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On the spectrum of a fourth order nonlinear eigenvalue problem with variable exponent and indefinite potential

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Abstract. The present paper deals with the spectrum of a fourth order nonlinear eigenvalue problem involving variable exponent conditions and a sign-changing potential. The main result of this paper establishes the existence of two positive constants λ_0 and λ_1 with $\lambda_0 \le \lambda_1$ such that every $\lambda \in [\lambda_1, +\infty)$ is an eigenvalue, while $\lambda \in (-\infty, \lambda_0)$ cannot be an eigenvalue of the above problem.

Keywords: fourth order elliptic equation, eigenvalue problem, variable exponent, indefinite potential.

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1 Introduction

The study of problems of elliptic equations and variational problems with $p(x)$-growth condition has attracted more and more attention in recent years. It possesses a solid background in physics and originates from the study on electrorheological fluids (see Růžička [12]) and elastic mechanics (see Zhikov [14]). It also has wide applications in different research fields, such as image processing models (see e.g. [5, 9], stationary thermorheological viscous flows (see [1]) and the mathematical description of the filtration processes of an ideal barotropic gas through a porous medium (see [2]).

In this paper, we are concerned with the study of the nonhomogeneous eigenvalue problem

$$\begin{cases} \Delta(|\Delta u|^{p_1(x)-2}\Delta u) + \Delta(|\Delta u|^{p_2(x)-2}\Delta u) + V(x)|u|^{\alpha(x)-2}u \\ \qquad = \lambda(|u|^{q_1(x)-2}u + |u|^{q_2(x)-2}u), \quad \text{in } \Omega, \\ u = \Delta u = 0, \quad \text{on } \partial\Omega, \end{cases} \tag{P}$$

where Ω is a bounded domain of $\mathbb{R}^N$ with smooth boundary, λ is a real number, V is an indefinite weight function, and p_1, p_2, q_1, q_2, α are continuous functions on $\overline{\Omega}$.

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The study of fourth order nonlinear eigenvalue problems involving variable exponents growth conditions has captured a special attention in the last few years; see e.g. [3,4,8,11]. We give in what follows a concise but complete image of the actual stage of research on this topic.

In the case that $q_1(x) = q_2(x) = q(x)$ for any $x \in \overline{\Omega}$ and $V \equiv 0$ in Ω, Ge and Zhou [8] established the existence of two positive constants λ_0 and λ_1 with $\lambda_0 \le \lambda_1$ such that any $\lambda \in [\lambda_1, +\infty)$ is an eigenvalue, while and $\lambda \in (0, \lambda_0)$ is not an eigenvalue of the above problem.

The same problem, for $V(x) = 0$, $p_1(x) = p_2(x)$ and $q_1(x) = q_2(x)$ is studied by Ayoujil and Amrouss in [3]. The authors established the existence of infinitely many eigenvalues for problem (P) by using an argument based on the Ljusternik–Schnirelmann critical point theory. Denoting by Λ the set of all nonnegative eigenvalues, they showed that $\sup \Lambda = +\infty$ and they pointed out that only under special conditions, which are somehow connected with a kind of monotony of the function $p(x)$, we have $\inf \Lambda > 0$ (this is in contrast with the case when $p(x)$ is a constant; then, we always have $\inf \Lambda > 0$).

This paper is a natural outgrowth of the results in [8]. We consider the eigenvalues for a fourth order nonlinear eigenvalue problem (P) with the potential $V \neq 0$ and $q_1 \neq q_2$ on $\overline{\Omega}$ in the right-hand side.

In this paper we study problem (P) under the following assumptions:

$\boldsymbol{H(p_1, p_2, q_1, q_2, \alpha)}$:

$$1 < p_2(x) < q_2^- \le q_2^+ \le \alpha(x) \le q_1^- \le q_1^+ < p_1(x) < \frac{N}{2}, \qquad \forall x \in \overline{\Omega},$$

$$q_1^+ < \frac{N p_2(x)}{N - 2p_2(x)}, \qquad \forall x \in \overline{\Omega},$$

where $q_i^- = \min_{x \in \overline{\Omega}} q_i(x)$ and $q_i^+ = \max_{x \in \overline{\Omega}} q_i(x)$ $(i = 1, 2)$;

$\boldsymbol{H(V)}$: $V \in L^{r(x)}(\Omega)$, with $r \in C(\overline{\Omega})$ and $r(x) > \frac{N}{\min_{\overline{\Omega}} \alpha}$, $\forall x \in \overline{\Omega}$.

Inspired by the above-mentioned papers, we study problem (P) from a more extensive viewpoint. More precisely, we will show the existence of two constants λ_0 and λ_1 with $\lambda_0 \le \lambda_1$ such that any $\lambda \ge \lambda_1$ is an eigenvalue of problem (P) while any $\lambda < \lambda_0$ is not an eigenvalue of problem (P).

This paper is composed of three sections. In Section 2, we recall the definition of variable exponent Lebesgue spaces, $L^{p(x)}(\Omega)$, as well as Sobolev spaces, $W^{1,p(x)}(\Omega)$. Moreover, some properties of these spaces will be also exhibited to be used later. In Section 3, we give the main results and their proofs.

2 Preliminaries

In this section we first recall some facts on variable exponent spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$. For details, we refer to [6,7,10].

Set

$$C_+(\overline{\Omega}) = \{h \in C(\overline{\Omega}) : h(x) > 1 \text{ for any } x \in \overline{\Omega}\}.$$

Define

$$h^- = \min_{x \in \overline{\Omega}} h(x), \qquad h^+ = \max_{x \in \overline{\Omega}} h(x) \quad \text{for any } h \in C_+(\overline{\Omega}).$$

For any $p \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space:

$$L^{p(x)}(\Omega) = \left\{ u : u \text{ is a measurable real value function } \int_\Omega |u(x)|^{p(x)} dx < +\infty \right\},$$

with the norm $|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)} dx \le 1\}$, and define the variable exponent Sobolev space

$$W^{k,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : D^\alpha u \in L^{p(x)}(\Omega), |\alpha| \le k\},$$

with the norm $\|u\|_{W^{k,p(x)}(\Omega)} = \|u\|_{k,p(x)} = \sum\limits_{|\alpha| \le k} |D^\alpha u|_{p(x)}$.

We remember that spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$ are separable and reflexive Banach spaces. Denoting by $W_0^{k,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{k,p(x)}(\Omega)$.

