



Boundary value problems on noncompact intervals for the n -th order vector differential inclusions

Jan Andres  and Martina Pavlačková

Department of Mathematical Analysis and Applications of Mathematics
Faculty of Science, Palacký University
17. listopadu 12, 771 46 Olomouc, Czech Republic

Received 14 June 2016, appeared 26 August 2016

Communicated by Alberto Cabada

Abstract. A general continuation principle for the n -th order vector asymptotic boundary value problems with multivalued right-hand sides is newly developed. This continuation principle is then applied to guarantee the existence and localization of solutions to the given asymptotic problems. The obtained results are finally supplied by two illustrative examples.

Keywords: asymptotic boundary value problems, n -th order differential inclusions, existence and localization results.

2010 Mathematics Subject Classification: 34A60, 34B15, 34B40, 47H04, 47H11.

1 Introduction


Asymptotic boundary value problems (b.v.p.) for higher-order differential equations and inclusions are important for many applications. For instance, they occur in the problems dealing with radially symmetric solutions of elliptic equations, semiconductor circuits and soil mechanics, fluid dynamics or in the boundary layer theory (see, e.g., [1, 2, 15], and the references therein).

Furthermore, it is well known (see, e.g., [5]) that the n -th order asymptotic control problems

$$\begin{aligned}x^{(n)}(t) &= f\left(t, x(t), \dots, x^{(n-1)}(t), u\right), \quad t \in [t_0, \infty), \quad u \in U, \\x &\in S,\end{aligned}$$

where S is a suitable constraint (e.g. asymptotic boundary conditions) and $u \in U$ are control parameters such that $u(t) \in \mathbb{R}^k$, for all $t \geq t_0$, can be converted into the equivalent multivalued problems

$$\begin{aligned}x^{(n)}(t) &\in F\left(t, x(t), \dots, x^{(n-1)}(t)\right), \\x &\in S,\end{aligned} \tag{1.1}$$

 Corresponding author. Email: jan.andres@upol.cz

where the multivalued mapping F , representing the right-hand side (r.h.s.), is defined by

$$F(t, x, \dots, x^{(n-1)}) := \{f(t, x, \dots, x^{(n-1)}, u)\}_{u \in U}.$$

Although boundary value problems for higher-order (mainly, the second-order) vector systems have been already intensively studied since the 70's (see e.g. [19, 21, 22, 30, 33]), there are only several papers devoted to noncompact (possibly infinite) intervals (see e.g. [4, 7, 10, 11, 14, 15, 18, 23, 24, 26–29, 31, 32], and the references therein). In these papers, various fixed point theorems, topological degree theory, shooting methods, upper and lower solution technique, etc., have been applied for the solvability of given problems. In the majority of mentioned papers, the second-order problems were considered and/or the right-hand sides of systems under consideration were single-valued, often even continuous.

The aim of this paper is to investigate the n -th order problem (1.1) on non-compact intervals with the right-hand sides governed by upper-Carathéodory multivalued mappings. Besides the existence of solutions, also their localization in a given set will be studied. Following the ideas in [3–5, 16–18], our approach is not sequential as traditionally, but direct. This means to consider the solutions as fixed points of the associated operators in Fréchet spaces. In this way, however, the bound sets technique like e.g. in [6] cannot be applied jointly with the degree arguments, because bounded subsets of nonnormable Fréchet spaces are, according to the Kolmogorov theorem, equal to their boundaries. On the other hand, a bigger variety of asymptotic boundary value problems can be so taken into account.

The paper is organized as follows. Firstly, the basic properties of multivalued mappings which are employed in the sequel are recalled. On this basis, we formulate the general continuation principle for the n -th order asymptotic boundary value problems with multivalued right-hand sides in a rather abstract way. Then this principle is applied in order to obtain the existence and localization of solutions. Finally, two illustrative examples are supplied.

2 Preliminaries

We start this section with some standard definitions and notations. At first, we recall some geometric notions of particular subsets of metric spaces and the notions of retracts. If (X, d) is an arbitrary metric space and $A \subset X$ its subset, we shall mean by $\text{Int } A$, \bar{A} and ∂A the interior, the closure and the boundary of A , respectively. For a subset $A \subset X$ and $\varepsilon > 0$, we define the set $N_\varepsilon(A) = \{x \in X \mid \exists a \in A : d(x, a) < \varepsilon\}$, i.e. $N_\varepsilon(A)$ is an open neighbourhood of the set A in X . A subset $A \subset X$ is called a *retract* of X if there exists a retraction $r: X \rightarrow A$, i.e. a continuous function satisfying $r(x) = x$, for every $x \in A$. Similarly, A is called a *neighbourhood retract* of X if there exists an open subset $U \subset X$ such that $A \subset U$ and A is a retract of U .

Let X, Y be two metric spaces. We say that X is an *absolute retract* (AR-space) if, for each Y and every closed $A \subset Y$, each continuous mapping $f: A \rightarrow X$ is extendable over Y . If f is extendable over some neighborhood of A , for each closed $A \subset Y$ and each continuous mapping $f: A \rightarrow X$, then X is an *absolute neighborhood retract* (ANR-space). Let us note that X is an ANR-space if and only if it is a retract of an open subset of a normed space and that X is an AR-space if and only if it is a retract of some normed space.

We say that a nonempty subset $A \subset X$ is *contractible*, provided there exist $x_0 \in A$ and a homotopy $h: A \times [0, 1] \rightarrow A$ such that $h(x, 0) = x$ and $h(x, 1) = x_0$, for every $x \in A$. A nonempty subset $A \subset X$ is called an R_δ -set if there exists a decreasing sequence $\{A_n\}_{n=1}^\infty$ of compact AR-spaces such that

$$A = \bigcap \{A_n; n = 1, 2, \dots\}.$$

Note that any R_δ -set is nonempty, compact and connected.

The following hierarchies hold for nonempty subsets of a metric space:

$$\text{compact} + \text{convex} \subset \text{compact AR} \subset \text{compact} + \text{contractible} \subset R_\delta\text{-set},$$

and all the above inclusions are proper.

A nonempty, compact subset A of a metric space X is called ∞ -proximally connected if, for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that, for every $n \in \mathbb{N}$ and for any map $g: \partial\Delta^n \rightarrow N_\delta(A)$, there exists a map $\tilde{g}: \Delta^n \rightarrow N_\varepsilon(A)$ such that $g(x) = \tilde{g}(x)$, for every $x \in \partial\Delta^n$, where $\partial\Delta^n := \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$ and $\Delta^n := \{x \in \mathbb{R}^{n+1} \mid |x| \leq 1\}$. On ANR-spaces, the notions of ∞ -proximally connected sets and R_δ -sets coincide. For more details about the above subsets of metric spaces, see, e.g., [5, 12, 20].

Our problems under consideration naturally lead to the notion of a Fréchet space. Let us recall that by a *Fréchet space*, we understand a complete (metrizable) locally convex vector space. Its topology can be generated by a countable family of seminorms or by a metric (see e.g. [5, Chapter I.1]). If a Fréchet space is normable, then it becomes a *Banach space*. Fréchet spaces considered below will be as follows:

- the space $C(J, \mathbb{R}^k)$ of continuous functions $x: J \rightarrow \mathbb{R}^k$ with the family of seminorms $p_i(q): C(J, \mathbb{R}^k) \rightarrow \mathbb{R}$ defined by

$$p_i(q) := \max_{t \in K_i} |q(t)|,$$

where $\{K_i\}$ is a sequence of compact subintervals of J such that

$$\bigcup_{i=1}^{\infty} K_i = J, \quad (2.1)$$

$$K_i \subset K_{i+1}, \quad \text{for all } i \in \mathbb{N}, \quad (2.2)$$

- the space $C^{n-1}(J, \mathbb{R}^k)$ of functions $x: J \rightarrow \mathbb{R}^k$ having continuous $(n-1)$ -st derivatives endowed with the system of seminorms $p_i^{n-1}(q): C^{n-1}(J, \mathbb{R}^k) \rightarrow \mathbb{R}$ defined by

$$p_i^{n-1}(q) := \max_{t \in K_i} |q(t)| + \max_{t \in K_i} |\dot{q}(t)| + \cdots + \max_{t \in K_i} |q^{(n-1)}(t)|,$$

where $\{K_i\}$ is a sequence of compact subintervals of J satisfying (2.1) and (2.2),

- the space $AC_{\text{loc}}^{n-1}(J, \mathbb{R}^n)$ of functions $x: J \rightarrow \mathbb{R}^n$ with locally absolutely continuous $(n-1)$ -st derivatives endowed with the family of seminorms $p_i^{n-1}_{\text{loc}}(q): AC_{\text{loc}}^{n-1}(J, \mathbb{R}^k) \rightarrow \mathbb{R}$ defined by

$$p_i^{n-1}_{\text{loc}}(q) := \max_{t \in K_i} |q(t)| + \max_{t \in K_i} |\dot{q}(t)| + \cdots + \max_{t \in K_i} |q^{(n-1)}(t)| + \int_{K_i} |q^{(n)}(t)| dt,$$

where $\{K_i\}$ is a sequence of compact subintervals of J satisfying (2.1) and (2.2).

