) \cup \{(\mu_k, 0)\}$ which meets $(\mu_k, 0)$ and ∞ in $\mathbb{R} \times E$.*

The proof of this theorem is similar to that of Theorem 2.3 of [19] (see also [10]), using the above arguments from Section 2 and relation (2.23).

4 Global bifurcation of solutions of problem (1.1)–(1.3) in the case $g \equiv 0$

We suppose that

$$g \equiv 0 \tag{4.1}$$

(in effect, we suppose that the nonlinearity h itself satisfies (1.4)). Then the problem (1.1)–(1.3) takes the form

$$\begin{aligned} &\ell w(x) = \lambda w(x) + f(x, w(x), \lambda), \qquad 0 < x < \pi, \\ &U(w) = 0. \end{aligned} \tag{4.2}$$

Together with (4.2), we consider the following approximation problem

$$\begin{aligned} &\ell w(x) = \lambda w(x) + f(x, |w(x)|^\varepsilon w(x), \lambda), \qquad 0 < x < \pi, \\ &U(w) = 0, \end{aligned} \tag{4.3}$$

where $\varepsilon \in (0, 1]$. By (1.6) the problem (4.3) is equivalent to the following system

$$\begin{gathered} v'(x) - p(x)u(x) = \lambda u(x) + f_1(x, |w(x)|^\varepsilon u(x), |w(x)|^\varepsilon v(x), \lambda), \\ u'(x) + r(x)v(x) = -\lambda v(x) - f_2(x, |w(x)|^\varepsilon u(x), |w(x)|^\varepsilon v(x), \lambda), \\ v(0)\cos\alpha + u(0)\sin\alpha = 0, \\ v(\pi)\cos\beta + u(\pi)\sin\beta = 0. \end{gathered} \tag{4.4}$$

Lemma 4.1. *For each integer k and each ν, and for any $0 < \varkappa < 1$ there exists solution $(\lambda_\varkappa, w_\varkappa)$ of problem (4.2) such that $\lambda_\varkappa \in J_k$, $w_\varkappa \in S_k^\nu$ and $\|w_\varkappa\| = \varkappa$, where $J_k = [\mu_k - ((K + M)/2 + c_k),$ $\mu_k + ((K + M)/2 + c_k)]$, and $c_k = O\left(\frac{1}{k}\right)$.*

Proof. By virtue of condition (1.4) we have

$$f(x, |w|^{\varepsilon} w, \lambda) = o\left(| w|\right) \quad \text{as } |w| \to 0, \tag{4.5}$$

uniformly with respect to $x \in [0, \pi]$ and $\lambda \in \Lambda$, for every compact interval $\Lambda \subset \mathbb{R}$. Then, by Theorem 3.1, for each integer k and each ν, there exists an unbounded continuum $C^{\nu}_{k,\varepsilon}$ of solutions of (4.3), such that

$$(\mu_k, 0) \in C^{\nu}_{k,\varepsilon} \subset (\mathbb{R} \times S^{\nu}_k) \cup \{(\mu_k, 0)\}\,.$$

Hence, for every $\varepsilon \in (0, 1]$ there exists a solution $(\lambda_\varepsilon, w_\varepsilon) \in \mathbb{R} \times S^{\nu}_k$ of problem (4.3) such that $\|w_\varepsilon\| \leq 1$. Then we have $|w_\varepsilon(x)| \leq 1$. We define the functions $\varphi_\varepsilon(x)$, $\psi_\varepsilon(x)$, $\phi_\varepsilon(x)$ and $\tau_\varepsilon(x)$ as follows:

$$\begin{aligned}
\varphi_\varepsilon(x) &= \frac{f_1(x,\, |w_\varepsilon(x)|^{\varepsilon} u_\varepsilon(x),\, |w_\varepsilon(x)|^{\varepsilon} v_\varepsilon(x),\, \lambda_\varepsilon)\, u_\varepsilon(x)}{u^2_\varepsilon(x) + v^2_\varepsilon(x)}\,,\\
\psi_\varepsilon(x) &= \frac{f_1(x,\, |w_\varepsilon(x)|^{\varepsilon} u_\varepsilon(x),\, |w_\varepsilon(x)|^{\varepsilon} v_\varepsilon(x),\, \lambda_\varepsilon)\, v_\varepsilon(x)}{u^2_\varepsilon(x) + v^2_\varepsilon(x)}\,,\\
\phi_\varepsilon(x) &= -\frac{f_2(x,\, |w_\varepsilon(x)|^{\varepsilon} u_\varepsilon(x),\, |w_\varepsilon(x)|^{\varepsilon} v_\varepsilon(x),\, \lambda_\varepsilon)\, u_\varepsilon(x)}{u^2_\varepsilon(x) + v^2_\varepsilon(x)}\,,\\
\tau_\varepsilon(x) &= -\frac{f_2(x,\, |w_\varepsilon(x)|^{\varepsilon} u_\varepsilon(x),\, |w_\varepsilon(x)|^{\varepsilon} v_\varepsilon(x),\, \lambda_\varepsilon)\, v_\varepsilon(x)}{u^2_\varepsilon(x) + v^2_\varepsilon(x)}\,.
\end{aligned} \tag{4.6}$$

From (4.4) and (4.6) it is seen that $(\lambda_\varepsilon,\, w_\varepsilon) = (\lambda_\varepsilon,\, \left(\begin{smallmatrix} u_\varepsilon \\ v_\varepsilon \end{smallmatrix}\right))$ is a solution of the linear eigenvalue problem

$$\begin{aligned}
v'(x) - p\,(x)\, u(x) &= \lambda u(x) + \varphi_\varepsilon(x)\, u(x) + \psi_\varepsilon(x)\, v(x),\\
u'(x) + r\,(x)\, v(x) &= -\,\lambda v(x) + \phi_\varepsilon(x)\, u(x) + \tau_\varepsilon(x)\, v(x),\\
v(0) \cos\alpha + u(0) \sin\alpha &= 0,\\
v(\pi) \cos\beta + u(\pi) \sin\beta &= 0.
\end{aligned} \tag{4.7}$$

Taking into account (1.4), from (4.6) we obtain

$$\begin{aligned}
|\varphi_\varepsilon(x)|,\ |\psi_\varepsilon(x)| &\leq M|w(x)|^{\varepsilon} \leq M, \qquad x \in [0, \pi],\\
|\phi_\varepsilon(x)|,\ |\tau_\varepsilon(x)| &\leq K|w(x)|^{\varepsilon} \leq K, \qquad x \in [0, \pi].
\end{aligned} \tag{4.8}$$

Since $w_\varepsilon \in S^{\nu}_k$, then λ_ε is a k-th eigenvalue of problem (4.7). Hence, by (2.15) (see Remark 2.6) we have the following asymptotic formula

$$\lambda_\varepsilon = k + \frac{\alpha - \beta - (1/2) \int_0^{\pi} \{p(t) + \varphi_\varepsilon(t) + r(t) - \tau_\varepsilon(t)\}\, dt}{\pi} + O\left(\frac{1}{k}\right). \tag{4.9}$$

Then, taking into account (4.8) from (4.9) we obtain

$$|\lambda_\varepsilon - \mu_k| \leq (K + M)/2\, +\, c_k, \tag{4.10}$$

where $c_k = O\left(\frac{1}{k}\right)$. Consequently, $\lambda_\varepsilon \in J_k$.