For $p(x) \in C_+(\overline{\Omega})$, by $L^{q(x)}(\Omega)$ we denote the conjugate space of $L^{p(x)}(\Omega)$ with $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$, then the Hölder's type inequality

$$\int_\Omega |uv| dx \le \left(\frac{1}{p^-} + \frac{1}{q^-} \right) |u|_{L^{p(x)}(\Omega)} |v|_{L^{q(x)}(\Omega)}, \qquad u \in L^{p(x)}(\Omega),\ v \in L^{q(x)}(\Omega) \tag{2.1}$$

holds. Furthermore, define mapping $\rho : L^{p(x)} \to \mathbb{R}$ by

$$\rho(u) = \int_\Omega |u|^{p(x)} dx,$$

then the following relations hold

$$|u|_{p(x)} < 1\ (= 1, > 1) \Leftrightarrow \rho(u) < 1\ (= 1, > 1), \tag{2.2}$$

$$|u|_{p(x)} > 1 \Rightarrow |u|_{p(x)}^{p^-} \le \rho(u) \le |u|_{p(x)}^{p^+}, \tag{2.3}$$

$$|u|_{p(x)} < 1 \Rightarrow |u|_{p(x)}^{p^+} \le \rho(u) \le |u|_{p(x)}^{p^-}. \tag{2.4}$$

$$|u_n - u|_{p(x)} \to 0 \Leftrightarrow \rho(u_n - u) \to 0. \tag{2.5}$$

Definition 2.1. Assume that spaces E, F are Banach spaces, we define the norm on the space $X := E \bigcap F$ as $\|u\|_X = \|u\|_E + \|u\|_F$.

In order to discuss problem (P), we need some theories on space

$$X_i := W_0^{1,p_i(x)}(\Omega) \bigcap W^{2,p_i(x)}(\Omega)$$

($i = 1, 2$). Since $p_1(x) > p_2(x)$ for any $x \in \overline{\Omega}$, so the space $W_0^{1,p_1(x)}(\Omega)$ is continuously embedded in $W_0^{1,p_2(x)}(\Omega)$, $W^{2,p_1(x)}(\Omega)$ is continuously embedded in $W^{2,p_2(x)}(\Omega)$, so X_1 is continuously embedded in X_2.

From the Definition 2.1, it follows that for any $u \in X_1$, $\|u\|_1 = \|u\|_{1,p(x)} + \|u\|_{2,p(x)}$, thus $\|u\|_1 = |u|_{p(x)} + |\nabla u|_{p(x)} + \sum_{|\alpha|=2} |D^\alpha u|_{p(x)}$.

In Zang and Fu [13], the equivalence of the norms was proved, and it was even proved that the norm $|\Delta u|_{p(x)}$ is equivalent to the norm $\|u\|_1$ (see [11, Theorem 4.4]).

Let us choose on X_1 the norm defined by $\|u\|_1 = |\Delta u|_{p(x)}$. Note that, $(X_1, \|\cdot\|_1)$ is also a separable and reflexive Banach space. Similar to (2.2),(2.3), (2.4) and (2.5), we have the following, define mapping $\rho_1 : X_1 \to \mathbb{R}$ by

$$\rho_1(u) = \int_\Omega |\Delta u|^{p(x)} dx,$$

then the following relations hold

$$\|u\|_1 < 1\ (=1, > 1) \Leftrightarrow \rho_1(u) < 1\ (=1, > 1), \tag{2.6}$$

$$\|u\|_1 > 1 \Rightarrow \|u\|_1^{p^-} \leq \rho_1(u) \leq \|u\|_1^{p^+}, \tag{2.7}$$

$$\|u\|_1 < 1 \Rightarrow \|u\|_1^{p^+} \leq \rho_1(u) \leq \|u\|_1^{p^-}. \tag{2.8}$$

$$\|u_n - u\|_1 \to 0 \Leftrightarrow \rho_1(u_n - u) \to 0. \tag{2.9}$$

Hereafter, let

$$p_1^*(x) = \begin{cases} \dfrac{Np_1(x)}{N - 2p_1(x)}, & p_1(x) < \dfrac{N}{2}, \\ +\infty, & p_1(x) \geq \dfrac{N}{2}, \end{cases}$$

and

$$p_2^*(x) = \begin{cases} \dfrac{Np_2(x)}{N - 2p_2(x)}, & p_2(x) < \dfrac{N}{2}, \\ +\infty, & p_2(x) \geq \dfrac{N}{2}. \end{cases}$$

Remark 2.2. If $h \in C_+(\overline{\Omega})$ and $h(x) < p_i^*(x)$ $(i = 1$ or $2)$ for any $x \in \overline{\Omega}$, by Theorem 3.2 in [3], we deduce that X_i is continuously and compactly embedded in $L^{h(x)}(\Omega)$.

Remark 2.3. Since $p_2(x) < p_1(x)$ for any $x \in \Omega$ it follows that $p_2^*(x) < p_1^*(x)$, using condition $\boldsymbol{H(p_1, p_2, q_1, q_2, \alpha)}$ we have a compact embedding $X_i \hookrightarrow L^{q_1(x)}(\Omega)$ and $X_i \hookrightarrow L^{q_2(x)}(\Omega)$ with $(i = 1, 2)$.

3 Main results and proofs

Since $p_2(x) < p_1(x)$ for any $x \in \Omega$ it follows that $W_0^{1,p_1(x)}(\Omega)$ and $W^{2,p_1(x)}(\Omega)$ are continuously embedded in $W_0^{1,p_2(x)}(\Omega)$ and $W^{2,p_2(x)}(\Omega)$ respectively. Thus, a solution for a problem of type (P) will be sought in the variable exponent space $X_1 = W_0^{1,p_1(x)}(\Omega) \bigcap W^{2,p_1(x)}(\Omega)$.

We say that $\lambda \in \mathbb{R}$ is an eigenvalue of problem (P) if there exists $u \in X_1 \backslash \{0\}$ such that

$$\int_\Omega \left(|\Delta u|^{p_1(x)-2} + |\Delta u|^{p_2(x)-2} \right) \Delta u \Delta v dx + \int_\Omega V(x) |u|^{\alpha(x)-2} uv dx \\ - \lambda \int_\Omega \left(|u|^{q_1(x)-2} + |u|^{q_2(x)-2} \right) uv dx = 0,$$

for all $v \in X_1$. We point out that if λ is an eigenvalue of problem (P), then the corresponding eigenfunction $u \in X_1 \backslash \{0\}$ is a weak solution of problem (P).

Define

$$\lambda_1 := \inf_{u\in X_1\setminus\{0\}} \frac{\int_\Omega \frac{1}{p_1(x)}|\Delta u|^{p_1(x)}dx + \int_\Omega \frac{1}{p_2(x)}|\Delta u|^{p_2(x)}dx + \int_\Omega \frac{V(x)}{\alpha(x)}|u|^{\alpha(x)}dx}{\int_\Omega \frac{1}{q_1(x)}|u|^{q_1(x)}dx + \int_\Omega \frac{1}{q_2(x)}|u|^{q_2(x)}dx}.$$

Our main result is given by the following theorem.

Theorem 3.1. *Assume that $\boldsymbol{H(p_1,p_2,q_1,q_2,\alpha)}$ holds. Then each $\lambda \in [\lambda_1, +\infty)$ is an eigenvalue of problem (P). Furthermore, there exists a constant λ_0 such that $\lambda_0 \le \lambda_1$ and any $\lambda \in (-\infty, \lambda_0)$ is not an eigenvalue of problem (P).*

Proof. The proof is divided into the following five steps.