The topologies in Fréchet spaces mentioned above can be generated by the metrics

$$d(x, y) := \sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \frac{p_i(x-y)}{1 + p_i(x-y)}$$

or

$$d^{n-1}(x, y) := \sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \frac{p_i^{n-1}(x-y)}{1 + p_i^{n-1}(x-y)}$$

or

$$d_{\text{loc}}^{n-1}(x, y) := \sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \frac{p_{i, \text{loc}}^{n-1}(x-y)}{1 + p_{i, \text{loc}}^{n-1}(x-y)},$$

respectively.

Let $J \subset \mathbb{R}$ be compact. By $H^{n,1}(J, \mathbb{R}^k)$, we will denote the Banach space of all C^{n-1} functions $x: J \rightarrow \mathbb{R}^k$ with absolutely continuous $(n-1)$ -st derivative.

In the sequel, we also need the following definitions and notions from the multivalued theory.

Let X, Y be two metric spaces. We say that F is a *multivalued mapping* from X to Y (written $F: X \multimap Y$) if, for every $x \in X$, a nonempty subset $F(x)$ of Y is given. We associate with F its graph Γ_F , i.e. the subset of $X \times Y$ defined by

$$\Gamma_F := \{(x, y) \in X \times Y \mid y \in F(x)\}.$$

If $X \cap Y \neq \emptyset$ and $F: X \multimap Y$, then a point $x \in X$ is called a *fixed point* of F , provided $x \in F(x)$. The set of all fixed points of F is denoted by $\text{Fix}(F)$, i.e.

$$\text{Fix}(F) := \{x \in X \mid x \in F(x)\}.$$

A multivalued mapping $F: X \multimap Y$ is called *upper semicontinuous* (shortly written u.s.c.) if, for each open set $U \subset Y$, the set $\{x \in X \mid F(x) \subset U\}$ is open in X .

The connections between upper semicontinuous mappings and mappings with closed graphs are summarized in the following propositions (see, e.g., [5, 20]).

Proposition 2.1. *Let X, Y be metric spaces and $F: X \multimap Y$ be a multivalued mapping with the closed graph Γ_F such that $F(X) \subset K$, where K is a compact set. Then F is u.s.c.*

Proposition 2.2. *Let X, Y be metric spaces and $F: X \multimap Y$ be an u.s.c. multivalued mapping with closed values, then Γ_F is a closed subset of $X \times Y$.*

A multivalued mapping $F: X \multimap Y$ is called *compact* if the set $F(X) = \bigcup_{x \in X} F(x)$ is contained in a compact subset of Y and it is called *closed* if the set $F(B)$ is closed in Y , for every closed subset B of X .

We say that a multivalued mapping $F: X \multimap Y$ is an R_δ -mapping if it is a u.s.c. mapping with R_δ -values.

We say that a multivalued map $\varphi: X \multimap Y$ is a J -mapping (written, $\varphi \in J(X, Y)$) if it is a u.s.c. mapping and $\varphi(x)$ is ∞ -proximally connected, for every $x \in X$. If the space Y is a neighbourhood retract of a Fréchet space (i.e. an ANR-space), then $\varphi \in J(X, Y)$, provided φ is an R_δ -mapping, as already pointed out (cf. [5, 20]).

Let Y be a separable metric space and $(\Omega, \mathcal{U}, \nu)$ be a *measurable space*, i.e. a nonempty set Ω equipped with a σ -algebra \mathcal{U} of its subsets and a countably additive measure ν on \mathcal{U} . A multivalued mapping $F: \Omega \multimap Y$ is called *measurable* if $\{\omega \in \Omega \mid F(\omega) \subset V\} \in \mathcal{U}$, for each open set $V \subset Y$.

We say that mapping $F: J \times \mathbb{R}^m \multimap \mathbb{R}^n$, where $J \subset \mathbb{R}$, is an *upper-Carathéodory mapping* if the map $F(\cdot, x): J \multimap \mathbb{R}^n$ is measurable on every compact subinterval of J , for all $x \in \mathbb{R}^m$, the map $F(t, \cdot): \mathbb{R}^m \multimap \mathbb{R}^n$ is u.s.c., for almost all (a.a.) $t \in J$, and the set $F(t, x)$ is compact and convex, for all $(t, x) \in J \times \mathbb{R}^m$.

We recall now some results which are employed in the sequel.

Proposition 2.3 (cf., e.g., [7]). *Let $F: [a, b] \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ be an upper-Carathéodory mapping satisfying $|y| \leq r(t)(1 + |x|)$, for every $(t, x) \in [a, b] \times \mathbb{R}^m$, and every $y \in F(t, x)$, where $r: [a, b] \rightarrow [0, \infty)$ is an integrable function. Then the composition $F(t, q(t))$ admits, for every $q \in C([a, b], \mathbb{R}^m)$, a single-valued measurable selection.*

Lemma 2.4 (cf. [8, Theorem 0.3.4]). *Let $J \subset \mathbb{R}$ be a compact interval. Assume that the sequence of absolutely continuous functions $x_n: J \rightarrow \mathbb{R}^k$ satisfies the following conditions:*

- (i) *the set $\{x_n(t) \mid n \in \mathbb{N}\}$ is bounded, for every $t \in J$,*
- (ii) *there exists a function $\alpha: J \rightarrow \mathbb{R}$, integrable in the sense of Lebesgue, such that*

$$|\dot{x}_n(t)| \leq \alpha(t), \quad \text{for a.a. } t \in J \text{ and for all } n \in \mathbb{N}.$$

Then there exists a subsequence of $\{x_n\}$ (for the sake of simplicity denoted as the sequence) convergent to an absolutely continuous function $x: J \rightarrow \mathbb{R}^k$ in the following sense:

- (iii) *$\{x_n\}$ converges uniformly to x ,*
- (iv) *$\{\dot{x}_n\}$ converges weakly in $L^1(J, \mathbb{R}^k)$ to \dot{x} .*

Lemma 2.5 (cf. [34]). *Let $[a, b] \subset \mathbb{R}$ be a compact interval, E_1, E_2 be Banach spaces and $F: [a, b] \times E_1 \rightarrow E_2$ be a multivalued mapping satisfying the following conditions:*

- (i) *$F(\cdot, x)$ has a strongly measurable selection, for every $x \in E_1$, i.e. there exists a sequence of step multivalued maps $F_n(\cdot, x): [a, b] \rightarrow E_2$ such that $d_H(F_n(\omega, x), F(\omega, x)) \rightarrow 0$, for a.a. $\omega \in \Omega$, as $n \rightarrow \infty$, for every $x \in E_1$,*
- (ii) *$F(t, \cdot)$ is u.s.c., for a.a. $t \in [a, b]$,*
- (iii) *the set $F(t, x)$ is compact and convex, for all $(t, x) \in [a, b] \times E_1$.*

Assume in addition that, for every nonempty, bounded set $\Omega \subset E_1$, there exists $\nu = \nu(\Omega) \in L^1([a, b], \mathbb{R}^+)$ such that $|F(t, x)| \leq \nu(t)$, for a.a. $t \in [a, b]$ and every $x \in \Omega$.

Let us define the Nemytskii operator $N_F: C([a, b], E_1) \rightarrow L^1([a, b], E_2)$ in the following way:

$$N_F(x) := \{f \in L^1([a, b], E_2) \mid f(t) \in F(t, x(t)), \text{ a.e. on } [a, b]\},$$

for every $x \in C([a, b], E_1)$. Then, if sequences $\{x_i\} \subset C([a, b], E_1)$ and $\{f_i\} \subset L^1([a, b], E_2)$, $f_i \in N_F(x_i)$, $i \in \mathbb{N}$, are such that $x_i \rightarrow x$ in $C([a, b], E_1)$ and $f_i \rightarrow f$ weakly in $L^1([a, b], E_2)$, then $f \in N_F(x)$.

For more details concerning multivalued theory see e.g. [8, 9, 20, 25].

