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Dynamics of a time-periodic and delayed reaction–diffusion model with a quiescent stage

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Abstract. In this paper, we study a time-periodic and delayed reaction–diffusion system with quiescent stage in both unbounded and bounded habitat domains. In unbounded habitat domain $\mathbb{R}$, we first prove the existence of the asymptotic spreading speed and then show that it coincides with the minimal wave speed for monotone periodic traveling waves. In a bounded habitat domain $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$, we obtain the threshold result on the global attractivity of either the zero solution or the unique positive time-periodic solution of the system.

Keywords: quiescent stage, time-periodic, delay, spreading speed, global attractivity.

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1 Introduction

In population ecology, dormancy or quiescence plays an important role in the growing process of some species such as reptiles and insects, which is an attractive biological phenomenon. A typical example is the growth of invertebrates living in small ponds in semi-arid region. Since the varying of growing environment subject to the disappear and reappear of rainfall, the individuals can be grouped into two parts: mobile sub-populations and non-mobile sub-populations. It means that the individuals switch between mobile and non-mobile states, while only the mobile sub-populations can reproduce.

It is well known that mathematical models have become basic tools in studying the evolution of population. Recently, considerable attentions have been paid to investigate population models with a quiescent stage or dormancy from the mathematical view (see e.g., [1, 4, 5, 20, 22]). Precisely speaking, a reaction-diffusion equation coupled with a quiescent stage can be used to describe aforementioned biological phenomena. Hadeler and Lewis [5] proposed the following basic model:

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) = d\Delta u(t,x) + f(u(t,x)) - \gamma u(t,x) + \beta v(t,x), \\ \frac{\partial}{\partial t}v(t,x) = \gamma u(t,x) - \beta v(t,x), \end{cases} \tag{1.1}$$

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where $u(t,x)$ and $v(t,x)$ are the densities of mobile and stationary sub-populations at time t and location x, respectively; f is the recruitment function and only depends on the density of mobile sub-populations; d is the diffusion rate of the mobile, γ and β are the switch rates between two states. Based on the mathematical analysis of (1.1), the authors provided some appropriate biological interpretation for their results. Zhang and Zhao [22] further investigated the asymptotic behavior of system (1.1) in both unbounded and bounded spatial domains. In the case where the habitat domain is $\mathbb{R}$, they established the existence of the asymptotic spreading speed which coincides with the minimal wave speed for monotone traveling waves. In the case where the habitat domain is bounded, they obtained a threshold result on the global attractivity of either zero or positive steady state. In addition, Zhang and Li [23] studied the monotonicity and uniqueness of traveling waves of (1.1).

As mentioned in [4], to study the effect of a quiescent phase, it is meaningful to incorporate the time delays, which can be caused by many factors such as hatching period or maturation period. Motivated by this, Wu and Zhao [20] studied the following time-delayed reaction-diffusion model with quiescent stage:

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) = d\Delta u(t,x) + f(u(t,x), u(t-\tau,x)) - \gamma u(t,x) + \beta v(t,x), \\ \frac{\partial}{\partial t}v(t,x) = \gamma u(t,x) - \beta v(t,x), \end{cases} \tag{1.2}$$

where $f(u(t,x), u(t-\tau,x))$ is the reproduction function, τ is a nonnegative constant. They established the existence of the minimal wave speed and further studied the asymptotic behavior, monotonicity and uniqueness of the traveling wave fronts. We mentioned that the analysis for (1.2) on bounded spatial domain remains open.

On the other hand, the effect from varying environment (e.g., the seasonal fluctuations and periodic availability of nutrient supplies) should not be ignored in reality. Therefore, it is more reasonable to assume that the reproduction rate and the two switching rates are time heterogeneous, especially, time periodic. More recently, Wang [19] considered a time-periodic version of (1.1):

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) = d\Delta u(t,x) + f(t, u(t,x)) - \gamma(t)u(t,x) + \beta(t)v(t,x), \\ \frac{\partial}{\partial t}v(t,x) = \gamma(t)u(t,x) - \beta(t)v(t,x), \end{cases} \tag{1.3}$$

where $f(t,\cdot) = f(t+\omega,\cdot)$, $\gamma(t) = \gamma(t+\omega)$, $\beta(t) = \beta(t+\omega)$, $\forall t > 0$, ω is a positive constant. For (1.3), the author [19] proved the existence of the spreading speed and showed that it coincides with the minimal wave speed of monotone periodic traveling waves. In the case where the spatial domain is bounded, a threshold result on the global attractivity of either zero or positive periodic solution was established.

Taking time delay and seasonality into consideration, in this paper, we consider the following time-periodic and delayed reaction-diffusion system:

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) = d(t)\Delta u(t,x) + f(t, u(t,x), u(t-\tau,x)) - \gamma(t)u(t,x) + \beta(t)v(t,x), \\ \frac{\partial}{\partial t}v(t,x) = \gamma(t)u(t,x) - \beta(t)v(t,x). \end{cases} \tag{1.4}$$

where $d(t) = d(t+\omega) \geq d > 0$, $f(t,u,w) = f(t+\omega,u,w)$, $\gamma(t) = \gamma(t+\omega) > 0$ and $\beta(t) = \beta(t+\omega) > 0$, $\forall t > 0$. Let $\partial_1 f(t,u,w) := \frac{\partial f(t,u,w)}{\partial u}$, $\partial_2 f(t,u,w) := \frac{\partial f(t,u,w)}{\partial w}$ for any $(t,u,w) \in \mathbb{R}^3_+$. Throughout this paper, we assume that the function $f \in C^1(\mathbb{R}^3_+, \mathbb{R}_+)$ satisfies:

(H1) $f(t,0,0) \equiv 0$ for all $t \geq 0$, $\partial_2 f(t,u,w) > 0, \forall (t,u,w) \in \mathbb{R}^3_+$, $|\partial_1 f(t,u,w)|$ is bounded in $\mathbb{R}^3_+$. Let $l := \sup\{|\partial_1 f(t,u,w)| : (t,u,w) \in \mathbb{R}^3\}$.

(H2) There exists a constant $L > 0$ such that

$$f(t,u,u) \leq \left(\min_{t\in[0,\omega]} \gamma(t)\right) u - \left(\max_{t\in[0,\omega]} \beta(t)\right) \left(\max_{t\in[0,\omega]} \frac{\gamma(t)}{\beta(t)}\right) u, \qquad \forall t > 0,\ u \geq L.$$

(H3) For each $t \geq 0$, $f(t,\cdot,\cdot)$ is strictly sub-homogeneous on $\mathbb{R}^2_+$ in the sense that $f(t,\theta u,\theta w) > \theta f(t,u,w)$ whenever $\theta \in (0,1)$, $\forall u,w > 0$.

The purpose of this paper is to investigate the asymptotic behavior of system (1.4). We first apply the results on monotone semiflow in [12–14] to obtain the spreading speed in a weak sense. Due to the zero diffusion arising from the quiescent stage, the system (1.4) has a weak regularity, which leads to a difficulty in obtaining the existence of traveling waves. To overcome this problem, we adopt the ideas involving the minimal wave speeds for monotone and "point-α-contraction" systems with monostable structure developed in [3].

The organization of this paper is as follows. Section 2 is devoted to obtain the existence of spreading speed and to show that the spreading speed exactly coincides with the minimal wave speed for monotone periodic traveling waves. In Section 3, we study the global dynamics of system (1.4) in a bounded domain $\Omega \subset \mathbb{R}^N$. In Section 4, we give the appendix on spreading speeds and periodic traveling waves for monotonic systems, which is used in Sections 2 and 3. The frameworks, concepts and results presented by this section are adapted from [3, 12, 13].

2 Dynamics in unbounded domain

In this section, we consider the system (1.3) on an unbounded spatial domain $\Omega = \mathbb{R}$:

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) = d(t)\frac{\partial^2 u(t,x)}{\partial x^2} + f(t,u(t,x),u(t-\tau,x)) - \gamma(t)u(t,x) + \beta(t)v(t,x), \\ \frac{\partial}{\partial t}v(t,x) = \gamma(t)u(t,x) - \beta(t)v(t,x), \quad t > 0,\ x \in \mathbb{R}, \\ u(s,x) = \phi_1(s,x), v(0,x) = \phi_2(x), \quad s \in [-\tau,0],\ x \in \mathbb{R}. \end{cases} \tag{2.1}$$

In the following, we are mainly concerned with the spreading speed and traveling wave solutions for (2.1). In the first subsection, we present some fundamental results, including the global dynamics of the spatially homogeneous system associated with (2.1), the existence of solutions to (2.1), a comparison principle and the properties of the periodic semiflow. In the second subsection, by appealing the abstract results established in [12, 13], we study the spreading speeds for (2.1). The third subsection is devoted to the existence of periodic traveling wave solutions for (2.1) by applying the results established in [3]. It needs to be noticed that due to the lack of compactness of the semiflow of (2.1), the abstract results on traveling waves in [12, 13] cannot be directly applied.

2.1 Preliminaries

Define $Y = C([-\tau,0],\mathbb{R})$ equipping with the usual supreme norm $\|\cdot\|_Y$. Then (Y,Y_+) is an ordered Banach space, where $Y_+ := C([-\tau,0],\mathbb{R}_+)$. For any $\varphi,\psi \in Y$, we write $\varphi \geq \psi$ if $\varphi - \psi \in Y_+$, $\varphi > \psi$ if $\varphi \geq \psi$ but $\varphi \neq \psi$, and $\varphi \gg \psi$ if $\varphi - \psi \in \mathrm{Int}(Y_+)$.

Let X be the set of all bounded and continuous functions from $\mathbb{R}$ to $\mathbb{R}$ and $X_+ = \{\varphi \in X : \varphi(x) \geq 0\ \forall x \in \mathbb{R}\}$. For any $\varphi,\psi \in X$, we write $\varphi \geq \psi(\varphi \gg \psi)$ if $\varphi(x) \geq \psi(x)(\varphi(x) > \psi(x))$ for all $x \in \mathbb{R}$. $\varphi > \psi$ if $\varphi \geq \psi$ but $\varphi \neq \psi$. Clearly, X_+ is a positive cone of X. We equip X with the compact open topology (i.e., $\varphi^m \to \varphi$ in X means that the sequence of $\varphi^m(x)$ converges to $\varphi(x)$ as $m \to \infty$ uniformly for x in any compact set on $\mathbb{R}$) induced by the following norm

$$\|\varphi\|_X = \sum_{k=1}^{\infty} \frac{\max_{|x|\le k} |\varphi(x)|}{2^k}, \qquad \forall \varphi \in X,$$

where $|\cdot|$ denotes the usual norm in $\mathbb{R}$.

Let $\mathcal{C}' := C([-\tau,0],X)$ and $\mathcal{C}'_+ = \{\varphi \in \mathcal{C}' : \varphi(s) \in X_+, s \in [-\tau,0]\}$. Then $(\mathcal{C}',\mathcal{C}'_+)$ is an ordered Banach space. For convenience, we identify an element $\varphi \in \mathcal{C}'$ as a function from $[-\tau,0] \times \mathbb{R}$ into $\mathbb{R}$ defined by $\varphi(s,x) = \varphi(s)(x)$ for any $s \in [-\tau,0]$ and $x \in \mathbb{R}$. For any continuous function $w(\cdot) : [-\tau,b) \to X, b > 0$, we define $w_t \in \mathcal{C}'$ by $w_t(s) = w(t+s)$ for all $t \in [0,b), s \in [-\tau,0]$. Clearly, $t \mapsto w_t$ is a continuous function from $[0,b)$ to $\mathcal{C}'$. Furthermore, we let $\mathcal{C} = \mathcal{C}' \times X$ and $\mathcal{C}_+ = \mathcal{C}'_+ \times X_+$, then $(\mathcal{C},\mathcal{C}_+)$ is an ordered Banach space. We equip $\mathcal{C}$ with the compact open topology and define the norm on $\mathcal{C}$

$$\|\phi\|_C = \sum_{k=1}^{\infty} \frac{\max_{s\in[-\tau,0],|x|\le k} |(\phi_1(s,x),\phi_2(x))|}{2^k}, \qquad \forall \phi = (\phi_1,\phi_2) \in \mathcal{C},$$

where $|\cdot|$ denote the usual norm in $\mathbb{R}^2$.

Let $\bar{\mathcal{C}} := Y \times \mathbb{R}$, then $(\bar{\mathcal{C}},\bar{\mathcal{C}}_+)$ is a ordered Banach space. For each $\mathbf{r} = (r_1,r_2) \in \bar{\mathcal{C}}$ with $r \gg 0$, define $\mathcal{C}_{\mathbf{r}} = \{\phi \in \mathcal{C} : \mathbf{r} \ge \phi \ge 0\}$ and $\bar{\mathcal{C}}_{\mathbf{r}} = \{\phi \in \bar{\mathcal{C}} : \mathbf{r} \ge \phi \ge 0\}$. For any positive vector $\mathbf{N} \in \mathbb{R}^2_+$, we let $\hat{\mathbf{N}}$ denote the constant function with vector value $\mathbf{N}$ in $\bar{\mathcal{C}},\mathcal{C}$.

Firstly, we consider the following spatial-independent system associated with (2.1),

$$\begin{cases} \frac{d\widehat{u}(t)}{dt} = f(t,\widehat{u}(t),\widehat{u}(t-\tau)) - \gamma(t)\widehat{u} + \beta(t)\widehat{v}, \\ \frac{d\widehat{v}(t)}{dt} = \gamma(t)\widehat{u} - \beta(t)\widehat{v}, \\ \widehat{u}(s) = \phi_1(s),\ \widehat{v}(0) = \phi_2, \quad s \in [-\tau,0],\ \phi = (\phi_1,\phi_2) \in \bar{\mathcal{C}}_+. \end{cases} \tag{2.2}$$

Note that $(0,0)$ is a solution of (2.2). Linearizing (2.2) at the zero solution, we get

$$\begin{cases} \frac{d\bar{u}(t)}{dt} = \partial_1 f(t,0,0)\bar{u}(t) + \partial_2 f(t,0,0)\bar{u}(t-\tau) - \gamma(t)\bar{u}(t) + \beta(t)\bar{v}(t), \\ \frac{d\bar{v}(t)}{dt} = \gamma(t)\bar{u}(t) - \beta(t)\bar{v}(t), \\ \bar{u}(s) = \phi_1(s),\ \bar{v}(0) = \phi_2, \quad s \in [-\tau,0],\ \phi = (\phi_1,\phi_2) \in \bar{\mathcal{C}}_+. \end{cases} \tag{2.3}$$

Due to the periodicity of f,γ,β, and assumptions (H1), we see that for any $\phi = (\phi_1,\phi_2) \in \bar{\mathcal{C}}_+$, (2.3) has a unique solution $\bar{U}(t,\phi) = (\bar{u}(t,\phi),\bar{v}(t,\phi))$ on $[0,\infty)$ with $\bar{U}(s,\phi) = \phi \in \bar{\mathcal{C}}_+$. Hence, we can define a solution semiflow $\{\Psi_t\}_{t\ge0}$ for (2.3) by

$$\Psi_t[\phi]_1(s) = \bar{u}(t+s,\phi),\ \Psi_t[\phi]_2 = \bar{v}(t,\phi).$$

Define the Poincaré map $\bar{P} : \bar{\mathcal{C}}_+ \to \bar{\mathcal{C}}_+$ by $\bar{P}(\phi) = (\bar{u}_\omega(\phi),\bar{v}_\omega(\phi))$ for all $\phi \in \bar{\mathcal{C}}_+$, and let $\bar{r} = r(\bar{P})$ be the spectral radius of $\bar{P}$. By arguments similar to [21, Proposition 2.1], we show the following results.

