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On a superlinear periodic boundary value problem with vanishing Green's function

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Abstract. We prove the existence of positive solutions for the boundary value problem

$$\begin{cases} y'' + a(t)y = \lambda g(t) f(y), & 0 \le t \le 2\pi, \\ y(0) = y(2\pi), & y'(0) = y'(2\pi), \end{cases}$$

where λ is a positive parameter, f is superlinear at ∞ and could change sign, and the associated Green's function may have zeros.

Keywords: superlinear, periodic, vanishing Green's function.

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1 Introduction

In this paper, we consider the existence of nonnegative solutions for the periodic boundary value problem

$$\begin{cases} y'' + a(t)y = \lambda g(t) f(y), & 0 \le t \le 2\pi, \\ y(0) = y(2\pi), & y'(0) = y'(2\pi), \end{cases} \tag{1.1}$$

where the associated Green's function is nonnegative and f is allowed to change sign. When $a(t) = m^2$, where m is a positive constant and $m \neq 1, 2, \dots,$ the Green's function for (1.1) is given by

$$G(t,s) = \frac{\sin(m|t-s|) + \sin m(2\pi - (|t-s|)}{2m(1-\cos 2m\pi)}, \qquad s,t \in [0, 2\pi].$$

Note that $G(t,s) > 0$ on $[0,2\pi] \times [0,2\pi]$ iff $m < 1/2$ and $G(t,s) \ge 0 = G(s,s)$ on $[0,2\pi] \times [0,2\pi]$ if $m = 1/2$. For a general nonnegative time-dependent $a \in L^p(0,2\pi)$, $1 \le p \le \infty$, Torres [14] showed that the Green's function for (1.1) is positive (resp. nonnegative) provided that $a >$

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0 on a set of positive measure, $\|a\|_p < K(2p^*)$ (resp. $\|a\|_p \le K(2p^*)$), where $p^* = p/(p-1)$ and

$$K(q) = \begin{cases} \frac{1}{q(2\pi)^{1/q}} \left(\frac{2}{2+q}\right)^{1-2/q} \left(\frac{\Gamma(\frac{1}{q})}{\Gamma(\frac{1}{2}+\frac{1}{q})}\right)^2 & \text{if } 1 \le q < \infty, \\ \frac{1}{2\pi} & \text{if } q = \infty. \end{cases}$$

In particular, when $a \in L^\infty(0, 2\pi)$, the Green's function is positive if $\|a\|_\infty < 1/4$ and nonnegative if $\|a\|_\infty \le 1/4$, which have been obtained in [12] when a is a constant. These conditions were extended to sign-changing $a(t)$ with nonnegative average in [5]. Existence results for positive solutions of (1.1) when the associated Green's function is positive have been obtained in [2,4,7,8,11,13,14,18] using Krasnosel'skii's fixed point theorem on the cone

$$K = \left\{ u \in C[0, 2\pi] : u(t) \ge \frac{A}{B} \|u\|_\infty \ \ \forall t \right\},$$

where A and B denote the minimum and maximum values of $G(t, s)$ on $[0, 2\pi] \times [0, 2\pi]$ respectively. When $A = 0$, this cone becomes the cone of nonnegative functions and is not effective in obtaining the desired estimates. The case when the Green's function $G(t, s)$ is nonnegative but $\beta = \min_{0 \le s \le 2\pi} \int_0^{2\pi} G(t, s)dt$ is positive was studied by Graef et al. in [6]. Specifically, assume g is continuous with $g(t) > 0 \ \forall t \in [0, 2\pi]$, they proved that (1.1) has a nonnegative solution for all $\lambda > 0$ when f is continuous, nonnegative with $f_0 = \infty$, $f_\infty = 0$ (sublinear), or when $f_0 = 0$, $f_\infty = \infty$ (superlinear) and f is convex. Here $f_0 = \lim_{u \to 0^+} \frac{f(u)}{u}$, $f_\infty = \lim_{u \to \infty} \frac{f(u)}{u}$. The method used in [6] is Krasnosel'skii's fixed point theorem on the cone

$$K = \left\{ u \in C[0, 2\pi] : u \ge 0 \text{ on } [0, 2\pi] \text{ and } \int_0^{2\pi} u(t)dt \ge \frac{\beta}{B} \|u\|_\infty \right\}.$$

The results in [6] were improved by Webb [16], in which g is allowed to be 0 at some points and the existence of nonnegative nontrivial solutions were obtained when $f \ge 0$ and either $f_\infty < \mu_{1,\lambda} < f_0$ (sublinear) or $f_0 < \mu_{1,\lambda}$, $\frac{f(R)}{R}$ is large enough and f is convex on $[0, T_\lambda]$ for a specific $T_\lambda > 0$ (superlinear), where $\mu_{1,\lambda}$ denote the principal characteristic value of the linear operator

$$L_\lambda u = \lambda \int_0^{2\pi} G(t, s)g(s)u(s)ds$$

on $C[0, 2\pi]$. The approach in [16] depends on fixed point theory on the modified cone