In order to develop the continuation principle for the n -th order asymptotic problems, the following important arguments will be also needed (cf. [4, 5, 16–18]). Let us assume that X is a retract of a Fréchet space E (by which X is an AR-space; cf. [12]) and D is an open subset of X (by which D is an ANR-space; cf. [12]). Let $G \in J(D, E)$ be locally compact, let $\text{Fix}(G)$ be compact and let the following condition hold:

$$\forall x \in \text{Fix}(G) \exists \text{ a set } U_x, \text{ open in } D, x \in U_x, \text{ such that } G(U_x) \subset X. \quad (2.3)$$

The class of locally compact J -mappings from D to E with a compact fixed point set and satisfying (2.3) will be denoted by $J_A(D, E)$.

We say that $G_1, G_2 \in J_A(D, E)$ are *homotopic in $J_A(D, E)$* if

1. there exists a homotopy $H \in J(D \times [0, 1], E)$ such that $H(\cdot, 0) = G_1$ and $H(\cdot, 1) = G_2$,
2. for every $x \in D$, there exists an open neighbourhood V_x of x in D such that $H|_{V_x \times [0, 1]}$ is a compact mapping,
3. for every $x \in D$ and every $t \in [0, 1]$, the following condition holds:

$$\begin{aligned} \text{If } x \in H(x, t), \text{ then there exists a set } U_x \text{ open in } D, x \in U_x, \\ \text{such that } H(U_x \times [0, 1]) \subset X. \end{aligned} \quad (2.4)$$

Remark 2.6. Note that condition (2.4) is equivalent to the following one:

$$\begin{aligned} \text{If } \{x_j\}_{j=1}^\infty \subset D \text{ converges to } x \in H(x, t), \text{ for some } t \in [0, 1], \\ \text{then } H(\{x_j\} \times [0, 1]) \subset X, \text{ for } j \text{ sufficiently large.} \end{aligned}$$

Remark 2.7 (see e.g. [3]). If $E = X$ is a Banach space, then condition (2.4) can be replaced by

$$\text{Fix}(H) \cap \partial D = \emptyset,$$

for all $t \in [0, 1]$, where $\text{Fix}(H) := \{x \in D \mid x \in H(x, t)\}$.

The following proposition, which will be applied below for obtaining the existence of a solution of the studied b.v.p., follows immediately from the results in [4, 5].

Proposition 2.8. *Let X be a retract of a Fréchet space E , D be an open subset of X and H be a homotopy in $J_A(D, E)$ such that*

- (i) $H(\cdot, 0)(D) \subset X$,
- (ii) *there exists $H_0 \in J(X)$ such that $H_0|_D = H(\cdot, 0)$, H_0 is compact and*

$$\text{Fix}(H_0) \cap (X \setminus D) = \emptyset.$$

Then there exists $x \in D$ such that $x \in H(x, 1)$.

As a direct consequence of Proposition 2.8, it is possible to obtain the following result.

Corollary 2.9. *Let X be a retract of a Fréchet space E , H be a homotopy in $J_A(X, E)$ such that $H(x, 0) \subset X$, for every $x \in X$, and let $H(\cdot, 0)$ be compact. Then $H(\cdot, 1)$ has a fixed point.*

3 Continuation principle

In this section, we consider the n -th order boundary value problem in the following form

$$\begin{aligned} x^{(n)}(t) \in F\left(t, x(t), \dot{x}(t), \dots, x^{(n-1)}(t)\right), \quad \text{for a.a. } t \in J, \\ x \in S, \end{aligned} \quad (3.1)$$

where J is a given real (possibly noncompact) interval, $F: J \times \mathbb{R}^{kn} \rightarrow \mathbb{R}^k$ is a multivalued upper-Carathéodory mapping and $S \subset AC_{\text{loc}}^{n-1}(J, \mathbb{R}^k)$.

By a *solution* of problem (3.1), we mean a function $x: J \rightarrow \mathbb{R}^k$ belonging to $AC_{\text{loc}}^{n-1}(J, \mathbb{R}^k)$ and satisfying (3.1), for almost all $t \in J$.

For our main result, the following proposition is pivotal.

Proposition 3.1. Let $H: J \times \mathbb{R}^{2kn} \rightarrow \mathbb{R}^k$ be an upper-Carathéodory mapping and let

$$S = \left\{ x \in AC_{\text{loc}}^{n-1}(J, \mathbb{R}^k) \mid l(x, \dot{x}, \dots, x^{(n-1)}) = 0 \right\},$$

where $l: C^{n-1}(J, \mathbb{R}^k) \times C^{n-2}(J, \mathbb{R}^k) \dots C(J, \mathbb{R}^k) \rightarrow \mathbb{R}^k$ is a linear bounded operator. Assume that

(i) there exists a subset Q of $C^{n-1}(J, \mathbb{R}^k)$ such that, for any $q \in Q$, the set $\mathfrak{T}(q)$ of all solutions of the boundary value problem

$$\begin{aligned} x^{(n)}(t) &\in H(t, x(t), \dot{x}(t), \dots, x^{(n-1)}(t), q(t), \dot{q}(t), \dots, q^{(n-1)}(t)), \quad \text{for a.a. } t \in J, \\ x &\in S \end{aligned}$$

is nonempty,

(ii) there exist $t^* \in J$ and a constant $M > 0$ such that

$$|x(t^*)| \leq M, \quad |\dot{x}(t^*)| \leq M, \quad \dots, \quad |x^{(n-1)}(t^*)| \leq M,$$

for all $x \in \mathfrak{T}(Q)$,

(iii) there exists a nonnegative, locally integrable function $\alpha: J \rightarrow \mathbb{R}$ such that

$$\left| H\left(t, x(t), \dots, x^{(n-1)}(t), q(t), \dots, q^{(n-1)}(t)\right) \right| \leq \alpha(t) \left(1 + |x(t)| + \dots + |x^{(n-1)}(t)|\right),$$

a.e. in J , for any $(q, x) \in \Gamma_{\mathfrak{T}}$.

Then $\mathfrak{T}(Q)$ is a relatively compact subset of $C^{n-1}(J, \mathbb{R}^k)$. Moreover, the multivalued operator $\mathfrak{T}: Q \rightarrow S$ is u.s.c. with compact values if the following condition is satisfied:

(iv) for each sequence $\{q_m, x_m\} \subset \Gamma_{\mathfrak{T}}$ satisfying

$$\left\{ \left(q_m, \dot{q}_m, \dots, q_m^{(n-1)}, x_m, \dot{x}_m, \dots, x_m^{(n-1)} \right) \right\} \rightarrow \left(q, \dot{q}, \dots, q^{(n-1)}, x, \dot{x}, \dots, x^{(n-1)} \right),$$

where $q \in Q$, it holds that $x \in S$.

Proof. From the well-known Arzelà–Ascoli lemma, it follows that the set $\mathfrak{T}(Q)$ is relatively compact in $C^{n-1}(J, \mathbb{R}^k)$ if and only if it is bounded and functions in $\mathfrak{T}(Q)$ and their derivatives are equicontinuous. Let us prove at first the boundedness of $\mathfrak{T}(Q)$. For this purpose, let $[a, b] \subset J$ be an arbitrary compact interval such that $t^* \in [a, b]$. Since

$$x^{(n-1)}(t) = x^{(n-1)}(t^*) + \int_{t^*}^t x^{(n)}(s) ds, \quad \text{for a.a. } t \in [a, b],$$

⋮

$$\dot{x}(t) = \dot{x}(t^*) + \int_{t^*}^t \ddot{x}(s) ds, \quad \text{for a.a. } t \in [a, b],$$

$$x(t) = x(t^*) + \int_{t^*}^t \dot{x}(s) ds, \quad \text{for a.a. } t \in [a, b],$$

it holds, according to conditions (ii) and (iii), that

$$\begin{aligned} &|x(t)| + |\dot{x}(t)| + \dots + |x^{(n-1)}(t)| \\ &\leq |x(t^*)| + |\dot{x}(t^*)| + \dots + |x^{(n-1)}(t^*)| + \int_{t^*}^t (|\dot{x}(s)| + |\ddot{x}(s)| + \dots + |x^{(n)}(s)|) ds \\ &\leq nM + \int_a^b (|\dot{x}(s)| + |\ddot{x}(s)| + \dots + \alpha(s)(1 + |x(s)| + \dots + |x^{(n-1)}(s)|)) ds \\ &\leq nM + \int_a^b \alpha(s) ds + \int_a^b (1 + \alpha(s))(1 + |x(s)| + |\dot{x}(s)| + \dots + |x^{(n-1)}(s)|) ds. \end{aligned}$$

Therefore, by Gronwall's lemma,

$$|x(t)| + |\dot{x}(t)| + \dots + |x^{(n-1)}(t)| \leq \left(nM + \int_a^b \alpha(s) ds \right) e^{\int_a^b (1+\alpha(s)) ds}, \quad \text{for a.a. } t \in [a, b]. \quad (3.2)$$

Since $[a, b] \subset J$ is arbitrary, it follows immediately from estimate (3.2) that $\mathfrak{T}(Q)$ is bounded in each seminorm, and so also bounded in $C^{n-1}(J, \mathbb{R}^k)$.

