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Precise asymptotic behavior of regularly varying solutions of second order half-linear differential equations

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Abstract. Accurate asymptotic formulas for regularly varying solutions of the second order half-linear differential equation

$$(|x'|^{\alpha}\text{sgn } x')' + q(t)|x|^{\alpha}\text{sgn } x = 0,$$

will be established explicitly, depending on the rate of decay toward zero of the function

$$Q_c(t) = t^{\alpha}\int_t^{\infty} q(s)ds - c$$

as $t \to \infty$, where $c < \alpha^{\alpha}(\alpha+1)^{-\alpha-1}$.

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1 Introduction

The second order half-linear differential equation

$$(|x'|^{\alpha}\text{sgn } x')' + q(t)|x|^{\alpha}\text{sgn } x = 0, \tag{A}$$

is considered under the assumption that

(a) $\alpha > 0$ is a constant, and (b) $q : [a, \infty) \to \mathbf{R}$, $a > 0$, is a continuous function.

Note that (A) can be expressed as

$$((x')^{\alpha *})' + q(t)x^{\alpha *} = 0,$$

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in terms of the asterisk notation

$$u^{\gamma *} = |u|^\gamma \operatorname{sgn} u, \quad u \in \mathbf{R}, \quad \gamma > 0.$$

In this paper we are concerned primarily with nontrivial solutions of (A) which exist in a neighborhood of infinity, that is, in an interval of the form $[t_0, \infty)$, $t_0 \geq a$. Such a solution is said to be oscillatory if it has a sequence of zeros clustering at infinity, and nonoscillatory otherwise.

Although equation (A) with $\alpha \neq 1$ is nonlinear, it has many qualitative properties in common with the linear differential equation $x'' + q(t)x = 0$. See Elbert [2] and Došly and Řehák [3]. For example, all nontrivial solutions of (A) are either oscillatory, in which case (A) is called oscillatory, or else nonoscillatory, in which case (A) is called nonoscillatory. Also, it is shown that (A) is nonoscillatory if and only if the generalized Riccati differential equation

$$u' + \alpha |u|^{1+\frac{1}{\alpha}} + q(t) = 0, \tag{B}$$

has a solution defined in some neighborhood of infinity.

In what follows our attention will be focused on the case where (A) is nonoscillatory. Since if $x(t)$ satisfies (A), so does $-x(t)$, it is natural to restrict our consideration to (eventually) positive solutions of (A).

The systematic study of equations of the form (A) by means of regularly varying functions (in the sense of Karamata) was proposed by Jaroš, Kusano and Tanigawa [5], who proved the following theorem.

Theorem A. *Assume that $q(t)$ is integrable (absolutely or conditionally). Let c be a constant such that*

$$c \in (-\infty, E(\alpha)), \quad \text{where} \quad E(\alpha) = \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}, \tag{1.1}$$

Let λ_1, λ_2 ($\lambda_1 < \lambda_2$) denote the two real roots of the equation

$$|\lambda|^{1+\frac{1}{\alpha}} - \lambda + c = 0. \tag{1.2}$$

Equation (A) possesses a pair of regularly varying solutions $x_i(t)$, $i = 1, 2$, such that

$$x_i \in RV(\lambda_i^{\frac{1}{\alpha}*}), \quad i = 1, 2, \tag{1.3}$$

if and only if

$$\lim_{t\to\infty} t^\alpha \int_t^\infty q(s)ds = c. \tag{1.4}$$

Recently, Řehák [10] considering only special case of the equation (A) with nonpositive differentiable coefficient $q(t)$ established a condition which guarantees that all eventually positive increasing solutions are regularly varying.

Theorem B. *Let q be negative differentiable function and*

$$\lim_{t\to\infty} q'(t)|q(t)|^{\frac{\alpha+1}{\alpha}} = C < 0\,. \tag{1.5}$$

Then, all positive eventually increasing solutions $x(t)$ of (A) are such that $\lim_{t\to\infty} x(t) = \infty$ and belongs to $\mathrm{RV}\big(-\alpha\rho_1^{\frac{1}{\alpha-1}}/C\big)$, where ρ_1 is the positive real root of the equation

$$|\rho|^{\frac{\alpha}{\alpha-1}} + \frac{C}{\alpha}\rho - \frac{1}{\alpha-1} = 0\,.$$

Although the integral condition (1.4) is more general then (1.5), Theorem A guarantees the existence of at least one positive increasing RV-solution, while Theorem B says that all positive increasing solutions are regularly varying.

A natural question arises about the possibility of acquiring detailed information on the asymptotic behavior at infinity of the solutions whose existence is assured by the above two theorems. This problem has been partially examined in [7, 11]. Namely, in [7] the equation (A) has been considered in the framework of regular variation, but only the case $c = 0$ in (1.4) has been considered, providing some asymptotic formulas for normalized slowly varying solutions of (A), while in [11] considering only special case of the equation (A) with negative differentiable coefficient $q(t)$, a condition is established which ensures that the equation (A) has exponentially increasing solutions and exponentially decreasing solutions, providing some asymptotic estimates for such solutions.

Therefore, the objective of this paper is to extend and improve results obtained in [7,11], by indicating assumptions that make it possible to determine the accurate asymptotic formulas for regularly varying solutions (1.3) of (A). This can be accomplished by elaborating the proof of Theorem A so as to gain insight into the interrelation between the asymptotic behavior of solutions of (A) and the rate of decay toward zero of the function

$$Q_c(t) = t^{\alpha} \int_t^{\infty} q(s)ds - c, \qquad c < E(\alpha), \tag{1.6}$$

as $t \to \infty$. In Section 2 we present the elaborated proof of Theorem A, thereby adding useful information to the exponential representations for regularly varying solutions (1.3) of (A) constructed in the paper [5]. Using the results of Section 2, we then specify in Section 3 some classes of equations of the form (A) having solutions (1.3) whose asymptotic behaviors are governed by the precise formulas. Examples illustrating the main results are provided in Section 4.

For the convenience of the reader the definition and some basic properties of regularly varying functions are summarized in the Appendix at the end of the paper.

2 Existence of regularly varying solutions

Let c be a constant satisfying (1.1) and let λ_i, $i = 1, 2$, $(\lambda_1 < \lambda_2)$ denote the real roots of the equation (1.2). It is clear that

$$0 < \lambda_1 < \lambda_2 \quad \text{if} \quad c \in (0, E(\alpha)); \qquad \lambda_1 < 0 < \lambda_2 \quad \text{if} \quad c \in (-\infty, 0)$$

$$\text{and} \quad 0 = \lambda_1 < \lambda_2 = 1 \quad \text{if } c = 0.$$

The purpose of this section is to prove variants of Theorem A ensuring the existence of regularly varying solutions $x_i \in \mathrm{RV}\big(\lambda_i^{\frac{1}{\alpha}*}\big)$, $i = 1, 2$, for equation (A), and utilize them for pointing out the cases where one can determine the asymptotic behavior of these solutions as $t \to \infty$. As in [5], the cases where $c = 0$ and $c \neq 0$ are examined separately.

2.1 The case where $c = 0$ in (1.2)

Let $c = 0$ in (1.2), so that its real roots are $\lambda_1 = 0$ and $\lambda_2 = 1$. Our task is to construct regularly varying solutions $x_i(t)$, $i = 1, 2$, of (A) such that $x_1 \in \mathrm{SV} = \mathrm{RV}(0)$ and $x_2 \in \mathrm{RV}(1)$ under certain conditions on $q(t)$ stronger than

$$Q(t) := t^{\alpha} \int_t^{\infty} q(s)ds \to 0, \qquad t \to \infty. \tag{2.1}$$

Our first result consists of the following two existence theorems indicative of how the asymptotic behavior of the SV- and RV(1)-solutions of (A) is affected by the decay property of $Q(t)$ as $t \to \infty$.

Theorem 2.1. *Suppose that there exists a continuous positive function $\phi(t)$ on $[a, \infty)$ which decreases to 0 as $t \to \infty$ and satisfies*

$$\left| t^{\alpha} \int_t^{\infty} q(s)ds \right| \leq \phi(t) \quad \text{for all large} \ \ t.$$

Then, equation (A) possesses a slowly varying solution $x_1(t)$ which is expressed in the form

$$x_1(t) = \exp\left\{ \int_T^t \left(\frac{v_1(s) + Q(s)}{s^{\alpha}} \right)^{\frac{1}{\alpha}*} ds \right\}, \qquad t \geq T, \tag{2.2}$$

for some $T > a$, with $v_1(t)$ satisfying

$$v_1(t) = O(\phi(t)^{1+\frac{1}{\alpha}}) \quad \text{as} \quad t \to \infty. \tag{2.3}$$

Proof. We seek a solution $x_1(t)$ of (A) expressed in the form (2.2). For $x_1(t)$ to be a solution of (A), it is necessary that $u(t) = (v_1(t) + Q(t))/t^{\alpha}$ satisfies the Riccati-type equation (B) for $t \geq T$. Further, if $v_1(t)$ tends to 0 as $t \to \infty$, $x_1(t)$ would be slowly varying solution. An elementary computation shows that equation (B) for $u(t)$ is transformed into the following differential equation for $v_1(t)$:

$$\left(\frac{v_1}{t^{\alpha}} \right)' + \frac{\alpha |v_1 + Q(t)|^{1+\frac{1}{\alpha}}}{t^{\alpha+1}} = 0, \qquad t \geq T, \tag{2.4}$$

the integrated version of which is

$$v_1(t) = \alpha t^{\alpha} \int_t^{\infty} \frac{|v_1(s) + Q(s)|^{1+\frac{1}{\alpha}}}{s^{\alpha+1}} ds, \qquad t \geq T. \tag{2.5}$$

With a help of fixed-point technique we show the existence of a solution of the integral equation (2.5).