Step 1. For each $\varepsilon > 0$, there exists $c_\varepsilon > 0$ such that

$$\left|\int_\Omega \frac{V(x)}{\alpha(x)}|u|^{\alpha(x)}dx\right| \le \varepsilon\left[\int_\Omega \frac{1}{p_1(x)}|\Delta u|^{p_1(x)}dx + \int_\Omega \frac{1}{p_2(x)}|\Delta u|^{p_2(x)}dx\right] + c_\varepsilon|V|_{r(x)}\int_\Omega\left(|u|^{\alpha^-}+|u|^{\alpha^+}\right)dx \tag{3.1}$$

for any $u \in X_1$.

Since $r(x) \ge r^-$ on $\overline{\Omega}$, it follows that $L^{r(x)}(\Omega) \subseteq L^{r(x)^-}(\Omega)$. Note that $r(x) > \frac{N}{\alpha^-}$ for each $x \in \overline{\Omega}$, which implies that $V \in L^{r^-}(\Omega)$ and $r^- > \frac{N}{\alpha^-}$.

Applying the Hölder's inequality, we get

$$\int_\Omega |V(x)||u|^{\alpha^-}dx \le |V|_{r^-}\left||u|^{\alpha^-}\right|_{(r^-)'} = |V|_{r^-}|u|^{\alpha^-}_{\alpha^-(r^-)'}, \tag{3.2}$$

where $\frac{1}{r^-} + \frac{1}{(r^-)'} = 1$.

Now for every $\varepsilon > 0$, we can show that for any $s \in (1, \frac{N\alpha^-}{N-2\alpha^-})$, there exists $C'_\varepsilon > 0$ such that

$$|u|_s \le \varepsilon|\Delta u|_{\alpha^-} + C'_\varepsilon|u|_{\alpha^-}, \qquad \forall u \in W_0^{1,\alpha^-}(\Omega)\cap W^{2,\alpha^-}(\Omega). \tag{3.3}$$

Indeed, assume it is not true for each $\varepsilon > 0$. Then there exists $\varepsilon_0 > 0$ and a sequence $\{v_n\} \subseteq W_0^{1,\alpha^-}(\Omega)\cap W^{2,\alpha^-}(\Omega)$ such that $|v_n|_s = 1$ and

$$\varepsilon_0|\Delta v_n|_{\alpha^-} + n|v_n|_{\alpha^-} < 1, \qquad \forall n \in N.$$

So, sequence $\{v_n\}$ is bounded in $W_0^{1,\alpha^-}(\Omega)\cap W^{2,\alpha^-}(\Omega)$ and $|v_n|_{\alpha^-} \to 0$, as $n \to +\infty$. Thus, up to a subsequence, still denote by $\{v_n\}$, we may assume that there exists $v \in W_0^{1,\alpha^-}(\Omega)\cap W^{2,\alpha^-}(\Omega)$, such that $v_n \rightharpoonup v$ in $W_0^{1,\alpha^-}(\Omega)\cap W^{2,\alpha^-}(\Omega)$ and actually $v = 0$. Since $s \in (1, \frac{N\alpha^-}{N-2\alpha^-})$, it follows by Remark 2.2 that $W_0^{1,\alpha^-}(\Omega)\cap W^{2,\alpha^-}(\Omega)$ is compactly embedded in $L^s(\Omega)$, which implies that $v_n \to 0$ in $L^s(\Omega)$. On the other hand, since $|v_n|_s = 1$ for each $n \in N$, we deduce that $|v|_s = 1$, which is a contradiction. Hence (3.3) is true.

Note that $r^- > \frac{N}{\alpha^-}$ and $(r^-)', \alpha^- < \frac{N\alpha^-}{N-2\alpha^-}$. From (3.2) and (3.3), we have

$$\begin{aligned}\int_\Omega |V(x)||u|^{\alpha^-}dx &\le |V|_{r^-}|u|^{\alpha^-}_{\alpha^-(r^-)'}\\ &\le |V|_{r^-}\left(\varepsilon|\Delta u|_{\alpha^-} + C'_\varepsilon|\Delta u|_{\alpha^-}\right)^{\alpha^-}\\ &\le 2^{\alpha^- -1}\varepsilon^{\alpha^-}|V|_{r^-}|\Delta u|^{\alpha^-}_{\alpha^-} + 2^{\alpha^- -1}(C'_\varepsilon)^{\alpha^-}|V|_{r^-}|u|^{\alpha^-}_{\alpha^-},\end{aligned} \tag{3.4}$$

for all $u \in W_0^{1,\alpha^-}(\Omega) \cap W^{2,\alpha^-}(\Omega)$.

Combining the similar arguments as those used in the proof of relation (3.4) with $r^- > \frac{N}{\alpha^+}$. We know that there exists $C''_\varepsilon > 0$ such that

$$\int_\Omega |V(x)||u|^{\alpha^+} dx \le 2^{\alpha^+-1}\varepsilon^{\alpha^+}|V|_{r^-}|\Delta u|_{\alpha^+}^{\alpha^+} + 2^{\alpha^+-1}(C''_\varepsilon)^{\alpha^+}|V|_{r^-}|u|_{\alpha^+}^{\alpha^+}, \tag{3.5}$$

for all $u \in W_0^{1,\alpha^+}(\Omega) \cap W^{2,\alpha^+}(\Omega)$.

By virtue of hypothesis $\boldsymbol{H(p_1,p_2,q_1,q_2,\alpha)}$, we deduce that $\alpha^- \le \alpha^+ < p_1(x)$ for any $x \in \overline{\Omega}$ and thus $X_1 \subseteq W_0^{1,\alpha^-}(\Omega) \cap W^{2,\alpha^-}(\Omega)$ and $X_1 \subseteq W_0^{1,\alpha^+}(\Omega) \cap W^{2,\alpha^+}(\Omega)$. Hence, relations (3.4) and (3.5) are true for any $u \in X_1$.

On the other hand, by $p_2(x) < \alpha^- \le \alpha(x) \le \alpha^+ < p_1(x)$ for each $x \in \overline{\Omega}$, we deduce that

$$\left|\int_\Omega V(x)|u|^{\alpha^+} dx\right| \le \frac{1}{\alpha^-}\int_\Omega |V(x)|(|u|^{\alpha^-} + |u|^{\alpha^+})dx, \qquad \forall u \in X_1 \tag{3.6}$$

and

$$\int_\Omega \left(|\Delta u|^{\alpha^-} + |\Delta u|^{\alpha^+}\right) dx \le 2p_1^+ \int_\Omega \left(\frac{1}{p_1(x)}|\Delta u|^{p_1(x)} + \frac{1}{p_2(x)}|\Delta u|^{p_2(x)}\right)dx. \tag{3.7}$$