Now, let us check the equicontinuity of $x, \dot{x}, \dots, x^{(n-1)}$, for each $x \in \mathfrak{T}(Q)$. Let $q \in Q$, $x \in \mathfrak{T}(q)$ and $t_1, t_2 \in J$ be arbitrary. Then, we obtain that

$$|x(t_1) - x(t_2)| \leq \left| \int_{t_1}^{t_2} |\dot{x}(\tau)| d\tau \right| \leq \left| \int_{t_1}^{t_2} \left(nM + \int_a^b \alpha(s) ds \right) e^{\int_a^b (1+\alpha(s)) ds} d\tau \right|. \quad (3.3)$$

Analogously, we can get, for each $k \in \{1, \dots, n-2\}$, that

$$|x^{(k)}(t_1) - x^{(k)}(t_2)| \leq \left| \int_{t_1}^{t_2} |x^{(k+1)}(\tau)| d\tau \right| \leq \left| \int_{t_1}^{t_2} \left(nM + \int_a^b \alpha(s) ds \right) e^{\int_a^b (1+\alpha(s)) ds} d\tau \right|. \quad (3.4)$$

Moreover,

$$\begin{aligned} |x^{(n-1)}(t_1) - x^{(n-1)}(t_2)| &\leq \left| \int_{t_1}^{t_2} |H(\tau, x(\tau), \dots, x^{(n-1)}(\tau), q(\tau), \dots, q^{(n-1)}(\tau))| d\tau \right| \\ &\leq \left| \int_{t_1}^{t_2} \alpha(\tau) \left(1 + |x(\tau)| + \dots + |x^{(n-1)}(\tau)| \right) d\tau \right| \\ &\leq \left| \int_{t_1}^{t_2} \alpha(\tau) \left(1 + \left(nM + \int_a^b \alpha(s) ds \right) e^{\int_a^b (1+\alpha(s)) ds} \right) d\tau \right|. \end{aligned} \quad (3.5)$$

The estimates (3.3)–(3.5) ensure the equicontinuity of $x, \dot{x}, \dots, x^{(n-1)}$. Thus, it is proven that $\mathfrak{T}(Q)$ is relatively compact.

Let us still show that the graph of the operator \mathfrak{T} is closed. Let $\{(q_m, x_m)\} \subset \Gamma_{\mathfrak{T}}$ be such that

$$\left\{ \left(q_m, \dot{q}_m, \dots, q_m^{(n-1)}, x_m, \dot{x}_m, \dots, x_m^{(n-1)} \right) \right\} \rightarrow \left(q, \dot{q}, \dots, q^{(n-1)}, x, \dot{x}, \dots, x^{(n-1)} \right),$$

where $q \in Q$, and let $[a, b] \subset J$ be an arbitrary compact interval such that $t^* \in [a, b]$.

By condition (iii) and estimate (3.2), the sequences $\{x_m\}, \{\dot{x}_m\}, \dots, \{x_m^{(n-1)}\}$ satisfy assumptions of Lemma 2.4. Thus, there exists a subsequence of $\{x_m\}$, for the sake of simplicity denoted as the sequence, uniformly convergent to x on $[a, b]$, such that $\{\dot{x}_m\}, \dots, \{x_m^{(n-1)}\}$ converges uniformly to $\dot{x}, \dots, x^{(n-1)}$ on $[a, b]$ and $\{x_m^{(n)}\}$ converges weakly to $x^{(n)}$ in $L^1([a, b], \mathbb{R}^k)$.

If we set

$$z_m := \left(x_m, \dot{x}_m, \dots, x_m^{(n-1)} \right),$$

then $\dot{z}_m \rightarrow (\dot{x}, \ddot{x}, \dots, x^{(n)})$ weakly in $L^1([a, b], \mathbb{R}^k)$. Let us consider the following system

$$\dot{z}_m(t) \in G \left(t, z_m(t), q_m(t), \dot{q}_m(t), \dots, q_m^{(n-1)}(t) \right), \quad \text{for a.a. } t \in [a, b],$$

where

$$G \left(t, z_m(t), q_m(t), \dot{q}_m(t), \dots, q_m^{(n-1)}(t) \right) = \left(\dot{x}_m, \dots, x_m^{(n)}, H(t, z_m(t), q_m(t), \dots, q_m^{(n-1)}(t)) \right).$$

Using Lemma 2.5, for $f_i := \dot{z}_m$, $f := (\dot{x}, \ddot{x}, \dots, x^{(n)})$, $x_i := (z_m, q_m, \dot{q}_m, \dots, q_m^{(n-1)})$, it follows that

$$(\dot{x}(t), \ddot{x}(t), \dots, x^{(n)}(t)) \in G \left(t, x(t), \dot{x}(t), \dots, x^{(n-1)}(t), q(t), \dot{q}(t), \dots, q^{(n-1)}(t) \right),$$

for a.a. $t \in [a, b]$, i.e.

$$x^{(n)}(t) \in H\left(t, x(t), \dot{x}(t), \dots, x^{(n-1)}(t), q(t), \dot{q}(t), \dots, q^{(n-1)}(t)\right), \quad \text{for a.a. } t \in [a, b].$$

Since $[a, b] \subset J$ is arbitrary,

$$x^{(n)}(t) \in H\left(t, x(t), \dot{x}(t), \dots, x^{(n-1)}(t), q(t), \dot{q}(t), \dots, q^{(n-1)}(t)\right), \quad \text{for a.a. } t \in J.$$

Condition (iv) implies that $x \in S$, and therefore $\Gamma_{\mathfrak{T}}$ is closed. Moreover, it follows immediately from Proposition 2.1 that the operator \mathfrak{T} is u.s.c.

Since \mathfrak{T} is a compact mapping, $\mathfrak{T}(q)$ is, for each $q \in Q$, a relatively compact set. Moreover, the operator \mathfrak{T} has a closed graph which implies that $\mathfrak{T}(q)$ is, for each $q \in Q$, closed, and therefore \mathfrak{T} has compact values. \square

As the main result of this section, we formulate the following theorem in which conditions ensuring the existence of a solution of the boundary value problem (3.1) are presented.

Theorem 3.2. *Let us consider the boundary value problem (3.1) and let $H: J \times \mathbb{R}^{2kn} \times [0, 1] \rightarrow \mathbb{R}^k$ be an upper-Carathéodory map such that*

$$H(t, c_1, \dots, c_n, c_1, \dots, c_n, 1) \subset F(t, c_1, \dots, c_n), \quad \text{for all } (t, c_1, \dots, c_n) \in J \times \mathbb{R}^{kn}. \quad (3.6)$$

Assume that

(i) *there exists a retract Q of $C^{n-1}(J, \mathbb{R}^k)$ and a closed subset S_1 of S such that the associated problem*

$$\begin{aligned} x^{(n)}(t) &\in H\left(t, x(t), \dots, x^{(n-1)}(t), q(t), \dots, q^{(n-1)}(t), \lambda\right), \quad \text{for a.a. } t \in J, \\ x &\in S_1 \end{aligned} \quad (3.7)$$

is solvable with an R_δ -set of solutions, for each $(q, \lambda) \in Q \times [0, 1]$,

(ii) *there exists a nonnegative, locally integrable function $\alpha: J \rightarrow \mathbb{R}$ such that*

$$\left| H(t, x(t), \dots, x^{(n-1)}(t), q(t), \dots, q^{(n-1)}(t), \lambda) \right| \leq \alpha(t) \left(1 + |x(t)| + \dots + |x^{(n-1)}(t)| \right),$$

a.e. in J , for any $(q, \lambda, x) \in \Gamma_{\mathfrak{T}}$, where \mathfrak{T} denotes the multivalued mapping which assigns to any $(q, \lambda) \in Q \times [0, 1]$ the set of solutions of (3.7),

(iii) $\mathfrak{T}(Q \times \{0\}) \subset Q$,

(iv) *there exist $t^* \in J$ and a constant $M > 0$ such that*

$$|x(t^*)| \leq M, \quad |\dot{x}(t^*)| \leq M, \dots, |x^{(n-1)}(t^*)| \leq M,$$

for any $x \in \mathfrak{T}(Q \times [0, 1])$,

(v) *if $q_j, q \in Q$, $q_j \rightarrow q$, $q \in \mathfrak{T}(q, \lambda)$, then there exists $j_0 \in \mathbb{N}$ such that, for every $j \geq j_0$, $\theta \in [0, 1]$ and $x \in \mathfrak{T}(q_j, \theta)$, we have $x \in Q$.*

Then problem (3.1) has a solution.