Choose $T \geq a$ so that

$$\left(1 + \frac{1}{\alpha} \right) (2\phi(t))^{\frac{1}{\alpha}} \leq \frac{1}{2} \quad \text{for} \quad t \geq T. \tag{2.6}$$

Let $C_0[T, \infty)$ denote the set of all continuous functions on $[T, \infty)$ tending to 0 as $t \to \infty$. $C_0[T, \infty)$ is a Banach space with the norm $\|v\|_0 = \sup\{|v(t)| : t \geq T\}$. Define the set $V_1 \subset C_0[T, \infty)$ and the integral operator $\mathcal{F}_1$ by

$$V_1 = \{v \in C_0[T, \infty) : 0 \leq v(t) \leq \phi(t), \quad t \geq T\},$$

and

$$\mathcal{F}_1 v(t) = \alpha t^{\alpha} \int_t^{\infty} \frac{|v(s) + Q(s)|^{1+\frac{1}{\alpha}}}{s^{\alpha+1}} ds, \quad t \geq T.$$

It is clear that V_1 is a closed convex subset of $C_0[T,\infty)$. It can be shown that $\mathcal{F}_1$ is a contraction mapping on V_1. In fact, if $v \in V_1$, then, using the decreasing nature of $\phi(t)$ and (2.6), we have

$$0 \le \mathcal{F}_1 v(t) \le \alpha t^{\alpha} \int_t^{\infty} \frac{(2\phi(s))^{1+\frac{1}{\alpha}}}{s^{\alpha+1}} ds \le (2\phi(t))^{1+\frac{1}{\alpha}} \le \phi(t), \qquad t \ge T, \tag{2.7}$$

implying that $\lim_{t\to\infty} \mathcal{F}_1 v(t) = 0$. It follows that $\mathcal{F}_1 v \in V_1$, so that $\mathcal{F}$ maps V_1 into itself. Moreover, if $v, w \in V_1$, then, noting that

$$\Big| |v(t) + Q(t)|^{1+\frac{1}{\alpha}} - |w(t) + Q(t)|^{1+\frac{1}{\alpha}} \Big| \le \left(1 + \frac{1}{\alpha}\right) (2\phi(t))^{\frac{1}{\alpha}} |v(t) - w(t)|,$$

we obtain

$$\begin{aligned} |\mathcal{F}_1 v(t) - \mathcal{F}_1 w(t)| &\le \alpha t^{\alpha} \int_t^{\infty} \frac{(1+\frac{1}{\alpha})(2\phi(s))^{\frac{1}{\alpha}} |v(s) - w(s)|}{s^{\alpha+1}} ds \\ &\le \left(1 + \frac{1}{\alpha}\right) (2\phi(t))^{\frac{1}{\alpha}} \|v - w\|_0 \le \frac{1}{2} \|v - w\|_0, \qquad t \ge T, \end{aligned}$$

implying that $\|\mathcal{F}_1 v - \mathcal{F}_1 w\|_0 \le \frac{1}{2}\|v - w\|_0$. This proves that $\mathcal{F}$ is a contraction mapping. It follows that $\mathcal{F}_1$ has a unique fixed point $v_1(t)$ in V_1, which clearly satisfies the integral equation (2.5), and hence the differential equation (2.4) on $[T,\infty)$. From (2.7) it follows that $v_1(t)$ satisfies (2.3). Moreover, the function $x_1(t)$ defined by (2.2), with this $v_1(t)$, is a slowly varying solution of equation (A). This completes the proof of Theorem 2.1. □

Theorem 2.2. *Suppose that there exists a continuous slowly varying function $\psi(t)$ on $[a,\infty)$ which decreases to 0 as $t \to \infty$ and satisfies*

$$\left| t^{\alpha} \int_t^{\infty} q(s) ds \right| \le \psi(t) \quad \textit{for all large } t.$$

Then, equation (A) possesses a regulary varying solution $x_2(t)$ of index 1, which is expressed in the form

$$x_2(t) = \exp\left\{ \int_T^t \left(\frac{1 + v_2(s) + Q(s)}{s^{\alpha}} \right)^{\frac{1}{\alpha}*} ds \right\}, \qquad t \ge T, \tag{2.8}$$

for some $T > a$, with $v_2(t)$ satisfying

$$v_2(t) = O(\psi(t)) \quad \textit{as} \quad t \to \infty. \tag{2.9}$$

Proof. The desired solution $x_2 \in \mathrm{RV}(1)$ is sought in the form (2.8). From the requirement that $u(t) = (1 + v_2(t) + Q(t))/t^{\alpha}$ satisfy (B) we obtain the differential equation for $v_2(t)$

$$t v_2' + \alpha \big(|1 + v_2 + Q(t)|^{1+\frac{1}{\alpha}} - v_2 - 1 \big) = 0,$$

which is transformed as

$$(t v_2)' + \alpha \left(|1 + v_2 + Q(t)|^{1+\frac{1}{\alpha}} - \left(1 + \frac{1}{\alpha}\right) v_2 - 1 \right) = 0, \qquad t \ge T. \tag{2.10}$$

It suffices to solve the special integrated version of (2.10)

$$v_2(t) = \frac{\alpha}{t} \int_T^t F(s, v_2(s)) ds, \qquad t \ge T, \tag{2.11}$$

under the condition $v_2(t) \to 0$ as $t \to \infty$, where

$$F(t,v) = 1 + \left(1 + \frac{1}{\alpha}\right)v - |1 + v + Q(t)|^{1+\frac{1}{\alpha}}. \tag{2.12}$$

For this purpose we need detailed information about $F(t,v)$. Let $T_0 \geq a$ be such that $\psi(t) \leq \frac{1}{4}$ for $t \geq T_0$, define $\mathcal{D} = \{(t,v) : t \geq T_0,\ |v| \leq \frac{1}{4}\}$ and consider $F(t,v)$ on the set $\mathcal{D}$. It will be convenient to decompose $F(t,v)$ as follows:

$$F(t,v) = G(t,v) + H(t,v) + k(t), \tag{2.13}$$

where

$$G(t,v) = (1+Q(t))^{1+\frac{1}{\alpha}} + \left(1+\frac{1}{\alpha}\right)(1+Q(t))^{\frac{1}{\alpha}}v - |1+v+Q(t)|^{1+\frac{1}{\alpha}},$$

$$H(t,v) = \left(1+\frac{1}{\alpha}\right)(1-(1+Q(t))^{\frac{1}{\alpha}})v, \qquad k(t) = 1-(1+Q(t))^{1+\frac{1}{\alpha}}.$$

Using the mean value theorem, for some $\theta \in (0,1)$ the following inequalities hold:

$$\begin{aligned}
|H(t,v)| &\leq \left(1+\frac{1}{\alpha}\right)\frac{1}{\alpha}\Big|1+\theta\,Q(t)\Big|^{\frac{1}{\alpha}-1}Q(t)|v|,\\
\left|\frac{\partial H(t,v)}{\partial v}\right| &= \left(1+\frac{1}{\alpha}\right)\left|1-(1+Q(t))^{\frac{1}{\alpha}}\right| \leq \left(1+\frac{1}{\alpha}\right)\frac{1}{\alpha}\Big|1+\theta\,Q(t)\Big|^{\frac{1}{\alpha}-1}Q(t),\\
\left|\frac{\partial G(t,v)}{\partial v}\right| &\leq \left(1+\frac{1}{\alpha}\right)\left|(1+Q(t))^{\frac{1}{\alpha}} - (1+v+Q(t))^{\frac{1}{\alpha}*}\right|\\
&\leq \left(1+\frac{1}{\alpha}\right)\frac{1}{\alpha}\Big|1+Q(t)+\theta\,v\Big|^{\frac{1}{\alpha}-1}|v|,\\
|k(t)| &\leq \left(1+\frac{1}{\alpha}\right)\Big|1+\theta\,Q(t)\Big|^{\frac{1}{\alpha}}Q(t)\,.
\end{aligned}$$

Also,

$$\begin{aligned}
\lim_{v\to 0}\frac{|G(t,v)|}{v^2} &= \frac{1}{2}\left(1+\frac{1}{\alpha}\right)\lim_{v\to 0}\frac{\left|(1+Q(t))^{\frac{1}{\alpha}} - (1+v+Q(t))^{\frac{1}{\alpha}}\right|}{v}\\
&= \frac{1}{2\alpha}\left(1+\frac{1}{\alpha}\right)\lim_{v\to 0}|1+Q(t)+\theta\,v|^{\frac{1}{\alpha}-1}\\
&= \frac{1}{2\alpha}\left(1+\frac{1}{\alpha}\right)\Big|1+Q(t)\Big|^{\frac{1}{\alpha}-1}.
\end{aligned}$$

Since,

$$\frac{3}{4} \leq \Big|1+\theta\,Q(t)\Big| \leq \frac{5}{4}, \quad \text{and} \quad \frac{1}{2} \leq \Big|1+Q(t)+\theta\,v\Big| \leq \frac{3}{2}, \qquad t \geq T_0,$$

we obtain that the following inequalities hold on $\mathcal{D}$:

$$\begin{aligned}
|G(t,v)| &\leq \frac{1}{\alpha}\left(1+\frac{1}{\alpha}\right)Av^2, & \left|\frac{\partial G(t,v)}{\partial v}\right| &\leq \frac{1}{\alpha}\left(1+\frac{1}{\alpha}\right)A|v|,\\
|H(t,v)| &\leq \frac{1}{\alpha}\left(1+\frac{1}{\alpha}\right)A|Q(t)|\,|v|, & \left|\frac{\partial H(t,v)}{\partial v}\right| &\leq \frac{1}{\alpha}\left(1+\frac{1}{\alpha}\right)A|Q(t)|,\\
|k(t)| &\leq \left(1+\frac{1}{\alpha}\right)A|Q(t)|,
\end{aligned} \tag{2.14}$$

where A is a positive constant such that

$$A = \left(\frac{3}{2}\right)^{\frac{1}{\alpha}-1} \quad \text{if} \quad \alpha \le 1; \qquad A = 2^{1-\frac{1}{\alpha}} \quad \text{if} \quad \alpha > 1.$$

We note that there exists a constant $\gamma > 0$ such that

$$\int_a^t \psi(s)ds \le \gamma t \psi(t), \qquad \int_a^t \sqrt{\psi(s)}ds \le \gamma t \sqrt{\psi(t)}, \qquad t \ge a. \tag{2.15}$$

This follows from the relations

$$\int_a^t \psi(s)ds \sim t\psi(t), \qquad \int_a^t \sqrt{\psi(s)}ds \sim t\sqrt{\psi(t)}, \qquad t \to \infty,$$

which are implied by the Karamata integration theorem applied to slowly varying functions $\psi(t)$ and $\sqrt{\psi(t)}$.