Proposition 2.1. *$\bar{r} = r(\bar{P})$ is positive and is an eigenvalue of $\bar{P}$ with a positive eigenfunction $\bar{\phi}^*$.*

Define $\widehat{B} : \mathbb{R}_+ \times \bar{\mathcal{C}} \to \mathbb{R}^2$ by

$$\widehat{B}(t,\phi) = \begin{pmatrix} \widehat{B}_1(t,\phi) \\ \widehat{B}_2(t,\phi) \end{pmatrix} = \begin{pmatrix} f(t,\phi_1(0),\phi_1(-\tau)) - \gamma(t)\phi_1(0) + \beta(t)\phi_2(0) \\ \gamma(t)\phi_1(0) - \beta(t)\phi_2(0) \end{pmatrix}.$$

Then, it is easy to verify that for each $t \geq 0, \bar{B}(t,\cdot)$ is cooperative on $\bar{\mathcal{C}}_+$. Clearly, the system (2.3) is irreducible. Moreover, for any $K > L$, $\mathbf{K} = \left(K, \max_{t\in[0,\omega]}\left(\frac{\gamma(t)}{\beta(t)}\right)K\right)$ is a super-solution for (2.2). Due to the strict subhomogeneity of $f(t,\cdot,\cdot)$, we have the following results about the global dynamics of (2.2).

Theorem 2.2. *Let* (H1)–(H3) *hold. The following statements are valid.*

(i) If $\bar{r} \leq 1$, then zero solution is globally asymptotically stable for (2.2) *with respect to $\bar{\mathcal{C}}_+$.*

(ii) If $\bar{r} > 1$, then (2.2) *has a unique positive ω-periodic solution $V^*(t) = (\widehat{u}^*(t), \widehat{v}^*(t))$, and $V^*(t)$ is globally asymptotically stable with respect to $\bar{\mathcal{C}}_+ \setminus \{\mathbf{0}\}$.*

Proof. Since f is strictly sub-homogeneous,

$$f(t,u,w) \leq \partial_1 f(t,0,0)u + \partial_2 f(t,0,0)w, \qquad \forall (t,u,w) \in \mathbb{R}^3_+.$$

Note that the solutions of system (2.3) exist globally on $[0,\infty)$. By the comparison theorem [18, Theorem 5.1.1] and the positivity theorem [18, Theorem 5.2.1], each solution $(\widehat{u}(t,\phi), \widehat{v}(t,\phi))$ of system (2.2) with initial value $\phi \in \bar{\mathcal{C}}_+$ exists globally, and $(\widehat{u}(t,\phi), \widehat{v}(t,\phi)) \geq (0,0), \forall t \geq -\tau$. Since system (2.2) is cooperative, it follows from [18, Theorem 5.1.1] that for any $\varphi, \psi \in \bar{\mathcal{C}}_+$ with $\varphi \leq \psi$, $(\widehat{u}_t(\varphi), \widehat{v}_t(\varphi)) \leq (\widehat{u}_t(\psi), \widehat{v}_t(\psi)), \forall t \geq 0$. Using the assumption (H1), in particular $\partial_2 f(t,u,w) > 0$ for $(t,u,w) \in \mathbb{R}^3_+$, we have $(\widehat{u}_t(\varphi), \widehat{v}_t(\varphi)) \ll (\widehat{u}_t(\psi), \widehat{v}_t(\psi)), \forall t \geq 2\tau$ for $\varphi < \psi$. Define $S : \bar{\mathcal{C}}_+ \to \bar{\mathcal{C}}_+$ by $S(\phi) = (\widehat{u}_\omega(\phi), \widehat{v}_\omega(\phi))$. Then S is monotone, and S^n is strongly monotone for $n\omega \geq 2\tau$. Moreover, it is easy to conclude from the sub-homogeneity of f that S is sub-homogeneous.

By the continuity and differentiability of solutions with respect to initial values, it follows that S is differentiable at zero, and $DS(0) = \bar{P}$. Furthermore, since $\partial_2 f(t,u,w) > 0$, [7, Theorem 3.6.1] and [18, Theorem 5.3.2] imply that $(DS(0))^n$ is compact and strongly positive for all $n\omega \geq 2\tau$. Consider $S^{n_0}, n_0\omega \geq 2\tau$. Then, S^{n_0} is strongly monotone, and $(DS(0))^{n_0}$ is compact and strongly positive.

In view of (H2), [18, Remark 5.2.1] implies that for any $h \geq 1, V_h = \{\phi \in \bar{\mathcal{C}}_+ : 0 \leq \phi_1(s) \leq hK, 0 \leq \phi_2 \leq h \cdot \max_{t\in[0,\omega]} \frac{\gamma(t)}{\beta(t)}K, s \in [-\tau, 0]\}$ is a positive invariant set for S. By [7, Theorem 3.6.1], for any fixed $h \geq 1$, $S^{n_0} : V_h \to V_h$ is compact. Then for any $\varphi, \psi \in V_h$ with $\varphi \leq \psi$, the closure of $S^{n_0}([\varphi,\psi])$ is a compact subset of V_h. Furthermore, $DS^{n_0}(0) = (DS(0))^{n_0}$ is compact and strongly positive. Since S is strictly sub-homogeneous, S^{n_0} is strongly monotone, and $r\{(DS(0))^{n_0}\} = r\{DS(0)\}^{n_0} = \bar{r}^{n_0}$, by [24, Theorem 2.3.4], we have the following conclusions.

(i) If $\bar{r} \leq 1$, then zero is a globally asymptotically stable fixed point of S^{n_0} with respect to V_h.

(ii) If $\bar{r} \geq 1$, then S^{n_0} has a unique positive fixed point $\widehat{\phi}^*$ in V_h, and $\widehat{\phi}^*$ is globally asymptotically stable with respect to $V_h \setminus \{0\}$.

Since $h \geq 1$ in the above discussion is arbitrary, we can conclude that zero solution of system (2.2) is globally asymptotically stable in case (i); and system (2.2) admits the unique, positive and $n_0\omega$-periodic solution $(\widehat{u}(t,\widehat{\phi}^*), \widehat{v}(t,\widehat{\phi}^*))$ in case (ii). At what follows, we further prove that $(\bar{u}(t,\phi^*), \bar{v}(t,\phi^*))$ is ω-periodic. According to Proposition 2.1, there exists a positive eigenfunction $\bar{\phi}^*$ such that $DS(0)(\bar{\phi}^*) = \bar{r}\bar{\phi}^*$. In case where $\bar{r} > 1$, for any small $\varsigma > 0$, by the monotonicity of S, we have

$$0 \ll (\varsigma,\varsigma) \ll S(\varsigma\bar{\phi}^*) \leq S^2(\varsigma\bar{\phi}^*) \leq \cdots \leq S^n(\varsigma\bar{\phi}^*) \leq \cdots.$$

Additionally, $S^{n_0 n}(\varsigma\bar{\phi}^*) \to \widehat{\phi}^*$, as $n \to \infty$. Since S is continuous and the sequence of $S^n(\varsigma\bar{\phi}^*)$ is monotone, $\widehat{\phi}^*$ is a fixed point of S, which implies that $(\widehat{u}(t,\widehat{\phi}^*), \widehat{v}(t,\widehat{\phi}^*))$ is a ω-periodic solution. The proof is complete. □

At what follows, we establish the existence, uniqueness and comparison principle for (2.1) with initial value $\phi = (\phi_1, \phi_2) \in \mathcal{C}$.

Consider the following time-periodic reaction-diffusion equation

$$\begin{cases} \partial_t w(t,x) = d(t)\frac{\partial^2 w(t,x)}{\partial x^2} - \gamma(t)w(t,x), \ t > 0, \ x \in \mathbb{R}, \\ w(0,x) = \varphi(x), \quad x \in \mathbb{R}, \ \varphi \in X. \end{cases} \tag{2.4}$$

By virtue of [9, Chapter II], it follows that (2.4) admits an evolution operator $T_1(t,s) : X \to X$, $0 \le s \le t$, that is, $T_1(t,t) = I, T_1(t,s)T_1(s,\rho) = T_1(t,\rho)$ for $0 \le \rho \le s \le t$ and $T_1(t,0)(\varphi)(x) = w(t,x,\varphi)$ for $t \ge 0, x \in \mathbb{R}$ and $\varphi \in X$, where $w(t,x,\varphi)$ is the solution of (2.4). Moreover, for any $0 \le s < t, T_1(t,s)$ is a compact and positive operator on X, and $T_1(t,s)(\varphi)(x) > 0$ for all $0 \le s < t, x \in \mathbb{R}$ and $\varphi \in X$, provided $\varphi(x) \ge 0$ and $\varphi \not\equiv 0$. Let $T_2(t,s)\phi_2 = e^{-\int_s^t \beta(\eta)d\eta}\phi_2, U = (u,v)$ and $\phi = (\phi_1,\phi_2) \in \mathcal{C}^+$. Integrating two equations of (2.1), we have

$$\begin{cases} u(t,\cdot,\phi) = T_1(t,0)\phi_1(0,\cdot) + \int_0^t T_1(t,s)(f(s,u(s,\cdot),u(s-\tau,\cdot)) + \beta(s)v(s,\cdot))ds, \\ v(t,\cdot,\phi) = T_2(t,0)\phi_2 + \int_0^t T_2(t,s)\gamma(s)u(s,\cdot)ds, \end{cases}$$

that is,

$$U(t,\phi) = T(t,0)\phi(0) + \int_0^t T(t,s)B(s,U_s)ds, \tag{2.5}$$

where

$$T(t,s) = \begin{pmatrix} T_1(t,s) & 0 \\ 0 & T_2(t,s) \end{pmatrix},$$

$$B(t,\phi) = \begin{pmatrix} B_1(\phi)(\cdot) \\ B_2(\phi)(\cdot) \end{pmatrix} = \begin{pmatrix} f(t,\phi_1(0,\cdot),\phi_1(-\tau,\cdot)) + \beta(t)\phi_2(\cdot) \\ \gamma(t)\phi_1(0,\cdot) \end{pmatrix}$$

for $t \in [0,+\infty)$. A function $\check{U}$ is said to be a lower solution of (2.1) if

$$\check{U}(t,\cdot) \le T(t,0)\check{U}(0,\cdot) + \int_0^t T(t,s)B(s,\check{U}_s)ds.$$

A function $\hat{U}$ is said to be an upper solution of (2.1) if

$$\hat{U}(t,\cdot) \ge T(t,0)\hat{U}(0,\cdot) + \int_0^t T(t,s)B(s,\hat{U}_s)ds.$$

Theorem 2.3. *Let* (H1)–(H4) *hold. For any $\phi = (\phi_1,\phi_2) \in \mathcal{C}_{\mathbf{K}}$, system* (2.1) *admits a unique mild solution $U(t,x,\phi)$ with $U_0(\cdot,\cdot,\phi) = \phi$ and $U_t(\cdot,\cdot,\phi) \in \mathcal{C}_{\mathbf{K}}$ for all $t \ge 0$, and $U(t,x,\phi)$ is a classic solution when $t > \tau$. Moreover, if $\check{U}(t,x)$ and $\hat{U}(t,x)$ are a pair of lower and upper solutions of* (2.1)*, respectively, with $\check{U}_0(\cdot,\cdot) \le \hat{U}_0(\cdot,\cdot)$, then $\check{U}_t(\cdot,\cdot) \le \hat{U}_t(\cdot,\cdot)$ for all $t \ge 0$.*

Proof. We first show that B is quasi-monotone on $[0,\infty) \times \mathcal{C}_{\mathbf{K}}$ in the sense that

$$\lim_{h\to 0^+} \text{dist}\left(\phi(0,\cdot) - \psi(0,\cdot) + h\left[B(t,\phi) - B(t,\psi)\right], X_+ \times X_+\right) = 0$$

for all $\phi, \psi \in \mathcal{C}_{\mathbf{K}}$ with $\phi_1(s,x) \geq \psi_1(s,x)$ and $\phi_2(x) \geq \psi_2(x)$, $\forall (s,x) \in [-\tau, 0] \times \mathbb{R}$. In fact, for any $\phi, \psi \in \mathcal{C}_{\mathbf{K}}$ with with $\phi_1(s,x) \geq \psi_1(s,x)$ and $\phi_2(x) \geq \psi_2(x)$, $\forall (s,x) \in [-\tau, 0] \times \mathbb{R}$, we have

$$\begin{aligned}
&\phi(0,\cdot) - \psi(0,\cdot) + h\,[B(t,\phi) - B(t,\psi)] \\
&\quad = \begin{pmatrix} \phi_1(0,\cdot) - \psi_1(0,\cdot) + h\,[f(t,\phi_1(0,\cdot),\phi_1(-\tau,\cdot)) - f(t,\psi_1(0,\cdot),\psi_1(-\tau,\cdot))] + h\beta(t)\,[\phi_2(\cdot) - \psi_2(\cdot)] \\ \phi_2(\cdot) - \psi_2(\cdot) + h\gamma(t)\,[\phi_1(0,\cdot) - \psi_1(0,\cdot)] \end{pmatrix} \\
&\quad \geq \begin{pmatrix} \phi_1(0,\cdot) - \psi_1(0,\cdot) + h\,[f(t,\phi_1(0,\cdot),\psi_1(-\tau,\cdot)) - f(t,\psi_1(0,\cdot),\psi_1(-\tau,\cdot))] + h\beta(t)\,[\phi_2(\cdot) - \psi_2(\cdot)] \\ \phi_2(\cdot) - \psi_2(\cdot) + h\gamma(t)\,[\phi_1(0,\cdot) - \psi_1(0,\cdot)] \end{pmatrix} \\
&\quad \geq \begin{pmatrix} \phi_1(0,\cdot) - \psi_1(0,\cdot) - lh\,[\phi_1(0,\cdot) - \psi_1(0,\cdot)] + h\beta(t)\,[\phi_2(\cdot) - \psi_2(\cdot)] \\ \phi_2(\cdot) - \psi_2(\cdot) + h\gamma(t)\,[\phi_1(0,\cdot) - \psi_1(0,\cdot)] \end{pmatrix}
\end{aligned}$$

Thus, we can choose h sufficiently small such that

$$\phi(0,\cdot) - \varphi(0,\cdot) + h[B(t,\phi) - B(t,\varphi)] \geq 0.$$

By [16, Corallary 5], (2.1) admits a unique mild solution $U(t,\cdot,\phi)$ on $[0,+\infty)$ for each $\phi \in \mathcal{C}_{\mathbf{K}}$, and the comparison principle holds for the lower and upper solutions. This completes the proof. □

Define a family of operators $\{Q_t\}_{t\geq 0}$ on $\mathcal{C}_{\mathbf{K}}$ by

$$Q_t[\phi](s,x) = U(t+s,x,\phi) = (u(t+s,x,\phi), v(t,x,\phi))$$

where $(u(t+s,x,\phi), v(t,x,\phi))$ is a solution of (2.1) with $(u(s,x), v(0,x)) = (\phi_1(s,x), \phi_2(x))$ for $s \in [-\tau, 0]$ and $x \in \mathbb{R}$. For any $(t_0, \phi^0) \in \mathbb{R}_+ \times \mathcal{C}_{\mathbf{K}}$, we have

$$\|Q_t(\phi) - Q_{t_0}(\phi^0)\|_{\mathcal{C}} \leq \|Q_t(\phi) - Q_t(\phi^0)\|_{\mathcal{C}} + \|Q_t(\phi^0) - Q_{t_0}(\phi^0)\|_{\mathcal{C}}.$$

Since $T(t,0)\varphi$ is continuous in $(t,\varphi) \in [0,\infty) \times X^2$ with respect to the compact open topology, by arguments similar to those in [15, Theorem 8.5.2], we know that $Q_t(\phi)$ is continuous at (t_0, ϕ^0) with respect to the compact open topology. According to the definition of ω-periodic semiflow (see Definition 4.7 in the Appendix), it follows that $\{Q_t\}_{t\geq 0}$ is an ω-periodic semiflow on $\mathcal{C}_{\mathbf{K}}$.