From (3.4), (3.5), (3.6) and (3.7), we have

$$\begin{aligned}
\left|\int_\Omega V(x)|u|^{\alpha^+} dx\right| &\le \frac{1}{\alpha^-}\int_\Omega |V(x)|(|u|^{\alpha^-} + |u|^{\alpha^+})dx \\
&\le \frac{1}{\alpha^-}\Big[2^{\alpha^--1}\varepsilon^{\alpha^-}|V|_{r^-}|\Delta u|_{\alpha^-}^{\alpha^-} + 2^{\alpha^--1}(C''_\varepsilon)^{\alpha^-}|V|_{r^-}|u|_{\alpha^-}^{\alpha^-} \\
&\qquad + 2^{\alpha^+-1}\varepsilon^{\alpha^+}|V|_{r^-}|\Delta u|_{\alpha^+}^{\alpha^+} + 2^{\alpha^+-1}(C''_\varepsilon)^{\alpha^+}|V|_{r^-}|u|_{\alpha^+}^{\alpha^+}\Big] \\
&\le \frac{2^{\alpha^+-1}|V|_{r^-}}{\alpha^-}\Big[\varepsilon^{\alpha^-}(|\Delta u|_{\alpha^-}^{\alpha^-} + |\Delta u|_{\alpha^+}^{\alpha^+}) + (C''_\varepsilon)^{\alpha^+}(|u|_{\alpha^-}^{\alpha^-} + |u|_{\alpha^+}^{\alpha^+})\Big] \\
&\le \frac{2^{\alpha^+-1}|V|_{r^-}}{\alpha^-}\Big[2p_1^+\varepsilon^{\alpha^-}\int_\Omega\left(\frac{1}{p_1(x)}|\Delta u|^{p_1(x)} + \frac{1}{p_2(x)}|\Delta u|^{p_2(x)}\right)dx \\
&\qquad + (C''_\varepsilon)^{\alpha^+}\int_\Omega\left(|u(x)|^{\alpha^-} + |u(x)|^{\alpha^+}\right)dx\Big].
\end{aligned} \tag{3.8}$$

Combining the last inequality with arbitrariness of ε, we infer that relation (3.1) is true.

Step 2. We show that

$$\lim_{\|u\|_1\to+\infty} \frac{\int_\Omega \frac{1}{p_1(x)}|\Delta u|^{p_1(x)}dx + \int_\Omega \frac{1}{p_2(x)}|\Delta u|^{p_2(x)}dx + \int_\Omega \frac{V(x)}{\alpha(x)}|u|^{\alpha(x)}dx}{\int_\Omega \frac{1}{q_1(x)}|u|^{q_1(x)}dx + \int_\Omega \frac{1}{q_2(x)}|u|^{q_2(x)}dx} = +\infty \tag{3.9}$$

and

$$\lim_{\|u\|_1\to 0} \frac{\int_\Omega \frac{1}{p_1(x)}|\Delta u|^{p_1(x)}dx + \int_\Omega \frac{1}{p_2(x)}|\Delta u|^{p_2(x)}dx + \int_\Omega \frac{V(x)}{\alpha(x)}|u|^{\alpha(x)}dx}{\int_\Omega \frac{1}{q_1(x)}|u|^{q_1(x)}dx + \int_\Omega \frac{1}{q_2(x)}|u|^{q_2(x)}dx} = +\infty. \tag{3.10}$$

Since $q_1(x) < \alpha^-, \alpha^+ < q_2(x), \forall x \in \overline{\Omega}$, we deduce that for every $x \in \overline{\Omega}$ and $u \in X_1$,

$$|u(x)|^{\alpha^-} + |u(x)|^{\alpha^+} \le 2(|u(x)|^{q_1(x)} + |u(x)|^{q_2(x)}),$$

which implies that

$$\frac{\int_\Omega (|u(x)|^{\alpha^-} + |u(x)|^{\alpha^+})dx}{\int_\Omega (|u(x)|^{q_1(x)} + |u(x)|^{q_2(x)})dx} \le 2. \tag{3.11}$$

From Step 1, we find that for $\varepsilon \in (0,1)$ there exists $c_\varepsilon > 0$ such that

$$\begin{aligned} &\frac{\int_\Omega \frac{1}{p_1(x)}|\Delta u|^{p_1(x)}dx + \int_\Omega \frac{1}{p_2(x)}|\Delta u|^{p_2(x)}dx + \int_\Omega \frac{V(x)}{\alpha(x)}|u|^{\alpha(x)}dx}{\int_\Omega \frac{1}{q_1(x)}|u|^{q_1(x)}dx + \int_\Omega \frac{1}{q_2(x)}|u|^{q_2(x)}dx} \\ &\qquad \ge \frac{\frac{1-\varepsilon}{p_1^+}\int_\Omega (|\Delta u|^{p_1(x)}dx + |\Delta u|^{p_2(x)})dx - c_\varepsilon |V|_{r(x)} \int_\Omega (|u|^{\alpha^-} + |u|^{\alpha^+})dx}{\frac{1}{q_2^-}\int_\Omega (|u|^{q_1(x)} + |u|^{q_2(x)})dx} \end{aligned} \tag{3.12}$$

for any $u \in X_1$.

By (3.11) and (3.12), there exist some positive constants c_1 and c_2, such that

$$\begin{aligned} &\frac{\int_\Omega \frac{1}{p_1(x)}|\Delta u|^{p_1(x)}dx + \int_\Omega \frac{1}{p_2(x)}|\Delta u|^{p_2(x)}dx + \int_\Omega \frac{V(x)}{\alpha(x)}|u|^{\alpha(x)}dx}{\int_\Omega \frac{1}{q_1(x)}|u|^{q_1(x)}dx + \int_\Omega \frac{1}{q_2(x)}|u|^{q_2(x)}dx} \\ &\qquad \ge c_1 \frac{\int_\Omega (|\Delta u|^{p_1(x)}dx + |\Delta u|^{p_2(x)})dx}{\int_\Omega (|u|^{q_1(x)} + |u|^{q_2(x)})dx} - c_2 |V|_{r(x)} \end{aligned} \tag{3.13}$$

for any $u \in X_1$.

Note that $q_1^+, q_2^+ < \frac{Np_2(x)}{N-2p_2(x)} < \frac{Np_1(x)}{N-2p_1(x)}, \forall x \in \overline{\Omega}$. So the embedding $X_1 \hookrightarrow L^{q_i^\pm}(\Omega)$ $(i = 1,2)$ is continuous and compact. Thus, for any $u \in X_1$, there positive constant c_3, such that

$$|u|_{q_i^\pm} \le c_3 \|u\|_1. \tag{3.14}$$

Thus, by (3.14), we have

$$\begin{aligned} &\int_\Omega \frac{1}{q_1(x)}|u|^{q_2(x)}dx + \int_\Omega \frac{1}{q_2(x)}|u|^{q_2(x)}dx \\ &\quad \le \int_\Omega (|u|^{q_1(x)} + |u|^{q_2(x)})dx \\ &\quad \le \int_\Omega (|u|^{q_1^-} + |u|^{q_1^+} + |u|^{q_2^-} + |u|^{q_2^+})dx \\ &\quad \le c_3^{q_1^-}\|u\|_1^{q_1^-} + c_3^{q_1^+}\|u\|_1^{q_1^+} + c_3^{q_2^-}\|u\|_1^{q_2^-} + c_3^{q_2^+}\|u\|_1^{q_2^+}. \end{aligned} \tag{3.15}$$