Choose $T \ge T_0$ so that

$$\frac{(\alpha+1)(\alpha+2)}{\alpha} A\gamma \sqrt{\psi(t)} \le l \quad t \ge T, \tag{2.16}$$

where $l \in (0,1)$ is a constant. Let V_2 denote the set

$$V_2 = \left\{ v \in C_0[T,\infty) : |v(t)| \le \sqrt{\psi(t)}, \quad t \ge T \right\}, \tag{2.17}$$

and define the integral operator $\mathcal{F}_2 : V_2 \to C_0[T,\infty)$ given by

$$\mathcal{F}_2 v(t) = \frac{\alpha}{t} \int_T^t F(s, v(s))ds, \quad t \ge T. \tag{2.18}$$

Using (2.13)–(2.18) and $\psi(t) \le \sqrt{\psi(t)}$, for $t \ge T$, we see that if $v \in V_2$, then

$$\begin{aligned}
|\mathcal{F}_2 v(t)| &\le \frac{\alpha}{t} \int_T^t (|G(s,v(s))| + |H(s,v(s))| + |k(s)|)ds \\
&\le \frac{\alpha}{t} \int_T^t \left[\frac{1}{\alpha}\left(1+\frac{1}{\alpha}\right) A\psi(s) + \frac{1}{\alpha}\left(1+\frac{1}{\alpha}\right) A\psi(s)\sqrt{\psi(s)} + \left(1+\frac{1}{\alpha}\right) A\psi(s) \right] ds \\
&\le \frac{\alpha}{t} A \int_T^t \frac{(\alpha+1)(\alpha+2)}{\alpha^2} \psi(s)ds \le \frac{(\alpha+1)(\alpha+2)}{\alpha} A\gamma\psi(t) \\
&\le \frac{(\alpha+1)(\alpha+2)}{\alpha} A\gamma \sqrt{\psi(t)}\sqrt{\psi(t)} \le \sqrt{\psi(t)}, \qquad t \ge T,
\end{aligned}$$

and that if $v, w \in V_2$, then

$$\begin{aligned}
|\mathcal{F}_2 v(t) - \mathcal{F}_2 w(t)| &\le \frac{\alpha}{t} \int_T^t [|G(s,v(s)) - G(s,w(s))| + |H(s,v(s)) - H(s,w(s))|]ds \\
&\le \frac{\alpha}{t} \int_T^t \left[\frac{1}{\alpha}\left(1+\frac{1}{\alpha}\right) A\sqrt{\psi(s)} + \frac{1}{\alpha}\left(1+\frac{1}{\alpha}\right) A\psi(s) \right] |v(s) - w(s)|ds \\
&\le \frac{2(\alpha+1)}{\alpha} A\gamma \sqrt{\psi(t)}\, \|v-w\|_0 \le l\|v-w\|_0, \qquad t \ge T,
\end{aligned}$$

which implies that $\|\mathcal{F}_2 v - \mathcal{F}_2 w\|_0 \le l\|v-w\|_0$. This shows that $\mathcal{F}_2$ is a contraction on V_2, and so there exists a fixed point v_2 in V_2, which satisfies the integral equation (2.11) and hence the differential equation (2.10) for $t \ge T$. Then, the function $x_2(t)$ defined by (2.8) with this $v_2(t)$ provides a solution of equation (A) on $[T,\infty)$. Since, $\lim_{t\to\infty}(v_2(t) + Q(t)) = 0$, we see that $x_2 \in \mathrm{RV}(1)$ as desired. □

2.2 The case where $c \neq 0$ in (1.2)

Let c be a nonzero number in the interval $(-\infty, E(\alpha))$ (cf. (1.1)). Then, the real roots λ_i, $i = 1, 2$, of (1.2) satisfy

$$0 < \lambda_1 < \lambda_2 \quad \text{if} \quad c > 0 \qquad \text{and} \qquad \lambda_1 < 0 < \lambda_2 \quad \text{if} \quad c < 0,$$

and

$$\lambda_1 < \Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha} < \lambda_2, \tag{2.19}$$

regardless of the sign of c. Our aim is to find regularly varying solutions $x_i(t)$, $i = 1, 2$, of (A) such that

$$x_1 \in \mathrm{RV}\Big(\lambda_1^{\frac{1}{\alpha}*}\Big) \quad \text{and} \quad x_2 \in \mathrm{RV}\Big(\lambda_2^{\frac{1}{\alpha}*}\Big)$$

under certain conditions on $q(t)$ stronger than (1.4). Since $\lambda_2 > 0$, the asterisk sign may be deleted from $\lambda_2^{\frac{1}{\alpha}*}$.

The extreme case where $Q_c(t) \equiv 0$ for all large t will be excluded from our consideration. Clearly, this case occurs only for the particular equation

$$(|x'|^{\alpha} \mathrm{sgn}\, x')' + \frac{\alpha c}{t^{\alpha+1}} |x|^{\alpha} \mathrm{sgn}\, x = 0,$$

which, as easily checked, has exact two trivial RV-solutions $x_i(t) = t^{\lambda_i^{\frac{1}{\alpha}*}}$, $i = 1, 2$.

The main results of this subsection are stated and proved as follows.

Theorem 2.3. *Let c be a nonzero constant in $(-\infty, E(\alpha))$. Suppose that there exists a continuous positive function $\phi(t)$ on $[a, \infty)$ which decreases to 0 as $t \to \infty$ and satisfies*

$$\Big| t^{\alpha} \int_t^{\infty} q(s)ds - c \Big| \le \phi(t) \quad \textit{for all large } t.$$

Then, equation (A) possesses a regularly varying solution $x_1 \in \mathrm{RV}(\lambda_1^{\frac{1}{\alpha}*})$ which is expressed in the form

$$x_1(t) = \exp\Big\{ \int_T^t \Big(\frac{\lambda_1 + v_1(s) + Q_c(s)}{s^{\alpha}} \Big)^{\frac{1}{\alpha}*} ds \Big\}, \qquad t \ge T, \tag{2.20}$$

for some $T > a$, where $v_1(t)$ satisfies

$$v_1(t) = O(\phi(t)) \quad \textit{as} \quad t \to \infty. \tag{2.21}$$

Proof. We construct a solution $x_1 \in \mathrm{RV}(\lambda_1^{\frac{1}{\alpha}*})$ of (A) having the representation (2.20). Substituting $u(t) = (\lambda_1 + v_1(t) + Q_c(t))/t^{\alpha}$ in the equation (B), we obtain the differential equation for $v_1(t)$

$$\frac{v_1'}{t^{\alpha}} - \frac{\alpha v_1}{t^{\alpha+1}} + \alpha \frac{|\lambda_1 + v_1 + Q_c(t)|^{1+\frac{1}{\alpha}} - |\lambda_1|^{1+\frac{1}{\alpha}}}{t^{\alpha+1}} = 0, \qquad t \ge T. \tag{2.22}$$

Using the notation

$$\mu_1 = (\alpha+1)\lambda_1^{\frac{1}{\alpha}*}, \tag{2.23}$$

we transform the above equation into

$$(t^{\mu_1 - \alpha} v_1)' + \alpha t^{\mu_1 - \alpha - 1} F_1(t, v_1) = 0, \qquad t \ge T, \tag{2.24}$$

where

$$F_1(t,v) = |\lambda_1 + v + Q_c(t)|^{1+\frac{1}{\alpha}} - \left(1+\frac{1}{\alpha}\right)\lambda_1^{\frac{1}{\alpha}*} v - |\lambda_1|^{1+\frac{1}{\alpha}}. \tag{2.25}$$

By (2.19) and (2.23), we see that $\mu_1 < \alpha$, so that it is natural to integrate (2.24) on $[t,\infty)$ to obtain the integral equation

$$v_1(t) = \alpha t^{\alpha-\mu_1}\int_t^\infty s^{\mu_1-\alpha-1}F_1(s,v_1(s))ds, \qquad t \geq T. \tag{2.26}$$

We consider $F_1(t,v)$ on the set

$$\mathcal{D}_1 = \left\{(t,v) : t \geq T_1,\ |v| \leq \frac{|\lambda_1|}{4}\right\},$$

where $T_1 > a$ is chosen so that $\psi(t) \leq \min\{\frac{|\lambda_1|}{4}, 1\}$ for $t \geq T_1$, and express it as

$$F_1(t,v) = G_1(t,v) + H_1(t,v) + k_1(t), \tag{2.27}$$

where

$$\begin{aligned} G_1(t,v) &= |\lambda_1 + v + Q_c(t)|^{1+\frac{1}{\alpha}} - \left(1+\frac{1}{\alpha}\right)(\lambda_1 + Q_c(t))^{\frac{1}{\alpha}*}v - |\lambda_1 + Q_c(t)|^{1+\frac{1}{\alpha}}, \\ H_1(t,v) &= \left(1+\frac{1}{\alpha}\right)\left[(\lambda_1+Q_c(t))^{\frac{1}{\alpha}*} - \lambda_1^{\frac{1}{\alpha}*}\right]v, \\ k_1(t) &= |\lambda_1 + Q_c(t)|^{1+\frac{1}{\alpha}} - |\lambda_1|^{1+\frac{1}{\alpha}}. \end{aligned} \tag{2.28}$$

By a similar procedure as in the proof of Theorem 2.2, using the mean value theorem the following inequalities are proved to hold in $\mathcal{D}_1$:

$$|G_1(t,v)| \leq \frac{1}{\alpha}\left(1+\frac{1}{\alpha}\right)A_1 v^2, \qquad |H_1(t,v)| \leq \frac{1}{\alpha}\left(1+\frac{1}{\alpha}\right)A_1|Q_c(t)||v|, \tag{2.29}$$

$$\left|\frac{\partial G_1(t,v)}{\partial v}\right| \leq \frac{1}{\alpha}\left(1+\frac{1}{\alpha}\right)A_1|v|, \qquad \left|\frac{\partial H_1(t,v)}{\partial v}\right| \leq \frac{1}{\alpha}\left(1+\frac{1}{\alpha}\right)A_1|Q_c(t)|, \tag{2.30}$$

$$|k_1(t)| \leq \left(1+\frac{1}{\alpha}\right)A_1|Q_c(t)|, \tag{2.31}$$

where A_1 is a positive constant such that

$$A_1 = \left(\frac{3|\lambda_1|}{2}\right)^{\frac{1}{\alpha}-1} \quad \text{if} \quad \alpha \leq 1, \qquad A_1 = \left(\frac{|\lambda_1|}{2}\right)^{\frac{1}{\alpha}-1} \quad \text{if} \quad \alpha > 1. \tag{2.32}$$

Let a constant $l \in (0,1)$ be given and let $T > T_1$ be large enough so that

$$\frac{(\alpha+1)(\alpha+2)}{\alpha(\alpha-\mu_1)}A_1\sqrt{\phi(t)} \leq l, \qquad t \geq T. \tag{2.33}$$

Define the set V_1 and the integral operator $\mathcal{F}_1$ by

$$V_1 = \left\{v \in C_0[T,\infty) : |v(t)| \leq \sqrt{\phi(t)},\ t \geq T\right\},$$

and

$$\mathcal{F}_1 v(t) = \alpha t^{\alpha-\mu_1} \int_t^\infty s^{\mu_1-\alpha-1} F_1(s, v(s))ds, \qquad t \geq T,$$

respectively. One can show that $\mathcal{F}_1$ is a contraction mapping on V_1 as follows. If $v \in V_1$, then using (2.29), (2.31) and (2.33), we have

$$\begin{aligned} |\mathcal{F}_1 v(t)| &\leq \alpha t^{\alpha-\mu_1} \int_t^\infty s^{\mu_1-\alpha-1} \frac{(\alpha+1)(\alpha+2)}{\alpha^2} A_1 \phi(s)ds \\ &\leq \frac{(\alpha+1)(\alpha+2)}{\alpha(\alpha-\mu_1)} A_1 \phi(t) \\ &= \frac{(\alpha+1)(\alpha+2)}{\alpha(\alpha-\mu_1)} A_1 \sqrt{\phi(t)}\sqrt{\phi(t)} \leq \sqrt{\phi(t)}, \qquad t \geq T, \end{aligned} \tag{2.34}$$

and if $v, w \in V_1$, then using (2.30) and (2.33), we see that

$$\begin{aligned} |\mathcal{F}_1 v(t) - \mathcal{F}_1 w(t)| &\leq \alpha t^{\alpha-\mu_1} \int_t^\infty \Big(\frac{1}{\alpha}\Big(1 + \frac{1}{\alpha}\Big) A_1 \sqrt{\phi(s)} + \frac{1}{\alpha}\Big(1+\frac{1}{\alpha}\Big) A_1 \phi(s) \Big) ds \\ &\leq \frac{2(\alpha+1)}{\alpha(\alpha-\mu_1)} A_1 \sqrt{\phi(t)}\, \|v-w\|_0 \leq l \|v-w\|_0, \qquad t \geq T, \end{aligned}$$

from which it follows that $\|\mathcal{F}_1 v - \mathcal{F}_1 w\|_0 \leq l\|v-w\|_0$. Therefore, there exists a unique fixed point $v_1 \in V_1$ of $\mathcal{F}_1$, which clearly satisfies the integral equation (2.26) on $[T, \infty)$. In view of (2.34) $v_1(t)$ has the property (2.21) as $t \to \infty$. The function $x_1(t)$ defined by (2.20) with this $v_1(t)$ then gives a solution of equation (A), which belongs to $\mathrm{RV}\big(\lambda_1^{\frac{1}{\alpha}*}\big)$, since $v_1(t) + Q_c(t) \to 0$ as $t \to \infty$. This completes the proof of Theorem 2.3. □