$$\tilde{K} = \left\{ u \in C[0, 2\pi] : u \ge 0 \text{ on } [0, 2\pi] \text{ and } \int_0^{2\pi} g(t)u(t)dt \ge B_0 \|u\|_\infty \right\},$$

where B_0 is a suitable positive constant. For results on the system

$$\begin{cases} y_i'' + a_i(t)y = \lambda g_i(t) f_i(y), \quad 0 \le t \le 2\pi, \\ y_i(0) = y_i(2\pi), \quad y_i'(0) = y_i'(2\pi), \quad i = 1, \dots, n, \end{cases}$$

see [9], where both the sublinear and superlinear cases were discussed. Note that convexity is needed for one of the f_i in the superlinear case. Related results in the sublinear case when the Green's function is nonnegative can be found in [4]. We refer to [10] for results in the case when the Green's function may change sign. In this paper, motivated by the results in [6,16], we shall establish the existence of positive solutions to (1.1) when the Green's function is nonnegative, and f is superlinear at ∞ without assuming convexity of f. We also allow the case when f can change sign. Note that nonnegative and convexity assumptions of f are essential for some of the proofs in [6, 16]. Our approach depends on a Krasnosel'skii type fixed point theorem in a Banach space.

We shall make the following assumptions:

(A1) $f : [0, \infty) \to \mathbb{R}$ is continuous;

(A2) $a : [0, 2\pi] \to [0, \infty)$ is continuous, $a(t) \leq 1/4$ for all t, and $a \not\equiv 0$;

(A3) $g \in L^1(0, 2\pi), g \geq 0$ and $g \not\equiv 0$ on any subinterval of $(0, 2\pi)$.

Our main result is the following.

Theorem 1.1. *Let (A1)–(A3) hold. Then*

(i) if $f_0 = 0$, $f_\infty = \infty$, and $f \geq 0$ then (1.1) has a positive solution for all $\lambda > 0$;

(ii) if $f_\infty = \infty$, then there exists a constant $\lambda^* > 0$ such that (1.1) has a positive solution y_λ for $\lambda < \lambda^*$. Furthermore $\|y_\lambda\|_\infty \to \infty$ as $\lambda \to 0^+$.

Example 1.2. Let c be a nonnegative constant, g satisfy (A3), and a satisfy (A2). Let $f(y) = y^\alpha \cos^2\left(\frac{1}{y}\right) - c$ for $y > 0$, $f(0) = -c$, where $\alpha > 1$. Then Theorem 1.1 (i) gives the existence of a positive solution to (1.1) for $c = 0$ and $\lambda > 0$, while if $c > 0$, Theorem 1.1 (ii) gives the existence of a large positive solution to (1.1) for $\lambda > 0$ small. Note that when $\alpha > 1, f$ is not convex on $[0, T)$ for any $T > 0$ since it is easy to see that $f\left(\frac{y}{2}\right) \not\leq \frac{1}{2}(f(y) + f(0)$ when $y = \left(\frac{\pi}{2} + 2n\pi\right)^{-1}$, $n \in \mathbb{N}$. Hence the results in [6, 16] cannot be applied here.

2 Preliminary results

Let $AC^1[0, 2\pi] = \{u \in C^1[0, 2\pi] : u'$ is absolutely continuous on $[0, 2\pi]\}$. We first recall the following fixed point result of Krasnosel'skii type in a Banach space (see e.g. [1, Theorem 12.3]).

Lemma A. Let X be a Banach space and $T : X \to X$ be a compact operator. Suppose there exist $h \in X, h \neq 0$and positive constants r, R with $r \neq R$ such that

(a) If $y \in X$ satisfies $y = \theta T y$ for some $\theta \in (0, 1]$, then $\|y\| \neq r$;

(b) If $y \in X$ satisfies $y = Ty + \xi h$ for some $\xi \geq 0$, then $\|y\| \neq R$.

Then T has a fixed point $y \in X$ with $\min(r, R) < \|y\| < \max(r, R)$.

Lemma 2.1. *Let $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$ and let $y \in AC^1[\alpha, \beta]$ be a nonnegative solution of*

$$y'' + \frac{1}{4} y \geq 0 \quad \textit{a.e. on } (\alpha, \beta). \tag{2.1}$$

Suppose one of the following conditions holds

(i) $y'(\alpha) = y(\beta) = 0$ or $y(\alpha) = y'(\beta) = 0$ and $\beta - \alpha < \pi$,

(ii) $y(\alpha) = y(\beta) = 0$ and $\beta - \alpha < 2\pi$,

(iii) $y(\alpha) = y(\beta) = 0$, $y'(\alpha) = y'(\beta)$, and $\beta - \alpha = 2\pi$.

Then $y \equiv 0$ on $[\alpha, \beta]$.