Let $\{\varepsilon_n\}_{n=1}^{\infty}$, $0 < \varepsilon_n < 1$, be a sequence converging to 0. Since $C^{\nu}_{k,\varepsilon_n}$ is unbounded continuum of the set of solutions of (4.3) containing the point $(\mu_k, 0)$, then for every ε_n and for any $\varkappa \in (0,1)$ there exists a solution $(\lambda_{\varepsilon_n}, w_{\varepsilon_n})$ of this problem such that $\lambda_{\varepsilon_n} \in J_k$, $w_{\varepsilon_n} \in S^{\nu}_k$ and $\|w_{\varepsilon_n}\| = \varkappa$. We may assume that $\lambda_{\varepsilon_n} \to \lambda_\varkappa \in J_k$. Since w_{ε_n} is bounded in $C\left([0,\pi];\mathbb{R}^2\right)$ and f is continuous in $C\left([0,\pi]\times\mathbb{R}^2\times\mathbb{R};\mathbb{R}^2\right)$, then from (4.3) (or (4.4)) implies that w_{ε_n} is bounded in $C^1\left([0,\pi];\mathbb{R}^2\right)$. Therefore, by the Arzelà–Ascoli theorem, we may assume that $w_{\varepsilon_n} \to w_\varkappa$ in $C\left([0,\pi];\mathbb{R}^2\right)$, and $\|w_\varkappa\| = \varkappa$. Passing to the limit as $n \to \infty$ in (4.3) we obtain that $(\lambda_\varkappa, w_\varkappa)$ is a solution of the nonlinear problem (4.2). For all n, $w_{\varepsilon_n} \in S^{\nu}_k$, hence $w_\varkappa$ lies in the closure of S^{ν}_k. Since $\|w_\varkappa\| = \varkappa$, then by virtue of Lemma 2.8 we have $w_\varkappa \in S^{\nu}_k$. □

We say that the point $(\lambda, 0)$ is a bifurcation point of problem (1.1)–(1.3) with respect to the set $\mathbb{R}\times S^{\nu}_k$, $k \in \mathbb{Z}$, if in every small neighborhood of this point there is solution to this problem which contained in $\mathbb{R}\times S^{\nu}_k$ (see [3]).

Corollary 4.2. *The set of bifurcation points of problem* (4.2) *is nonempty, and if* $(\lambda, 0)$ *is a bifurcation point of* (4.2) *with respect to the set* $\mathbb{R}\times S^{\nu}_k$*, then* $\lambda \in J_k$.

For each $k \in \mathbb{Z}$ and each ν, we define the set $\widetilde{D}^{\nu}_k \subset Y$ to be the union of all the components $D^{\nu}_{k,\lambda}$ of Y which bifurcating from the bifurcation points $(\lambda, 0)$ of (4.2) with respect to the set $\mathbb{R}\times S^{\nu}_k$. By Lemma 4.1 and Corollary 4.2 the set $\widetilde{D}^{\nu}_k$ is nonempty. Let $D^{\nu}_k = \widetilde{D}^{\nu}_k \cup (J_k \times \{0\})$. Note that the set D^{ν}_k is connected in $\mathbb{R}\times E$, but $\widetilde{D}^{\nu}_k$ may not be connected in $\mathbb{R}\times E$.

Theorem 4.3. *For each* $k \in \mathbb{Z}$ *and each* ν*, the connected component* D^{ν}_k *of* Y *lies in* $(\mathbb{R}\times S^{\nu}_k) \cup (J_k \times \{0\})$ *and is unbounded in* $\mathbb{R}\times E$.

Proof. By Lemma 4.1, Corollary 4.2 and an argument similar to that of [13, Theorem 2.1], we can obtain the desired conclusion. □

Assume that the function $f(x, w, \lambda)$ satisfies the condition (1.4) for all $x \in [0,\pi]$ and $(w, \lambda) \in \mathbb{R}^2 \times \mathbb{R}$. Thus we have the following result.

Lemma 4.4. *Let* $(\lambda, w) \in \mathbb{R}\times E$ *be a solution of problem* (4.2). *Then* $w \in \bigcup_{k=-\infty}^{\infty} S_k$*, and if* $w \in S_k$*, then* $\lambda \in J_k$.