Lemma 2.4. *For each $t > 0, Q_t$ is strictly subhomogeneous in the sense that $Q_t(\theta\phi) > \theta Q_t(\phi)$ for any fixed $\theta \in (0,1)$.*

Proof. For any $\phi := (\phi_1, \phi_2) \in \mathcal{C}_{\mathbf{K}}$ with $\phi \not\equiv 0$. Let $(u(t,x,\phi), v(t,x,\phi))$ be the solution of (1.4) with $u(s,x) = \phi_1(s,x), v(0,x) = \phi_2(x)$ for $s \in [-\tau, 0]$ and $x \in \mathbb{R}$. Fix $\theta \in (0,1)$. Since f is subhomogeneous, we have

$$\begin{aligned}
\frac{\partial(\theta u(t,x))}{\partial t} &= \theta\left[d(t)\frac{\partial^2 u(t,x)}{\partial x^2} + f(t,u(t,x),u(t-\tau,x)) - \gamma(t)u(t,x) + \beta(t)v(t,x)\right] \\
&\leq d(t)\theta\frac{\partial^2 u(t,x)}{\partial x^2} + f(t,\theta u(t,x),\theta u(t-\tau,x)) - \gamma(t)\theta u(t,x) + \beta(t)\theta v(t,x)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial(\theta v(t,x))}{\partial t} &= \theta\,[\gamma(t)u(t,x) - \beta(t)v(t,x)] \\
&= \gamma(t)\theta u(t,x) - \beta(t)\theta v(t,x).
\end{aligned}$$

Thus, $(\theta u(t,x,\phi), \theta v(t,x,\phi))$ is a lower solution of (2.1) with $\theta u(s,x,\phi) = \theta\phi_1(s,x)$, $\theta v(0,x) = \theta\phi_2(x)$ for $s \in [-\tau, 0]$ and $x \in \mathbb{R}$. Then $(\theta u(t,x,\phi), \theta v(t,x,\phi)) \leq (u(t,x,\theta\phi), v(t,x,\theta\phi))$, where $(u(t,x,\theta\phi), v(t,x,\theta\phi))$ is a solution of (2.1) with $u(s,x) = \theta\phi_1(s,x), v(0,x) = \theta\phi_2(x)$ for $s \in [-\tau, 0]$ and $x \in \mathbb{R}$.

Let $(w_1, w_2) := (u(t,x,\theta\phi) - \theta u(t,x,\phi), v(t,x,\theta\phi) - \theta v(t,x,\phi))$, then $w_1(s,x) = 0$ for $(s,x) \in [-\tau, 0] \times \mathbb{R}$, $w_2(0,x) = 0$ for $x \in \mathbb{R}$, and $w_i(t,x) \geq 0$, $\forall (t,x) \in [0,\infty) \times \mathbb{R}$, $i = 1, 2$. We further show that $w_i(t,x) > 0$, $\forall (t,x) \in [0,\infty) \times \mathbb{R}$, $i = 1, 2$. Direct calculation yields

$$\begin{aligned}
\frac{\partial w_1}{\partial t} &= \frac{\partial u(t,x,\theta\phi)}{\partial t} - \theta \frac{\partial u(t,x,\phi)}{\partial t} \\
&= d(t)\frac{\partial^2 w_1}{\partial x^2} + f(t, u(t,x,\theta\phi), u(t-\tau,x,\theta\phi)) - \theta f(t, u(t,x,\phi), u(t-\tau,x,\phi)) \\
&\quad - \gamma(t)w_1 + \beta(t)w_2 \\
&= d(t)\frac{\partial^2 w_1}{\partial x^2} + f(t, u(t,x,\theta\phi), u(t-\tau,x,\theta\phi)) - f(t, \theta u(t,x,\phi), \theta u(t-\tau,x,\phi)) \\
&\quad + f(t, \theta u(t,x,\phi), \theta u(t-\tau,x,\phi)) - \theta f(t, u(t,x,\phi), u(t-\tau,x,\phi)) - \gamma(t)w_1 + \beta(t)w_2 \\
&\geq d(t)\frac{\partial^2 w_1}{\partial x^2} + f(t, u(t,x,\theta\phi), \theta u(t-\tau,x,\theta\phi)) - f(t, \theta u(t,x,\phi), \theta u(t-\tau,x,\phi)) \\
&\quad + g(t,x) - \gamma(t)w_1 \\
&\geq d(t)\frac{\partial^2 w_1}{\partial x^2} - l w_1 + g(t,x) - \gamma(t)w_1,
\end{aligned}$$

where

$$g(t,x) := f\left(t, \theta u(t,x,\phi), \theta u(t-\tau,x,\phi)\right) - \theta f\left(t, u(t,x,\phi), u(t-\tau,x,\phi)\right).$$

Following the assumption (H3), we have $g(t,x) > 0$ for $t > 0$ and $x \in \mathbb{R}$. Consider the following equation

$$\begin{cases} \frac{\partial \tilde{w}_1}{\partial t} = d(t)\frac{\partial^2 \tilde{w}_1}{\partial x^2} - l\tilde{w}_1 + g(t,x) - \gamma(t)\tilde{w}_1, & t > 0, \\ \tilde{w}_1(0,x) = 0, & x \in \mathbb{R}. \end{cases} \tag{2.6}$$

Then, we can rewrite (2.6) as

$$\tilde{w}_1(t,\cdot,\varphi) = \int_0^t \tilde{T}_1(t,s)g(s,\cdot)ds, \qquad t \geq 0, \tag{2.7}$$

where the evolution operator $\tilde{T}_1(t,s) : X \to X$, $0 \leq s \leq t$ is defined by $\tilde{T}_1(t,s)\varphi = e^{-l(t-s)}T_1(t,s)\varphi$ for all $t \geq s \geq 0$. Since $g(t,x) > 0$, $\forall t > 0, x \in \mathbb{R}$, it follows from the strong positivity of $T_1(t,s)$ that the solution of (2.6) satisfies $w_1(t,x) > 0$ for all $t > 0$ and $x \in \mathbb{R}$. Consequently, the comparison principle implies $w_1(t,x) \geq \tilde{w}_1(t,x) > 0$ for all $t > 0$ and $x \in \mathbb{R}$. Due to $w_1(t,x) > 0$ for $t > 0$ and $x \in \mathbb{R}$, similarly, we have $w_2(t,x) > 0$ for $t > 0$ and $x \in \mathbb{R}$.

Hence, $(u(t,x,\theta\phi), v(t,x,\theta\phi)) > (\theta u(t,x,\phi), \theta v(t,x,\phi))$, $\forall t > 0, x \in \mathbb{R}$, which indicates that for each $t > 0$, Q_t is strictly subhomogeneous. This completes the proof. □

Lemma 2.5. *For any $\phi = (\phi_1, \phi_2) \in \mathcal{C}_{\mathbf{K}}$ with $\phi \not\equiv 0$, $U(t,x,\phi) = (u(t,x,\phi), v(t,x,\phi)) > 0$ for $t > \tau$ and $x \in \mathbb{R}$.*

Proof. For any $\phi = (\phi_1, \phi_2) \in \mathcal{C}_{\mathbf{K}}$ with $\phi \not\equiv 0$, by Theorem 2.3, we have $U(t,x,\phi) \geq 0$ for all $t \geq 0$ and $x \in \mathbb{R}$. In what follows, we show that $U(t,x,\phi) > 0, t > \tau, x \in \mathbb{R}$. Since $u(t,x,\phi)$ satisfies

$$\begin{aligned}\frac{\partial u(t,x,\phi)}{\partial t} &= d(t)\frac{\partial^2 u(t,x,\phi)}{\partial x^2} + f(t,u(t,x,\phi),u(t-\tau,x,\phi)) - \gamma(t)u(t,x,\phi) + \beta(t)v(t,x,\phi)\\ &\geq d(t)\frac{\partial^2 u(t,x,\phi)}{\partial x^2} + f(t,u(t,x,\phi),u(t-\tau,x,\phi)) - \gamma(t)u(t,x,\phi)\\ &\geq d(t)\frac{\partial^2 u(t,x,\phi)}{\partial x^2} + f(t,u(t,x,\phi),0) - \gamma(t)u(t,x,\phi)\\ &\geq d(t)\frac{\partial^2 u(t,x,\phi)}{\partial x^2} - lu(t,x,\phi) - \gamma(t)u(t,x,\phi),\end{aligned}$$

it follows from the parabolic comparison principle that $u(t,x,\phi) \geq e^{-lt}T_1(t,0)\phi_1(0,\cdot), t > 0$. Thus, by the strong positivity of $T_1(t,s)$ for $t > s \geq 0$, we see that $u(t,x,\phi) > 0$ for all $t > 0, x \in \mathbb{R}$, provided that $\phi_1(0,\cdot) \not\equiv 0$.

In the following, we show that for any $\phi_1 \not\equiv 0$ and $\phi_1(0,\cdot) = 0$, there exists $t_0 \in [0,\tau]$ such that $u(t_0,\cdot,\phi_1) > 0$. Suppose, by contradiction, that for some $\phi_1' \not\equiv 0$ and $\phi_1'(0,\cdot) = 0$, $u(t,\cdot,\phi_1') \equiv 0$ for $t \in [0,\tau]$. In view of (2.5), we obtain that

$$0 = \int_0^t T_1(t,s)\left[f(s,0,u(s-\tau,x)) + \beta(s)v(s,x)\right]ds, \qquad t \in [0,\tau].$$

Since $T_1(t,s)$ is strongly positive for $t > s \geq 0$, $f(s,0,u(s-\tau,\cdot)) \geq f(s,0,0) = 0, s \in [0,\tau]$, and $\beta(s)v(s,\cdot) \geq 0, s \in [0,\tau]$, we have $f(s,0,u(s-\tau,x)) = 0$ for any $s \in [0,\tau]$ and $x \in \mathbb{R}$. Hence, by (H1), it follows that $u(s-\tau,x) = 0$ for any $s \in [0,\tau]$ and $x \in \mathbb{R}$, which means $\phi_1 \equiv 0$, a contradiction. Consequently, we have $u(t_0,\cdot,\phi_1) > 0$ for some $t_0 \in [0,\tau]$. Applying the comparison principle and strong positivity of $T_1(t,s), t > s \geq 0$ again, for any $t > t_0$, we see that $u(t,\cdot,\phi) \geq e^{-l(t-t_0)}T_1(t,t_0)u(t_0,\cdot,\phi_1) > 0$ for $t > t_0$.

Since $v(t,\cdot,\phi)$ satisfies

$$v(t,\cdot) = T_2(t,t_0)v(t_0,\cdot,\phi) + \int_{t_0}^t T_2(t,s)u(s,\cdot,\phi)ds \geq \int_{t_0}^t T_2(t,s)u(s,\cdot,\phi)ds,$$

it follows from the strong positivity of $T_2(t,s), t > s \geq 0$ that $v(t,x) > 0$ for $t > t_0$ and $x \in \mathbb{R}$.

In the case where $\phi_2(x) \not\equiv 0$, by the strong positivity of $T_2(t,s), t > s \geq 0$, we have $v(t,x) > 0$ for $t > 0$ and $x \in \mathbb{R}$. Since $u(t,x,\phi)$ satisfies

$$\frac{\partial u(t,x,\phi)}{\partial t} \geq d(t)\frac{\partial^2 u(t,x,\phi)}{\partial x^2} - lu(t,x,\phi) - \gamma(t)u(t,x,\phi) + \beta(t)v(t,x,\phi),$$

by an argument similar to (2.6), we can prove that $u(t,x,\phi) > 0$ for all $t > 0$ and $x \in \mathbb{R}$.

Therefore, for any $\phi \in \mathcal{C}_{\mathbf{K}}$ with $\phi \not\equiv 0$, we have $U(t,x,\phi) > 0$ for all $t > \tau$ and $x \in \mathbb{R}$. This completes the proof. □

2.2 Spreading speed

In what follows, the theory for spreading speeds for monotone autonomous semiflows and periodic semiflows in the monostable case developed in [12, 13] will be used to study the spatial dynamics of (2.1). For the sake of convenience, the abstract results in [12, 13] can be found in the Appendix.

Define $V_0^* \in \bar{\mathcal{C}}$ as $V_0^*(s) = (\bar{u}_0^*(s), \bar{v}_0^*) = (\bar{u}^*(s), \bar{v}^*(0))$ for all $s \in [-\tau, 0]$. Throughout this subsection, we further assume that

(H4) $\bar{r} > 1$.

We first show the spreading properties of the Poincaré map.

Proposition 2.6. *Assume that* (H1)–(H4) *hold. The Poincaré map* $Q_\omega : \mathcal{C}_{V_0^*} \to \mathcal{C}_{V_0^*}$ *admits a spreading speed* c_ω^*.

Proof. Clearly, Q_ω satisfies (A1), (A2) and (A4) in the Appendix. It suffices to show that (A3) and (A5) of the Appendix are satisfied.