Proof. (i) Suppose $y'(\alpha) = y(\beta) = 0$. Multiplying (2.1) by $\sin\left(\frac{\pi(\beta-t)}{2(\beta-\alpha)}\right)$ and integrating on $[\alpha, \beta]$, we obtain

$$0 \geq \left(\frac{1}{4} - \left(\frac{\pi}{2(\beta-\alpha)}\right)^2\right) \int_\alpha^\beta y(t) \sin\left(\frac{\pi(\beta-t)}{2(\beta-\alpha)}\right) dt \geq 0,$$

which implies $y \equiv 0$ on $[\alpha, \beta]$. On the other hand, if $y(\alpha) = y'(\beta) = 0$ then the function $\tilde{y}(t) = y(\beta + \alpha - t)$ satisfies $\tilde{y}'(\alpha) = \tilde{y}(\beta) = 0$ and (2.1). Hence $\tilde{y} \equiv 0$ i.e. $y \equiv 0$ on $[\alpha, \beta]$, which completes the proof.

(ii) Multiplying (2.1) by $\sin\left(\frac{\pi(\beta-t)}{\beta-\alpha}\right)$ and integrating on $[\alpha, \beta]$, we obtain

$$0 \geq \left(\frac{1}{4} - \left(\frac{\pi}{\beta-\alpha}\right)^2\right) \int_\alpha^\beta y(t) \sin\left(\frac{\pi(\beta-t)}{\beta-\alpha}\right) dt \geq 0,$$

which implies $y \equiv 0$ on $[\alpha, \beta]$.

(iii) Let $\tau \in [\alpha, \beta]$ and $h(t) = y''(t) + \frac{1}{4}y(t)$.

Multiplying the equation

$$y'' + \frac{1}{4}y = h(t) \tag{2.2}$$

by $\sin\left(\frac{\tau-t}{2}\right)$ and integrating on $[\alpha, \tau]$ gives

$$\frac{1}{2}y(\tau) - y'(\alpha)\sin\left(\frac{\tau-\alpha}{2}\right) = \int_\alpha^\tau h(t)\sin\left(\frac{\tau-t}{2}\right) dt. \tag{2.3}$$

Next, multiplying (2.2) by $\sin\left(\frac{t-\tau}{2}\right)$ and integrating on $[\tau, \beta]$ gives

$$\frac{1}{2}y(\tau) + y'(\beta)\sin\left(\frac{\beta-\tau}{2}\right) = \int_\tau^\beta h(t)\sin\left(\frac{t-\tau}{2}\right) dt. \tag{2.4}$$

Adding (2.3), (2.4) and using $y'(\alpha) = y'(\beta)$ together with $\beta = \alpha + 2\pi$, we obtain

$$y(\tau) = \int_\alpha^\tau h(t)\sin\left(\frac{\tau-t}{2}\right) dt + \int_\tau^\beta h(t)\sin\left(\frac{t-\tau}{2}\right) dt. \tag{2.5}$$

Since $y(\alpha) = 0$ and $h(t)\sin\left(\frac{t-\alpha}{2}\right) \geq 0$ on (α, β), it follows that $h(t)\sin\left(\frac{t-\alpha}{2}\right) = 0$ for a.e. $t \in (\alpha, \beta)$. Hence $h \equiv 0$ and therefore (2.5) implies $y(\tau) = 0$ for all $\tau \in [\alpha, \beta]$, which completes the proof. □

As a consequence of Lemma 2.1, we have the following result, which was obtained in [15] (see also [12] when a is a constant). However, our proof is new and simple. We refer to [17] for related results when $a \in L^1(\mathbb{S}, \mathbb{R})$, where $\mathbb{S}$ is the circle of length 1.

Corollary 2.2. *Let $y \in AC^1[0, 2\pi]$ satisfy*

$$\begin{cases} y'' + a(t)y \geq 0 \quad \text{a.e. on } [0, 2\pi], \\ y(0) = y(2\pi), \quad y'(0) = y'(2\pi). \end{cases} \tag{2.6}$$

Then either $y > 0$ on $[0, 2\pi]$ or $y \equiv 0$ on $[0, 2\pi]$. In particular, if y_i, $i = 1, 2$, satisfy

$$\begin{cases} y_1'' + a(t)y_1 \geq y_2'' + a(t)y_2 \quad \text{a.e. on} [0, 2\pi], \\ y_i(0 = y_i(2\pi), \quad y_i'(0) = y_i'(2\pi), \quad i = 1, 2, \end{cases}$$

then $y_1 \geq y_2$ on $[0, 2\pi]$.

Proof. Extend y to be a 2π-periodic function on $\mathbb{R}$. Then $y \in C^1(\mathbb{R})$ and y' is absolutely continuous on $\mathbb{R}$. Suppose $y(\tau) > 0$ for some $\tau \in [0, 2\pi]$. We claim that $y > 0$ on $[0, 2\pi]$. Suppose to the contrary that $y(\tau_0) \le 0$ for some $\tau_0 \in [0, 2\pi]$. Since $y(\tau_0) = y(\tau_0 \pm 2\pi)$, there exists an interval (α, β) containing τ such that $y > 0$ on (α, β), $y(\alpha) = y(\beta) = 0$, $0 < \beta - \alpha \le 2\pi$, and (2.1) holds, which contradicts Lemma 2.1(ii) and (iii). Hence $y > 0$ on $[0, 2\pi]$ as claimed. On the other hand, if $y \le 0$ on $[0, 2\pi]$ then $y'' \ge 0$ a.e. on $[0, 2\pi]$. Let $y(\tau_1) = \max_{t\in[0,2\pi]} y(t)$. Then $y'(\tau_1) = 0$, and hence $y(t) = y(\tau_1)$ for all $t \in [0, 2\pi]$. Hence (2.6) immediately gives $y \ge 0$ on $[0, 2\pi]$. Consequently $y \equiv 0$, which completes the proof of the first part. The second part follows by using the first part with $y = y_1 - y_2$. □