Proof. Suppose that $(\lambda, w(x)) \in \mathbb{R}\times E$ is a solution of problem (4.2). Let

$$\begin{aligned} \varphi(x) &= \frac{f_1(x, u(x), v(x), \lambda)\, u(x)}{u^2(x)+v^2(x)}, \qquad & \psi(x) &= \frac{f_1(x, u(x), v(x), \lambda)\, v(x)}{u^2(x)+v^2(x)}, \\ \phi(x) &= -\frac{f_2(x, u(x), v(x), \lambda) u(x)}{u^2(x)+v^2(x)}, \qquad & \tau(x) &= -\frac{f_2(x, u(x), v(x), \lambda) v(x)}{u^2(x)+v^2(x)}. \end{aligned} \tag{4.11}$$

Then (λ, w) is a solution of the following eigenvalue problem

$$\begin{gathered} v'(x) - p(x)\, u(x) = \lambda u(x) + \varphi(x)\, u(x) + \psi(x)\, v(x), \\ u'(x) + r(x)\, v(x) = -\lambda v(x) + \phi(x)\, u(x) + \tau(x)\, v(x), \\ v(0)\cos\alpha + u(0)\sin\alpha = 0, \\ v(\pi)\cos\beta + u(\pi)\sin\beta = 0. \end{gathered} \tag{4.12}$$

Hence, by Theorem 2.5, we have $w(x) \in \bigcup_{k=-\infty}^{\infty} S_k$.

Let $w(x) \in S_k$ for some $k \in \mathbb{Z}$. According to Theorem 2.5 λ is a k-th eigenvalue of problem (4.12). Taking into account (1.4), from (4.11) we obtain

$$\begin{aligned} |\varphi(x)|,\ |\psi(x)| &\le M, \qquad x \in [0,\pi], \\ |\phi(x)|,\ |\tau(x)| &\le K, \qquad x \in [0,\pi]. \end{aligned} \tag{4.13}$$

Then, by (4.13) it follows from (2.15) that $\lambda \in J_k$. □

By virtue of Lemma 4.4 from Theorem 4.3 we obtain the following result.

Theorem 4.5. *Let the function $f(x,w,\lambda)$ satisfies the condition (1.4) for all $(x,w,\lambda) \in [0,\pi] \times \mathbb{R}^2 \times \mathbb{R}$. Then for each $k \in \mathbb{Z}$ and each ν, the connected component D_k^ν of Y lies in $J_k \times S_k^\nu$ and is unbounded in $\mathbb{R} \times E$.*

5 Global bifurcation of solutions of problem (1.1)–(1.3) in the general case

Lemma 5.1. *For each $k \in \mathbb{Z}$ and each ν, and for sufficiently small $\tau > 0$ there exists a solution (λ_τ, w_τ) of problem (1.1)–(1.3) such that $w_\tau \in S_k^\nu$ and $\|w_\tau\| = \tau$.*

Proof. Alongside with the problem (1.1)–(1.3) we shall consider the following approximate problem

$$\begin{aligned} &\ell w(x) = \lambda w(x) + f(x, |w(x)|^\varepsilon w(x), \lambda) + g(x, w(x), \lambda), \qquad 0 < x < \pi, \\ &U(w) = 0, \end{aligned} \tag{5.1}$$

where $\varepsilon \in (0,1]$.

By (1.4) the function $f(x,|w|^\varepsilon w,\lambda)$ satisfies the condition (4.5). Then, by Theorem 3.1, for each integer k and each ν there exists an unbounded continuum $A_{k,\varepsilon}^\nu$ of solutions of (5.1) such that

$$(\mu_k, 0) \in A_{k,\varepsilon}^\nu \subset (\mathbb{R} \times S_k^\nu) \cup \{(\mu_k, 0)\}.$$