Proof. At first, we show that all the assumptions of Proposition 3.1 are satisfied. Conditions (i), (ii) and (iv) in Theorem 3.1 guarantee conditions (i), (ii) and (iii) in Proposition 3.1.

Let $\{(q_m, \lambda_m, x_m)\} \subset \Gamma_{\mathfrak{T}}$, $(q_m, \lambda_m, x_m) \rightarrow (q, \lambda, x)$, $(q, \lambda) \in Q \times [0, 1]$ be arbitrary. Then, since $x_m \in S_1$, $x_m \rightarrow x$ and S_1 is closed, it holds that $x \in S_1$. Therefore, condition (iv) from Proposition 3.1 holds as well. Thus, $\mathfrak{T}: Q \times [0, 1] \rightarrow S_1$ is, according to Proposition 3.1, a compact u.s.c. mapping with compact values. According to assumption (i), \mathfrak{T} has R_δ -values, and so it belongs to the class $J(Q \times [0, 1], C^{n-1}(I, \mathbb{R}^k))$. Assumption (v) implies that \mathfrak{T} is a homotopy in $J_A(Q, C^{n-1}(I, \mathbb{R}^k))$. From Corollary 2.9, it follows that there exists a fixed point of $\mathfrak{T}(\cdot, 1)$ in Q . Moreover, by the inclusion (3.6) and since $S_1 \subset S$, the fixed point of $\mathfrak{T}(\cdot, 1)$ is a solution of the original b.v.p. (3.1). \square

4 Existence and localization results

Let us consider the b.v.p.

$$x^{(n)}(t) + \sum_{i=1}^n A_i(t, x(t), \dots, x^{(n-1)}(t))x^{(n-i)}(t) \in F\left(t, x(t), \dots, x^{(n-1)}(t)\right), \quad (4.1)$$

for a.a. $t \in J$, $x \in S$,

where

(i) $J \subset \mathbb{R}$,

(ii) $A_i: J \times \mathbb{R}^{kn} \rightarrow \mathbb{R}^k \times \mathbb{R}^k$ are, for all $i = 1, \dots, n$, Carathéodory matrix functions such that

$$|A_i(t, q(t), \dots, q^{(n-1)}(t))| \leq a(t),$$

for a.a. $t \in J$, a suitable locally integrable function $a: J \rightarrow [0, \infty)$, and all $q \in Q$, where $Q \subset C^{n-1}(J, \mathbb{R}^k)$,

(iii) $F: J \times \mathbb{R}^{kn} \rightarrow \mathbb{R}^k$ is an upper-Carathéodory mapping,

(iv) S is a subset of $AC_{\text{loc}}^{n-1}(J, \mathbb{R}^k)$.

If the problems associated to (4.1) are fully linearized, we obtain the following result when applying the continuation principle from the previous section.

Theorem 4.1. *Let us consider the b.v.p. (4.1) and assume that*

(i) *there exists a nonnegative, locally integrable function $\alpha: J \rightarrow \mathbb{R}$ such that*

$$\left| F\left(t, q(t), \dot{q}(t), \dots, q^{(n-1)}(t)\right) \right| \leq \alpha(t), \quad \text{a.e. in } J,$$

for any $q \in Q$, where Q is a retract of $C^{n-1}(J, \mathbb{R}^k)$,

(ii) *there exist $t^* \in J$ and a constant $M > 0$ such that*

$$|x(t^*)| \leq M, |\dot{x}(t^*)| \leq M, \dots, |x^{(n-1)}(t^*)| \leq M,$$

for any $x \in \mathfrak{T}(Q \times [0, 1])$, where \mathfrak{T} denotes the mapping which assigns to each $(q, \lambda) \in Q \times [0, 1]$ the set of solutions of fully linearized problems

$$x^{(n)}(t) + \sum_{i=1}^n A_i(t, q(t), \dots, q^{(n-1)}(t))x^{(n-i)}(t) \in \lambda F(t, \dot{q}(t), \dots, q^{(n-1)}(t)), \quad \text{for a.a. } t \in J,$$

$x \in S_1$,

(iii) S_1 is a closed convex subset of S ,

(iv) $\mathfrak{T}(q, \lambda) \neq \emptyset$, for all $(q, \lambda) \in Q \times [0, 1]$, and $\mathfrak{T}(Q \times \{0\}) \subset Q$,

(v) if $q_j, q \in Q$, $q_j \rightarrow q$, $q \in \mathfrak{T}(q, \lambda)$, then there exists $j_0 \in \mathbb{N}$ such that, for every $j \geq j_0$, $\theta \in [0, 1]$ and $x \in \mathfrak{T}(q_j, \theta)$, we have $x \in Q$.

Then the b.v.p. (4.1) has a solution in $S_1 \cap Q$.

Proof. Since the associated problems are, for all $(q, \lambda) \in Q \times [0, 1]$, fully linearized, the mapping F has convex values and S_1 is convex, the set $\mathfrak{T}(q, \lambda)$ is also convex, for all $(q, \lambda) \in Q \times [0, 1]$. Therefore, all assumptions of Theorem 3.2 are satisfied, and so the problem (4.1) has a solution in $S_1 \cap Q$. \square

Making use of the result in [13], dealing with the equivalency of norms in the Banach space $H^{n,1}(J, \mathbb{R}^k)$, we are able to improve condition (ii) from Theorem 4.1 as follows.

Corollary 4.2. *Let us consider the b.v.p. (4.1) and let conditions (i), (iii), (iv) from Theorem 4.1 hold. Moreover, instead of condition (ii), let us assume that $\mathfrak{T}(Q \times [0, 1])$ is bounded in $C(J, \mathbb{R}^k)$. Then the b.v.p. (4.1) has a solution in $S_1 \cap Q$.*

Proof. Since $\mathfrak{T}(Q \times [0, 1])$ is bounded in $C(J, \mathbb{R}^k)$, there exists a continuous function $m: J \rightarrow \mathbb{R}$ such that

$$|x(t)| \leq m(t), \text{ for all } x \in \mathfrak{T}(Q \times [0, 1]) \text{ and all } t \in J.$$

Let us show that, for any compact interval $I \subset J$, there exists a constant $M > 0$ such that

$$p_I(x) := \sum_{i=1}^{n-1} \sup_{t \in I} |x^{(i)}(t)| \leq M, \text{ for all } x \in \mathfrak{T}(Q \times [0, 1]).$$

According to Lemma 2.36 in [13] and the remarks below that lemma, the following two norms in $H^{n,1}(I, \mathbb{R}^k)$

$$\|x\| := \sum_{i=1}^{n-1} \sup_{t \in I} |x^{(i)}(t)| + \int_I |x^{(n)}(t)| dt$$

and

$$\|x\|_Q := \sup_{t \in I} |x(t)| + \sup_{x \in Q} \int_I \left| x^{(n)}(t) + \sum_{i=1}^n A_i(t, q(t), \dots, q^{(n-1)}(t)) x^{(i)}(t) \right| dt$$

are equivalent.

It is obvious that $p_I(x) \leq \|x\|$ and, by the above mentioned equivalency of norms, there exists a constant $c > 0$ such that

$$\|x\| \leq c \|x\|_Q \leq c \left(\max_{t \in I} m(t) + \int_I \alpha(t) dt \right) \leq M.$$

Therefore, $\mathfrak{T}(Q \times [0, 1])$ is also bounded in $C^{n-1}(J, \mathbb{R}^k)$ which, in particular, ensures the validity of condition (ii) from Theorem 4.1. \square

Remark 4.3. Let us note that Corollary 4.2 cannot be deduced by a simple transformation of a studied problem to the first-order problem. The obtained result is a vector generalization of Corollary 2.37 in [4] and it also generalizes the vector result for the second-order b.v.p. in [7], where A_i did not depend on x . Moreover, a more restrictive condition (ii) was used there.

Observe that condition (v) in Theorem 4.1 hold when $S_1 \subset Q$, by which Theorem 4.1 can be simplified in the following way, suitable for practical applications.