Theorem 2.4. *Let c be a nonzero constant in $(-\infty, E(\alpha))$. Suppose that there exists a continuous slowly varying function $\psi(t)$ on $[a, \infty)$ which tends to 0 as $t \to \infty$ and satisfies*

$$\left| t^\alpha \int_t^\infty q(s)ds - c \right| \leq \psi(t) \quad \text{for all large } t.$$

Then, equation (A) possesses a regulary varying solution $x_2 \in \mathrm{RV}\big(\lambda_2^{\frac{1}{\alpha}*}\big)$ which is expressed in the form

$$x_2(t) = \exp\Big\{ \int_T^t \Big(\frac{\lambda_2 + v_2(s) + Q_c(s)}{s^\alpha} \Big)^{\frac{1}{\alpha}*} ds \Big\}, \qquad t \geq T, \tag{2.35}$$

for some $T > a$, where $v_2(t)$ satisfies

$$v_2(t) = O(\psi(t)) \quad as \quad t \to \infty. \tag{2.36}$$

Proof. Note that the function $x_2(t)$ defined by (2.35) is a regularly varying solution of index $\lambda_2^{\frac{1}{\alpha}*}$ of (A) if $v_2(t)$ tends to 0 as $t \to \infty$ and has the property that the function $u(t) = (\lambda_2 + v_2(t) + Q_c(t))/t^\alpha$ satisfies the equation (B) for all large t. The existence of such a $v_2(t)$ is equivalent to the solvability of the differential equation

$$\frac{v_2'}{t^\alpha} - \frac{\alpha v_2}{t^{\alpha+1}} + \alpha \frac{|\lambda_2 + v_2 + Q_c(t)|^{1+\frac{1}{\alpha}} - |\lambda_2|^{1+\frac{1}{\alpha}}}{t^{\alpha+1}} = 0,$$

in the class of continuously differentiable functions tending to 0 as $t \to \infty$. Exactly as in the proof of Theorem 2.3 this equation is transformed into

$$(t^{\mu_2-\alpha} v_2)' + t^{\mu_2-\alpha-1} F_2(t, v_2) = 0, \tag{2.37}$$

where

$$\mu_2 = (\alpha+1)\lambda_2^{\frac{1}{\alpha}}, \tag{2.38}$$

and

$$F_2(t,v) = |\lambda_2 + v + Q_c(t)|^{1+\frac{1}{\alpha}} - \left(1+\frac{1}{\alpha}\right)\lambda_2^{\frac{1}{\alpha}} - \lambda_2^{1+\frac{1}{\alpha}}. \tag{2.39}$$

Here the variable of $F_2(t,v)$ is restricted to the domain

$$\mathcal{D}_2 = \left\{(t,v) : t \geq T_2,\ |v| \leq \frac{|\lambda_2|}{4}\right\},$$

where $T_2 > a$ is chosen so that $\psi(t) \leq \min\{\frac{|\lambda_2|}{4}, 1\}$ for $t \geq T_2$.

Noting that the constant μ_2 in (2.38) satisfies $\mu_2 > \alpha$ because of (2.19), we form the following integrated version of (2.37)

$$v_2(t) = -\alpha\, t^{\mu_2-\alpha} \int_T^t s^{\mu_2-\alpha-1} F_2(s, v_2(s))ds, \qquad t \geq T, \tag{2.40}$$

and solve it in the space $C_0[T,\infty)$ for some suitably chosen $T > a$. For this purpose use is made of the fact that there exists a constant $\gamma > 0$ such that

$$\begin{aligned} t^{\alpha-\mu_2} \int_a^t s^{\mu_2-\alpha-1}\psi(s)ds &\leq \frac{\gamma}{\mu_2-\alpha}\psi(t), \\ t^{\alpha-\mu_2} \int_a^t s^{\mu_2-\alpha-1}\sqrt{\psi(s)}ds &\leq \frac{\gamma}{\mu_2-\alpha}\sqrt{\psi(t)}, \end{aligned} \qquad t \geq a\,. \tag{2.41}$$

This is an immediate consequence of the Karamata integration theorem applied to $t^{\mu_2-\alpha-1}f(t)$ for any $f \in \mathrm{SV}$.

In order to solve the integral equation (2.40) it is convenient to use the decomposition of $F_2(t,v)$ corresponding precisely to (2.27)

$$F_2(t,v) = G_2(t,v) + H_2(t,v) + k_2(t), \tag{2.42}$$

where G_2, H_2 and k_2 stand, respectively, for G_1, H_1 and k_1 in (2.28) with λ_1 replaced with λ_2. Naturally, as regards G_2, H_2 and k_2 exactly the same type of estimates as (2.29)–(2.31) hold true in $\mathcal{D}_2$ provided λ_1 in (2.32) is replaced by λ_2.

Let a constant $l \in (0,1)$ be given and choose $T > T_2$ so that

$$\frac{(\alpha+1)(\alpha+2)}{\alpha(\mu_2-\alpha)} A_1\gamma\sqrt{\psi(t)} \leq l, \qquad t \geq T. \tag{2.43}$$

Consider the integral operator

$$\mathcal{F}_2 v(t) = -\alpha t^{\alpha-\mu_2} \int_T^t s^{\mu_2-\alpha-1} F_2(s, v(s))ds, \qquad t \geq T,$$

and the set

$$V_2 = \left\{v \in C_0[T,\infty) : |v(t)| \leq \sqrt{\psi(t)},\ t \geq T\right\}.$$

Using the estimates corresponding to (2.29)-(2.31) in combination with (2.41) and (2.43), we can show that if $v \in V_2$, then

$$\begin{aligned} |\mathcal{F}_2 v(t)| &\leq \alpha t^{\alpha-\mu_2} \int_T^t s^{\mu_2-\alpha-1} \frac{(\alpha+1)(\alpha+2)}{\alpha^2} A_1\psi(s)ds \\ &\leq \frac{(\alpha+1)(\alpha+2)}{\alpha(\mu_2-\alpha)} A_1\gamma\psi(t) \leq \sqrt{\psi(t)}, \qquad t \geq T, \end{aligned} \tag{2.44}$$

and if $v, w \in V_2$, then

$$|\mathcal{F}_2 v(t) - \mathcal{F}_2 w(t)| \le \alpha t^{\alpha-\mu_2} \int_T^t s^{\mu_2-\alpha-1} \frac{2(\alpha+1)}{\alpha^2} A_1 \sqrt{\psi(s)}\, |v(s) - w(s)| ds$$
$$\le \frac{(\alpha+1)(\alpha+2)}{\alpha(\mu_2-\alpha)} A_1 \gamma \sqrt{\psi(t)}\, \|v - w\|_0 \le l \|v - w\|_0, \qquad t \ge T,$$

implying that $\|\mathcal{F}_2 v - \mathcal{F}_2 w\|_0 \le l\|v - w\|_0$. This confirms that $\mathcal{F}_2$ is a contraction on V_2, and consequently $\mathcal{F}_2$ has a fixed point $v_2(t) \in V_2$ which solves the integral equation (2.40). The property (2.36) of $v_2(t)$ follows from (2.44). The function $x_2(t)$ defined by (2.35) with this $v_2(t)$ gives the desired solution in $\mathrm{RV}(\lambda_2^{\frac{1}{\alpha}})$ of (A). This completes the proof of Theorem 2.4. □

3 Asymptotic behavior of regularly varying solutions

It is natural to ask whether one can accurately determine the asymptotic behavior at infinity of the regularly varying solutions of equation (A) whose existence was established in the above four theorems. An answer to this question is provided in this section by way of the exponential representations for the solutions which, in some cases, make it possible to reveal the effect of the functions $Q(t)$ or $Q_c(t)$ upon the behavior of the solutions under study.

We begin by indicating the situation in which the asymptotic behavior of the SV- and RV(1)-solutions of (A) described in Theorems 2.1 and 2.2 can be determined precisely.

Throughout the text "$t \ge T$" means that t is sufficiently large, so that T need not to be the same at each occurrence.

Theorem 3.1. *Let $\phi(t)$ be a positive continuous function on $[a, \infty)$ which decreases to 0 as $t \to \infty$ and satisfies*

$$\int_a^\infty \frac{\phi(t)^{\frac{1}{\alpha}}}{t} dt = \infty, \qquad \int_a^\infty \frac{\phi(t)^{\frac{2}{\alpha}}}{t} dt < \infty. \tag{3.1}$$

Suppose that the function $Q(t)$ defined by (2.1) is eventually of one-signed and satisfies

$$|Q(t)| = \phi(t) + O\big(\phi(t)^{1+\frac{1}{\alpha}}\big), \qquad t \to \infty. \tag{3.2}$$

Then, equation (A) possesses a nontrivial slowly varying solution $x_1(t)$ such that

$$x_1(t) \sim c \exp\left\{ \operatorname{sgn} Q \int_a^t \frac{\phi(s)^{\frac{1}{\alpha}}}{s} ds \right\}, \qquad t \to \infty. \tag{3.3}$$

for some constant $c > 0$.