Hence, it follows that for any $\varepsilon \in (0,1]$ there exists a solution $(\lambda_{\tau,\varepsilon}, w_{\tau,\varepsilon})$ of problem (5.1) such that $w_{\tau,\varepsilon} \in S_k^\nu$ and $\|w_{\tau,\varepsilon}\| = \tau$. It is obvious that $(\lambda_{\tau,\varepsilon}, w_{\tau,\varepsilon})$ is a solution of the nonlinear problem

$$\begin{aligned} &\ell w(x) = \lambda w(x) + P_\varepsilon(x) w(x) + g(x, w(x), \lambda), \qquad 0 < x < \pi, \\ &U(w) = 0. \end{aligned} \tag{5.2}$$

where

$$P_\varepsilon(x) = \begin{pmatrix} \varphi_\varepsilon(x) & \psi_\varepsilon(x) \\ \phi_\varepsilon(x) & \tau_\varepsilon(x) \end{pmatrix}$$

and the functions $\varphi_\varepsilon(x)$, $\psi_\varepsilon(x)$, $\phi_\varepsilon(x)$ and $\tau_\varepsilon(x)$ are determined of right hand sides of (4.6) with $(\lambda_{\tau,\varepsilon}, w_{\tau,\varepsilon})$ instead of $(\lambda_\varepsilon, w_\varepsilon)$.

Taking into account condition (1.4) we have

$$\begin{aligned} |\varphi_\varepsilon(x)|,\ |\psi_\varepsilon(x)| &\le M, \qquad x \in [0,\pi], \\ |\phi_\varepsilon(x)|,\ |\tau_\varepsilon(x)| &\le K, \qquad x \in [0,\pi]. \end{aligned}$$

Therefore, by virtue of (2.15), the k-th eigenvalue $\lambda_{k,\varepsilon}$ of the linear problem

$$\begin{aligned} &\ell w(x) = \lambda w(x) + P_\varepsilon(x) w(x), \qquad 0 < x < \pi, \\ &U(w) = 0, \end{aligned} \tag{5.3}$$

is contained in J_k. By [11, Ch. 4, §2, Theorem 2.1] and Theorems 2.4, 2.5 the point $(\lambda_{k,\varepsilon}, 0)$ is a only bifurcation point of problem (5.2) with respect to the set $\mathbb{R} \times S_k^\nu$, and this point corresponds to a continuous branch of nontrivial solutions. Consequently, each sufficiently small $\tau > 0$ responds arbitrarily small $\rho_{\tau,\varepsilon}$ such that

$$\lambda_{\tau,\varepsilon} \in (\lambda_{k,\varepsilon} - \rho_{\tau,\varepsilon}, \lambda_{k,\varepsilon} + \rho_{\tau,\varepsilon}) \subset [\mu_k - \tilde{c}_k - \rho_0, \mu_k + \tilde{c}_k + \rho_0], \tag{5.4}$$

where $\tilde{c}_k = (K + M)/2 + c_k$, $\rho_0 = \sup_{\varepsilon,\tau} \rho_{\tau,\varepsilon} > 0$.

Since the set $\{w_{\tau,\varepsilon} \in E : 0 < \varepsilon \le 1\}$ is bounded in $C([0,\pi]; \mathbb{R}^2)$, the functions f and g are continuous in $[0,\pi] \times \mathbb{R}^2 \times \mathbb{R}$ and $\{\lambda_{\tau,\,\varepsilon} \in R : 0 < \varepsilon \le 1\}$ is bounded in $\mathbb{R}$ (see (5.4)), then by (5.2) the set $\{w_{\tau,\varepsilon} \in E : 0 < \varepsilon \le 1\}$ is also bounded in $C^1([0,\pi]; \mathbb{R}^2)$. Hence, by the Arzelà–Ascoli theorem this set is compact in E.