Hence, from (3.13) and (3.15), for any $u \in X_1$ with $\|u\|_1 > 1$, there exists positive constant c_4, such that

$$\begin{aligned} &\frac{\int_\Omega \frac{1}{p_1(x)}|\Delta u|^{p_1(x)}dx + \int_\Omega \frac{1}{p_2(x)}|\Delta u|^{p_2(x)}dx + \int_\Omega \frac{V(x)}{\alpha(x)}|u|^{\alpha(x)}dx}{\int_\Omega \frac{1}{q_1(x)}|u|^{q_1(x)}dx + \int_\Omega \frac{1}{q_2(x)}|u|^{q_2(x)}dx} \\ &\ge c_1 \frac{\int_\Omega |\Delta u|^{p_1(x)}dx}{\int_\Omega \frac{1}{q_1(x)}|u|^{q_1(x)}dx + \int_\Omega \frac{1}{q_2(x)}|u|^{q_2(x)}dx} - c_2|V|_{r(x)} \\ &\ge c_4 \frac{\int_\Omega |\Delta u|^{p_1(x)}dx}{\|u\|_1^{q_1^-} + \|u\|_1^{q_1^+} + \|u\|_1^{q_2^-} + \|u\|_1^{q_2^+}} - c_2|V|_{r(x)} \\ &\ge c_4 \frac{\|u\|_1^{p_1^-}}{\|u\|_1^{q_1^-} + \|u\|_1^{q_1^+} + \|u\|_1^{q_2^-} + \|u\|_1^{q_2^+}} - c_2|V|_{r(x)} \\ &\to +\infty, \quad \text{as } \|u\|_1 \to +\infty, \end{aligned} \tag{3.16}$$

because $p_1^- > q_1^+ \geq q_1^- \geq q_2^+ \geq q_2^-$. So, the relation (3.9) holds.

On the other hand, using (3.13) and (3.14), for any $u \in X_1$ with $\|u\|_1 < 1$ small enough, there exists positive constant c_5, such that

$$\begin{aligned}
&\frac{\int_\Omega \frac{1}{p_1(x)}|\Delta u|^{p_1(x)}dx + \int_\Omega \frac{1}{p_2(x)}|\Delta u|^{p_2(x)}dx + \int_\Omega \frac{V(x)}{\alpha(x)}|u|^{\alpha(x)}dx}{\int_\Omega \frac{1}{q_1(x)}|u|^{q_1(x)}dx + \int_\Omega \frac{1}{q_2(x)}|u|^{q_2(x)}dx}\\
&\geq c_1\frac{\int_\Omega |\Delta u|^{p_2(x)}dx}{\int_\Omega \frac{1}{q_1(x)}|u|^{q_1(x)}dx + \int_\Omega \frac{1}{q_2(x)}|u|^{q_2(x)}dx} - c_2|V|_{r(x)}\\
&\geq c_5\frac{\int_\Omega |\Delta u|^{p_2(x)}dx}{\|u\|_1^{q_1^+} + \|u\|_1^{q_1^-} + \|u\|_1^{q_2^+} + \|u\|_1^{q_2^-}} - c_2|V|_{r(x)}\\
&\geq c_4\frac{\|u\|_1^{p_2^+}}{\|u\|_1^{q_1^-} + \|u\|_1^{q_1^+} + \|u\|_1^{q_2^-} + \|u\|_1^{q_2^+}} - c_2|V|_{r(x)}.
\end{aligned} \tag{3.17}$$

Since $p_2^+ < q_2^- \leq q_2^+ \leq q_1^- \leq q_1^+$, passing to the limit as $\|u\|_1 \to 0$ in the above inequality we deduce that relation (3.10) holds true.

Step 3. There exists $u_0 \in X_1 \backslash \{0\}$ such that

$$\frac{\int_\Omega \frac{1}{p_1(x)}|\Delta u_0|^{p_1(x)}dx + \int_\Omega \frac{1}{p_2(x)}|\Delta u_0|^{p_2(x)}dx + \int_\Omega \frac{V(x)}{\alpha(x)}|u_0|^{\alpha(x)}dx}{\int_\Omega \frac{1}{q_1(x)}|u_0|^{q_1(x)}dx + \int_\Omega \frac{1}{q_2(x)}|u_0|^{q_2(x)}dx} = \lambda_1.$$

In fact, let $\{u_n\} \subseteq X_1 \setminus \{0\}$ be a minimizing sequence for λ_1, that is,

$$\lim_{n\to\infty}\frac{\int_\Omega \frac{1}{p_1(x)}|\Delta u_n|^{p_1(x)}dx + \int_\Omega \frac{1}{p_2(x)}|\Delta u_n|^{p_2(x)}dx + \int_\Omega \frac{V(x)}{\alpha(x)}|u_n|^{\alpha(x)}dx}{\int_\Omega \frac{1}{q_1(x)}|u_n|^{q_1(x)}dx + \int_\Omega \frac{1}{q_2(x)}|u_n|^{q_2(x)}dx} = \lambda_1. \tag{3.18}$$

By Step 2, we have $\{u_n\}$ is bounded in X_1. Since X_1 is reflexive it follows that there exists $u_0 \in X_1$ such that $u_n \rightharpoonup u_0$ in X_1. On the other hand, the function $\int_\Omega \frac{1}{p_1(x)}|\Delta u|^{p_1(x)}dx + \int_\Omega \frac{1}{p_2(x)}|\Delta u|^{p_2(x)}dx : X_1 \to \mathbb{R}$ is a convex function, hence it is weakly lower semi-continuous, that is,

$$\begin{aligned}
&\liminf_{n\to\infty}\left(\int_\Omega \frac{1}{p_1(x)}|\Delta u_n|^{p_1(x)}dx + \int_\Omega \frac{1}{p_2(x)}|\Delta u_n|^{p_2(x)}dx\right)\\
&\quad\geq \left(\int_\Omega \frac{1}{p_1(x)}|\Delta u_0|^{p_1(x)}dx + \int_\Omega \frac{1}{p_2(x)}|\Delta u_0|^{p_2(x)}dx\right).
\end{aligned} \tag{3.19}$$

Note that X_1 is compactly embedded in $L^{q_i(x)}(\Omega)(i = 1, 2)$, thus,

$$u_n \to u_0 \quad \text{in } L^{q_i(x)}(\Omega)\ (i = 1, 2). \tag{3.20}$$

By (2.9) and (3.20), it is easy to see that

$$\begin{aligned}
&\lim_{n\to\infty}\int_\Omega\left[\frac{1}{q_1(x)}|u_n|^{q_1(x)}dx + \frac{1}{q_2(x)}|u_n|^{q_2(x)}dx\right]\\
&\qquad = \int_\Omega \frac{1}{q_1(x)}|u_0|^{q_1(x)}dx + \int_\Omega \frac{1}{q_2(x)}|u_0|^{q_2(x)}dx.
\end{aligned} \tag{3.21}$$