As in the proof of Theorem 2.9, $Q_t = L_t + S_t$ for $t > 0$. When $t > \tau, L_t = 0$, and hence, $Q_t = S_t$. Since the derivatives $(\partial_t u(t,x,\phi), \partial_t v(t,x,\phi))$ are uniformly bounded for $t > 0, x \in \mathbb{R}$ and $\phi \in \mathcal{C}_\mathbf{K}$, the set $\{Q_t[U](\cdot,x) : U \in \mathcal{C}_\mathbf{K}, x \in \mathbb{R}$ is precompact in $\bar{\mathcal{C}}_\mathbf{K}$, that is Q_t satisfies (A3)(a) with $\beta = \mathbf{K}$ when $t > \tau$. In view of the abstract integral equation (2.5) and the compactness of $T_1(t,s)$ for $0 \le s < t$, it is not difficult to see that $\{U_{t_1}(\phi)(0,\cdot), \phi \in \mathcal{C}_\mathbf{K}\} = \{U(t_1,\cdot,\phi), \phi \in \mathcal{C}_\mathbf{K}\}$ is percompact in X^2 for $t_1 > 0$ (see, e.g. [11, Lemma 2.6]). Additionally, Q_t satisfies (A3)(b′) with $\beta = \mathbf{K}$ and $\varsigma = t$ for $t \in (0,\tau]$ (see also the proof of [7, Theorem 3.6.1]).

Let $\hat{Q}_t$ be the restriction of Q_t to $\bar{\mathcal{C}}_\mathbf{K}$. It is easy to see that $\hat{Q}_t : \bar{\mathcal{C}}_\mathbf{K} \to \bar{\mathcal{C}}_\mathbf{K}$ is an ω-periodic semiflow generated by (2.2) with initial date $\bar{V}_0 = \phi \in \bar{\mathcal{C}}_\mathbf{K}$. Moreover, $\hat{Q}_t$ is strictly monotone for any $t > \tau$ and strongly monotone for any $t \ge 2\tau$ on $\bar{\mathcal{C}}_\mathbf{K}$. By (H4) and Theorem 2.2, we can conclude from Dancer–Hess connecting orbit lemma (see, e.g., [24]) that $\hat{Q}_\omega$ admits a strongly monotone full orbit connecting $\mathbf{0}$ to V_0^*. Note that $V_0^* \in \bar{\mathcal{C}}_\mathbf{K}$. Hence, (A5) with $\beta = V_0^*$ holds for Q_ω.

Therefore, it follows from Theorem 4.2 and Remark 4.3 in the Appendix that Q_ω has an asymptotic speed of spread c_ω^*. This completes the proof. □

In the following, we show the explicit formula of c_ω^*.

Clearly, $(0,0)$ is a solution of (2.1). Consider the following linearization problem of (2.1) at the zero solution

$$\begin{cases} \partial_t u(t,x) = d(t)\frac{\partial^2 u(t,x)}{\partial x^2} + \partial_1 f(t,0,0)u(t,x) \\ \qquad\qquad + \partial_2 f(t,0,0)u(t-\tau,x) - \gamma(t)u(t,x) + \beta(t)v(t,x), \\ \partial_t v(t,x) = \gamma(t)u(t,x) - \beta(t)v(t,x), \\ \quad u(s,x) = \phi_1(s,x),\ v(0,x) = \phi_2(x), \quad s \in [-\tau,0],\ x \in \mathbb{R}. \end{cases} \tag{2.8}$$

For $\rho > 0$, substituting $(u(t,x),v(t,x)) = (e^{-\rho x}z_1(t), e^{-\rho x}z_2(t))$ into (2.8), we get

$$\begin{cases} \frac{d}{dt}z_1(t) = \left[d(t)\rho^2 + \partial_1 f(t,0,0) - \gamma(t)\right] z_1(t) + \partial_2 f(t,0,0)z_1(t-\tau) + \beta(t)z_2(t), \\ \frac{d}{dt}z_2(t) = \gamma(t)z_1(t) - \beta(t)z_2(t). \end{cases} \tag{2.9}$$

Then, $(u(t,x),v(t,x)) = (e^{-\rho x}z_1(t), e^{-\rho x}z_2(t))$ is a solution of (2.8) with $(u(s,x),v(0,x)) = (e^{-\rho x}z_1(s), e^{-\rho x}z_2(0))$ for $s \in [-\tau,0]$ and $x \in \mathbb{R}$, provided that $(z_1(t),z_2(t))$ is a solution of (2.9).

Define the linear solution map of (2.8) as M_t and let $Z(t,z^0) = (z_1(t,z^0), z_2(t,z^0))$ be the solution of (2.9) with $(z_1(s,z^0), z_2(0,z^0)) = (z_1^0(s), z_2^0) = Z^0 \in \bar{\mathcal{C}}$ for $s \in [-\tau,0]$. Define $B_\rho^t(Z^0) = M_t(Z^0 e^{-\rho x})(0)$ for $Z^0 = (z_1^0, z_2^0) \in \bar{\mathcal{C}}$. Thus, it is easy to see that $B_\rho^t(Z^0) = Z(t,z^0)$, that is, $B_\rho^t(Z^0)$ is the solution map of (2.9).

Let $r(\rho)$ be the spectral radius of the Poincaré map associated with (2.9). Since system (2.9) is linear periodic cooperative and irreducible, similarly to Proposition 2.1, it follows that there is a positive ω-periodic function $W(t) = (w_1(t), w_2(t))$ such that $Z(t) = e^{\lambda(\rho)t}W(t)$ is a solution of (2.9), where $\lambda(\rho) = \frac{\ln r(\rho)}{\omega}$.

Define $\psi = (\psi_1, \psi_2) \in \bar{\mathcal{C}}$ by $(\psi_1(\theta), \psi_2) = \left(e^{\lambda(\rho)\theta}w_1(\theta), w_2(0)\right)$ for all $\theta \in [-\tau, 0]$. It is easy to see that $(z_1(t,\psi), z_2(t,\psi)) = \left(e^{\lambda(\rho)t}w_1(t), e^{\lambda(\rho)t}w_2(t)\right)$ for all $t \geq 0$. Thus, we have

$$\begin{aligned} B_\rho^t(\psi)(\theta) &= (z_1(t+\theta,\psi), z_2(t,\psi)) \\ &= \left(e^{\lambda(\rho)t} \cdot e^{\lambda(\rho)\theta}w_1(t+\theta), e^{\lambda(\rho)t}w_2(t)\right), \qquad \forall \theta \in [-\tau, 0],\ t \geq 0. \end{aligned}$$

By the periodicity of $W(t)$, it follows that

$$\begin{aligned} B_\rho^\omega(\psi)(\theta) &= \left(e^{\lambda(\rho)\omega} \cdot e^{\lambda(\rho)\theta}w_1(\theta), e^{\lambda(\rho)\omega}w_2(0)\right) \\ &= e^{\lambda(\rho)\omega}\left(\psi_1(\theta), \psi_2\right) \\ &= e^{\lambda(\rho)\omega}\psi(\theta), \qquad \forall \theta \in [-\tau, 0], \end{aligned}$$

that is, $B_\rho^\omega(\psi) = e^{\lambda(\rho)\omega}\psi$. This means that $e^{\lambda(\rho)\omega}$ is the principal eigenvalue of B_ρ^ω with positive eigenfunction ψ. Then we have

$$\Phi(\rho) := \frac{1}{\rho}\ln(e^{\lambda(\rho)\omega}) = \frac{\lambda(\rho)\omega}{\rho} = \frac{\ln r(\rho)}{\rho}.$$

In order to apply Theorem 4.5 in the Appendix, we need to show that $\Phi(\infty) = \infty$. For the first equation in (2.9), we have

$$\frac{d}{dt}z_1(t) \geq \left[d(t)\rho^2 + \partial_1 f(t,0,0) - \gamma(t)\right] z_1(t),$$

and hence,

$$\frac{w_1'(t)}{w_1(t)} \geq d(t)\rho^2 + \partial_1 f(t,0,0) - \gamma(t) - \lambda(\rho).$$

Then

$$0 = \int_0^\omega \frac{w_1'(t)}{w_1(t)}dt \geq \int_0^\omega \left[d(t)\rho^2 + \partial_1 f(t,0,0) - \gamma(t)\right] dt - \lambda(\rho)\omega.$$

Consequently,

$$\Phi(\rho) = \frac{\lambda(\rho)\omega}{\rho} \geq \rho\int_0^\omega d(t)dt + \frac{\int_0^\omega \left[\partial_1 f(t,0,0) - \gamma(t)\right] dt}{\rho},$$

which means that $\Phi(\infty) = \infty$.

On the other hand, when $\rho = 0$, (2.9) reduces to (2.3), and hence, it follows from (H4) that $\lambda(0) = \bar{r} > 1$, that is, (C7) in Theorem 4.5 in the Appendix is valid. Then we have the following result.

Proposition 2.7. *Assume that* (H1)–(H4) *hold. Let c_ω^* be the asymptotic speed of spread of Q_ω. Then* $c_\omega^* = \inf_{\rho>0}\Phi(\rho) = \inf_{\rho>0}\frac{\ln r(\rho)}{\rho}$.

Proof. Since $f(t,\cdot,\cdot)$ is subhomogeneous, it follows from [24, Lemma 2.3.2] that

$$f(t,u,w) \leq \partial_1 f(t,0,0)u + \partial_2 f(t,0,0)w,$$

and hence,

$$\begin{aligned} & f(t,u(t,x),u(t-\tau,x)) - \gamma(t)u(t,x) + \beta(t)v(t,x) \\ & \quad \le \partial_1 f(t,0,0)u(t,x) + \partial_2 f(t,0,0)u(t-\tau,x) - \gamma(t)u(t,x) + \beta(t)v(t,x). \end{aligned}$$

It then follows from comparison principle that $Q_t(\phi) \le M_t(\phi)$, $\forall \phi \in \mathcal{C}_{V_0^*}$. Consequently, we can conclude from Theorem 4.5(1) in the Appendix that $c_\omega^* \le \inf_{\rho>0} \Phi(\rho)$.

By virtue of (H1), there exists a positive number ς such that $\varsigma + \partial_1 f(t,0,0) > 0, \forall t \in [0,\omega]$. Let $\bar{f}(t,u,w) := \varsigma u + f(t,u,w)$. Then

$$\partial_1 \bar{f}(t,0,0) > 0, \qquad \partial_2 \bar{f}(t,0,0) > 0, \qquad \forall t \in [0,\omega].$$

It is not difficult to see that for any $\epsilon \in (0,1)$, there is $\delta = \delta(\epsilon) \in (0,K)$ such that

$$\bar{f}(t,u,w) \ge (1-\epsilon)\partial_1 \bar{f}(t,0,0)u + (1-\epsilon)\partial_2 \bar{f}(t,0,0)w, \qquad \forall (u,w) \in [0,\delta]^2,$$

and hence,

$$\begin{aligned} f(t,u,w) &= -\varsigma u + \bar{f}(t,u,w) \\ & \ge \left[(1-\epsilon)\partial_1 \bar{f}(t,0,0) - \epsilon\varsigma\right] u + (1-\epsilon)\partial_2 \bar{f}(t,0,0)w, \qquad \forall (u,w) \in [0,\delta]^2. \end{aligned}$$

Let $r^\epsilon(\rho)$ be the spectral radius of the Poincaré map associated with the following linear periodic cooperative and irreducible system

$$\begin{cases} \frac{d}{dt} z_1(t) = \left[d(t)\rho^2 + (1-\epsilon)\partial_1 f(t,0,0) - \epsilon\varsigma - \gamma(t)\right] z_1(t) \\ \qquad\qquad + (1-\epsilon)\partial_2 f(t,0,0) z_1(t-\tau) + \beta(t) z_2(t), \\ \frac{d}{dt} z_2(t) = \gamma(t) z_1(t) - \beta(t) z_2(t). \end{cases} \tag{2.10}$$

Let M_t^ϵ be the solution map associated with the linear periodic system

$$\begin{cases} \partial_t u = d(t)\frac{\partial^2 u}{\partial x^2} + \left[(1-\epsilon)\partial_1 f(t,0,0) - \epsilon\varsigma - \gamma(t)\right] u(t) \\ \qquad + (1-\epsilon)\partial_2 f(t,0,0) u(t-\tau) + \beta(t) v(t), \\ \partial_t v = \gamma(t) u(t) - \beta(t) v(t). \end{cases} \tag{2.11}$$

With the aim of the comparison principle, there is $\xi = (\xi_1, \xi_2) = \xi(\delta) \gg 0$ in $\bar{\mathcal{C}}$ such that for any $\phi = (\phi_1, \phi_2) \in \mathcal{C}_\xi$,

$$Q_t(\phi)(x) \le \widehat{U}(t,\xi) \le \left(\delta, \max_{t\in[0,\omega]} \frac{\gamma(t)}{\beta(t)}\delta\right), \qquad \forall t \in [0,\omega],\ x \in \mathbb{R},$$

where $\widehat{U}(t,\xi) = (\widehat{u}(t,\xi), \widehat{v}(t,\xi))$ is the solution of (2.2) with $\widehat{U}(s,\xi) = (\widehat{u}(s,\xi), \widehat{v}(0,\xi)) = \xi$. Thus, for any $\phi \in \mathcal{C}_\xi$, $U(t,x,\phi)$ satisfies

$$\begin{cases} \partial_t u \ge d(t)\frac{\partial^2 u}{\partial x^2} + \left[(1-\epsilon)\partial_1 f(t,0,0) - \epsilon\varsigma - \gamma(t)\right] u(t) \\ \qquad + (1-\epsilon)\partial_2 f(t,0,0) u(t-\tau) + \beta(t) v(t), \\ \partial_t v = \gamma(t) u(t) - \beta(t) v(t), \quad \forall t \in [0,\omega], \end{cases} \tag{2.12}$$

and hence, the comparison principle implies that $Q_t(\phi) \ge M_t^\epsilon(\phi)$, $\forall \phi \in \mathcal{C}_\xi, t \in [0,\omega]$. By an argument for M_t^ϵ similar to that for M_t, from Theorem 4.5 (2) in the Appendix, we get that $c_\omega^* \ge \inf_{\rho>0} \frac{\ln r^\epsilon(\rho)}{\rho}$, and hence, letting $\epsilon \to 0$, yields $c_\omega^* \ge \inf_{\rho>0} \frac{\ln r(\rho)}{\rho}$. Therefore, $c_\omega^* = \inf_{\rho>0} \frac{\ln r(\rho)}{\rho}$. □

The following result shows that $c^* := \frac{c_\omega^*}{\omega}$ is the spreading speed for solutions of (2.1) with initial functions having compact support.