Proof. Since (3.2) implies the existence of a constant $\kappa \ge 1$ such that $|Q(t)| \le \kappa\phi(t)$ for all large t, from Theorem 2.1 (with $\phi(t)$ replaced by $\kappa\phi(t)$) it follows that (A) has an SV-solution $x_1(t)$ represented with (2.2), where $v_1(t)$ is of the form (2.5) and satisfies (2.3). Suppose that $Q(t)$ is one-signed on $[T, \infty)$ for some $T > a$. Noting that (3.2) is rewritten as

$$Q(t) = \widetilde{Q}\, \phi(t) + O(\phi(t)^{1+\frac{1}{\alpha}}), \quad \text{for } t \ge T,$$

where $\widetilde{Q} = \operatorname{sgn} Q$, using (2.3) we see that

$$(v_1(t) + Q(t))^{\frac{1}{\alpha}*} = \widetilde{Q}\, \phi(t)^{\frac{1}{\alpha}} (1 + O(\phi(t)^{\frac{1}{\alpha}})) = \widetilde{Q}\, \phi(t)^{\frac{1}{\alpha}} + O(\phi(t)^{\frac{2}{\alpha}}), \qquad t \ge T. \tag{3.4}$$

Combining (2.2) and (3.4), we obtain for $t \geq T$

$$x_1(t) = \exp\left\{ \widetilde{Q} \int_T^t \frac{\phi(s)^{\frac{1}{\alpha}}}{s} ds \right\} \exp\left\{ \int_T^t \frac{O(\phi(s)^{\frac{2}{\alpha}})}{s}\, ds \right\},$$

from which the precise asymptotic behavior (3.3) of $x_1(t)$ follows due to the second condition in (3.1). □

Theorem 3.2. *Let $\psi(t)$ be a continuously differentiable function on $[a, \infty)$ which is slowly varying, decreases to 0 as $t \to \infty$ and satisfies*

$$\int_a^\infty \frac{\psi(t)}{t} dt = \infty, \qquad \int_a^\infty \frac{\psi(t)^2}{t} dt < \infty. \tag{3.5}$$

Suppose that the function $Q(t)$ defined by (2.1) is eventually one-signed and satisfies

$$|Q(t)| = \psi(t) + O(\psi(t)^2), \qquad t \to \infty. \tag{3.6}$$

Then, equation (A) possesses a nontrivial regularly varying solution $x_2(t)$ of index 1 such that

$$x_2(t) \sim c\, t \exp\left\{ -\mathrm{sgn}\, Q \int_a^t \frac{\psi(s)}{s} ds \right\}, \qquad t \to \infty, \tag{3.7}$$

for some constant $c > 0$.

Proof. Because of (3.6) there is a constant $\kappa \geq 1$ such that $|Q(t)| \leq \kappa\psi(t)$ for all large t, and so applying Theorem 2.2 (with $\psi(t)$ replaced by $\kappa\psi(t)$), we see that (A) has an RV(1)-solution $x_2(t)$ expressed in the form (2.8), where $v_2(t)$ satisfies the decay condition (2.9) and the integral equation (2.11), with $F(t, v)$ being given by (2.12). Suppose that $Q(t)$ defined by (2.1) is one-signed on $[T, \infty)$ for some $T > a$.

For more information about the decay of $v_2(t)$ we are going to use the decomposition (2.13) of $F(t, v)$ and the estimates for $G(t, v)$, $H(t, v)$ and $k(t)$ obtained in (2.14), which we may assume holding on $[T, \infty)$. First, note that (2.9) and (2.14) implies

$$G(t, v(t)) = O(\psi(t)^2), \qquad H(t, v(t)) = O(\psi(t)^2), \qquad t \to \infty, \tag{3.8}$$

while denoting by $\widetilde{Q} = \mathrm{sgn}\, Q$ and rewriting (3.6) as $Q(t) = \widetilde{Q}\,\psi(t) + O(\psi(t)^2)$, $t \geq T$, we see that

$$k(t) = -\left(1 + \frac{1}{\alpha}\right) Q(t) + O(Q(t)^2) = -\left(1 + \frac{1}{\alpha}\right) \widetilde{Q}\psi(t) + O(\psi(t)^2), \qquad t \to \infty\,. \tag{3.9}$$

Using (3.8) and (3.9) in (2.11) and taking into account the relation

$$\frac{1}{t} \int_T^t O(\psi(s)^2) ds = O(\psi(t)^2), \qquad t \geq T,$$

which follows from the Karamata integration theorem, we obtain

$$v_2(t) = -(\alpha + 1)\widetilde{Q}\, \frac{1}{t} \int_T^t \psi(s) ds + O(\psi(t)^2), \qquad t \geq T.$$

This, combined with

$$\int_T^t \psi(s) ds = t\psi(t) - T\psi(T) + \int_T^t s|\psi'(s)| ds, \qquad t \geq T,$$

gives

$$v_2(t) = -(\alpha+1)\widetilde{Q}\,\psi(t) + O\Big(\frac{1}{t}\Big) + O\Big(\frac{1}{t}\int_T^t s|\psi'(s)|ds\Big) + O(\psi(t)^2), \qquad t \ge T,$$

which implies that

$$v_2(t) + Q(t) = -\alpha\widetilde{Q}\,\psi(t) + O\Big(\frac{1}{t}\Big) + O\Big(\frac{1}{t}\int_T^t s|\psi'(s)|ds\Big) + O(\psi(t)^2), \qquad t \ge T.$$

On the other hand it is clear that $v_2(t) + Q(t) = O(\psi(t)^2)$ as $t \to \infty$. Bringing the above observations together, we find

$$\begin{aligned}(1 + v_2(t) + Q(t))^{\frac{1}{\alpha}} &= 1 + \frac{1}{\alpha}(v_2(t) + Q(t)) + O((v_2(t) + Q(t))^2)\\ &= 1 - \widetilde{Q}\,\psi(t) + O\Big(\frac{1}{t}\Big) + O\Big(\frac{1}{t}\int_T^t s|\psi'(s)|ds\Big) + O(\psi(t)^2), \qquad t \ge T.\end{aligned} \tag{3.10}$$

We now combine (2.8) with (3.10) to obtain for $t \ge T$

$$\begin{aligned}x_2(t) = \frac{t}{T}\exp\Big\{-\widetilde{Q}\int_T^t \frac{\psi(s)}{s}ds\Big\}\\ \times\exp\Big\{\int_T^t \Big[O\Big(\frac{\psi(s)^2}{s}\Big) + O\Big(\frac{1}{s^2}\Big) + O\Big(\frac{1}{s^2}\int_T^s r|\psi'(r)|dr\Big)\Big]ds\Big\}.\end{aligned} \tag{3.11}$$

Notice that $O(\psi(t)^2/t)$ is integrable on $[T, \infty)$ by (3.5), while the integrability of $O\big(t^{-2}\int_T^t s|\psi'(s)|ds\big)$ follows from

$$\int_T^t \frac{1}{s^2}\int_T^s r|\psi'(r)|drds \le \int_T^t |\psi'(s)|ds = \psi(T) - \psi(t), \qquad t \ge T.$$

Therefore,

$$\exp\Big\{\int_T^t \Big[O\Big(\frac{\psi(s)^2}{s}\Big) + O\Big(\frac{1}{s^2}\Big) + O\Big(\frac{1}{s^2}\int_T^s r|\psi'(r)|dr\Big)\Big]ds\Big\} \sim C > 0, \qquad t \to \infty,$$

implying from (3.11) the desired asymptotic formula (3.7) for $x(t)$. □

Our next task is to establish the accurate asymptotic formulas for the regularly varying solutions of (A) constructed in Theorems 2.3 and 2.4. The non-zero constant c satisfying (1.1), the function $Q_c(t)$ defined by (1.6), the real roots λ_i, $i = 1, 2$, of (1.2) satisfying (2.19) and the constants μ_i, $i = 1, 2$, given by (2.23) and (2.38) will be used below.

Theorem 3.3. *Let $\phi(t)$ be a positive continuously differentiable function on $[a, \infty)$ which decreases to 0 as $t \to \infty$, has the property that $t|\phi'(t)|$ is decreasing and satisfies*

$$\int_a^\infty \frac{\phi(t)}{t}dt = \infty, \qquad \int_a^\infty \frac{\phi(t)^2}{t}dt < \infty. \tag{3.12}$$

Suppose that the function $Q_c(t)$ defined by (1.6) is eventually one-signed and satisfies

$$|Q_c(t)| = \phi(t) + O(\phi(t)^2), \qquad t \to \infty. \tag{3.13}$$

Then, equation (A) possesses a nontrivial regularly varying varying solution $x_1(t)$ of index $\lambda_1^{\frac{1}{\alpha}*}$ such that

$$x_1(t) \sim c\, t^{\lambda_1^{\frac{1}{\alpha}*}} \exp\Big\{\frac{|\lambda_1|^{\frac{1}{\alpha}-1}}{\alpha - \mu_1}\,\mathrm{sgn}\,Q_c\int_a^t \frac{\phi(s)}{s}ds\Big\}, \qquad t \to \infty. \tag{3.14}$$

for some constant $c > 0$.

Proof. Suppose that the function $Q_c(t)$ defined by (1.6) is one-signed on $[T,\infty)$, for some $T > a$, so that we may rewrite (3.13) as

$$Q_c(t) = \widetilde{Q_c}\,\phi(t) + O(\phi(t)^2), \qquad t \geq T, \tag{3.15}$$

where $\widetilde{Q_c} = \operatorname{sgn} Q_c$. Since (3.13) implies the existence of a constant $\kappa \geq 1$ such that $|Q_c(t)| \leq \kappa\phi(t)$ for all large t, by Theorem 2.3 (with $\phi(t)$ replaced by $\kappa\phi(t)$) there exists an $\mathrm{RV}(\lambda_1^{\frac{1}{\alpha}*})$-solution $x_1(t)$ of (A) which is expressed as (2.20), where $v_1(t)$ is a solution of the integral equation (2.26) satisfying (2.21) with $F_1(t,v)$ defined by (2.25). As in the proof of Theorem 2.3 we express $F_1(t,v)$ as in (2.27) and utilize estimates presented in (2.29), which without lost of generality is assumed to be valid on $[T,\infty)$. By combining (2.29) with (3.15) we obtain

$$G_1(t,v(t)) = O(\phi(t)^2), \qquad H_1(t,v(t)) = O(\phi(t)^2), \qquad t \to \infty\,. \tag{3.16}$$

Also, since $Q_c(t) \to 0$ and $v_1(t) \to 0$ as $t \to \infty$, for large enough t we have that

$$\operatorname{sgn}\,(Q_c(t) + \lambda_1) = \operatorname{sgn}\lambda_1 = \operatorname{sgn}\,(Q_c(t) + v_1(t) + \lambda_1). \tag{3.17}$$

Thus,

$$k_1(t) = |\lambda_1|^{1+\frac{1}{\alpha}}\left(\left|1 + \frac{Q_c(t)}{\lambda_1}\right|^{1+\frac{1}{\alpha}} - 1\right) = |\lambda_1|^{1+\frac{1}{\alpha}}\left(\left(1 + \frac{Q_c(t)}{\lambda_1}\right)^{1+\frac{1}{\alpha}} - 1\right),$$

implying using (2.23) and (3.15)

$$k_1(t) = \left(1 + \frac{1}{\alpha}\right)\lambda_1^{\frac{1}{\alpha}*} Q_c(t) + O(Q_c(t)^2) = \frac{\mu_1}{\alpha}\widetilde{Q_c}\,\phi(t) + O(\phi(t)^2), \qquad t \geq T. \tag{3.18}$$