Let $I_1 = [\frac{\pi}{2}, \frac{3\pi}{4}]$, $I_2 = [\pi, \frac{5\pi}{4}]$, $I_3 = [\frac{3\pi}{2}, \frac{7\pi}{4}]$, $I_4 = [\frac{5\pi}{4}, \frac{3\pi}{2}]$ and $J_1 = [0\frac{\pi}{2}]$, $J_2 = [\frac{\pi}{2}, \pi]$, $J_3 = [\pi, \frac{3\pi}{2}]$, $J_4 = [\frac{3\pi}{2}, 2\pi]$. The next result plays an important role in the proof of the main results.

Lemma 2.3. *There exists a positive constant m such that all solutions $y \in AC^1[0, 2\pi]$ of (2.6) satisfy*

$$y(t) \ge m\|y\|$$

for $t \in I_i$ for some $i \in \{1, 2, 3, 4\}$.

Proof. Let $y \in AC^1[0, 2\pi]$ be a solution of (2.6). Then $y \ge 0$ on $[0, 2\pi]$ by Corollary 2.2. Let $\|y\| = y(\tau)$ for some $\tau \in [0, 2\pi]$. Then $y'(\tau) = 0$. Let z_τ satisfy

$$\begin{cases} z_\tau'' + a(t)z_\tau = 0 \quad \text{on } [0, 2\pi], \\ z_\tau(\tau) = 1, \quad z_\tau'(\tau) = 0. \end{cases} \tag{2.7}$$

Note that the existence of a unique solution $z_\tau \in C^2[0, 2\pi]$ follows from the basic theory for linear differential equations (see e.g. [3, Theorem 3.7.1]). We shall verify that z_τ is bounded in $C^2[0, 2\pi]$ by a constant independent of $\tau \in [0, 2\pi]$. Indeed, by integrating the equation in (2.7), we get

$$z_\tau(t) = 1 - \int_\tau^t (t - s)a(s)z_\tau(s)ds$$

for $t \in [0, 2\pi]$, which, together with (A2), implies

$$|z_\tau(t)| \le 1 + \frac{\pi}{2}\int_\tau^t |z_\tau(s)|ds \quad \text{for } t \ge \tau,$$

and

$$|z_\tau(t)| \le 1 + \frac{\pi}{2}\int_t^\tau |z_\tau(s)|ds \quad \text{for } t \le \tau.$$

Hence Gronwall's inequality gives

$$|z_\tau(t)| \le e^{(\pi/2)|t-\tau|} \le e^{\pi^2} \tag{2.8}$$

for $t \in [0, 2\pi]$. Since $z_\tau'(t) = -\int_\tau^t a(s)z_\tau(s)ds$ and $z_\tau'' = -a(t)z_\tau$ on $[0, 2\pi]$, it follows from (2.8) that z_τ is bounded in $C^2[0, 2\pi]$ by a constant independent of $\tau \in [0, 2\pi]$.

Claim 1: *There exists a constant $m > 0$ such that $z_\tau(t) \ge m$ for all $\tau \in J_i$ and $t \in I_i$, $i \in \{1, 2, 3, 4\}$.*

Suppose to the contrary that there exists $i \in \{1, 2, 3, 4\}$ and sequences $(\tau_n) \subset J_i$, $(t_n) \subset I_i$, $(z_n) \subset C^2[0, 2\pi]$ such that $z_n(t_n) \le \frac{1}{n}$ for all n and

$$\begin{cases} z_n'' + a(t)z_n = 0 \quad \text{on } [0, 2\pi], \\ z_n(\tau_n) = 1, \quad z_n'(\tau_n) = 0. \end{cases}$$

Since (z_n) is bounded in $C^2[0,2\pi]$ by the above discussion, and $(\tau_n),(t_n)$ are bounded in J_i, I_i respectively, by passing to a subsequence if necessary, we can assume that there exist $\tau_i \in J_i, t_i \in I_i$, and $z \in C^1[0,2\pi]$ such that $\tau_n \to \tau_i, t_n \to t_i$, and $z_n \to z$ in $C^1[0,2\pi]$. Note that $t_n \geq \tau_n$ for $i < 4$ and $n \in \mathbb{N}$, and so $t_i \geq \tau_i$ for $i < 4$. Since

$$z_n(t) = 1 - \int_{\tau_n}^{t} (t-s)a(s)z_n(s)ds,$$

by passing to the limit as $n \to \infty$, we obtain

$$z(t) = 1 - \int_{\tau_i}^{t} (t-s)a(s)z(s)ds,$$

i.e. z satisfies

$$\begin{cases} z'' + a(t)z = 0 \quad \text{on } [0,2\pi], \\ z(\tau_i) = 1, \quad z'(\tau_i) = 0. \end{cases}$$