Let $\{\varepsilon_n\}_{n=1}^\infty$, $0 < \varepsilon_n < 1$, be a sequence converging to 0, and such that $(\lambda_{\tau,\varepsilon_n}, w_{\tau,\varepsilon_n}) \to (\lambda_\tau, w_\tau)$ in $\mathbb{R} \times E$. Passing to the limit as $n \to \infty$ in (5.2) we obtain that (λ_τ, w_τ) is a solution of the nonlinear problem (1.1)–(1.3). Since $\|w_\tau\| = \tau$ then by Lemma 2.8 we have $w_\tau \in S_k^\nu$. □

Corollary 5.2. *The set of bifurcation points of problem* (1.1)–(1.3) *with respect to the set* $\mathbb{R} \times S_k^\nu$ *is nonempty.*

Lemma 5.3. *Let* ε_n, $0 \le \varepsilon_n \le 1$, $n = 1, 2, \ldots$, *be a sequence converging to* 0*. If* $(\lambda_{\varepsilon_n}, w_{\varepsilon_n})$ *is a solution of problem* (5.1) *corresponding to* $\varepsilon = \varepsilon_n$*, and sequence* $\{(\lambda_{\varepsilon_n},\ w_{\varepsilon_n})\}_{n=1}^\infty$ *converges to* $(\xi,\ 0)$ *in* $\mathbb{R} \times E$*, then* $\xi \in J_k$*.*

Proof. Assume the contrary, i.e. let $\xi \notin J_k$. We denote $\sigma = \operatorname{dist}\{\xi, J_k\}$. Since $\lambda_{\varepsilon_n} \to \xi$, then there exists $n_\sigma \in \mathbb{N}$ such that for all $n > n_\sigma$ we have the inequality $|\lambda_{\varepsilon_n} - \xi| < \sigma/2$. Hence, $\operatorname{dist}\{\lambda_{\varepsilon_n}, J_k\} > \sigma/2$ at $n > n_\sigma$.

Note that $(\lambda_{\varepsilon_n}, w_{\varepsilon_n})$ is a solution of nonlinear problem (5.2) for $\varepsilon = \varepsilon_n$. Since $(\lambda_{k,\varepsilon_n}, 0)$ is a only bifurcation point of problem (5.2) with respect to the set $\mathbb{R} \times S_k^\nu$, then every sufficiently large $n > n_\sigma$ corresponds to a arbitrarily small $\rho_n > 0$ that $\rho_n < \sigma/2$ and $\lambda_{\varepsilon_n} \in (\lambda_{k,\varepsilon_n} - \rho_n, \lambda_{k,\varepsilon_n} + \rho_n)$, where $\lambda_{k,\varepsilon_n}$ is the k-th eigenvalue of the linear problem (5.3) for $\varepsilon = \varepsilon_n$. Consequently, $\lambda_{\varepsilon_n} \in (\lambda_{k,\varepsilon_n} - \sigma/2, \lambda_{k,\varepsilon_n} + \sigma/2)$. From the proof of Lemma 4.1 we have $\lambda_{k,\varepsilon_n} \in J_k$, whence it follows inequality $\operatorname{dist}\{\lambda_{\varepsilon_n}, J_k\} < \sigma/2$ which contradicts $\operatorname{dist}\{\lambda_{\varepsilon_n}, J_k\} > \sigma/2$. □

Corollary 5.4. *If* $(\lambda, 0)$ *is a bifurcation point of problem* (1.1)–(1.3) *with respect to the set* S_k^ν*, then* $\lambda \in J_k$*.*

For each $k \in \mathbb{Z}$ and each ν, we define the set $\widetilde{T}_k^\nu \subset Y$ to be the union of all the components $T_{k,\lambda}^\nu$ of Y which bifurcating from the bifurcation points $(\lambda, 0)$ of (1.1)–(1.3) with respect to the set $\mathbb{R} \times S_k^\nu$. Let $T_k^\nu = \widetilde{T}_k^\nu \cup (J_k \times \{0\})$.

Theorem 5.5. *For each* $k \in \mathbb{Z}$ *and each* ν*, the connected component* T_k^ν *of* Y *lies in* $(\mathbb{R} \times S_k^\nu) \cup (J_k \times \{0\})$ *and is unbounded in* $\mathbb{R} \times E$*.*

The proof of Theorem 5.5 is similar to that of [13, Theorem 2.1] using Lemmas 5.1, 5.3 and Corollaries 5.2, 5.4.

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