Theorem 2.8. *Assume that* (H1)–(H4) *hold and let* $c^* = \frac{c_\omega^*}{\omega}$ *and* $U(t,x,\phi)$ *be a solution of* (2.1) *with* $U(\cdot,\cdot,\phi) \in \mathcal{C}_{V_0^*}$*. Then the following statements are valid.*

(i) For any $c > c^*$*, if* $\phi \in \mathcal{C}_{V_0^*}$ *with* $0 \le \phi \ll V_0^*$ *and* $\phi(\cdot,x) = 0$ *for* x *outside a bounded interval, then*

$$\lim_{t\to\infty,\ |x|\ge ct} U(t,x,\phi) = 0;$$

(ii) For any $c < c^*$*, if* $\phi \in \mathcal{C}_{V_0^*}$ *with* $\phi \not\equiv 0$*, then*

$$\lim_{t\to\infty,\ |x|\le ct} (U(t,x,\phi) - V^*(t)) = 0.$$

Proof. By Theorem 4.8(1) in the Appendix, the conclusion (i) is valid.

(ii) According to the strict subhomogeneity of Q_t and Theorem 4.8 (2) in the Appendix, for any $c < c^*$, there exists a positive number σ such that, if $\phi \in \mathcal{C}_{V_0^*}$ with $\phi(\cdot,x) > 0$ for x on an interval of length 2σ, then

$$\lim_{t\to\infty,\ |x|\le ct} (U(t,x,\phi) - V^*(t)) = 0.$$

Furthermore, in view of the strong positivity of Q_t for all $t > 2\tau$ (see Lemma 2.5), it follows that for a fixed $t_0 > 2\tau$, $Q_{t_0}(\phi) \gg 0$. Taking $Q_{t_0}(\phi)$ as an initial value for $U(t,x,\phi)$ and by the above analysis, we complete the proof of part (ii). □

2.3 Periodic traveling waves

In this subsection, we show that $c^* = \frac{c_\omega^*}{\omega}$ is the minimal wave speed. Due to the lack of compactness of system (2.1), we apply the abstract results on traveling waves in [3], which are presented in the Appendix. To do it, we introduce a new space. Let $\mathcal{M}$ be the space consisting of all monotone functions from $\mathbb{R}$ to $\bar{\mathcal{C}}$. For any $\phi, \psi \in \mathcal{M}$, we write $w \ge z$ if $w(x) \ge z(x)$ for $x \in \mathbb{R}$ and $w > z$ if $w \ge z$ but $w \ne z$. Equip $\mathcal{M}$ with the compact open topology. Similar to $\mathcal{C}_{\mathbf{r}}$, we can define $\mathcal{M}_{\mathbf{r}} = \{\phi \in \mathcal{M} : \phi \in \bar{\mathcal{C}}_{\mathbf{r}}\}$. Giving a subset $A \subseteq \mathcal{M}$ and $p \in \mathbb{R}$, we define $A(p) := \{W(p) : W \in A\}$.

Applying Proposition 4.12 in the Appendix, we have the following results, which assert that the spreading speed c_ω^* established in Proposition 2.7 coincides with the minimal wave speed of traveling waves for $\{Q_\omega^n\}_{n\ge 0}$ on $\mathcal{M}_{V_0^*}$.

Proposition 2.9. *Assume that* (H1)–(H4) *hold. Let* c_ω^* *is the spreading speed established in Proposition 2.7. Then the following statements are valid.*

(i) For any $c \ge c_\omega^*$*, there is a traveling wave* $W(x - cn)$ *connecting* V_0^* *and* $\mathbf{0}$*.*

(ii) For any $c < c_\omega^*$*, there is no traveling wave connecting* V_0^* *and* $\mathbf{0}$*.*

Proof. By Proposition 4.12, it suffices to verify that $Q_\omega : \mathcal{M}_{V_0^*} \to \mathcal{M}_{V_0^*}$ satisfies (B1)–(B5) of the Appendix. In view of Definition 4.7 in the Appendix, $\{Q_t\}_{t\ge 0}$ is an ω-periodic semiflow on $\mathcal{M}_{V_0^*}$ generated by $Q_t(\phi)(\theta,x) = U(t+\theta,x,\phi)$, where $U(\cdot,\cdot,\phi)$ is the solution of (2.1) with $\phi = (\phi_1,\phi_2) \in \mathcal{M}_{V_0^*}$. Similar to the previous subsection, we know that Q_ω satisfies (B1), (B2), (B4) and (B5). In the remainder of proof, we only need to show that (B3) is valid for Q_ω on $\mathcal{M}_{V_0^*}$. In order to realize this, we define (see, e.g., [6,14])

$$L_t[\phi](\theta,x)=\begin{cases}\phi(t+\theta,x)-\phi(0,x), & t+\theta<0,\\ 0, & t+\theta\geq 0\end{cases}$$

and

$$S_t[\phi](\theta,x)=\begin{cases}\phi(0,x), & t+\theta<0,\\ Q_t[\phi](\theta,x), & t+\theta\geq 0.\end{cases}$$

Obviously, $Q_t=L_t+S_t$ for $t>0$. As introduced in [6, Theorem 4.1.11] and [14, Section 3], for any bounded set $\mathcal{W}$ in $\mathcal{M}_{V_0^*}$, we have $\alpha\left(L_t[\mathcal{W}](0)\right)\leq e^{-\nu t}\alpha\left(\mathcal{W}(0)\right)$ for some positive number ν. Since the derivatives $(\partial_t u(t,0,\phi),\partial_t v(t,0,\phi))$ are uniformly bounded for $t>0$ and $\phi\in\mathcal{W}$, the set $S_t[\mathcal{W}](\cdot,0)$ is compact. Hence, we get

$$\alpha\left(Q_t[\mathcal{W}](0)\right)\leq\alpha\left(L_t[\mathcal{W}](0)\right)+\alpha\left(S_t[\mathcal{W}](0)\right)\leq e^{-\nu t}\alpha\left(\mathcal{W}(0)\right),$$

which further implies that Q_ω satisfies (B3). Thus, it follows from Proposition 4.12 in the Appendix that c_ω^* is the minimal wave speed for traveling waves of $\{Q_\omega^n\}$ connecting V_0^* to $\mathbf{0}$. □

Theorem 2.10. *Assume that* (H1)–(H4) *hold. Let* $c^*:=\frac{c_\omega^*}{\omega}$. *Then the following statements are valid.*

(i) For any $c\geq c^*$, *system* (2.1) *admits an* ω*-periodic traveling wave solution* $\mathcal{U}(t,x-ct)$ *connecting* $V^*(t)$ *and* 0.

(ii) For any $c<c^*$, *there is no* ω*-periodic traveling wave connecting* $V^*(t)$ *and* 0.

Proof. Motivated by the proof of [12, Theorem 2.3], we define $P_t=T_{-ct}Q_t$, where $\{Q_t\}_{t\geq 0}$ is the ω-periodic semiflow on $\mathcal{M}_{V_0^*}$ generated by (2.1). Thus, $\{P_t\}_{t\geq 0}$ is an ω-periodic semiflow on $\mathcal{M}_{V_0^*}$. Therefore, by Proposition 2.9 and arguments similar to those in [12, Theorem 2.2 and 2.3], we complete the proof of the conclusions. □

3 Dynamics in a bounded domain

In this section, we consider system (1.4) in a bounded spatial domain:

$$\begin{cases}\frac{\partial}{\partial t}u(t,x)=d(t)\Delta u(t,x)+f(t,u(t,x),u(t-\tau,x))-\gamma(t)u(t,x)+\beta(t)v(t,x),\\ \frac{\partial}{\partial t}v(t,x)=\gamma(t)u(t,x)-\beta(t)v(t,x), \quad (t,x)\in(0,\infty)\times\Omega,\\ Bu=Bv=0, \quad (t,x)\in(0,\infty)\times\partial\Omega,\\ u(s,x)=\phi_1(s,x),v(0,x)=\phi_2(x), \quad (s,x)\in[-\tau,0]\times\Omega.\end{cases} \tag{3.1}$$

where $\Omega\subset\mathbb{R}^N(N\geq 1)$ is bounded domain with boundary $\partial\Omega$ of class $C^{2+\theta}(0\leq\theta\leq 1)$. The boundary operator is either $Bu=u$ (Dirichlet boundary condition) or $Bu=\frac{\partial u}{\partial \nu}+\alpha(x)u$ (Robin type boundary condition) for some non-negative function $\alpha\in C^{1+\theta}(\partial\Omega,\mathbb{R})$, and $\frac{\partial u}{\partial \nu}$ is the differentiation in the direction of outward normal ν to $\partial\Omega$.

Let $\mathbb{X}=L^p(\Omega)$ and $p\in(N,+\infty)$ be fixed. For any $\rho\in\left(\frac{1}{2}+\frac{N}{2p},1\right)$, let $\mathbb{X}_\rho$ be the fractional power space of $\mathbb{X}$ with respect to $-\Delta$ and the boundary condition $Bu=0$ (see, e.g., [8]). Then $\mathbb{X}_\rho$ is an ordered Banach space with the order cone $\mathbb{X}_\rho^+$ consisting of all non-negative functions in $\mathbb{X}_\rho$, and $\mathbb{X}_\rho^+$ has non-empty interior $\mathrm{Int}(\mathbb{X}_\rho^+)$. Moreover, $\mathbb{X}_\rho \subset C^{1+m}(\bar{\Omega})$ with continuous inclusion for $m \in [0, 2\rho - 1 - \frac{N}{p})$. Let $\mathcal{X} = \mathbb{X}_\rho \times \mathbb{X}_\rho$ and $\mathcal{X}^+ = \mathbb{X}_\rho^+ \times \mathbb{X}_\rho^+$. Then $(\mathcal{X}, \mathcal{X}^+)$ is an ordered Banach space with the order cone $\mathcal{X}^+$ consisting of all non-negative functions in $\mathcal{X}$, and $\mathcal{X}^+$ has non-empty interior $\mathrm{Int}(\mathcal{X}^+)$. Denote $\|\cdot\|_\rho$ as the norm on $\mathbb{X}_\rho$. Then there exists a constant $l_\rho > 0$ such that $\|\phi\|_\infty := \max_{x\in\bar{\Omega}} |(\phi_1(x), \phi_2(x))| \le l_\rho \|\phi\|_\rho$ for all $\phi \in \mathcal{X}$. Let $\mathcal{E} = C([-\tau, 0], \mathbb{X}_\rho) \times \mathbb{X}_\rho$ and $\mathcal{E}^+ = C([-\tau, 0], \mathbb{X}_\rho^+) \times \mathbb{X}_\rho^+$. For convenience, we will identify an element $\psi \in C([-\tau, 0], \mathbb{X}_\rho)$ as a function from $[-\tau, 0] \times \bar{\Omega}$ to $\mathbb{R}$ defined by $\psi(s, x) = \psi(s)(x)$. For any function $y(\cdot) : [-\tau, b) \to \mathbb{X}_\rho$, where $b > 0$, we define $y_t \in C([-\tau, 0], \mathbb{X}_\rho^+)$ by $y_t(s) = y(t+s)$ for all $s \in [-\tau, 0], t \in [0, b)$. For any $K > L$, let $\mathcal{E}_{\mathbf{K}} = \{\phi \in \mathcal{E} : \mathbf{K} \ge \phi \ge 0\}$.

Note that the differential operator Δ generates an analytic semigroup $\bar{T}_0(t)$ on $L^p(\Omega)$ and standard parabolic maximum principle (see, e.g., [18, Corollary 7.2.3]) implies that the semigroup $\bar{T}_0(t) : \mathbb{X}_\rho \to \mathbb{X}_\rho$ is strongly positive in the sense that $\bar{T}_0(t)(\mathbb{X}_\rho^+ \setminus \{0\}) \subseteq \mathrm{Int}(\mathbb{X}_\rho^+)$ for all $t > 0$. By similar analysis to that in section 2, system (3.1) can be written as an integral equation (2.5) with $U_0(\cdot, \cdot, \phi) \in \mathcal{E}^+$. For any $\phi \in \mathcal{E}_{\mathbf{K}}$, it follows from [16, Corollary 5] that (3.1) admits a unique mild solution $U(t, \cdot, \phi)$ with $U_0(\cdot, \cdot, \phi) = \phi$ on $[0, \infty)$. Moreover, $U(t, x, \phi)$ is a classic solution when $t > \tau$ and the comparison theorem holds for system (3.1).

Define a family of operator $\{Q_t\}_{t\ge 0}$ on $\mathcal{E}^+$ by

$$Q_t[\phi](s, x) := U(t+s, x, \phi) = (u(t+s, x, \phi), v(t, x, \phi)), \qquad \forall \phi \in \mathcal{E}^+,\ x \in \Omega,\ t \ge 0.$$

where $(u(t, x, \phi), v(t, x, \phi))$ is a solution of (3.1) with $(u(s, x), v(0, x)) = (\phi_1(s, x), \phi_2(x))$ for $s \in [-\tau, 0]$ and $x \in \Omega$. Similarly as in section 2, it is easy to see that $\{Q_t\}_{t\ge 0}$ is a monotone periodic semiflow on $\mathcal{E}^+$, and Q_t is strictly subhomogeneous for each $t > 0$. When $t > \tau$, $U(t, x, \phi) > 0$, $\forall x \in \bar{\Omega}$, $\forall \phi \in \mathcal{E}^+$ with $\phi \not\equiv 0$, and hence, Q_t is strongly positive for $t > 2\tau$. Let $n_0 := \inf\{n \in \mathbb{N}_+ : n\omega > 2\tau\}$. Then $Q_{n_0\omega}$ is strongly positive on $\mathcal{E}^+$.

Recall the definition of a global attractor (see, e.g., [24, Chapter 1]). Let $(\mathbf{G}, \rho)$ be a metric space with metric ρ, $\digamma$ is a continuous map. A bounded set $\mathcal{A}$ is said to attract a bounded set $\mathcal{B}$ in $\mathbf{G}$ if $\lim_{n\to\infty} \sup_{x\in\mathcal{B}} d(\digamma^n(x), \mathcal{A}) = 0$. A global attractor for $\digamma : \mathbf{G} \to \mathbf{G}$ is an attractor that attracts every point in $\mathbf{G}$.