Since $z(t_i) = \lim_{n\to\infty} z_n(t_n) \leq 0$, we obtain for $i < 4$ that $t_i > \tau_i$ (since $t_i \neq \tau_i$), and there exists $\tilde{t}_i \in (\tau_i, t_i]$ such that $z > 0$ on $(\tau_i, \tilde{t}_i)$ and $z(\tilde{t}_i) = 0$. Since $\tilde{t}_i - \tau_i \leq \frac{3\pi}{4}$, Lemma 2.1 (i) gives $z = 0$ on $(\tau_i, \tilde{t}_i)$, a contradiction. On the other hand, if $i = 4$ then $t_4 < \tau_4$ and there exists $\tilde{t}_4 \in [t_4, \tau_4)$ such that $z > 0$ on $(\tilde{t}_4, \tau_4)$ and $z(\tilde{t}_4) = 0$. Since $\tau_4 - \tilde{t}_4 \leq \frac{3\pi}{4}$, we obtain a contradiction with Lemma 2.1 (i). This proves the claim.

Let $u = y - \|y\| z_\tau$. Then u satisfies

$$\begin{cases} u'' + a(t)u \geq 0 \quad \text{a.e. on } [0,2\pi], \\ u(\tau) = 0, \quad u'(\tau) = 0. \end{cases}$$

Claim 2: $u \geq 0$ on $[0,2\pi]$.

Indeed, suppose $u(\tilde{\tau}) < 0$ for some $\tilde{\tau} \in [0,2\pi]$ with $\tilde{\tau} < \tau$. Then there exists $\tilde{\tau}_0 \in (\tilde{\tau}, \tau]$ such that $u < 0$ on $(\tilde{\tau}, \tilde{\tau}_0)$ and $u(\tilde{\tau}_0) = 0$. Hence

$$u'' \geq -a(t)u \geq 0 \quad \text{a.e. on } (\tilde{\tau}, \tilde{\tau}_0]. \tag{2.9}$$

If $u'(\tilde{\tau}_0) \leq 0$, then (2.9) implies $u' \leq 0$ on $(\tilde{\tau}, \tilde{\tau}_0]$ and so $u(t) \geq u(\tilde{\tau}_0) = 0$ on $(\tilde{\tau}, \tilde{\tau}_0]$, a contradiction. On the other hand, if $u'(\tilde{\tau}_0) > 0$ then there exists $\tilde{\tau}_1 \in (\tilde{\tau}_0, \tau]$ such that $u > 0$ on $(\tilde{\tau}_0, \tilde{\tau}_1)$ and $u(\tilde{\tau}_1) = 0$. Since $\tilde{\tau}_1 - \tilde{\tau}_0 < 2\pi$, Lemma 2.1 (ii) implies $u \equiv 0$ on $(\tilde{\tau}_0, \tilde{\tau}_1)$, a contradiction. Similarly, we reach a contradiction in the case $\tilde{\tau} > \tau$, which proves claim 2.

Since $\tau \in \cup_{i=1}^4 J_i$, it follows from claims 1 and 2 that there exists $i \in \{1,2,3,4\}$ such that

$$y(t) \geq \|y\| z_\tau(t) \geq m\|y\|$$

for all $t \in I_i$, which completes the proof of Lemma 2.3. □

By Lemma 2.6 below, there exists $z \in AC^1[0,2\pi]$ satisfying

$$\begin{cases} z'' + a(t)z = g(t) \quad \text{a.e. on } [0,2\pi], \\ z(0) = z(2\pi), \quad z'(0) = z'(2\pi). \end{cases} \tag{2.10}$$

Since $g \not\equiv 0$, Corollary 2.2 gives $z > 0$ on $[0,2\pi]$.

Corollary 2.4. *Let k be a positive constant and $y \in AC^1[0,2\pi]$ satisfy*

$$\begin{cases} y'' + a(t)y \geq -\lambda k g(t) & \text{a.e. on } [0,2\pi], \\ y(0) = y(2\pi), \quad y'(0) = y'(2\pi). \end{cases} \tag{2.11}$$

Then

(i) $y \geq -\lambda k z$ *on* $[0,2\pi]$

(ii) If $\|y\| \geq 2\lambda k \|z\|(m+1)m^{-1}$ *then*

$$y(t) \geq m_0 \|y\| \tag{2.12}$$

for $t \in I_i$ *for some* $i \in \{1,2,3,4\}$*, where* $m_0 = m/2$ *and* m *is given by Lemma 2.3.*

Proof. Let $u = y + \lambda k z$. Then u satisfies

$$u'' + a(t)u \geq 0 \quad \text{a.e. on } [0,2\pi],$$

from which Corollary 2.2 and Lemma 2.3 give $u \geq 0$ on $[0,2\pi]$ and

$$y(t) + \lambda k z(t) = u(t) \geq \|u\|m = \|y + \lambda k z\|m$$

for $t \in I_i$ for some $i \in \{1,2,3,4\}$. Thus $y \geq -\lambda k z$ on $[0,2\pi]$ and

$$y(t) \geq \|y\|m - \lambda k \|z\|(m+1),$$

from which (2.12) follows if $\|y\| \geq 2\lambda k\|z\|(m+1)m^{-1}$. □

Lemma 2.5. *Let $U, V \in C^2[0,2\pi]$ be the solutions of*

$$\begin{cases} U'' + a(t)U = 0 & \text{on } [0,2\pi], \\ U(0) = 1, \quad U'(0) = 0, \end{cases}$$

and

$$\begin{cases} V'' + a(t)V = 0 & \text{on } [0,2\pi], \\ V(0) = 0, \quad V'(0) = 1. \end{cases}$$

Then $U(2\pi), V'(2\pi) < 1$.