Corollary 4.4. *Let us consider the b.v.p. (4.1), where J is a given real interval, $F: J \times \mathbb{R}^{kn} \rightarrow \mathbb{R}^k$ is an upper-Carathéodory mapping and S is a subset of $AC_{\text{loc}}^{n-1}(J, \mathbb{R}^k)$.*

Assume that

- (i) *there exists a retract Q of $C^{n-1}(J, \mathbb{R}^k)$ such that $S \cap Q$ is closed and convex and that the associated problem*

$$\begin{aligned} x^{(n)}(t) + A_1(t)x^{(n-1)}(t) + \dots + A_n(t)x(t) &\in F\left(t, \dot{q}(t), \dots, q^{(n-1)}(t)\right), \quad \text{for a.a. } t \in J, \\ x &\in S \cap Q \end{aligned}$$

is solvable, for each $q \in Q$,

- (ii) *there exists a nonnegative, locally integrable function $\alpha: J \rightarrow \mathbb{R}$ such that*

$$\left| F\left(t, q(t), \dot{q}(t), \dots, q^{(n-1)}(t)\right) \right| \leq \alpha(t), \quad \text{a.e. in } J,$$

for any $q \in Q$,

- (iii) *there exist $t^* \in J$ and a constant $M > 0$ such that*

$$|x(t^*)| \leq M, \quad |\dot{x}(t^*)| \leq M, \dots, |x^{(n-1)}(t^*)| \leq M,$$

for any $x \in \mathfrak{T}(Q)$ (or $\mathfrak{T}(Q)$ is bounded in $C(J, \mathbb{R}^k)$).

Then the b.v.p. (4.1) has a solution in $S \cap Q$.

Remark 4.5. Let us note that Corollary 4.4 generalizes the results in [4] and [7] as well as Proposition 2.1 in [18] which was (unlike our result) obtained only as a vector modification of the scalar result in [4].

5 Illustrative examples

Let us finally illustrate the application of Corollary 4.4 by two examples. The first one concerns the n -th order vector target (terminal) problem.

Example 5.1. Let us consider the n -th order target problem

$$\left. \begin{aligned} x_1^{(n)}(t) &\in F_1(t, x_1(t), \dots, x_k(t)), \quad \text{for a.a. } t \in [0, \infty), \\ &\vdots \\ x_k^{(n)}(t) &\in F_k(t, x_1(t), \dots, x_k(t)), \quad \text{for a.a. } t \in [0, \infty), \\ &\lim_{t \rightarrow \infty} x_1(t) = l_1, \\ &\quad \quad \quad \vdots \\ &\lim_{t \rightarrow \infty} x_k(t) = l_k, \end{aligned} \right\} \quad (5.1)$$

where, for all $i = 1, \dots, k$, $F_i: [0, \infty) \times \mathbb{R}^k \rightarrow \mathbb{R}$ are upper-Carathéodory mappings and $l_i \in \mathbb{R}$. Moreover, let there exist $K > 0$ such that, for all $i = 1, \dots, k$,

$$\int_0^\infty t^{n-1} \cdot \alpha_i(t) dt < (n-1)!(K - |l_i|), \quad (5.2)$$

where

$$\alpha_i(t) := \sup_{|x_i| \leq K, \text{ for all } i=1, \dots, k} |F_i(t, x_1, \dots, x_k)|.$$

Then it is possible to apply Corollary 4.4 and obtain that the target problem (5.1) has a solution $x = (x_1, \dots, x_k)$ satisfying $|x_i(t)| \leq K$, for all $i = 1, \dots, k$ and all $t \in [0, \infty)$. More concretely, let us define the set Q of candidate solutions as

$$Q := \left\{ (q_1, \dots, q_k) \in C^{n-1}([0, \infty), \mathbb{R}^k) \mid |q_i(t)| \leq K, \text{ for all } t \in [0, \infty) \text{ and all } i = 1, \dots, k \right\},$$

and let us consider the family of fully linearized associated problems

$$\left. \begin{aligned} x_1^{(n)}(t) &\in F_1(t, q_1(t), \dots, q_k(t)), \text{ for a.a. } t \in [0, \infty), \\ &\vdots \\ x_k^{(n)}(t) &\in F_k(t, q_1(t), \dots, q_k(t)), \text{ for a.a. } t \in [0, \infty), \\ \lim_{t \rightarrow \infty} x_1(t) &= l_1, \\ &\vdots \\ \lim_{t \rightarrow \infty} x_k(t) &= l_k. \end{aligned} \right\} \quad (5.3)$$

At first, let us verify condition (i) from Corollary 4.4. If $q = (q_1, \dots, q_k) \in Q$ is arbitrary, then $F_i(t, q(t))$ admits, for all $i = 1, \dots, k$, according to Proposition 2.3, a single-valued selection $f_{q,i}(t)$, measurable on every compact subinterval of $[0, \infty)$. The corresponding problem

$$\left. \begin{aligned} x_1^{(n)}(t) &= f_{q,1}(t), \text{ for a.a. } t \in [0, \infty), \\ &\vdots \\ x_k^{(n)}(t) &= f_{q,k}(t), \text{ for a.a. } t \in [0, \infty), \\ \lim_{t \rightarrow \infty} x_1(t) &= l_1, \\ &\vdots \\ \lim_{t \rightarrow \infty} x_k(t) &= l_k, \end{aligned} \right\} \quad (5.4)$$

has a solution $x = (x_1, \dots, x_k)$ such that

$$x_i(t) = l_i + \frac{1}{(n-1)!} \int_t^\infty (s-t)^{n-1} \cdot f_{q,i}(s) ds, \quad \text{for all } i = 1, \dots, k \text{ and a.a. } t \in [0, \infty).$$

This solution belongs to Q , according to (5.2), and so the assumption (i) from Corollary 4.4 is satisfied.

The validity of assumption (ii) from Corollary 4.4 follows immediately from the properties of functions α_i and the definition of the set Q . Moreover, all solutions of (5.3) belong, for arbitrary $q \in Q$, to the closed, bounded subset of $C^{n-1}([0, \infty), \mathbb{R}^k)$, namely

$$\left\{ (x_1, \dots, x_k) \in C^{n-1}([0, \infty), \mathbb{R}^k) \mid |x_i(t)| \leq |l_i| + \frac{1}{(n-1)!} \int_t^\infty s^{n-1} \cdot \alpha_i(s) ds; \right.$$

$$\left. |\dot{x}_i(t)| \leq t^{n-1} \cdot \alpha_i(t), \dots, |x_i^{(n-1)}(t)| \leq \frac{d^{n-2}}{dt^{n-2}} \left(t^{n-1} \cdot \alpha_i(t) \right), \quad i = 1, \dots, k, t \in [0, \infty) \right\}.$$

In order to verify assumption (iii) from Corollary 4.4, let us observe that it follows from the boundary conditions $\lim_{t \rightarrow \infty} x_1(t) = l_1, \dots, \lim_{t \rightarrow \infty} x_k(t) = l_k$ that $\lim_{t \rightarrow \infty} \dot{x}_i(t) = 0, \dots,$

$\lim_{t \rightarrow \infty} x_i^{(n-1)}(t) = 0$, for all $i = 1, \dots, k$. Therefore, there exist $t_1, \dots, t_{n-1} \in [0, \infty)$ such that, for all $j = 1, \dots, n-1$ and $i = 1, \dots, k$, it holds that $|x_i^{(j)}(t)| \leq K$, for all $t \geq t_j$. Therefore, assumption (iii) holds with $t^* = \max\{t_1, \dots, t_{n-1}\}$ and $M = \sqrt{k}K$.

Summing up, all assumptions of previous corollary are satisfied, by which the target problem (5.1) admits a solution in Q .

As the second illustrative example, let us study the n -th order multivalued Sturm–Liouville b.v.p.

Example 5.2. Let us consider the n -th order Sturm–Liouville b.v.p. on the half-line

$$\left. \begin{aligned} -x^{(n)}(t) &\in F(t, x(t)), \text{ for a.a. } t \in [0, \infty), \\ x^{(i)}(0) &= A_i, \quad i = 0, 1, \dots, n-3, \\ x^{(n-2)}(0) - ax^{(n-1)}(0) &= B, \\ \lim_{t \rightarrow \infty} x^{(n-1)}(t) &= C, \end{aligned} \right\} \quad (5.5)$$

where $F: [0, \infty) \times \mathbb{R} \rightrightarrows \mathbb{R}$ is an upper-Carathéodory mapping, $a > 0, A_i, B, C \in \mathbb{R}$, $i = 0, \dots, n-3$. Moreover, let there exist $M > 0$ such that, for all $i = 0, 1, \dots, n-1$,

$$\int_0^\infty \alpha(t) dt < \frac{M - L_i}{K_i}, \quad (5.6)$$

where

$$\alpha(t) := \sup_{|x| \leq M} |F(t, x)|,$$

$$K_i := \frac{a}{(n-1-i)!} (n-2-i)^{\frac{n-2-i}{n-1-i}} + \frac{n-1-i}{(n-2-i)!},$$

$$L_i := \sum_{k=i}^{n-3} \frac{|A_k|(n-1-k)}{(k-i)!(n-1-i)} \cdot \left(\frac{k-i}{n-1-k}\right)^{\frac{k-i}{n-1-i}} + \frac{|aC+B|}{(n-3-i)!} \cdot \left(\frac{1}{n-2-i}\right)^{\frac{1}{n-1-i}} + \frac{|C|}{(n-1-i)!}.$$