Theorem 3.1. *Assume that* (H1)–(H4) *hold. Then $Q_{n_0\omega}$ admits a connected global attractor on $\mathcal{E}^+$.*

Proof. Firstly, we prove that $\{Q_t\}_{t\ge 0}$ is point dissipative on $\mathcal{E}^+$. It suffices to prove that there exists a positive number L such that for any $\phi \in \mathcal{E}^+$,

$$\lim_{t\to\infty} \|U(t, \cdot, \phi)\|_\rho \le L.$$

In the following, we use the arguments similar to the proof of [22, Theorem 3.1](see also [19, Theorem 3.1]). For any $\phi \in \mathcal{E}^+$ and $s \in [-\tau, 0]$, let $W_0 = (\max_{x\in\bar{\Omega}} \phi_1(s, x), \max_{x\in\bar{\Omega}} \phi_2(x))$ and $\bar{U}(t, W_0)$ be the solution of (2.2) with $\bar{U}_0(W_0) = W_0$. Clearly, $\bar{U}(t, W_0)$ is a upper solution of (3.1). By the comparison theorem, we have $U(t, \cdot, \phi) \le \bar{U}(t, W_0)$ for $t > 0$ and $x \in \bar{\Omega}$. In view of Theorem 2.2 (ii), we have $\lim_{t\to\infty} \|\bar{U}(t, W_0)\|_\infty < 2\sigma$, where $\sigma := \max_{t\in[0,\omega]} \|V^*(t)\|_\infty$, and $V^*(t)$ is the unique positive and globally asymptotically stable ω-periodic solution of (2.2). Thus, $\limsup_{t\to\infty} \|U(t, \cdot, \phi)\|_\infty < 2\sigma$ for any $\phi \in \mathcal{E}^+$, which means that there exists a $t_0 > 0$ such that

$$\|U(t, \cdot, \phi)\|_\infty < 2\sigma, \qquad \forall t > t_0, \text{ for any } \phi \in \mathcal{E}^+.$$

Let $\|\cdot\|_0$ is the norm on $\mathbb{X} \times \mathbb{X} := L^p \times L^p$. Then, there is a positive number k such that $\|U\|_0 \le k \|U\|_\infty$. Therefore, for $t \ge t_0$, we see that

$$\|U(t, \cdot, \phi)\|_0 \le k \|U(t, \cdot, \phi)\|_\infty < 2k\sigma,$$

and hence

$$\|U(t+t_0,\cdot,\phi)\|_0 < 2k\sigma, \qquad \forall t \geq 0.$$

By virtue of [9, Lemma 19.3], it follows that there exists two constants γ^* and m such that

$$\|U(t+t_0,\cdot,\phi)\|_\rho \leq mt^{-\gamma^*} \|U(t_0,\cdot,\phi)\|_0 < 2t^{-\gamma^*}mk\sigma, \qquad \forall t \geq 1,$$

where $\rho < \gamma^* < 1$, m depends on γ^*, ρ and σ. Consequently, we have

$$\lim_{t\to\infty} \|U(t,\cdot,\phi)\|_\rho \leq L := 2mk\sigma, \qquad \forall \phi \in \mathcal{E}^+.$$

On the other hand, similar to the proof of Proposition 2.9 in section 2, we can prove that for each $t > 0$, Q_t is α-contracting on $\mathcal{E}^+$ with contracting function $e^{-\nu t}$, where ν is a positive constant. Note that $\mathcal{E}_\mathbf{K}$ is positively invariant and the orbits of bounded sets are bounded. By the continuous-time version of Zhao [24, Theorem 1.1.2], $Q_{n_0\omega}$ admits a connected global attractor which attracts each bounded set in $\mathcal{E}^+$. This completes the proof. □

Consider the linearized system of (3.1) at $(0,0)$ as follows:

$$\begin{cases} \frac{\partial}{\partial t}\tilde{u}(t,x) = d(t)\Delta\tilde{u}(t,x) + \partial_1 f(t,0,0)\tilde{u}(t,x) + \partial_2 f(t,0,0)\tilde{u}(t-\tau,x) \\ \qquad\qquad - \gamma(t)\tilde{u}(t,x) + \beta(t)\tilde{v}(t,x), \\ \frac{\partial}{\partial t}\tilde{v}(t,x) = \gamma(t)\tilde{u}(t,x) - \beta(t)\tilde{v}(t,x), \quad (t,x) \in (0,\infty)\times\Omega, \\ B\tilde{u}(t,x) = B\tilde{v}(t,x) = 0, \quad (t,x) \in (0,\infty)\times\partial\Omega, \\ \tilde{u}(s,x) = \phi_1(s,x), \tilde{v}(0,x) = \phi_2(x), \quad (s,x) \in [-\tau,0]\times\Omega. \end{cases} \tag{3.2}$$

Similar to the proof of Theorem 2.3, we can show that the comparison principle holds for system (3.2).

Now we consider (1.4) and (3.2) as $n_0\omega$-periodic systems. Define the Poincaré map of (3.2) $P : \mathcal{E} \to \mathcal{E}$ by $P(\phi) = \tilde{U}_{n_0\omega}(\phi)$, where $\tilde{U}_{n_0\omega}(\phi)(s,x) = \tilde{U}(n_0\omega + s, x, \phi)$, $\forall (s,x) \in [-\tau,0]\times\Omega$, and $\tilde{U} := (\tilde{u},\tilde{v})$ is the solution of (3.2) with $\tilde{u}(s,x) = \phi_1(s,x)$ and $\tilde{v}(0,x) = \phi_2(x)$, $\forall (s,x) \in [-\tau,0]\times\Omega$. Let r be the spectral radius of P. Similar to the proof Proposition 2.9, we can show that P is α-contracting on $\mathcal{E}^+$. By the strong positivity of P and the generalized Krein–Rutman theorem (see, e.g. [17] or [10, Lemma 2.2]), $r > 0$ and P has a positive eigenfunction $\bar{\phi} := (\bar{\phi}_1,\bar{\phi}_2) \in \mathrm{Int}(\mathcal{E}^+)$ corresponding to r. By a similar argument to that in [11, Lemma 3.2], we can prove the following result.

Lemma 3.2. *Let $\lambda = -\frac{1}{n_0\omega}\ln r$. Then there exists a positive $n_0\omega$-periodic function $\check{V}(t,x)$ such that $e^{-\lambda t}\check{V}(t,x)$ is a solution of (3.2).*

Proof. By the definitions of r and $\bar{\phi}$, we have $P(\bar{\phi}) = r\bar{\phi}$. Let $\bar{U}(t,x,\bar{\phi}) := (\bar{u}(t,x,\bar{\phi}),\bar{v}(t,x,\bar{\phi}))$ be the solution of (3.2) with $\bar{u}(s,x) = \bar{\phi}_1(s,x), \bar{v}(0,x) = \bar{\phi}_2(x)$, $\forall (s,x) \in [-\tau,0]\times\Omega$. Since $\bar{\phi} \gg 0$, we have $\bar{U}(t,x,\bar{\phi}) \gg 0$. Let $\lambda = -\frac{1}{n_0\omega}\ln r$ and $(\check{u}(t,x),\check{v}(t,x)) =: \check{V}(t,x) = e^{\lambda t}\bar{U}(t,x,\bar{\phi})$, $\forall (s,x) \in [-\tau,0]\times\Omega$. Then $r = e^{-n_0\omega\lambda}$ and $\check{V}(t,x) > 0$, $\forall (s,x) \in [-\tau,0]\times\Omega$. Moreover,

$$\frac{\partial}{\partial t}\check{V}(t,x) = e^{\lambda t}\frac{\partial}{\partial t}\bar{U}(t,x,\bar{\phi}) + \lambda e^{\lambda t}\bar{U}(t,x,\bar{\phi}). \tag{3.3}$$

Thus, we have

$$\begin{aligned}\frac{\partial}{\partial t}\check{u}(t,x) &= e^{\lambda t}\frac{\partial}{\partial t}\bar{u}(t,x,\bar{\phi}) + \lambda e^{\lambda t}\bar{u}(t,x,\bar{\phi}) \\ &= e^{\lambda t}[d(t)\Delta\bar{u}(t,x) + \partial_1 f(t,0,0)\bar{u}(t,x) + \partial_2 f(t,0,0)\bar{u}(t-\tau,x) \\ &\quad - \gamma(t)\bar{u}(t,x) + \beta(t)\bar{v}(t,x)] + \lambda\check{u} \\ &= d(t)\Delta\check{u}(t,x) + \partial_1 f(t,0,0)\check{u}(t,x) + e^{\lambda\tau}\partial_2 f(t,0,0)\check{u}(t-\tau,x) \\ &\quad - \gamma(t)\check{u}(t,x) + \beta(t)\check{v}(t,x) + \lambda\check{u},\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial t}\check{v}(t,x) &= e^{\lambda t}\frac{\partial}{\partial t}\bar{v}(t,x,\bar{\phi}) + \lambda e^{\lambda t}\bar{v}(t,x,\bar{\phi}) \\ &= e^{\lambda t}[\gamma(t)\bar{u}(t,x) - \beta(t)\bar{v}(t,x)] + \lambda\check{v} \\ &= \gamma(t)\check{u}(t,x) - \beta(t)\check{v}(t,x) + \lambda\check{v}.\end{aligned}$$

Thus, $\check{V}$ is a solution of $n_0\omega$-periodic system (3.3) with $B\check{V} = 0$ on $(0,+\infty)\times\partial\Omega$ and $\check{u}(s,x) = e^{\lambda s}\bar{\phi}_1(s,x), \check{v}(0,x) = \bar{\phi}_2(x)$, $\forall(s,x)\in[-\tau,0]\times\Omega$.

For any $\theta\in[-\tau,0]$ and $x\in\Omega$, we have

$$\begin{aligned}\check{u}(n_0\omega+\theta,x) &= e^{\lambda(n_0\omega+\theta)}P[\bar{\phi}]_1(\theta,x) = e^{\lambda(n_0\omega+\theta)}r\bar{\phi}_1(\theta,x) \\ &= e^{\lambda\theta}\check{u}(\theta,x,\bar{\phi}) = \check{u}(\theta,x)\end{aligned}$$

and

$$\begin{aligned}\check{v}(n_0\omega,x) &= e^{\lambda n_0\omega}P[\bar{\phi}]_2(x) = e^{\lambda n_0\omega}r\bar{\phi}_2(x) \\ &= \bar{\phi}_2(x) = \check{v}(0,x).\end{aligned}$$

Therefore, $\check{V}(\theta,\cdot) = \check{V}(n_0\omega+\theta,\cdot), \forall\theta\in[-\tau,0]$. It then follows from the existence and uniqueness of solutions of (3.3) that $\check{V}(t,x) = \check{V}(n_0\omega+t,x)$, $\forall(t,x)\in(-\tau,0)\times\Omega$, that is, $\check{V}(t,x)$ is a $n_0\omega$-periodic solution of (3.3). It is easy to see that $e^{-\lambda t}\check{V}(t,x)$ is a solution of (3.2). □

Define $P_0 : \mathcal{E}\to\mathcal{E}$ by $P_0(\phi) = \tilde{U}_\omega(\phi)$ for all $\phi\in\mathcal{E}$, where $\tilde{U}_\omega(\phi)(s,x) = \tilde{U}(\omega+s,x,\phi)$, $\forall(s,x)\in\Omega\times[-\tau,0]$, where $\tilde{U} := (\tilde{u},\tilde{v})$ is the solution of (3.2) with $\tilde{u}(s,x) = \phi_1(s,x), \tilde{v}(0,x) = \phi_2(x)$, $\forall(s,x)\in[-\tau,0]\times\Omega$. Let $r_0 = r(P_0)$ be the spectral radius of P_0.

Theorem 3.3. *Let* (H1)–(H3) *hold and $U(t,x,\phi)$ be the solution of* (3.1) *for any $\phi\in\mathcal{E}^+$. Then the following statements are valid:*

(1) If $r_0 < 1$, then $\lim_{t\to\infty}\|U(t,\cdot,\phi)\|_\rho = 0$ for any $\phi\in\mathcal{E}^+$;

(2) If $r_0 > 1$, then (3.1) *admit a unique positive ω-periodic solution $U^*(t,x)$ and $\lim_{t\to\infty}\|U(t,\cdot,\phi) - U^*(t,\cdot)\|_\rho = 0$ for any $\phi\in\mathcal{E}^+\setminus\{\mathbf{0}\}$.*

Proof. Since $P = \tilde{U}_{n_0\omega}$, $P_0 = \tilde{U}_\omega$, we have $P = P_0^{n_0}$. By the properties of spectral radius of linear operators, it follows that $r(P) = (r(P_0))^{n_0}$, i.e., $r = r_0^{n_0}$. As mentioned in [11, Theorem 3.3], the qualitative solutions of (3.1) and (3.2) do not change whether we consider them as $n_0\omega$-periodic systems or ω-periodic systems. Thus the conditions in Theorem 3.3 can be replaced by $r<1$ and $r>1$, respectively. In what follows, we consider (3.1) and (3.2) as $n_0\omega$-periodic systems and prove the conclusions (1) and (2) under the conditions of $r<1$ and $r>1$, respectively.

In the case where $r < 1$, we have $\lambda = -\frac{1}{n_0\omega}\ln r > 0$. Let $\bar{\phi}$ be the positive eigenfunction of P corresponding to r. In view of Lemma 3.2, it follows that (3.2) admits a solution $\bar{U}(t,x) := \bar{U}(t,x,\bar{\phi}) = e^{-\lambda t}\check{V}(t,x)$ with $\bar{u}(s,x) = \bar{\phi}_1(s,x), \bar{v}(x,0) = \bar{\phi}_2(x)$, $\forall(s,x) \in [-\tau,0]\times\Omega$, where $\check{V}(t,x)$ is $n_0\omega$-periodic in $t > -\tau$. Then $\check{V}(t,x)$ is bounded on $[-\tau,+\infty)\times\Omega$, and hence, there exists an $\eta > 0$ such that $\|\bar{U}(t,\cdot)\|_\infty \le \eta$ for all $t > -\tau$. Thus $\lim_{t\to\infty}\|\bar{U}(t,\cdot)\|_\infty = 0$. Similarly as in the proof of Theorem 3.1, we have $\lim_{t\to\infty}\|\bar{U}(t,\cdot)\|_\rho = 0$.

For a given $\phi \in \mathcal{E}^+$, we can choose a sufficiently small number δ such that $\bar{\phi} \ge \delta\phi$ in $\mathcal{E}^+$. By the comparison principle, we have that $\bar{U}(t,x) \ge \delta\tilde{U}(t,x,\phi)$, $\forall(t,x) \in [-\tau,0]\times\Omega$, where $\tilde{U}(t,x,\phi)$ is is the solution of (3.2) with $\tilde{U}(s,x) = \phi(s,x)$, $\forall(s,x) \in [-\tau,0]\times\Omega$. Thus we have $\lim_{t\to\infty}\|\tilde{U}(\cdot,\phi)\|_\infty = 0$, and hence $\lim_{t\to\infty}\|\tilde{U}(t,\cdot,\phi)\|_\rho = 0$, $\forall\phi \in \mathcal{E}^+$.

Since $U(t,x) = (u(t,x),v(t,x))$ satisfies

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) \le d(t)\Delta u(t,x) + \partial_1 f(t,0,0)u(t,x) + \partial_2 f(t,0,0)u(t-\tau,x) \\ \qquad\qquad - \gamma(t)u(t,x) + \beta(t)v(t,x), \\ \dfrac{\partial}{\partial t}v(t,x) = \gamma(t)u(t,x) - \beta(t)v(t,x), \quad t > 0,\ x \in \Omega, \end{cases}$$

by the comparison theorem for abstract functional differential equations (see, e.g., [16, Proposition 3]), for any $\phi \in \mathcal{E}^+$, we have $U(t,\cdot,\phi) \le \tilde{U}(t,\cdot,\phi)$ for all $t > -\tau$, where $U(t,\cdot,\phi)$ and $\tilde{U}(t,\cdot,\phi)$ are solutions of (3.1) and (3.2), respectively. Therefore, $\lim_{t\to\infty}\|U(t,\cdot,\phi)\|_\rho = 0$ for all $\phi \in \mathcal{E}^+$.