Proof. Suppose $U(2\pi) \geq 1$. If there exists $\tau \in (0,2\pi)$ such that $U(\tau) < 0$ then, since $U(0) > 0$, there exists an interval $[\alpha,\beta] \subset (0,2\pi)$ such that $U < 0$ on (α,β) and $U(\alpha) = U(\beta) = 0$. Since $a(t) \leq 1/4$, it follows from Lemma 2.1 (ii) with $y = -U$ that $U = 0$ on (α,β), a contradiction. On the other hand, if $U \geq 0$ on $(0,2\pi)$ then $U'' \leq 0$ on $(0,2\pi)$ i.e. U' is nonincreasing on $[0,2\pi]$. Hence $U' \leq 0$ on $[0,2\pi]$, which implies $U(2\pi) \leq U(0) = 1$. Thus $U(2\pi) = 1 = U(0)$ and since U is nonincreasing, we deduce that $U = 1$ on $[0,2\pi]$. Consequently, the equation in U gives $a(t) = 0$ for all $t \in [0,2\pi]$, a contradiction. Hence $U(2\pi) < 1$. Next, we show that $V'(2\pi) < 1$. Since $V(0) = 0$ and $V'(0) > 0$, it follows that $V(t) > 0$ for $t > 0$ near 0. Hence if $V(\tau_0) < 0$ for some $\tau_0 \in (0,2\pi)$ then there exists $\beta \in (0,\tau_0)$ such that $V > 0$ on $(0,\beta)$ and $V(\beta) = 0 = V(0)$, a contradiction with Lemma 2.1 (ii). Hence $V \geq 0$ on $(0,2\pi)$, which implies $V'' \leq 0$ on $(0,2\pi)$. Consequently, $V'(2\pi) \leq V'(0) = 1$. If $V'(2\pi) = 1$ then $V' = 1$ on $[0,2\pi]$, which implies $V(t) = t$ for $t \in [0,2\pi]$. Using the equation in V, we see that $a(t) = 0$ for all $t \in [0,2\pi]$, a contradiction. Hence $V'(2\pi) < 1$, which completes the proof.

Lemma 2.6. *Let $h \in L^1(0, 2\pi)$. Then the problem*

$$\begin{cases} y'' + a(t)y = h(t) & \text{a.e. on } [0, 2\pi], \\ y(0) = y(2\pi), \quad y'(0) = y'(2\pi) \end{cases} \tag{2.13}$$

has a unique solution $y \in AC^1[0, 2\pi]$, which is given by

$$y(t) = \int_0^{2\pi} G(t, s)h(s)ds, \tag{2.14}$$

where

$$G(t, s) = c_1 V(t)V(s) - c_2 U(t)U(s) + \begin{cases} c_3 U(s)V(t) - c_4 U(t)V(s), & 0 \le s \le t \le 2\pi, \\ c_3 U(t)V(s) - c_4 U(s)V(t), & 0 \le t \le s \le 2\pi, \end{cases}$$

$c_1 = \frac{U'(2\pi)}{D}, c_2 = \frac{V(2\pi)}{D}, c_3 = \frac{U(2\pi)-1}{D}, c_4 = \frac{V'(2\pi)-1}{D}$, $D = U(2\pi) + V'(2\pi) - 2$, *and U, V are defined in Lemma 2.5.*

Proof. By Corollary 2.2, the only solution of

$$\begin{cases} y'' + a(t)y = 0 & \text{a.e. on } [0, 2\pi], \\ y(0) = y(2\pi), \quad y'(0) = y'(2\pi), \end{cases}$$

is the trivial one. Hence Fredholm's alternative theorem implies that the inhomogeneous problem (2.13) has a unique solution, which is given by (2.14) (see [2, Theorem 2.4]). Note that $G(t, s)$ is defined since $D < 0$ in view of Lemma 2.5. From (2.14), a calculation shows that

$$\begin{aligned} y'(t) = {} & c_1 \left(\int_0^{2\pi} V(s)h(s)ds \right) V'(t) - c_2 \left(\int_0^{2\pi} U(s)h(s)ds \right) U'(t) \\ & + c_3 \left(\int_0^{t} U(s)h(s)ds \right) V'(t) - c_4 \left(\int_0^{t} V(s)h(s)ds \right) U'(t) \\ & + c_3 \left(\int_t^{2\pi} V(s)h(s)ds \right) U'(t) - c_4 \left(\int_t^{2\pi} U(s)h(s)ds \right) V'(t), \end{aligned}$$