In order to show that under these conditions problem (5.5) admits at least one solution, let us consider, instead of $C^{n-1}([0, \infty), \mathbb{R})$, the Banach space $(X, \|\cdot\|)$, where

$$X := \left\{ x \in C^{n-1}([0, \infty), \mathbb{R}) \mid \lim_{t \rightarrow \infty} \frac{x^{(i)}(t)}{1+t^{n-1-i}} \text{ exists, for all } i = 0, 1, \dots, n-1 \right\}$$

and

$$\|x\| := \max\{\|x\|_0, \|x\|_1, \dots, \|x\|_{n-1}\}$$

with

$$\|x\|_i := \sup_{t \in [0, \infty)} \left| \frac{x^{(i)}(t)}{1+t^{n-1-i}} \right|, \quad i = 0, 1, \dots, n-1.$$

Let us define the closed convex set Q of candidate solutions by

$$Q := \{q \in X \mid |q(t)| \leq M, \text{ for all } t \in [0, \infty)\}$$

and let us consider the fully linearized associated problems

$$\left. \begin{aligned} -x^{(n)}(t) &\in F(t, q(t)), \text{ for a.a. } t \in [0, \infty), \\ x^{(i)}(0) &= A_i, \quad i = 0, 1, \dots, n-3, \\ x^{(n-2)}(0) - ax^{(n-1)}(0) &= B, \\ \lim_{t \rightarrow \infty} x^{(n-1)}(t) &= C. \end{aligned} \right\} \quad (5.7)$$

Moreover, let us define the solution mapping $\mathfrak{T}: Q \rightarrow C^{n-1}([0, \infty), \mathbb{R})$ which assigns to each $q \in Q$, the set of solutions of (5.7). At first, let us verify that \mathfrak{T} maps Q into the space X . Let $q \in Q$ be arbitrary. Then, according to Proposition 2.3, $F(\cdot, q(\cdot))$ admits a single-valued selection $f_q(\cdot)$, measurable on every compact subinterval of $[0, \infty)$. It follows from the computations made in [26] that the corresponding problem

$$\left. \begin{aligned} -x^{(n)}(t) &= f_q(t), \text{ for a.a. } t \in [0, \infty), \\ x^{(i)}(0) &= A_i, \quad i = 0, 1, \dots, n-3, \\ x^{(n-2)}(0) - ax^{(n-1)}(0) &= B, \\ \lim_{t \rightarrow \infty} x^{(n-1)}(t) &= C \end{aligned} \right\} \quad (5.8)$$

has a unique solution x given by

$$x(t) = l(t) + \int_0^\infty G(t, s) f_q(s) ds, \quad \text{for a.a. } t \in [0, \infty),$$

where

$$l(t) := \sum_{k=0}^{n-3} \frac{A_k}{k!} t^k + \frac{aC + B}{(n-2)!} t^{n-2} + \frac{C}{(n-1)!} t^{n-1}$$

and

$$G(t, s) := \begin{cases} \frac{a}{(n-2)!} t^{n-2} + \sum_{k=0}^{n-2} \frac{(-1)^k}{(k+1)!(n-2-k)!} s^{k+1} t^{n-2-k}, & \text{for } 0 \leq s \leq t < \infty; \\ \frac{a}{(n-2)!} t^{n-2} + \frac{1}{(n-1)!} t^{n-1}, & \text{for } 0 \leq t \leq s < \infty. \end{cases}$$

By the direct computation, we obtain that, for all $i = 0, \dots, n-1$, and a.a. $t \in [0, \infty)$,

$$x^{(i)}(t) = \sum_{k=i}^{n-3} \frac{A_k t^{k-i}}{(k-i)!} + \frac{aC + B}{(n-2-i)!} t^{n-2-i} + \frac{C t^{n-1-i}}{(n-1-i)!} + \int_0^\infty g_i(t, s) f_q(s) ds,$$

where

$$g_i(t, s) := \begin{cases} \frac{a}{(n-2-i)!} t^{n-2-i} + \sum_{k=0}^{n-2-i} \frac{(-1)^k}{(k+1)!(n-2-k-i)!} s^{k+1} t^{n-2-k-i}, & \text{for } 0 \leq s \leq t < \infty; \\ \frac{a}{(n-2-i)!} t^{n-2-i} + \frac{1}{(n-1-i)!} t^{n-1-i}, & \text{for } 0 \leq t \leq s < \infty, \end{cases}$$

and so

$$\lim_{t \rightarrow \infty} \frac{x^{(i)}(t)}{1 + t^{n-1-i}} = \frac{C}{(n-1-i)!} + \int_0^\infty \frac{f_q(s)}{(n-1-i)!} ds < \infty.$$

This implies that $\mathfrak{T}(Q) \subset X$.

Let us still verify that $\mathfrak{T}(Q) \subset Q$. For all $k, l \in \mathbb{N}$, it holds that

$$\sup_{t \in [0, \infty)} \frac{t^k}{1+t^l} = \begin{cases} \frac{l-k}{l} \left(\frac{k}{l-k} \right)^{\frac{k}{l}}, & \text{for } k < l, \\ 1, & \text{for } k = l, \\ \infty, & \text{for } k > l, \end{cases}$$

and so, for $s \leq t$, and all $i = 0, 1, \dots, n-1$,

$$\begin{aligned} & \sup_{t \in [0, \infty)} \left| \frac{g_i(t, s)}{1+t^{n-1-i}} \right| \\ & \leq \sup_{t \in [0, \infty)} \frac{at^{n-2-i}}{(n-2-i)!(1+t^{n-1-i})} + \sup_{t \in [0, \infty)} \sum_{k=0}^{n-2-i} \frac{t^{n-1-i}}{(k+1)!(n-2-k-i)!(1+t^{n-1-i})} \\ & \leq \sup_{t \in [0, \infty)} \frac{at^{n-2-i}}{(n-2-i)!(1+t^{n-1-i})} + \sup_{t \in [0, \infty)} \frac{(n-1-i)t^{n-1-i}}{(n-2-i)!(1+t^{n-1-i})} \\ & = \frac{a}{(n-1-i)!} (n-2-i)^{\frac{n-2-i}{n-1-i}} + \frac{n-1-i}{(n-2-i)!} = K_i. \end{aligned}$$

For $t > s$, and all $i = 0, 1, \dots, n-1$,

$$\begin{aligned} \sup_{t \in [0, \infty)} \left| \frac{g_i(t, s)}{1+t^{n-1-i}} \right| & \leq \sup_{t \in [0, \infty)} \frac{at^{n-2-i}}{(n-2-i)!(1+t^{n-1-i})} + \sup_{t \in [0, \infty)} \frac{t^{n-1-i}}{(n-1-i)!(1+t^{n-1-i})} \\ & = \frac{a}{(n-1-i)!} (n-2-i)^{\frac{n-2-i}{n-1-i}} + \frac{1}{(n-1-i)!} \leq K_i. \end{aligned}$$

Therefore, for all $i = 0, 1, \dots, n-1$,

$$\begin{aligned} \|x\|_i & \leq \sum_{k=i}^{n-3} \frac{|A_k|}{(k-i)!} \sup_{t \in [0, \infty)} \frac{t^{k-i}}{1+t^{n-1-i}} + \frac{|aC+B|}{(n-2-i)!} \sup_{t \in [0, \infty)} \frac{t^{n-2-i}}{1+t^{n-1-i}} \\ & \quad + \frac{|C|}{(n-1-i)!} \sup_{t \in [0, \infty)} \frac{t^{n-1-i}}{1+t^{n-1-i}} + \int_0^\infty \sup_{t \in [0, \infty)} \left| \frac{g_i(t, s)}{1+t^{n-1-i}} \right| \alpha(s) ds \leq L_i + K_i \int_0^\infty \alpha(t) dt, \end{aligned}$$

and so $x \in Q$, according to (5.6). Therefore, the assumption (i) from Corollary 4.4 is satisfied.

The validity of assumption (ii) from Corollary 4.4 follows immediately from the properties of mapping F and the definition of the set Q . Moreover, all solutions of (5.8) belong, for arbitrary $q \in Q$, to the closed, bounded subset of X , namely

$$\left\{ x \in X \mid \|x\|_i \leq L_i + K_i \int_0^\infty \alpha(t) dt, \quad i = 0, 1, \dots, n-1 \right\},$$

which implies that $\mathfrak{T}(Q)$ is bounded in X . Therefore, assumption (iii) from Corollary 4.4 is satisfied as well.