On the other hand, by $X_1 \hookrightarrow L^{\alpha(x)r'(x)}(\Omega)$, we know that

$$\begin{aligned}\left|\int_\Omega V(x)|u_n-u_0|^{\alpha(x)}dx\right| &\le 2|V|_{r(x)}\left||u_n-u_0|^{\alpha(x)}\right|_{r'(x)}\\ &\le 2|V|_{r(x)}|u_n-u_0|^{\alpha^\tau}_{\alpha(x)r'(x)}\\ &\to 0, \quad \text{as } n\to\infty,\end{aligned}$$

where

$$\tau=\begin{cases}+, & |u_n-u_0|_{\alpha(x)r'(x)}\ge 1,\\ -, & |u_n-u_0|_{\alpha(x)r'(x)}<1,\end{cases}$$

which implies that

$$\lim_{n\to\infty}\int_\Omega V(x)|u_n|^{\alpha(x)}dx=\int_\Omega V(x)|u_0|^{\alpha(x)}dx. \tag{3.22}$$

In view of (3.19), (3.21) and (3.22), we obtain

$$\frac{\int_\Omega \frac{1}{p_1(x)}|\Delta u_0|^{p_1(x)}dx+\int_\Omega \frac{1}{p_2(x)}|\Delta u_0|^{p_2(x)}dx+\int_\Omega \frac{V(x)}{\alpha(x)}|u_0|^{\alpha(x)}dx}{\int_\Omega \frac{1}{q_1(x)}|u_0|^{q_1(x)}dx+\int_\Omega \frac{1}{q_2(x)}|u_0|^{q_2(x)}dx}=\lambda_1,$$

if $u_0\neq 0$.

It remains to show that u_0 is nontrivial. Assume the contrary. Then $u_n \rightharpoonup 0$ in X_1 and $u_n\to 0$ in $L^{\theta(x)}(\Omega)$ for any $\theta\in C(\overline{\Omega})$ with $1<\theta(x)<p_1^*(x)$ on $\overline{\Omega}$. Thus, we have

$$\lim_{n\to\infty}\left[\int_\Omega \frac{1}{q_1(x)}|u_n|^{q_1(x)}dx+\int_\Omega \frac{1}{q_2(x)}|u_n|^{q_2(x)}dx\right]=0 \tag{3.23}$$

and

$$\lim_{n\to\infty}\int_\Omega \frac{V(x)}{\alpha(x)}|u_n|^{\alpha(x)}dx=0. \tag{3.24}$$

Let $\varepsilon\in(0,\lambda_1)$ be fixed. By (3.18) we deduce that for n large enough we have

$$\begin{aligned}\Bigg|\int_\Omega \frac{1}{p_1(x)}|\Delta u_n|^{p_1(x)}dx+\int_\Omega \frac{1}{p_2(x)}|\Delta u_n|^{p_2(x)}dx+\int_\Omega \frac{V(x)}{\alpha(x)}|u_n|^{\alpha(x)}dx\\ -\lambda_1\left(\int_\Omega \frac{1}{q_1(x)}|u_n|^{q_1(x)}dx+\int_\Omega \frac{1}{q_2(x)}|u_n|^{q_2(x)}dx\right)\Bigg|\\ <\varepsilon\left(\int_\Omega \frac{1}{q_1(x)}|u_n|^{q_1(x)}dx+\int_\Omega \frac{1}{q_2(x)}|u_n|^{q_2(x)}dx\right),\end{aligned}$$

which yields that

$$\begin{aligned}&(\lambda_1-\varepsilon)\left(\int_\Omega \frac{1}{q_1(x)}|u_n|^{q_1(x)}dx+\int_\Omega \frac{1}{q_2(x)}|u_n|^{q_2(x)}dx\right)\\ &\quad<\int_\Omega \frac{1}{p_1(x)}|\Delta u_n|^{p_1(x)}dx+\int_\Omega \frac{1}{p_2(x)}|\Delta u_n|^{p_2(x)}dx+\int_\Omega \frac{V(x)}{\alpha(x)}|u_n|^{\alpha(x)}dx\\ &\quad<(\lambda_1+\varepsilon)\left(\int_\Omega \frac{1}{q_1(x)}|u_n|^{q_1(x)}dx+\int_\Omega \frac{1}{q_2(x)}|u_n|^{q_2(x)}dx\right).\end{aligned} \tag{3.25}$$

Combining (3.23), (3.24) and (3.25), we have

$$\lim_{n\to\infty}\int_\Omega \frac{1}{p_1(x)}|\Delta u_n|^{p_1(x)}dx+\int_\Omega \frac{1}{p_2(x)}|\Delta u_n|^{p_2(x)}dx=0. \tag{3.26}$$

By (2.9) and (3.26), we have

$$u_n \to 0 \quad \text{in } X_1, \quad \text{that is } \|u_n\|_1 \to 0.$$

From this information and relation (3.10), we get

$$\lim_{\|u_n\|_1 \to 0} \frac{\int_\Omega \frac{1}{p_1(x)}|\Delta u_n|^{p_1(x)}dx + \int_\Omega \frac{1}{p_2(x)}|\Delta u_n|^{p_2(x)}dx + \int_\Omega \frac{V(x)}{\alpha(x)}|u_n|^{\alpha(x)}dx}{\int_\Omega \frac{1}{q_1(x)}|u_n|^{q_1(x)}dx + \int_\Omega \frac{1}{q_2(x)}|u_n|^{q_2(x)}dx} = +\infty$$

and this is a contradiction. Thus $u_0 \neq 0$.

By Step 3, we conclude that there exists $u_0 \in X_1 \setminus \{0\}$ such that

$$\begin{aligned}\lambda_1 &= \frac{\int_\Omega \frac{1}{p_1(x)}|\Delta u_0|^{p_1(x)}dx + \int_\Omega \frac{1}{p_2(x)}|\Delta u_0|^{p_2(x)}dx + \int_\Omega \frac{V(x)}{\alpha(x)}|u_0|^{\alpha(x)}dx}{\int_\Omega \frac{1}{q_1(x)}|u_0|^{q_1(x)}dx + \int_\Omega \frac{1}{q_2(x)}|u_0|^{q_2(x)}dx} \\ &= \inf_{u \in X_1 \setminus \{0\}} \frac{\int_\Omega \frac{1}{p_1(x)}|\Delta u|^{p_1(x)}dx + \int_\Omega \frac{1}{p_2(x)}|\Delta u|^{p_2(x)}dx + \int_\Omega \frac{V(x)}{\alpha(x)}|u|^{\alpha(x)}dx}{\int_\Omega \frac{1}{q_1(x)}|u|^{q_1(x)}dx + \int_\Omega \frac{1}{q_2(x)}|u|^{q_2(x)}dx}.\end{aligned}$$