In the case where $r > 1$, we have $\lambda < 0$. Let $M_0 = \{\phi \in \mathcal{E}^+ : \phi \not\equiv \mathbf{0}\}$, $\partial M_0 = \mathcal{E}^+\backslash M_0 = \{\mathbf{0}\}$. By argument similar to the proof of Lemma 2.5, we can show that for any $\phi \in M_0$, the solution of (3.1) satisfies $U(t,x,\phi) > 0$, $\forall(t,x) \in [\tau,\infty)\times\Omega$. Further, $Q_t(M_0) \subset \mathrm{Int}(\mathcal{E}^+)$ for all $t > 2\tau$. Clearly, $Q_t(\mathbf{0}) = \mathbf{0}$, $\forall t \ge 0$. In the following, we show the following claim.

Claim. $\{\mathbf{0}\}$ is a uniform weak repeller for (3.1) in the sense that there exists $\eta_0 > 0$ such that $\limsup_{t\to\infty}\|Q_t(\phi)\|_\rho \ge \eta_0$ holds for any $\phi \in M_0$.

We consider the following $n_0\omega$-periodic system:

$$\begin{cases} \frac{\partial}{\partial t}u^\varepsilon(t,x) = d(t)\Delta u^\varepsilon(t,x) + (\partial_1 f(t,0,0)-\varepsilon)u^\varepsilon(t,x) + (\partial_2 f(t,0,0)-\varepsilon)u^\varepsilon(t-\tau,x) \\ \qquad\qquad - \gamma(t)u^\varepsilon(t,x) + \beta(t)v^\varepsilon(t,x), \\ \frac{\partial}{\partial t}v^\varepsilon(t,x) = \gamma(t)u^\varepsilon(t,x) - \beta(t)v^\varepsilon(t,x), \quad (t,x) \in (0,\infty)\times\Omega, \\ Bu^\varepsilon = Bv^\varepsilon = 0, \quad (t,x) \in (0,\infty)\times\partial\Omega, \\ u^\varepsilon(s,x) = \phi_1(s,x),\ v^\varepsilon(o,x) = \phi_2(x), \quad (t,x) \in [-\tau,0]\times\Omega. \end{cases} \tag{3.4}$$

Define the Poincaré map of (3.4) $P_\varepsilon : \mathcal{E} \to \mathcal{E}$ by

$$P_\varepsilon(\phi) = U^\varepsilon_{n_0\omega}(\phi), \qquad \forall\phi \in \mathcal{E},$$

where $U^\varepsilon_{n_0\omega}(\phi)(s,x) = U^\varepsilon(n_0\omega + s,x,\phi)$, $\forall(s,x) \in [-\tau,0]\times\Omega$, and $U^\varepsilon(t,x,\phi) := (u^\varepsilon(t,x,\phi), v^\varepsilon(t,x,\phi))$ is the solutions of (3.4) with $u^\varepsilon(s,x) = \phi_1(s,x)$ and $v^\varepsilon(0,x) = \phi_2(x)$, $\forall(s,x) \in [-\tau,0]\times\Omega$. Let $r_\varepsilon = r(P_\varepsilon)$ be the spectral radius of P_ε. Since $r = r(P) > 1$, there exists a sufficiently small positive number ε_0 such that $r_\varepsilon > 1$, $\forall\varepsilon \in [0,\varepsilon_0)$. We fix an $\varepsilon \in (0,\varepsilon_0)$. Since

$$\lim_{u\to 0^+,\ w\to 0^+} f(t,u,w) = \partial_1 f(t,0,0)u + \partial_2 f(t,0,0)w$$

uniformly for $t \in [0,n_0\omega]$, there exists $\delta_\varepsilon > 0$ such that

$$f(t,u(t,x),u(t-\tau,x)) > (\partial_1 f(t,0,0)-\varepsilon)u(t,x) + (\partial_2 f(t,0,0)-\varepsilon)u(t-\tau,x)$$

for $t \in [0, n_0\omega]$, $(u,v) \in (0,\delta_\varepsilon) \times (0,\delta_\varepsilon)$. Let $\eta_0 = \delta_\varepsilon / l_\rho$. Suppose, by contradiction, that there exists $\phi^0 \in M_0$ such that $\limsup_{t\to\infty} \|Q_t(\phi^0)\|_\rho < \eta_0$. Then there exists $t_0 > 2\tau$ such that $\|U(t,\cdot,\phi^0)\|_\infty \le l_\rho \|U(t,\cdot,\phi^0)\|_\rho < \delta_\varepsilon$ for all $t \ge t_0$. Thus, $U(t,x,\phi^0)$ satisfies

$$\begin{cases} \frac{\partial}{\partial t} u(t,x) > d(t)\Delta u(t,x) + (\partial_1 f(t,0,0) - \varepsilon) u(t,x) + (\partial_2 f(t,0,0) - \varepsilon) u(t-\tau,x) \\ \qquad\qquad - \gamma(t) u(t,x) + \beta(t) v(t,x), \\ \frac{\partial}{\partial t} v(t,x) = \gamma(t) u(t,x) - \beta(t) v(t,x), \quad (t,x) \in (t_0,\infty) \times \Omega. \end{cases} \tag{3.5}$$

Let $\lambda_\varepsilon = -\frac{1}{n_0\omega} \ln r_\varepsilon$ and $\bar{\phi}_\varepsilon$ be the positive eigenfunction of P_ε associated with r_ε. Similar to the proof of Lemma 3.2, there exists a positive $n_0\omega$-periodic function $\check{V}^\varepsilon(t,x)$ such that $\bar{U}^\varepsilon(t,x,\bar{\phi}_\varepsilon) = e^{-\lambda_\varepsilon t} \check{V}^\varepsilon(t,x)$, $\forall t \ge 0$, where $\bar{U}^\varepsilon(t,x,\bar{\phi}_\varepsilon)$ is a solution of system (3.4) with $\bar{u}^\varepsilon(s,x) = \bar{\phi}_{\varepsilon,1}(s,x)$, $\bar{v}^\varepsilon(0,x) = \bar{\phi}_{\varepsilon,2}(x)$, $\forall (s,x) \in [-\tau,0] \times \Omega$. Since $U(t,x,\phi^0) \in \mathrm{Int}(\mathcal{E}^+)$ for $t \ge 2\tau$ and $x \in \Omega$, there exists $\varrho > 0$ such that

$$U(t_0+s,x,\phi^0) \ge \varrho \bar{U}^\varepsilon(s,x,\bar{\phi}_\varepsilon) = \varrho \bar{\phi}_\varepsilon, \qquad \forall (s,x) \in [-\tau,0] \times \bar{\Omega}.$$

Then, it follows from the comparison theorem and (3.5) that

$$U(t,x,\phi^0) \ge \varrho \bar{U}^\varepsilon(t-t_0,x,\bar{\phi}_\varepsilon) = \varrho e^{-\lambda_\varepsilon (t-t_0)} \check{V}^\varepsilon(t,x), \qquad \forall (t,x) \in [t_0,\infty] \times \bar{\Omega}.$$

Thus, $U(t,x,\phi^0)$ is unbounded due to $\lambda_\varepsilon < 0$, which is a contradiction.

By the claim above, $Q_{n_0\omega}$ is weakly uniformly persistent with respect to $(M_0, \partial M_0)$. As $Q_{n_0\omega}$ admits a global attractor on $\mathcal{E}^+$, according to [24, Theorem 1.3.3], then $Q_{n_0\omega}$ is uniformly persistent with respect to $(M_0, \partial M_0)$.

Note that $Q_{n_0\omega}$ is α-contracting, point dissipative and uniformly persistent. It follows from [24, Theorem 1.3.6] that $Q_{n_0\omega} : M_0 \to M_0$ admits a global attractor $\mathcal{A}_0$ and has a fixed point $\hat{\phi}$ in $\mathcal{A}_0$. Since $Q_{n_0\omega}$ is strictly subhomogeneous, then $Q_{n_0\omega}$ has at most one fixed point according to [25, Lemma 1]. Thus, $Q_{n_0\omega}$ has a unique fixed point $\hat{\phi} \in M_0$. According to the strong monotonicity of $Q_{n_0\omega}$, we have $\hat{\phi} \in \mathrm{Int}(\mathcal{E}^+)$. Due to the strong monotonicity and strict sub-homogeneity of $Q_{n_0\omega}$, it follows from [24, Theorem 2.3.2] that $\mathcal{A}_0 = \{\hat{\phi}\}$. Consequently, $\hat{\phi}$ is globally attractive in M_0 for $Q_{n_0\omega}$.

Let $U(t,x,\hat{\phi})$ be the solution of (3.1) with $u(s,x) = \hat{\phi}_1(s,x)$ and $v(0,x) = \hat{\phi}_2(x)$ for all $(s,x) \in [-\tau,0] \times \Omega$. Since $\hat{\phi}$ is a fixed point of $Q_{n_0\omega}$ and is globally attractive in M_0, $U(t,x,\hat{\phi})$ is a $n_0\omega$-periodic solution of (3.1) which attracts each solution of (3.1) in $\mathcal{E}^+ \setminus \{\mathbf{0}\}$. That is

$$\lim_{t\to\infty} \|U(t,\cdot,\phi) - U(t,\cdot,\hat{\phi})\|_\rho = 0, \qquad \forall \phi \in \mathcal{E}^+ \setminus \{\mathbf{0}\}.$$

It is only necessary to show that $U(t,x,\hat{\phi})$ is also ω-periodic. Since $Q_{n_0\omega}(\hat{\phi}) = \hat{\phi}$, we have $Q_\omega(Q_{n_0\omega}(\hat{\phi})) = Q_\omega(\hat{\phi})$, and hence, $Q_{n_0\omega}(Q_\omega(\hat{\phi})) = Q_\omega(\hat{\phi})$. This implies that $Q_\omega(\hat{\phi})$ is also a fixed point of $Q_{n_0\omega}$. On the other hand, due to the strong positivity of $\hat{\phi}$ and the monotonicity of Q_ω, we have $Q_\omega(\hat{\phi}) \gg 0$. Thus, it follows that $Q_\omega(\hat{\phi}) = \hat{\phi}$ from the uniqueness of fixed point of $Q_{n_0\omega}$, and hence, $U(t,x,\hat{\phi})$ is the ω-periodic solution of (3.1). Therefore, $U^*(t,x) := U(t,x,\hat{\phi}) = (u(t,x,\hat{\phi}), v(t,x,\hat{\phi}))$ is the ω-periodic solution of (3.1) which attracts all the solutions of (3.1). This completes the proof. □

4 Appendix

In this Appendix, we present some results of [3,12,13] about spreading speeds and traveling waves for monotone evolution systems.

Definition 4.1. Let $(\mathbb{X}, \|\cdot\|)$ be a Banach space over $\mathbb{R}$. For a bounded subset B of $\mathbb{X}$, the Kuratowski measure of noncompactness of B is defined as

$$\alpha(B) = \inf\{r > 0 : B \text{ has a finite cover of diameter } r\}.$$

Let X be the set of all bounded and continuous functions from $\mathcal{H}$ to $\mathbb{R}^k$, where $\mathcal{H} = \mathbb{R}$ or $\mathbb{Z}$. Let τ be a nonnegative real number and $\mathbb{C}$ be the set of all bounded and continuous functions from $[-\tau, 0] \times \mathcal{H}$ to $\mathbb{R}^k$. Clearly, any vector in $\mathbb{R}^k$ and any element in the space $\bar{\mathbb{C}} := C([-\tau, 0], \mathbb{R}^k)$ can be regarded as a function in $\mathbb{C}$.

For $u = (u^1, \ldots, u^k)$ and $v = (v^1, \ldots, v^k) \in \mathbb{C}$, we write $u \geq v (u \gg v)$ provided $u^i(\theta, x) \geq v^i(\theta, x)(u^i(\theta, x) > v^i(\theta, x)), \forall i = 1, \ldots, k, \theta \in [-\tau, 0]$, and $x \in \mathcal{H}$; and $u > v$ provided $u \geq v$ but $u \neq v$. For any two vectors a and b in $\mathbb{R}^k$ or two functions $a, b \in \bar{\mathbb{C}}$, we can define $a \geq (>, \gg) b$ similarly. For any $r \in \bar{\mathbb{C}}$ with $r \gg 0$, we define $\mathbb{C}_r := \{u \in \mathbb{C} : r \geq u \geq 0\}$ and $\bar{\mathbb{C}}_r := \{u \in \bar{\mathbb{C}} : r \geq u \geq 0\}$.

We always equip $\mathbb{C}$ with the maximum norm $\|\cdot\|$ and the positive cone $\bar{\mathbb{C}}_+ := \{\phi \in \bar{\mathbb{C}} : \phi(\theta) \geq 0, \ \forall \theta \in [-\tau, 0]\}$ so that $\mathbb{C}$ is an ordered Banach space. We also equip $\mathbb{C}$ with the compact open topology, that is, $v^n \to v$ in $\mathbb{C}$ means that the sequence of functions $v^n(\theta, x)$ converges to $v(\theta, x)$ uniformly for (θ, x) in every compact set. Moreover, we can define the metric function $d(\cdot, \cdot)$ in $\mathbb{C}$ with respect to this topology by

$$d(u, v) = \sum_{k=0}^{\infty} \frac{\max_{|x| \leq k, \theta \in [-\tau, 0]} |u(\theta, x) - v(\theta, x)|}{2^k} \qquad \forall u, v \in \mathbb{C}.$$

So that $(\mathbb{C}, d)$ is a metric space.

Define the reflection operator R by $R[u](\theta, x) := u(\theta, -x)$, and the translation operator T_y by $T_y[u](\theta, x) := u(\theta, x - y)$ for all $\theta \in [-\tau, 0], x \in \mathbb{R}$. Let $\beta \in \bar{\mathbb{C}}$ with $\beta \gg 0$, and $Q : \mathbb{C}_\beta \to \mathbb{C}_\beta$ be a map. We impose the following hypotheses on Q.

(A1) $Q[R[u]] = R[Q[u]], T_y[Q[u]] = Q[T_y[u]], \ \forall y \in \mathcal{H}$.

(A2) $Q : \mathbb{C}_\beta \to \mathbb{C}_\beta$ is continuous with respect to the compact open topology.

(A3) One of the following two conditions holds:

- **(a)** $\{Q[u](\cdot, x) : u \in \mathbb{C}_\beta, x \in \mathcal{H}\}$ is a precompact subset of $\bar{\mathbb{C}}$.
- **(b′)** The set $Q[\mathbb{C}_\beta](0, \cdot)$ is precompact in X, and there is a positive number $\varsigma \leq \tau$ such that $Q[u](\theta, x) = u(\theta + \varsigma, x)$ for $-\tau \leq \theta < -\varsigma$, and the operator

$$S[u](\theta, x) := \begin{cases} u(0, x), & -\tau \leq \theta < -\varsigma, \\ Q[u](\theta, x), & -\varsigma \leq \theta \leq 0 \end{cases}$$

  has the property that $S[D](\cdot, 0)$ is precompact in $\bar{\mathbb{C}}_\beta$ for any T-invariant set $D \subset \mathbb{C}_\beta$ with $D(0, \cdot)$ precompact in X. A set $D \subset \mathbb{C}_\beta$ is said to be T-invariant if $T_y D = D$ for all $y \in \mathbb{R}$.