Summing up, all assumptions of Corollary 4.4 are satisfied, by which the Sturm–Liouville problem (5.5) admits a solution in Q .

Remark 5.3. Let us note that, instead of (5.5), we can consider a more general Sturm–Liouville problem with the r.h.s. also depending on the derivatives

$$\left. \begin{aligned} -x^{(n)}(t) & \in F(t, x(t), \dot{x}(t), \dots, x^{(n-1)}(t)), \quad \text{for a.a. } t \in [0, \infty), \\ x^{(i)}(0) & = A_i, \quad i = 0, 1, \dots, n-3, \\ x^{(n-2)}(0) - ax^{(n-1)}(0) & = B, \\ \lim_{t \rightarrow \infty} x^{(n-1)}(t) & = C, \end{aligned} \right\}$$

where $F: [0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}$ is an upper-Carathéodory mapping, $a > 0$, $A_i, B, C \in \mathbb{R}$, $i = 0, \dots, n-3$.

In this case, the same conclusion holds, provided condition (5.6) is satisfied with

$$\alpha(t) := \sup_{|x| \leq M, (y_1, \dots, y_n) \in \mathbb{R}^n} |F(t, x, y_1, \dots, y_n)|.$$

Acknowledgements

Supported by the grant No. 14-06958S “Singularities and impulses in boundary value problems for nonlinear ordinary differential equations” of the Grant Agency of the Czech Republic.

References

- [1] R. P. AGARWAL, D. O'REGAN, *Infinite interval problems for differential, difference and integral equations*, Kluwer, Dordrecht, 2001. [MR1845855](#); [url](#)
- [2] R. P. AGARWAL, D. O'REGAN, Infinite interval problems modeling phenomena which arise in the theory of plasma and electrical potential theory, *Stud. Appl. Math.* **111**(2003), 339–358. [MR1999644](#); [url](#)
- [3] J. ANDRES, Topological principles for ordinary differential equations, in: *Handbook of differential equations, Vol. III*, Elsevier, Amsterdam, 2006, pp. 1–101. [MR2457612](#); [url](#)
- [4] J. ANDRES, G. GABOR, L. GÓRNIOWICZ, Boundary value problems on infinite intervals, *Trans. Amer. Math. Soc.* **351**(1999), No. 12, 4861–4903. [MR1603870](#)
- [5] J. ANDRES, L. GÓRNIOWICZ, *Topological fixed point principles for boundary value problems*, Topological Fixed Point Theory and its Applications, Vol. 1, Kluwer, Dordrecht, 2003. [MR1998968](#); [url](#)
- [6] J. ANDRES, L. MALAGUTI, V. TADDEI, Bounded solutions of Carathéodory differential inclusions: a bound sets approach, *Abstr. Appl. Anal.* **9**(2003), 547–571. [MR1981485](#); [url](#)
- [7] J. ANDRES, M. PAVLAČKOVÁ, Asymptotic boundary value problems for second-order differential systems, *Nonlin. Anal.* **71**(2009), No. 5–6, 1462–1473. [MR2524361](#); [url](#)
- [8] J.-P. AUBIN, A. CELLINA, *Differential inclusions*, Springer-Verlag, Berlin, 1984. [MR0755330](#); [url](#)
- [9] J.-P. AUBIN, H. FRANKOWSKA, *Set-valued analysis*, Birkhäuser, Boston, 1990. [MR1048347](#); [url](#)
- [10] C. BAI, C. LI, Unbounded upper and lower solution method for third-order boundary-value problems on the half-line, *Electron. J. Differ. Equ.* **119**(2009), 1–12. [MR2550081](#)
- [11] B. BIANCONI, F. PAPALINI, Non-autonomous boundary value problems on the real line, *Discr. Cont. Dynam. Syst.* **15**(2006), No. 3, 759–776. [MR2220747](#); [url](#)
- [12] K. BORSUK, *Theory of retracts*, Vol. 44, PWN, Warsaw, 1967. [MR0216473](#)

- [13] M. CECCHI, M. FURI, M. MARINI, About the solvability of ordinary differential equations with asymptotic boundary conditions, *Boll. Un. Mat. Ital. C (6)* **4**(1985), No. 1, 329–345. [MR805224](#)
- [14] E. ÇETIN, R. P. AGARWAL, Existence of solutions for fourth order three-point boundary value problems on a half-line, *Electron. J. Qual. Theory Differ. Equ.* **2015**, No. 62, 1–23. [MR3418562](#); [url](#)
- [15] M. COUNTRYMAN, R. KANNAN, Nonlinear boundary value problems on semi-infinite intervals, *Comput. Math. Appl.* **28**(1994), 59–75. [MR1300682](#); [url](#)
- [16] M. FURI, P. PERA, A continuation method on locally convex spaces and applications to ODE on noncompact intervals, *Annal. Polonici Math.* **47**(1987), 331–346. [MR927581](#)
- [17] M. FURI, P. PERA, On the fixed point index in locally convex spaces, *Proc. Roy. Soc. Edinburgh Sect. A* **106**(1987), 161–168. [MR899950](#)
- [18] G. GABOR, Some results on existence and structure of solution sets to differential inclusions on the halfline, *Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8)* **5**(2002), No. 2, 431–446. [MR1911199](#)
- [19] R. E. GAINES, J. MAWHIN, Ordinary differential equations with nonlinear boundary conditions, *J. Differential Equations* **26**(1977), 200–222. [MR0463557](#); [url](#)
- [20] L. GÓRNIOWICZ, *Topological fixed point theory of multivalued mappings*, Kluwer, Dordrecht, 1999. [MR1748378](#); [url](#)
- [21] A. GRANAS, R. B. GUENTHER, J. W. LEE, Some existence results for the differential inclusions $y^{(k)} \in F(x, y, \dots, y^{(k-1)})$, $y \in B, C$. *R. Acad. Sci. Paris Sér. I Math.* **307**(1988), 391–396. [MR0965805](#)
- [22] A. GRANAS, R. B. GUENTHER, J. W. LEE, Some general existence principles in the Carathéodory theory of nonlinear differential systems, *J. Math. Pures Appl.* **70**(1991), 153–196. [MR1103033](#)
- [23] G. SH. GUSEINOV, I. YASLAN, Boundary value problems for second order nonlinear differential equations on infinite intervals, *J. Math. Anal. Appl.* **290**(2004), 620–638. [MR2033047](#); [url](#)
- [24] G. HERZOG, On bounded solutions of second order systems, *Demonstratio Math.* **39**(2006), No. 4, 793–801. [MR2296714](#)
- [25] S. HU, N. S. PAPAGEORGIU, *Handbook of multivalued analysis, Vol. II. Applications*, Mathematics and its Applications, Vol. 500, Kluwer Academic Publishers, Dordrecht, 2000. [MR1741926](#); [url](#)
- [26] H. LIAN, J. ZHAO, R. P. AGARWAL, Upper and lower solution method for n th-order BVPs on an infinite interval, *Bound. Value Probl.* **2014**, 2014:100, 17 pp. [MR3352641](#); [url](#)
- [27] Y. LIU, Boundary value problem for second order differential equations on unbounded domains, *Acta Anal. Funct. Appl.* **4**(2002), 211–216. [MR1956716](#)

- [28] R. MA, Existence of positive solution for second-order boundary value problems on infinite intervals, *Appl. Math. Lett.* **16**(2003), 33–39. [MR1938189](#); [url](#)
- [29] J. MAWHIN, Boundary value problems for nonlinear second order vector differential equations, *J. Differential Equations* **16**(1974), 257–269. [MR0361250](#); [url](#)
- [30] P. OMARI, F. ZANOLIN, On forced nonlinear oscillations in n -th order differential systems with geometric conditions, *Nonlinear Anal.* **8**(1984), No. 7, 723–748. [MR0750046](#)
- [31] C. G. PHILOS, A boundary value problem on the half-line for higher-order nonlinear differential equations, *Proc. Roy. Soc. Edinburgh Sect. A* **139**(2009), 1017–1035. [MR2545991](#); [url](#)
- [32] S. UMAMAHESWARAN, M. VENKATA RAMA, Multipoint focal boundary value problems on infinite interval, *J. Appl. Math. Stoch. Anal.* **5**(1992), 283–289. [MR1183601](#); [url](#)
- [33] G. VIDOSSICH, A general existence theorem for boundary value problems for ordinary differential equations, *Nonlinear Anal.* **15**(1990), No. 10, 897–914. [MR1081661](#)
- [34] I. I. VRABIE, *Compactness methods for nonlinear evolutions*, 2nd ed., Longman House, Burn Mill, Harlow, 1995. [MR1375237](#)