Using (3.16) and (3.18) in (2.26) we obtain

$$\begin{aligned} v_1(t) &= \alpha t^{\alpha-\mu_1}\int_t^\infty s^{\mu_1-\alpha-1}\left[\frac{\mu_1}{\alpha}\widetilde{Q_c}\,\phi(s) + O(\phi(s)^2)\right]ds \\ &= \mu_1\widetilde{Q_c}\,t^{\alpha-\mu_1}\int_t^\infty s^{\mu_1-\alpha-1}\phi(s)ds + O(\phi(t)^2), \qquad t \geq T, \end{aligned}$$

from which, via integration by parts, it follows that

$$v_1(t) = \frac{\mu_1}{\alpha-\mu_1}\,\widetilde{Q_c}\,\phi(t) + O(J(t)) + O(\phi(t)^2), \qquad t \geq T, \tag{3.19}$$

where

$$J(t) = t^{\alpha-\mu_1}\int_t^\infty s^{\mu_1-\alpha}|\phi'(s)|ds.$$

Combining (3.15) and (3.19) we obtain

$$v_1(t) + Q_c(t) = \frac{\alpha}{\alpha-\mu_1}\widetilde{Q_c}\,\phi(t) + O(J(t)) + O(\phi(t)^2), \qquad t \geq T,$$

which due to (3.17) gives

$$(\lambda_1 + v_1(t) + Q_c(t))^{\frac{1}{\alpha}*} = \lambda_1^{\frac{1}{\alpha}*} + \frac{\lambda_1^{\frac{1}{\alpha}*}}{\lambda_1(\alpha-\mu_1)}\,\widetilde{Q_c}\,\phi(t) + O(J(t)) + O(\phi(t)^2), \qquad t \geq T.$$

Therefore, the representation formula (2.20) for $x_1(t)$ becomes

$$x_1(t) = \left(\frac{t}{T}\right)^{\lambda_1^{\frac{1}{\alpha}*}}\exp\left\{\frac{|\lambda_1|^{\frac{1}{\alpha}-1}}{\alpha-\mu_1}\widetilde{Q_c}\int_T^t\frac{\phi(s)}{s}ds\right\}\exp\left\{\int_T^t\left[O\left(\frac{J(s)}{s}\right) + O\left(\frac{\phi(s)^2}{s}\right)\right]ds\right\}. \tag{3.20}$$

Since $O(\phi(t)^2/t)$ is integrable on $[T,\infty)$ by (3.12) as well as $O(J(t)/t)$ because

$$\frac{|J(t)|}{t} = \frac{\mu_1}{\alpha-\mu_1}\, t^{\alpha-\mu_1-1}\int_t^\infty s^{\mu_1-\alpha-1} s|\phi'(s)|ds \le \frac{\mu_1}{(\alpha-\mu_1)^2}\,|\phi'(t)|, \qquad t\ge T,$$

the desired asymptotic formula (3.14) for $x_1 \in \mathrm{RV}\big(\lambda_1^{\frac{1}{\alpha}*}\big)$ follows from (3.20). This completes the proof of Theorem 3.3. □

Theorem 3.4. *Let $\phi(t)$ be a positive continuously differentiable slowly varying function on $[a,\infty)$ which decreases to 0 as $t\to\infty$, has the property that $t|\psi'(t)|$ is slowly varying and satisfies*

$$\int_a^\infty \frac{\psi(t)}{t}dt = \infty, \qquad \int_a^\infty \frac{\psi(t)^2}{t}dt < \infty. \tag{3.21}$$

Suppose that the function $Q_c(t)$ defined by (1.6) is eventually one-signed and satisfies

$$|Q_c(t)| = \psi(t) + O(\psi(t)^2), \qquad t\to\infty. \tag{3.22}$$

Then, equation (A) possesses a nontrivial regularly varying solution $x_2(t)$ of index $\lambda_2^{\frac{1}{\alpha}}$ such that

$$x_2(t) \sim c\, t^{\lambda_2^{\frac{1}{\alpha}}} \exp\left\{ \frac{\lambda_2^{\frac{1}{\alpha}-1}}{\alpha-\mu_2}\,\mathrm{sgn}\, Q_c \int_a^t \frac{\psi(s)}{s}ds \right\}, \qquad t\to\infty. \tag{3.23}$$

for some constant $c>0$.

Proof. Suppose that the function $Q_c(t)$ defined by (1.6) is one-signed on $[T,\infty)$ for some $T>a$. By (3.22) there is a constant $\kappa\ge 1$ such that $|Q_c(t)|\le \kappa\psi(t)$ for all large t. Consequently, Theorem 2.4 ensures that (A) has a solution $x\in \mathrm{RV}(\lambda_2^{\frac{1}{\alpha}})$ of the form (2.35), where $v_2(t)$ satisfies (2.36) and the integral equation (2.40) with $F_2(t,v)$ defined by (2.39). As in the proof of Theorem 2.4 we express $F_2(t,v)$ as in (2.42) where G_2, H_2 and k_2 stand, respectively, for G_1, H_1 and k_1 in (2.28) with λ_1 replaced with λ_2. As regards G_2 and H_2 exactly the same type of estimates as (2.29) hold for all large t provided λ_1 is replaced by λ_2, which together with (2.36) gives

$$G_2(t,v_2(t)) = O(\psi(t)^2), \qquad H_2(t,v_2(t)) = O(\psi(t)^2), \qquad t\to\infty. \tag{3.24}$$

Moreover, as regards

$$k_2(t) = (\lambda_2 + Q_c(t))^{1+\frac{1}{\alpha}} - \lambda_2^{1+\frac{1}{\alpha}},$$

from (2.36) it follows that

$$k_2(t) = \left(1+\frac{1}{\alpha}\right)\lambda_2^{\frac{1}{\alpha}} Q_c(t) + O(Q_c(t)^2) = \frac{\mu_2}{\alpha}\,\widetilde{Q_c}\,\psi(t) + O(\psi(t)^2), \qquad t\ge T, \tag{3.25}$$

where $\widetilde{Q_c} = \mathrm{sgn}\, Q_c$. Using (3.24) and (3.25) in (2.40), we obtain

$$\begin{aligned} v_2(t) &- \mu_2\widetilde{Q_c}\, t^{\alpha-\mu_2}\int_T^t \Big(s^{\mu_2-\alpha-1}\psi(s) + O(\psi(s)^2)\Big)ds \\ &= -\mu_2\widetilde{Q_c}\, t^{\alpha-\mu_2}\int_T^t s^{\mu_2-\alpha-1}\psi(s)ds + O(\psi(t)^2), \qquad t\ge T. \end{aligned}$$

Since

$$t^{\alpha-\mu_2}\int_T^t s^{\mu_2-\alpha-1}\psi(s)ds = \frac{\psi(t)}{\mu_2-\alpha} - \frac{\psi(T)}{\mu_2-\alpha}\left(\frac{t}{T}\right)^{\alpha-\mu_2} + \frac{t^{\alpha-\mu_2}}{\mu_2-\alpha}\int_T^t s^{\mu_2-\alpha}|\psi'(s)|ds,$$

it follows that

$$v_2(t) = -\frac{\mu_2}{\mu_2-\alpha}\widetilde{Q_c}\,\psi(t) + O(t^{\alpha-\mu_2}) + O(J(t)) + O(\psi(t)^2), \qquad t \geq T, \tag{3.26}$$

where

$$J(t) = t^{\alpha-\mu_2}\int_T^t s^{\mu_2-\alpha}|\psi'(s)|ds.$$

Our final step is to show that (3.26) is crucial in determining the asymptotic behavior of $x_2(t)$ given by (3.23). Employing (3.26), we find that

$$\begin{aligned}(\lambda_2 + v_2(t) + Q_c(t))^{\frac{1}{\alpha}} &= \lambda_2^{\frac{1}{\alpha}} + \frac{1}{\alpha}\lambda_2^{\frac{1}{\alpha}-1}(v(t)+Q_c(t)) + O((v(t)+Q_c(t))^2)\\ &= \lambda_2^{\frac{1}{\alpha}} - \frac{\lambda_2^{\frac{1}{\alpha}-1}}{\mu_2-\alpha}\widetilde{Q_c}\,\psi(t) + O(t^{\alpha-\mu_2}) + O(J(t)) + O(\psi(t)^2), \quad t \geq T,\end{aligned}$$

which, substituted for (2.35), shows that $x_2(t)$ is expressed as

$$\begin{aligned}x_2(t) = \left(\frac{t}{T}\right)^{\lambda_2^{\frac{1}{\alpha}}} &\exp\left\{-\frac{\lambda_2^{\frac{1}{\alpha}-1}}{\mu_2-\alpha}\mathrm{sgn}\,Q_c\int_T^t \frac{\psi(s)}{s}ds\right\}\\ &\times\exp\left\{\int_T^t\left[O(s^{\alpha-\mu_2-1}) + O\left(\frac{J(s)}{s}\right) + O\left(\frac{\psi(s)^2}{s}\right)\right]ds\right\}.\end{aligned} \tag{3.27}$$

The integrability of $O(\psi(t)^2/t)$ on $[T,\infty)$ follows from assumption (3.21), while the integrability of $O(t^{\alpha-\mu_2-1})$ is obvious since $\alpha-\mu_2<0$. Finally, the integrability of $O(J(t)/t)$ on $[T,\infty)$ follows from the relation

$$\frac{|J(t)|}{t} = \frac{\mu_2}{\alpha-\mu_2}t^{\alpha-\mu_2-1}\int_T^t s^{\mu_2-\alpha-1}s|\psi'(s)|ds \leq \frac{\mu_2}{(\alpha-\mu_2)^2}|\psi'(t)|, \qquad t \geq T,$$

which is a consequence of the Karamata integration theorem. Therefore,

$$\exp\left\{\int_T^t\left[O(s^{\alpha-\mu_2-1}) + O\left(\frac{J(s)}{s}\right) + O\left(\frac{\psi(s)^2}{s}\right)\right]ds\right\} \sim C > 0, \qquad t \to \infty,$$

implying from (3.27) the precise asymptotic formula (3.23) for $x_2 \in \mathrm{RV}(\lambda_2^{\frac{1}{\alpha}})$. The proof of Theorem 3.4 has thus been completed. □

4 Examples and concluding remarks

We now present some examples illustrating our main results and showing that our results extend and improve results obtained in [7,11].