Then for any $v \in X_1$ we have

$$\frac{d}{dt}\left(\frac{\int_\Omega \frac{1}{p_1(x)}|\Delta(u_0+tv)|^{p_1(x)}dx + \int_\Omega \frac{1}{p_2(x)}|\Delta(u_0+tv)|^{p_2(x)}dx + \int_\Omega \frac{V(x)}{\alpha(x)}|(u_0+tv)|^{\alpha(x)}dx}{\int_\Omega \frac{1}{q_1(x)}|u_0+tv|^{q_1(x)}dx + \int_\Omega \frac{1}{q_2(x)}|u_0+tv|^{q_2(x)}dx} \right)\Bigg|_{t=0} = 0.$$

A simple computation yields

$$\begin{aligned}&\left(\int_\Omega (|\Delta u_0|^{p_1(x)-2} + |\Delta u_0|^{p_2(x)-2})\Delta u_0 \Delta v dx + \int_\Omega V(x)|u_0|^{\alpha(x)-2}u_0 v dx \right) \\ &\qquad \times \left(\int_\Omega \frac{1}{q_1(x)}|u_0|^{q_1(x)}dx + \int_\Omega \frac{1}{q_2(x)}|u_0|^{q_2(x)}dx \right) \\ &= \left(\int_\Omega \frac{1}{p_1(x)}|\Delta u_0|^{p_1(x)}dx + \int_\Omega \frac{1}{p_2(x)}|\Delta u_0|^{p_2(x)}dx + \int_\Omega \frac{1}{\alpha(x)}|u_0|^{\alpha(x)}dx \right) \\ &\qquad \times \left(\int_\Omega |u_0|^{q_1(x)-2}u_0 v dx + \int_\Omega |u_0|^{q_2(x)-2}u_0 v dx \right)\end{aligned} \tag{3.27}$$

for any $v \in X_1$.

Returning to (3.27) and using

$$\frac{\int_\Omega \frac{1}{p_1(x)}|\Delta u_0|^{p_1(x)}dx + \int_\Omega \frac{1}{p_2(x)}|\Delta u_0|^{p_2(x)}dx + \int_\Omega \frac{V(x)}{\alpha(x)}|u_0|^{\alpha(x)}dx}{\int_\Omega \frac{1}{q_1(x)}|u_0|^{q_1(x)}dx + \int_\Omega \frac{1}{q_2(x)}|u_0|^{q_2(x)}dx} = \lambda_1$$

and $\int_\Omega \frac{1}{q_1(x)}|u_0|^{q_1(x)}dx + \int_\Omega \frac{1}{q_2(x)}|u_0|^{q_2(x)}dx \neq 0$, we obtain λ_1 is an eigenvalue of problem (P).

Step 4. We will show that any $\lambda \in (\lambda_1, +\infty)$ is an eigenvalue of problem (P).

Let $\lambda \in (\lambda_1, +\infty)$ be arbitrary but fixed. Define $D_\lambda : X_1 \to \mathbb{R}$ by

$$\begin{aligned}D_\lambda(u) = \int_\Omega \frac{1}{p_1(x)}|\Delta u|^{p_1(x)}dx + \int_\Omega \frac{1}{p_2(x)}|\Delta u|^{p_2(x)}dx + \int_\Omega \frac{V(x)}{\alpha(x)}|u|^{\alpha(x)}dx \\ - \lambda\left(\int_\Omega \frac{1}{q_1(x)}|u|^{q_1(x)}dx + \int_\Omega \frac{1}{q_2(x)}|u|^{q_2(x)}dx \right).\end{aligned}$$

Clearly, $D_\lambda \in C^1(X_1, \mathbb{R})$ with

$$\langle D'_\lambda(u), v\rangle = \int_\Omega (|\Delta u|^{p_1(x)-2} + |\Delta u|^{p_2(x)-2})\Delta u \Delta v dx) + \int_\Omega V(x)|u|^{\alpha(x)-2}uvdx - \lambda \int_\Omega (|u|^{q_1(x)-2} + |u|^{q_2(x)-2})uvdx, \qquad \forall u, v \in X_1.$$

Thus, λ is an eigenvalue of problem (P) if and only if there exists $u_\lambda \in X_1 \setminus \{0\}$ a critical point of D_λ.

It follows from Step 1 and (3.15) that

$$\begin{aligned}
D_\lambda(u) = & \int_\Omega \frac{1}{p_1(x)}|\Delta u|^{p_1(x)}dx + \int_\Omega \frac{1}{p_2(x)}|\Delta u|^{p_2(x)}dx + \int_\Omega \frac{V(x)}{\alpha(x)}|u|^{\alpha(x)}dx \\
& - \lambda\left(\int_\Omega \frac{1}{q_1(x)}|u|^{q_1(x)}dx + \int_\Omega \frac{1}{q_2(x)}|u|^{q_2(x)}dx\right) \\
\geq & \frac{1}{p_1^+}\int_\Omega (|\Delta u|^{p_1(x)} + |\Delta u|^{p_2(x)})dx + \int_\Omega \frac{V(x)}{\alpha(x)}|u|^{\alpha(x)}dx \\
& - \lambda\left(\int_\Omega \frac{1}{q_1(x)}|u|^{q_1(x)}dx + \int_\Omega \frac{1}{q_2(x)}|u|^{q_2(x)}dx\right) \\
\geq & \frac{1-\varepsilon}{p_1^+}\int_\Omega (|\Delta u|^{p_1(x)} + |\Delta u|^{p_2(x)})dx - c_\varepsilon |V|_{r(x)}\int_\Omega (|u|^{\alpha^-} + |u|^{\alpha^+})dx \\
& - \lambda c_3^{q_1^-}\|u\|_1^{q_1^-} + c_3^{q_1^+}\|u\|_1^{q_1^+} + c_3^{q_2^-}\|u\|_1^{q_2^-} + c_3^{q_2^+}\|u\|_1^{q_2^+} \\
\geq & \frac{1-\varepsilon}{p_1^+}\int_\Omega |\Delta u|^{p_1(x)}dx - c_\varepsilon |V|_{r(x)}(|u|_{\alpha^-}^{\alpha^-} + |u|_{\alpha^+}^{\alpha^+}) \\
& - \lambda c_3^{q_1^-}\|u\|_1^{q_1^-} + c_3^{q_1^+}\|u\|_1^{q_1^+} + c_3^{q_2^-}\|u\|_1^{q_2^-} + c_3^{q_2^+}\|u\|_1^{q_2^+} \\
\geq & \frac{1-\varepsilon}{p_1^+}\|u\|_1^{p_1^-} - c_\varepsilon |V|_{r(x)}(c\|u\|_1^{\alpha^-} + c\|u\|_1^{\alpha^+}) - \lambda c_3^{q_1^-}\|u\|_1^{q_1^-} + c_3^{q_1^+}\|u\|_1^{q_1^+} \\
& + c_3^{q_2^-}\|u\|_1^{q_2^-} + c_3^{q_2^+}\|u\|_1^{q_2^+} \\
\to & \infty, \quad \text{as } \|u\|_1 \to +\infty,
\end{aligned}$$

since $1 < q_2^- \leq q_2^+ \leq \alpha^- \leq \alpha^+ \leq q_1^- \leq q_1^+ < p_1^-$, i.e. $\lim_{\|u\|_1 \to +\infty} D_\lambda(u) = \infty$.