from which we see that $y \in AC^1[0, 2\pi]$ and satisfies (2.13). □

3 Proof of the main results

Let X be the Banach space $C[0, 2\pi]$ equipped with the norm $\|u\| = \sup_{t \in [0, 2\pi]} |u(t)|$. For $u \in X$, define

$$Tu(t) = \lambda \int_0^{2\pi} G(t, s)g(s)f(|u(s)|)ds$$

for $t \in [0, 2\pi]$, where $G(t, s)$ is the Green's function of $y'' + a(t)y$ with the periodic boundary conditions in (1.1) given by Lemma 2.6. Then $y = Tu \in AC^1[0, 1]$ satisfies

$$\begin{cases} y'' + a(t)y = \lambda g(t)f(|u|) & \text{a.e. on } [0, 2\pi], \\ y(0) = y(2\pi), \quad y'(0) = y'(2\pi). \end{cases}$$

It is easy to see that $T : X \to X$ is continuous and since T maps bounded sets in X into bounded sets in $C^1[0, 2\pi]$, T is a compact operator. For the rest of the paper, we shall use the following notations:

$$f^{0,z} = \sup_{0 \le t \le z} |f(t)| \quad \text{and} \quad f_{z,\infty} = \inf_{t \ge z} f(t) \quad \text{for } z \ge 0.$$

Note that $f^{0,z}$ and $f_{z,\infty}$ are nondecreasing on $[0, \infty)$.

Proof of Theorem 1.1. (i) By Corollary 2.2, $Tu \ge 0$ for all u. Let $0 < \varepsilon < \frac{1}{\lambda \|z\|}$, where z is defined by (2.10). Since $f_0 = 0$, there exists a constant $r > 0$ such that

$$f(z) < \varepsilon z \quad \text{for } z \in (0, r].$$

We shall verify that the conditions of Lemma A with $h \equiv 1$ are satisfied.

(a) *Let $y \in X$ satisfy $y = \theta T y$ for some $\theta \in (0, 1]$. Then $\|y\| \ne r$.*

Indeed, suppose to the contrary that $\|y\| = r$. Then

$$y'' + a(t)y = \lambda\theta g(t) f(|y|) \le \lambda\varepsilon g(t)\|y\| \quad \text{a.e. on } [0, 2\pi],$$

from which Corollary 2.2 implies

$$y \le \lambda\varepsilon z\|y\| \quad \text{on } [0, 2\pi].$$

Hence $\lambda\varepsilon\|z\| \ge 1$, a contradiction with the choice of ε.

(b) *Let $y \in X$ satisfy $y = Ty + \xi$ for some $\xi \ge 0$. Then $\|y\| < R$ for $R >> 1$.*

Note that y satisfies

$$y'' + a(t)y = a(t)\xi + \lambda g(t) f(|y|) \quad \text{a.e. on } [0, 2\pi].$$

Let M be a constant such that $\lambda M m c > \pi/2$, where $c = \min_{1 \le i \le 4} \int_{I_i} g(t)dt$ and m is given by Lemma 2.3. Since $f_\infty = \infty$, there exists a constant $A > 0$ such that

$$f(z) > Mz \quad \text{for } z \ge A.$$

We claim that $\|y\| < R$ for $R > A/m$. Indeed, suppose $\|y\| \ge R > A/m$. By Lemma 2.3, there exists $i \in \{1, 2, 3, 4\}$ such that

$$y(t) \ge \|y\| m \ge Rm > A$$

for $t \in I_i$, which implies

$$f(y(t)) > My(t) \ge Mm\|y\|$$

for $t \in I_i$. Thus

$$y'' + a(t)y \ge \begin{cases} \lambda M m \|y\| g(t), & t \in I_i, \\ 0 & t \notin I_i \end{cases} \quad \text{a.e. on } [0, 2\pi],$$

and upon integrating on $[0, 2\pi]$, we get

$$\int_0^{2\pi} a(t)y(t)dt \ge \lambda M m \|y\| \int_{I_i} g(t)dt \ge \lambda M m c \|y\|.$$

Since $a \le 1/4$ on $[0, 2\pi]$, this implies

$$\frac{\pi}{2}\|y\| \ge \lambda M m c \|y\|,$$

i.e. $\pi/2 \geq \lambda Mmc$, a contradiction with the choice of M. Hence $\|y\| < R$ as claimed.

By Lemma A, T has a fixed point y with $r < \|y\| < R$. By Corollary 2.2, $y > 0$ on $[0, 2\pi]$.