Example 4.1. Consider the half-linear differential equation

$$(|x'|^\alpha \operatorname{sgn} x')' + q_1(t)|x|^\alpha \operatorname{sgn} x = 0, \quad q_1(t) = \frac{\alpha\,\theta^\alpha}{t^{\alpha+1}(\log t)^{\alpha(1-\theta)}}\left(1 + \frac{A}{(\log t)^{1-\theta}} + \frac{B}{\log t}\right), \quad (\mathrm{E}_1)$$

on $[1,\infty)$, where $\theta \in (0, \frac{1}{2})$, A and B are constants. Put

$$\phi(t) = \frac{\theta^\alpha}{(\log t)^{\alpha(1-\theta)}}.$$

An easy calculation shows that

$$q_1(t) = -\left(\frac{\phi(t)}{t^\alpha}\right)' + O\left(\left(\frac{\phi(t)}{t^\alpha}\right)^{1+\frac{1}{\alpha}}\right), \quad \text{for all large } t \geq 1,$$

and so integrating the above from t to ∞ and multiplying with t^α, we see that

$$Q(t) = \phi(t) + O\big(\phi(t)^{1+\frac{1}{\alpha}}\big) \quad \text{for all large } t \geq 1, \tag{4.1}$$

implying that $Q(t)$ is eventually positive. Moreover, because of $\theta \in (0, \frac{1}{2})$, it holds that

$$\int^\infty \frac{\phi(s)^{\frac{1}{\alpha}}}{s} ds = \int^\infty \frac{\theta}{s(\log s)^{1-\theta}} ds = \infty, \qquad \int^\infty \frac{\phi(s)^{\frac{2}{\alpha}}}{s} ds = \int^\infty \frac{\theta^2}{s(\log s)^{2(1-\theta)}} ds < \infty.$$

Thus all the hypotheses of Theorem 3.1 are fulfilled for equation (E_1), and so there exists a nontrivial SV-solution $x_1(t)$ of (E_1) having the precise asymptotic behavior

$$x_1(t) \sim c\exp\left\{\int_1^t \frac{\phi(s)^{\frac{1}{\alpha}}}{s} ds\right\} = c\,\exp\left\{\int_1^t \frac{\theta}{s(\log s)^{1-\theta}} ds\right\} = c\exp\big((\log t)^\theta\big), \qquad t \to \infty,$$

for some $c > 0$.

Note that if in particular $A = -\theta$ and $B = 1-\theta$, (E_1) has an exact SV-solution $x(t) = \exp((\log t)^\theta)$.

It should be noticed that if $\alpha \leq 1$, then (4.1) implies $Q(t) = \phi(t) + O\big(\phi(t)^2\big)$ as $t \to \infty$. Also, if $\theta \in \big(0, 1 - \frac{1}{2\alpha}\big)$ condition (3.5) of Theorem 3.2 is satisfied with $\psi(t) = \phi(t)$. Thus, Theorem 3.2 is applicable to (E_1) and ensures the existence of its nontrivial RV(1)-solution $x_2(t)$ with the precise asymptotic behavior

$$x_2(t) \sim c_2 t \exp\Big(\frac{\theta^\alpha}{\alpha(1-\theta)-1}\,(\log t)^{1-\alpha(1-\theta)}\Big), \qquad t \to \infty.$$

for some constant $c_2 > 0$.

Example 4.2. Consider the half-linear equation

$$(|x'|^\alpha \operatorname{sgn} x')' + q_2(t)|x|^\alpha \operatorname{sgn} x = 0, \qquad q_2(t) = -\frac{\alpha}{t^{\alpha+1}\log t}\left(1 + \frac{A}{\log t} + \frac{B}{(\log t)^2}\right)^\gamma, \qquad (\mathrm{E}_2)$$

on $[1,\infty)$, where $A \geq 0$, $B \geq 0$ and γ are constants. Putting $\psi(t) = 1/\log t$, it is shown that

$$q_2(t) = \left(\frac{\psi(t)}{t^\alpha}\right)' + O\left(\frac{\psi(t)^2}{t^{\alpha+1}}\right) \quad \text{for all large } t,$$

from which we see that

$$Q(t) = -\psi(t) + O(\psi(t)^2), \qquad t \to \infty, \tag{4.2}$$

and $Q(t)$ is eventually negative. Since $\psi(t)$ satisfies

$$\int^{\infty} \frac{\psi(t)}{t} dt = \int^{\infty} \frac{dt}{t \log t} = \infty, \qquad \int^{\infty} \frac{\psi(t)^2}{t} dt = \int^{\infty} \frac{dt}{t(\log t)^2} < \infty,$$

applying Theorem 3.2 to equation (E$_2$), we conclude that (E$_2$) possesses a nontrivial RV(1)-solution $x_2(t)$ with the precise asymptotic behavior

$$x_2(t) \sim c\, t \exp\left\{ \int_1^t \frac{\psi(s)}{s} ds \right\} = c\, t \exp\left\{ \int_1^t \frac{ds}{s \log s} ds \right\} = c\, t \log t, \qquad t \to \infty.$$

Note that if in particular $A = 1$, $B = 0$ and $\gamma = \alpha - 1$, then (E$_2$) has an exact RV(1)-solution $x(t) = t \log t$.

We remark that if $\alpha \in (1, 2)$, then (4.2) implies $Q(t) = -\psi(t) + O\big(\psi(t)^{1+\frac{1}{\alpha}}\big)$ as $t \to \infty$. Since $\psi(t)$ satisfies

$$\int^{\infty} \frac{\psi(t)^{\frac{1}{\alpha}}}{t} dt = \int^{\infty} \frac{(\log t)^{-\frac{1}{\alpha}}}{t} dt = \infty, \qquad \int^{\infty} \frac{\psi(t)^{\frac{2}{\alpha}}}{t} dt = \int^{\infty} \frac{(\log t)^{-\frac{2}{\alpha}}}{t} dt < \infty,$$

it follows from Theorem 3.1 (with $\phi(t) = \psi(t)$) that (E$_2$) admits a nontrivial SV-solution $x_1(t)$ such that

$$x_1(t) \sim c_1 \exp\left(\frac{\alpha}{1-\alpha} (\log t)^{\frac{\alpha-1}{\alpha}} \right), \qquad t \to \infty.$$

for some constant $c_1 > 0$.

Example 4.3. Consider the half-linear equation

$$((x')^3)' + q_3(t)x^3 = 0, \qquad q_3(t) = -\frac{6}{t^4}\left(1 + \frac{1}{\log t}\right)^2 \left(1 + \frac{3}{2\log t} + \frac{A}{(\log t)^2}\right), \tag{E$_3$}$$

on $[1, \infty)$, where A is a constant. As is easily checked, $q_3(t)$ satisfies

$$t^3 \int_t^{\infty} q_3(s) ds + 2 = -\frac{7}{\log t} + O\left(\frac{1}{(\log t)^2} \right)$$

for all large t, which implies that the hypotheses of Theorems 3.3 and 3.4 are fulfilled with the choice

$$\alpha = 3, \qquad c = -2, \qquad \phi(t) = \psi(t) = \frac{7}{\log t},$$

while $Q_c(t)$ is eventually negative. We need to find the two real roots of the equation $\lambda^{\frac{4}{3}} - \lambda - 2 = 0$. Its smaller root is $\lambda_1 = -1$, while the approximate value of the larger root is $\lambda_2 \approx 3.67857$. Since $\lambda_1^{\frac{1}{\alpha}*} = -1$ and $\mu_1 = -4$, by Theorem 3.3 equation (E$_3$) possesses a nontrivial RV(-1)-solution $x_1(t)$ with the asymptotic behavior

$$x_1(t) \sim c_1 t^{-1} \exp\left\{ -\int_1^t \frac{ds}{s \log s} \right\} = \frac{c_1}{t \log t}, \qquad t \to \infty,$$

for some constant $c_1 > 0$. If in particular $A = 1$, then (E$_3$) has an exact RV(-1)-solution $x(t) = 1/t \log t$.

From Theorem 3.4, noting that $\mu_2 = 4\lambda_2^{\frac{1}{3}}$, it follows that equation (E_3) also possesses a nontrivial $\mathrm{RV}(\lambda_2^{\frac{1}{3}})$-solution $x_2(t)$ such that

$$x_2(t) \sim c_2\, t^{\lambda_2^{\frac{1}{3}}} \exp\left\{-\frac{\lambda_2^{-\frac{2}{3}}}{3-\mu_2}\int_1^t \frac{\psi(s)}{s}ds\right\} \sim c_2 t^{\lambda_2^{\frac{1}{3}}} (\log t)^{\nu_2}, \qquad t \to \infty,$$

for some constant $c_2 > 0$, where $\lambda_2^{\frac{1}{3}} \approx 1.54369$ and $\nu_2 = 7\big(4\lambda_2 - 3\lambda_2^{\frac{2}{3}}\big)^{-1} \approx 0.925269$.

Example 4.4. Consider the half-linear equation

$$\big(|x'|^{\frac{1}{2}}\mathrm{sgn}\, x'\big)' + q_4(t)|x|^{\frac{1}{2}}\mathrm{sgn}\, x = 0, \tag{E_4}$$

$$q_4(t) = \frac{5}{27}t^{-\frac{3}{2}}\left(1+\frac{9}{4\log t}\right)^{-\frac{1}{2}}\left(1+\frac{9}{20\log t}+\frac{A}{(\log t)^2}\right), \qquad t \geq 1,$$

where A is a constant. A simple calculation shows that $q_4(t)$ satisfies

$$t^{\frac{1}{2}}\int_t^\infty q_4(s)ds - \frac{10}{27} = -\frac{1}{4\log t} + O\left(\frac{1}{(\log t)^2}\right), \qquad t \to \infty$$

and this confirms that the hypotheses of Theorems 3.3 and 3.4 are fulfilled with the choice

$$\alpha = \frac{1}{2}, \qquad c = \frac{10}{27}, \qquad \phi(t) = \psi(t) = \frac{1}{4\log t},$$

and $Q_c(t)$ is eventually negative. The equation $|\lambda|^3 - \lambda + \frac{10}{27} = 0$ has two real roots $\lambda_1 = \frac{\sqrt{6}-1}{3} < \frac{2}{3} = \lambda_2$. Noting that $\lambda_2^{\frac{1}{\alpha}-1} = \frac{2}{3}$ and $\alpha - \mu_2 = -\frac{1}{6}$, one can assert from Theorem 3.4 that there exists a nontrivial $\mathrm{RV}(\frac{4}{9})$-solution $x_2(t)$ with the asymptotic behavior

$$x_2(t) \sim c_2 t^{\frac{4}{9}} \exp\left\{4\int_1^t \frac{\psi(s)ds}{s}\right\} = c_2 t^{\frac{4}{9}}\log t, \qquad t \to \infty,$$

for some constant $c_2 > 0$. If $A = 0$, then (E_4) has an exact $\mathrm{RV}(\frac{4}{9})$-solution $x(t) = t^{\frac{4}{9}}\log t$.