(A4) $Q : \mathbb{C}_\beta \to \mathbb{C}_\beta$ is monotone in the sense that $Q[u_1] \geq Q[u_2]$ whenever $u_1 \geq u_2$ in $\mathbb{C}_\beta$.

(A5) $Q : \bar{\mathbb{C}}_\beta \to \bar{\mathbb{C}}_\beta$ admits exactly two fixed points 0 and β, and for any positive number ϵ, there is an $\alpha \in \bar{\mathbb{C}}_\beta$ with $\|\alpha\|_{\bar{\mathbb{C}}} < \epsilon$ such that $Q[\alpha] \gg \alpha$.

Theorem 4.2 ([13, Theorems 2.11, 2.15 and Corollary 2.16])**.** *Suppose that Q satisfies* (A1)–(A5). *Let $u_0 \in \mathbb{C}_\beta$ and $u_n = Q[u_{n-1}]$ for $n \geq 1$. Then there is a real number c_+ such that the following statements are valid.*

(1) For any $c > c_+$, if $0 \leq u_0 \ll \beta$ and $u_0(\cdot, x) = 0$ for x outside a bounded interval, then $\lim_{n\to\infty, |x|\geq nc} u_n(\theta, x) = 0$ uniformly for $\theta \in [-\tau, 0]$.

(2) For any $c < c_+$ and any $\sigma \in \bar{\mathbb{C}}_\beta$ with $\sigma \gg 0$, there exists r_σ such that if $u_0(\cdot, x) \geq \sigma(\cdot)$ for x on an interval of length $2r_\sigma$, then $\lim_{n\to\infty, |x|\leq nc} u_n(\theta, x) = \beta(\theta)$ uniformly for $\theta \in [-\tau, 0]$. If, in addition, Q is subhomogeneous on $\mathbb{C}_\beta$, then r_σ can be chosen to be independent of $\sigma \gg 0$.

Remark 4.3. We call c_+ the asymptotic speed of spread (in short, spreading speed) of a discrete-time semiflow $\{Q^n\}_{n=0}^\infty$ on $\bar{\mathbb{C}}_\beta$ provided that Theorem 4.2 hold (see [13, Section 2]).

Remark 4.4. Theorem 4.2 is still valid if we replace (A3)(a) with the following weaker assumption.

(A3) (a′) There is a number $l \in [0, 1)$ such that for any $A \subset \mathbb{C}_\beta$ and $x \in \mathcal{H}$, $\alpha(\{Q[u](\cdot, x) : u \in A\}) \leq l\alpha(\{u(\cdot, x) : u \in A\})$.

A linear operators approach was also developed in [13] to estimate the spreading speed c of Q. Let $M : \mathbb{C} \to \mathbb{C}$ be a linear operator with the following properties.

(C1) M is continuous with respect to the compact open topology.

(C2) M is a positive operator, that is, $M[v] \geq 0$ whenever $v > 0$.

(C3) M satisfies (A3) with $\mathbb{C}_\beta$ replaced by any subset of $\mathbb{C}$ consisting of uniformly bounded functions.

(C4) $M[R[u]] = R[M[u]], T_y[M[u]] = M[T_y[u]] \quad \forall u \in \mathbb{C}, y \in \mathcal{H}$.

(C5) M can be extended to a linear operator on the linear space $\tilde{C}$ of all function $v \in C([-\tau, 0] \times \mathcal{H}, \mathbb{R}^k)$ having the form

$$v(\theta, x) = v_1(\theta, x)e^{\mu_1 x} + v_2(\theta, x)e^{\mu_2 x}, \qquad v_1, v_2 \in \mathbb{C}, \mu_1, \mu_2 \in \mathbb{R},$$

such that if $v_n, v \in \mathbb{C}$ and $v_n(\theta, x) \to v(\theta, x)$ uniformly on any bounded set, then $M[v_n](\theta, x) \to M[v](\theta, x)$ uniformly on any bounded set. We note that hypothesis (C4) implies that $M[v] \in \mathbb{C}$ whenever $v \in \mathbb{C}$, and hence, M is also a linear operator on $\bar{\mathcal{C}}$.

Define the linear map $B_\mu : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ by

$$B_\mu[\alpha](\theta) = M[\alpha e^{-\mu x}](\theta, 0), \qquad \forall \theta \in [-\tau, 0].$$

In particular, $B_0 = M$ on $\bar{\mathbb{C}}$. If $\alpha_n, \alpha \in \bar{\mathbb{C}}$ and $\alpha_n \to \alpha$ as $n \to \infty$, then $\alpha_n(\theta)e^{-\mu x} \to \alpha(\theta)e^{-\mu x}$ uniformly on any bounded subset of $[-\tau, 0] \times \mathcal{H}$. Thus, we have $B_\mu[\alpha_n] = M[\alpha_n e^{-\mu x}](\cdot, 0) \to M[\alpha e^{-\mu x}](\cdot, 0) = B_\mu[\alpha]$, and hence B_μ is continuous. Moreover, B_μ is a positive operator on $\mathbb{C}$. We assume that

(C6) For any $\mu \geq 0$, B_μ is a positive operator, and there is n_0 such that

$$B_\mu^{n_0} = \underbrace{B_\mu \circ \cdots \circ B_\mu}_{n_0}$$

is a compact and strongly positive linear operator on $\mathbb{C}$.

It then follows from that B_μ has a principal eigenvalue $\lambda(\mu)$ with a strongly positive eigenfunction. The following condition is needed for the estimate of the spreading speed c^*.

(C7) The principal eigenvalue $\lambda(0)$ of B_0 is larger than 1.

We say that M has compact support provided there is some ρ such that for any $\alpha \in \mathbb{C}$, $M[\alpha](\theta, x)$ depends only on the value of ęÁ in $[-\tau, 0] \times [x-\rho, \rho + x]$.

Theorem 4.5 ([13, Theorem 3.10]). *Let Q be an operator on $\mathbb{C}_\beta$ satisfying* (A1)–(A5) *and c^* be its asymptotic speed of spread. Assume that the linear operator M satisfies* (C1)–(C7) *and that either M has compact support, or the infimum of $\Phi(\mu) := \frac{1}{\mu}\ln\lambda(\mu)$ is attained at some finite value μ^* and $\Phi(+\infty) > \Phi(\mu^*)$. Then the following statements are valid.*

(1) If $Q[u] \le M[u]$ for all $u \in \mathbb{C}_\beta$, then $c^* \le \inf_{\mu>0}\Phi(\mu)$.

(2) If there is some $\eta \in \bar{\mathbb{C}}$ with $\eta \gg 0$ such that $Q[u] \ge M[u]$ for any $u \in \mathbb{C}_\eta$, then $c^* \ge \inf_{\mu>0}\Phi(\mu)$.

Remark 4.6. Theorem 4.5 is still valid if we replace (C6) with the following assumption.

(C6′) For any $\mu \ge 0$, B_μ is a positive operator, and there exist n_0 and $l \in [0,1)$ such that

$$B_\mu^{n_0} = \underbrace{B_\mu \circ \cdots \circ B_\mu}_{n_0}$$

is a strongly positive linear operator on $\bar{\mathbb{C}}$ and $\alpha(B_\mu^{n_0}(A)) \le l\alpha(A)$ for any bounded subset A of $\bar{\mathbb{C}}$.

Definition 4.7. Let $\omega > 0$ and $r \in \bar{\mathbb{C}}$ with $r \gg 0$ be given. A family of mappings $\{Q_t\}_{t\ge0}$ is said to be an ω-periodic semiflow on $\mathbb{C}_r$ provided Q_t has the following properties.

(i) $Q_0[v] = v, \forall v \in \mathbb{C}_r$.

(ii) $Q_{t+\omega}[v] = Q_t[Q_\omega[v]],\ \forall t \ge 0,\ v \in \mathbb{C}_r$.

(iii) $Q(t, v) := Q_t(v)$ is continuous in (t, v) on $[0,\infty) \times \mathbb{C}_r$.

The mapping Q_ω is called the Poincaré (or periodic) map associated with this periodic semiflow.

It is easy to see that property (iii) holds if $Q(\cdot, v)$ is continuous on $[0, +\infty)$ for each $v \in \mathbb{C}_r$, and $Q(t, \cdot)$ is continuous uniformly for t in bounded intervals in the sense that for any $v_0 \in \mathbb{C}_r$, bounded interval I and $\epsilon > 0$, there exists $\delta = \delta(v_0, I, \epsilon) > 0$ such that if $d(v, v_0) < \delta$, then $d(Q_t[v], Q_t[v_0]) < \epsilon$ for all $t \in I$.

Theorem 4.8 ([12, Theorem 2.1]). *Let $r \in \bar{\mathbb{C}}$ with $r \gg 0$, $\{Q_t\}_{t\ge0}$ be an ω-periodic semiflow on C_r with two x-independent ω-periodic orbits $0 \ll \beta(t)$. Suppose that the Poincaré map $Q = Q_\omega$ satisfies all hypotheses* (A1)–(A5) *with $\beta = \beta(0)$, and Q_t satisfies* (A1) *for any $t > 0$. Let c_ω^* be the asymptotic speed of spread for Q_ω. Then the following statements are valid.*

(1) For any $c > c_\omega^*/\omega$, if $v \in \mathbb{C}_\beta$ with $0 \le v \ll \beta$, and $v(\cdot, x) = 0$ for x outside a bounded interval. then $\lim_{t\to\infty, |x|\ge ct} Q_t[v](\theta, x) = 0$ uniformly for $\theta \in [-\tau, 0]$.

(2) For any $c < c^*_\omega/\omega$ and $\sigma \in \bar{\mathbb{C}}_\beta$ with $\sigma \gg 0$, there is a positive number r_σ such that if $v \in \mathbb{C}_\beta$and $v(\cdot, x) \gg \sigma(\cdot)$ for x on an interval of length $2r_\sigma$, then $\lim_{t\to\infty,|x|\le ct}(Q_t[v](\theta, x) - \beta(t)(\theta)) = 0$ uniformly for $\theta \in [-\tau, 0]$. If, in addition, Q_ω is subhomogeneous, then r_σ can be chosen to be independent of $\sigma \gg 0$.

Definition 4.9. We say that $W(\theta, t, x - ct)$ is a periodic traveling wave of the ω-periodic semiflow $\{Q_t\}_{t\ge0}$ if the vector-valued function $W(\theta, t, z)$ is ω-periodic in t and $Q_t[W(\cdot, 0, \cdot)](\theta, x) = W(\theta, t, x - ct)$, and that $W(\theta, t, x - ct)$ connects $\beta(t)$ to 0 if $W(\cdot, t, -\infty) = \beta(t)$ and $W(\cdot, t, +\infty) = 0$.

Theorem 4.10 ([12, Theorems 2.2, 2.3]). *Suppose that $\mathcal{H} = \mathbb{R}$ and Q_ω satisfies hypothesis* (A1)–(A6) *with $\beta = \beta(0)$, and let c^*_ω be the asymptotic speed of spread of Q_ω.*

(1) For any $0 < c < c^*_\omega/\omega$, $\{Q_t\}_{t\ge0}$ has no ω-periodic traveling wave $W(\theta, t, x - ct)$ connecting $\beta(t)$ to 0.

(2) If Q_t satisfies (A1) *and* (A4) *for each $t > 0$. Then for any $c \ge c^*_\omega/\omega$, $\{Q_t\}_{t\ge0}$ has an ω-periodic traveling wave $U(\theta, t, x - ct)$ connecting $\beta(t)$ to 0 such that $U(\theta, t, s)$ is continuous, and nonincreasing in $s \in \mathbb{R}$.*

In the following, we collect the abstract results on traveling waves in [3] (see also e.g., [2]). Let $\mathcal{M}$ be the space consisting of all monotone functions from $\mathbb{R}$ to $\bar{\mathbb{C}}$. For any $\phi, \psi \in \mathcal{M}$, we write $w \ge z$ if $w(x) \ge z(x)$ for $x \in \mathbb{R}$ and $w > z$ if $w \ge z$ but $w \ne z$. Equip $\mathcal{M}$ with the compact open topology. Similar to $\mathbb{C}_r$, we can define $\mathcal{M}_r = \{\phi \in \mathcal{M} : \phi \in \bar{\mathbb{C}}_r\}$. Giving a subset $A \subseteq \mathcal{M}$ and $p \in \mathbb{R}$, we define $A(p) := \{W(p) : W \in A\}$. In the following, we make some assumptions for a given operator $Q : \mathcal{M}_\beta \to \mathcal{M}_\beta$ (see [2,3]).

(B1) $T_y \circ Q = Q \circ T_y$ for $y \in \mathbb{R}$;

(B2) if $W_k \to W$ in $\mathcal{M}$, then $Q[W_k](x) \to Q[W](x)$ almost everywhere in $\bar{\mathbb{C}}$;

(B3) there exists $k_\alpha \in [0, 1]$ such that $\alpha(Q[\mathcal{W}](0)) \le k_\alpha \alpha(\mathcal{W}(0))$ for $\mathcal{W} \subseteq \mathcal{M}_\beta$;

(B4) $Q : \mathcal{M}_\beta \to \mathcal{M}_\beta$ is monotone in the sense that $Q[W_1] \ge Q[W_2]$ whenever $W_1 \ge W_2$ in $\mathcal{M}_\beta$;

(B5) $Q : \bar{\mathbb{C}}_\beta \to \bar{\mathbb{C}}_\beta$ admits two fixed points 0 and β in $\bar{\mathbb{C}}^+$, and for $\varpi \in \bar{\mathbb{C}}^+$ with $0 \ll \varpi \ll \beta$, $\lim_{n\to\infty} Q^n[\varpi] = \beta$.

Definition 4.11. For any number c, we say that $W : [-\tau, 0] \times \mathbb{R} \to \mathbb{R}^k (k \ge 1)$ is a traveling wave of the map Q connecting β to 0 with the wave speed c if $Q^n[W](\theta, x) = W(\theta, x - cn)$, $W(\cdot, -\infty) = \beta$ and $W(\cdot, \infty) = 0$.

In the case where the system admits no advection, the upper and lower bounds of rightward spreading speeds in [3, Theorem 3.8] are same, hence we have the following observation.

Proposition 4.12. *Assume that Q satisfies* (B1)–(B5). *Let c_+ be the spreading speed of Q, then the following statements are valid:*

(1) for any $c > c_+$, there exists a continuous traveling wave $W(x - cn)$ connecting β to 0;

(2) for any $c < c_+$, there is no traveling wave connecting β to 0.

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