(ii) Let k be a positive constant such that $f(z) \geq -k$ for all $z \geq 0$. By Lemma 2.6, there exist $z_i, \tilde{z}_i \in AC^1[0, 2\pi]$ satisfying

$$z_i'' + a(t)z_i = \begin{cases} g(t) & t \in I_i, \\ 0, & t \notin I_i \end{cases} \quad z_i(0) = z_i(2\pi),\ z_i'(0) = z_i'(2\pi),$$

and

$$\tilde{z}_i'' + a(t)\tilde{z}_i = \begin{cases} 0, & t \in I_i, \\ kg(t), & t \notin I_i, \end{cases} \quad \tilde{z}_i(0) = \tilde{z}_i(2\pi),\ \tilde{z}_i'(0) = \tilde{z}_i'(2\pi),$$

for $i \in \{1, 2, 3, 4\}$. Note that $z_i > 0$ on $[0, 2\pi]$ for all i by Corollary 2.2. Choose $r > 0$ so that

$$f_{m_0 r,\infty} \min_{1 \leq i \leq 4, t \in [0, 2\pi]} z_i(t) > \max_{1 \leq i \leq 4} \|\tilde{z}_i\|, \tag{3.1}$$

where m_0 is given by Corollary 2.4. Let $\lambda > 0$ be such that

$$\lambda \max\{f^{0,r}\|z\|, 2k\|z\|(m+1)m^{-1}\} < r. \tag{3.2}$$

We shall verify that

(a) *Let $y \in X$ satisfy $y = \theta Ty$ for some $\theta \in (0, 1]$. Then $\|y\| \neq r$.*

Suppose to the contrary that $\|y\| = r$. Then

$$-\lambda f^{0,r} g(t) \leq y'' + a(t)y \leq \lambda f^{0,r} g(t) \quad \text{a.e. on } (0, 2\pi),$$

from which it follows that

$$|y(t)| \leq \lambda f^{0,r} z(t),$$

for $t \in [0, 2\pi]$, where z is defined in (2.10). Hence

$$r = \|y\| \leq \lambda f^{0,r}\|z\|,$$

a contradiction with (3.2), which proves (a).

(b) *There exists a constant $R_\lambda > r$ such that any solution $y \in X$ of $y = Ty + \xi$ for some $\xi \geq 0$ satisfies $\|y\| \neq R_\lambda$.*

Let $y \in X$ satisfy $y = Ty + \xi$ for some $\xi \geq 0$. Since $\lim_{z \to \infty} \frac{f_{z,\infty}}{z} = \infty$, there exists a constant $R_\lambda > r$ be such that

$$\lambda \left(f_{m_0 R_\lambda,\infty} \min_{1 \leq i \leq 4, t \in [0, 2\pi]} z_i(t) - \max_{1 \leq i \leq 4} \|\tilde{z}_i\| \right) > R_\lambda. \tag{3.3}$$

Suppose $\|y\| = R_\lambda$. Since $\|y\| \geq 2\lambda k\|z\|(m+1)m^{-1}$ and

$$y'' + a(t)y \geq \lambda g(t) f(|y|) \geq -\lambda kg(t) \quad \text{a.e. on } [0, 2\pi],$$

it follows from Corollary 2.4 that $y \geq -\lambda kz$ on $[0, 2\pi]$ and $y(t) \geq m_0\|y\|$ for $t \in I_i$ for some $i \in \{1, 2, 3, 4\}$. Hence

$$\begin{aligned} y'' + a(t)y &\geq \lambda g(t) f(|y|) \geq \lambda g(t) f_{|y|,\infty} \\ &\geq \lambda \left(f_{m_0\|y\|,\infty} \begin{cases} g(t), & t \in I_i, \\ 0, & t \notin I_i, \end{cases} - \begin{cases} 0, & t \in I_i \\ kg(t), & t \notin I_i \end{cases} \right) \quad \text{a.e. on } (0, 2\pi). \end{aligned}$$

By Corollary 2.2,

$$y \geq \lambda(f_{m_0\|y\|,\infty} z_i - \tilde{z}_i) \quad \text{on } [0, 2\pi], \tag{3.4}$$

which implies by (3.3) that

$$R_\lambda = \|y\| \geq \lambda \left(f_{m_0 R_\lambda,\infty} \min_{1\leq i\leq 4, t\in[0,2\pi]} z_i(t) - \max_{1\leq i\leq 4} \|\tilde{z}_i\| \right) > R_\lambda,$$

a contradiction. Hence $\|y\| \neq R_\lambda$, which proves (b).

By Lemma A, T has a fixed point $y_\lambda \in X$ with $r < \|y_\lambda\| < R$. Since (3.4) holds, we obtain from (3.1) that

$$y_\lambda \geq \lambda \left(f_{m_0 r,\infty} \min_{1\leq i\leq 4, t\in[0,2\pi]} z_i(t) - \max_{1\leq i\leq 4} \|\tilde{z}_i\| \right) > 0 \quad \text{on } [0, 2\pi].$$

It remains to show that $\|y_\lambda\| \to \infty$ as $\lambda \to 0^+$. Since

$$y_\lambda'' + a(t)y_\lambda = \lambda g(t) f(y_\lambda) \leq \lambda g(t) f^{0,\|y_\lambda\|} \quad \text{a.e. on } (0, 2\pi),$$

it follows that

$$y_\lambda \leq \lambda f^{0,\|y_\lambda\|} z \quad \text{on } [0, 2\pi],$$

which implies

$$\frac{f^{0,\|y_\lambda\|}}{\|y_\lambda\|} \geq \frac{1}{\lambda\|z\|}.$$

Since $\|y_\lambda\| > r$, it follows that $\|y_\lambda\| \to \infty$ as $\lambda \to 0^+$, which completes the proof of Theorem 1.1. □

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