On the other hand, by application of Theorem 3.3 there exists a nontrivial regularly varying solution $x_1(t)$ of index λ_1^2 enjoying the asymptotic behavior

$$x_1(t) \sim c_1 t^{\lambda_1^2}(\log t)^{\nu_1}, \qquad t \to \infty,$$

for some constant $c_1 > 0$, where

$$\lambda_1^2 = \frac{7-2\sqrt{6}}{9}, \quad \nu_1 = \frac{\lambda_1}{2(3\lambda_1^2 - 1)} = -\frac{4+\sqrt{6}}{8}.$$

Concluding remarks

(1) In this paper the equation (A) in which $q(t)$ satisfies condition (1.4) with $c = E(\alpha)$, i.e.,

$$\lim_{t\to\infty} t^\alpha \int_t^\infty q(s)ds = \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \tag{4.3}$$

has been excluded. It should be noted that equation (A) may be oscillatory or nonoscillatory depending on the choice of $q(t)$ satisfying (4.3) (see e.g. the paper [8]). Such equations are often said to be in the border case. Since in this case the equation $|\lambda|^{1+\frac{1}{\alpha}} - \lambda + E(\alpha) = 0$ has the only one real root $(\frac{\alpha}{\alpha+1})^{\alpha}$, the regularity index of regularly varying solutions of (A), if exists, must be equal to $\frac{\alpha}{\alpha+1}$. In [5] a sufficient condition is presented for equation (A) in the border case possesses a trivial regularly varying solution of index $\frac{\alpha}{\alpha+1}$. It would be of interest to answer the question: Is it possible to find conditions under which equation (A) in the border case possesses nontrivial RV($\frac{\alpha}{\alpha+1}$)-solutions and to determine their precise asymptotic behavior as $t \to \infty$?

(2) In the paper [6] an attempt is made to generalize the results for (A) obtained in [5] to the half-linear differential equations of the form

$$(p(t)|x'|^{\alpha}\text{sgn } x')' + q(t)|x|^{\alpha}\text{sgn } x = 0, \tag{4.4}$$

where $\alpha > 0$ is a constant and $p(t) > 0$, $q(t)$ are continuous functions on $[a, \infty)$. Naturally the qualitative properties of positive solutions of (4.4) depend heavily on the coefficient $p(t)$. In order to precisely describe the effect of the function $p(t)$ upon the behavior of positive solutions of (4.4) the authors of [6] used the class of *generalized Karamata functions*, introduced in [4], as the framework for the asymptotic analysis of (4.4), and demonstrate how to build in the new framework the existence theory of generalized Karamata solutions for (4.4) which extends the results on regularly varying solutions of (A) developed in [5]. It is expected that one can possibly indicate a class of equations of the form (4.4) possessing generalized Karamata solutions whose asymptotic behaviors at infinity are determined accurately and explicitly.

Appendix: Regularly varying functions

Definition 4.5. A measurable function $f : [0, \infty) \to (0, \infty)$ is called *regularly varying of index* $\rho \in \mathbf{R}$ if

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \lambda^{\rho} \quad \text{for all} \;\; \lambda > 0.$$

The set of all regularly varying functions of index ρ is denoted by RV(ρ). The symbol SV is often used to denote RV(0), in which case members of SV are called *slowly varying functions*. Since any function $f(t) \in$ RV(ρ) is expressed as $f(t) = t^{\rho} g(t)$ with $g(t) \in$ SV, the class SV of slowly varying functions is of fundamental importance in the theory of regular variation.

Definition 4.6. If $f \in$ RV(ρ) has the property that

$$\lim_{t\to\infty} \frac{f(t)}{t^{\rho}} = \text{const} > 0,$$

then it is called a *trivial regularly varying function of index* ρ and is denoted by $f \in$ tr-RV(ρ). Otherwise $f(t)$ is called a *nontrivial regularly varying solution of index* ρ and is denoted by $f \in$ ntr-RV(ρ).

One of the most important properties of regularly varying functions is the following *representation theorem*.

Proposition 4.7. *$f(t) \in RV(\rho)$ if and only if $f(t)$ is represented in the form*

$$f(t) = c(t) \exp\left\{ \int_{t_0}^{t} \frac{\delta(s)}{s} ds \right\}, \qquad t \geq t_0, \tag{A.1}$$

for some $t_0 > 0$ *and for some measurable functions* $c(t)$ *and* $\delta(t)$ *such that*

$$\lim_{t\to\infty} c(t) = c_0 \in (0,\infty) \quad \textit{and} \quad \lim_{t\to\infty} \delta(t) = \rho.$$

If in particular $c(t) \equiv c_0$ in (A.1), then $f(t)$ is said to be a *normalized* regularly varying function of index ρ.

Typical examples of slowly varying functions are: all functions tending to some positive constants as $t \to \infty$,

$$\prod_{n=1}^{N} (\log_n t)^{\alpha_n}, \ \ \alpha_n \in \mathbf{R}, \quad \text{and} \quad \exp\left\{\prod_{n=1}^{N} (\log_n t)^{\beta_n}\right\}, \ \ \beta_n \in (0,1),$$

where $\log_n t$ denotes the n-th iteration of the logarithm. It is known that the functions

$$L(t) = 2 + \sin(\log_n t), \quad n \geq 2, \ \text{ and } \ M(t) = \exp\left\{(\log t)^\theta \cos\,(\log t)^\theta\right\}, \quad \theta \in \left(0, \frac{1}{2}\right),$$

are slowly varying. They are oscillating in the sense that

$$\limsup_{t\to\infty} L(t) = 3 \quad \text{and} \quad \liminf_{t\to\infty} L(t) = 1,$$

and

$$\limsup_{t\to\infty} M(t) = \infty \quad \text{and} \quad \liminf_{t\to\infty} M(t) = 0.$$

It should be noted that $2 + \sin t$ and $2 + \sin(\log t)$ are not slowly varying.

The following result illustrates operations which preserve slowly variation.

Proposition 4.8. *Let* $L(t)$*,* $L_1(t)$ *and* $L_2(t)$ *be slowly varying. Then,* $L(t)^\alpha$ *for any* $\alpha \in \mathbf{R}$*,* $L_1(t) + L_2(t)$*,* $L_1(t)L_2(t)$ *and* $L_1(L_2(t))$ *(if* $L_2(t) \to \infty$*) are slowly varying.*

The operations given in the above proposition preserve regular variation in the following sense.

Proposition 4.9. *Let* $f(t)$*,* $f_1(t)$ *and* $f_2(t)$ *be regularly varying of indices* ρ*,* ρ_1 *and* ρ_2*, respectively. Then, for any* $\alpha \in \mathbf{R}$ $f(t)^\alpha$ *is regularly varying of index* $\alpha\rho$*,* $f_1(t) + f_2(t)$ *is regularly varying of index* $\max\{\rho_1, \rho_2\}$*,* $f_1(t)f_2(t)$ *is regularly varying of index* $\rho_1 + \rho_2$ *and* $f_1(f_2(t))$ *is regularly varying of index* $\rho_1\rho_2$ *provided* $\lim_{t\to\infty} f_2(t) = \infty$*.*

A slowly varying function may grow to infinity or decay to 0 as $t \to \infty$. But its order of growth or decay is severely limited as is shown in the following

Proposition 4.10. *Let* $L \in \mathrm{SV}$*. Then, for any* $\varepsilon > 0$*,*

$$\lim_{t\to\infty} t^\varepsilon L(t) = \infty, \qquad \lim_{t\to\infty} t^{-\varepsilon} L(t) = 0.$$

It can be shown that any regularly varying function of non-zero index is asymptotic to a monotone regularly varying function of the same index.

Proposition 4.11. *Let* $f \in \mathrm{RV}(\rho)$*.*

(i) If $\rho > 0$*, then*

$$\sup\{f(s) : t_0 \leq s \leq t\} \sim f(t), \qquad \inf\{f(s) : s \geq t\} \sim f(t), \qquad t \to \infty.$$

(ii) If $\rho < 0$*, then*

$$\sup\{f(s) : s \geq t\} \sim f(t), \qquad \inf\{f(s) : t_0 \leq s \leq t\} \sim f(t), \qquad t \to \infty.$$

The following result called Karamata's integration theorem is of highest importance in handling slowly and regularly varying functions analytically.

Proposition 4.12. *Let* $L(t) \in \mathrm{SV}$. *Then,*

(i) if $\alpha > -1$,

$$\int_a^t s^\alpha L(s) ds \sim \frac{1}{\alpha+1} t^{\alpha+1} L(t), \qquad t \to \infty;$$

(ii) if $\alpha < -1$,

$$\int_t^\infty s^\alpha L(s) ds \sim -\frac{1}{\alpha+1} t^{\alpha+1} L(t), \qquad t \to \infty;$$

(iii) if $\alpha = -1$,

$$l(t) = \int_a^t \frac{L(s)}{s} ds \in SV \quad \text{and} \quad \lim_{t\to\infty} \frac{L(t)}{l(t)} = 0,$$

$$m(t) = \int_t^\infty \frac{L(s)}{s} ds \in SV, \quad \text{and} \quad \lim_{t\to\infty} \frac{L(t)}{m(t)} = 0.$$

Here in defining $m(t)$ *it is assumed that* $L(t)/t$ *is integrable on a neighborhood of infinity.*

For the almost complete exposition of theory of regular variation and its applications the reader is referred to the treatise of Bingham et al. [1]. See also Seneta [12]. A comprehensive survey of results up to the year 2000 on the asymptotic analysis of second order ordinary differential equations by means of regular variation can be found in the monograph of Marić [9].

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References

[1] N. H. Bingham, C. M. Goldie, J. L. Teugels, *Regular variation*, Encyclopedia of Mathematics and its Applications, Vol. 27, Cambridge University Press, 1987. MR898871; url

[2] Á. Elbert, A half-linear second order differential equation, *Colloq. Math. Soc. János Bolyai* **30**(1979), 158–180. MR0680591

[3] O. Došlý, P. Řehák, *Half-linear differential equations*, North-Holland Mathematics Studies, Vol. 202, Elsevier, North Holland, 2005. MR2158903

[4] J. Jaroš, T. Kusano, Self-adjoint differential equations and generalized Karamata functions, *Bull. Cl. Sci. Math. Nat. Sci. Math.* **29**(2004), 25–60. MR2099615; url

[5] J. Jaroš, T. Kusano, T. Tanigawa, Nonoscillation theory for second order half-linear differential equations in the framework of regular variation, *Results Math.* **43**(2003), 129–149. MR1962855

[6] J. Jaroš, T. Kusano, T. Tanigawa, Nonoscillatory half-linear differential equations and generalized Karamata functions, *Nonlinear Anal.* **64**(2006), 762–787. MR2197094

[7] T. Kusano, V. Marić, T. Tanigawa, Asymptotics of some classes of nonoscillatory solutions of second order half-linear differential equations, *Bull. Cl. Sci. Math. Nat. Sci. Math.* **28**(2003), 61–74. MR2041180

[8] T. Kusano, Y. Naito, A. Ogata, Strongly oscillation and nonoscillation of quasilinear differential equations of second order, *Differ. Equ. Dyn. Syst.* **2**(1994), 1–10.

[9] V. Marić, *Regular variation and differential equations*, Lecture Notes in Mathematics, Vol. 1726, Springer-Verlag, Berlin, 2000. MR1753584; url

[10] P. Řehák, De Haan type increasing solutions of half-linear differential equations, *J. Math. Anal. Appl.* **412**(2014), 236–243. MR3145797

[11] P. Řehák, Exponential estimates for solutions of half-linear differential equations, *Acta Math. Hungar.*, **147**(2015), No. 1, 158–171. MR3391519

[12] E. Seneta, *Regularly varying functions*, Lecture Notes in Mathematics, Vol. 508, Springer-Verlag, Berlin, 1976. MR0453936; url