On the other hand, with similar arguments as in the proof of Step 3, we can show that the functional D_λ is weakly lower semi-continuous. So by the Weierstrass theorem, there exists $u_\lambda \in X_1$ a global minimum point of D_λ and thus, a critical point of D_λ. Next we prove that u_λ is nontrivial.

Indeed, since

$$\lambda_1 = \inf_{u \in X_1 \setminus \{0\}} \frac{\int_\Omega \frac{1}{p_1(x)}|\Delta u|^{p_1(x)}dx + \int_\Omega \frac{1}{p_2(x)}|\Delta u|^{p_2(x)}dx + \int_\Omega \frac{V(x)}{\alpha(x)}|u|^{\alpha(x)}dx}{\int_\Omega \frac{1}{q_1(x)}|u|^{q_1(x)}dx + \int_\Omega \frac{1}{q_2(x)}|u|^{q_2(x)}dx}$$

and $\lambda > \lambda_1$ it follows that there exists $v_\lambda \in X_1$ such that $D_\lambda(v_\lambda) < 0$, that is

$$\int_\Omega \frac{1}{p_1(x)}|\Delta v_\lambda|^{p_1(x)}dx + \int_\Omega \frac{1}{p_2(x)}|\Delta v_\lambda|^{p_2(x)}dx + \int_\Omega \frac{V(x)}{\alpha(x)}|v_\lambda|^{\alpha(x)}dx < \lambda\left(\int_\Omega \frac{1}{q_1(x)}|v_\lambda|^{q_1(x)}dx + \int_\Omega \frac{1}{q_2(x)}|v_\lambda|^{q_2(x)}dx\right).$$

Thus, we conclude that

$$\inf_{u\in X_1} D_\lambda(u) < 0$$

and u_λ is a nontrivial critical point of D_λ, and then λ is an eigenvalue of problem (P).

Step 5. We will show that any $\lambda \in (0, \lambda_0)$ is not an eigenvalue of problem (P), where λ_0 is given by

$$\lambda_0 := \inf_{u\in X_1\setminus\{0\}} \frac{\int_\Omega (|\Delta u|^{p_1(x)}dx + |\Delta u|^{p_2(x)})dx \int_\Omega V(x)|u|^{\alpha(x)}dx}{\int_\Omega |u|^{q_1(x)}dx + \int_\Omega |u|^{q_2(x)}dx}.$$

Firstly, we verify that $\lambda_0 \le \lambda_1$. Due to Step 2, λ_1 and u_0 is an eigenvalue and is an eigenfunction corresponding to λ_1 of (P), then for every $v \in X_1$ we have

$$\begin{aligned}\int_\Omega (|\Delta u_0|^{p_1(x)-2} + |\Delta u_0|^{p_2(x)-2})\Delta u_0 \Delta v dx + \int_\Omega V(x)|u_0|^{\alpha(x)-2}u_0 v dx \\ = \lambda_1\Big(\int_\Omega |u_0|^{q_1(x)-2}u_0 v dx + \int_\Omega |u_0|^{q_2(x)-2}u_0 v dx\Big),\end{aligned} \tag{3.28}$$

which implies that

$$\begin{aligned}\int_\Omega (|\Delta u_0|^{p_1(x)-2} + |\Delta u_0|^{p_2(x)-2})\Delta u_0 \Delta u_0 dx + \int_\Omega V(x)|u_0|^{\alpha(x)-2}u_0 u_0 dx \\ = \lambda_1\Big(\int_\Omega |u|^{q_1(x)-2}u_0 u_0 dx + \int_\Omega |u|^{q_2(x)-2}u_0 u_0 dx\Big),\end{aligned}$$

that is

$$\int_\Omega (|\Delta u_0|^{p_1(x)} + |\Delta u_0|^{p_2(x)})dx + \int_\Omega V(x)|u_0|^{\alpha(x)}dx = \lambda_1\Big(\int_\Omega |u|^{q_1(x)}dx + \int_\Omega |u|^{q_2(x)}dx\Big).$$

Then, it follows that $\lambda_0 \le \lambda_1$.

Now we prove our assertion. Arguing by contradiction: assume that there exists $\lambda \in (0, \lambda_0)$ is an eigenvalue of problem (P). Thus, there exists $u_\lambda \in X_1 \setminus \{0\}$ such that

$$\begin{aligned}\int_\Omega (|\Delta u_\lambda|^{p_1(x)-2} + |\Delta u_\lambda|^{p_2(x)-2})\Delta u_\lambda \Delta v dx + \int_\Omega V(x)|u_\lambda|^{\alpha(x)-2}u_\lambda v dx \\ = \lambda\Big(\int_\Omega |u_\lambda|^{q_1(x)-2}u_\lambda v dx + \int_\Omega |u_\lambda|^{q_2(x)-2}u_\lambda v dx\Big)\end{aligned} \tag{3.29}$$

for any $v \in X_1$. Thus, for $v = u_\lambda$ we have

$$\int_\Omega (|\Delta u_\lambda|^{p_1(x)} + |\Delta u_\lambda|^{p_2(x)})dx + \int_\Omega V(x)|u_\lambda|^{\alpha(x)}dx = \lambda\Big(\int_\Omega |u_\lambda|^{q_1(x)}dx + \int_\Omega |u_\lambda|^{q_2(x)}dx\Big).$$

The fact that $u_\lambda \in X_1 \setminus \{0\}$ assures that $\int_\Omega |u_\lambda|^{q_1(x)}dx + \int_\Omega |u_\lambda|^{q_2(x)}dx > 0$. Since $\lambda < \lambda_0$, the above information yields

$$\begin{aligned}&\int_\Omega (|\Delta u_\lambda|^{p_1(x)} + |\Delta u_\lambda|^{p_2(x)})dx + \int_\Omega V(x)|u_\lambda|^{\alpha(x)}dx \\ &\ge \lambda_0 \int_\Omega (|u_\lambda|^{q_1(x)} + |u_\lambda|^{q_2(x)})dx \\ &> \lambda \int_\Omega (|u_\lambda|^{q_1(x)} + |u_\lambda|^{q_2(x)})dx \\ &= \int_\Omega (|\Delta u_\lambda|^{p_1(x)} + |\Delta u_\lambda|^{p_2(x)})dx + \int_\Omega V(x)|u_\lambda|^{\alpha(x)}dx.\end{aligned}$$

Obviously, this is contradiction. Therefore, we deduce that each $\lambda \in (-\infty, \lambda_0)$ is not an eigenvalue of problem (P). □

Remark 3.2. In Theorem 3.1, we are not able to deduce whether $\lambda_0 = \lambda_1$ or $\lambda_0 < \lambda_1$. In the latter case an interesting question concerns the existence of eigenvalue of the problem (P) in the interval $[\lambda_0, \lambda_1)$.

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