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On solutions of space-fractional diffusion equations by means of potential wells

Dedicated to Professor Tibor Krisztin on the occasion of his 60th birthday

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Abstract. In this paper, we study the initial boundary value problem of space-fractional diffusion equations. First, we introduce a family of potential wells. Then we show the existence of global weak solutions, provided that the initial energy $J(u_0)$ is positive and less than the potential well depth d. Finally, we establish the vacuum isolating and blow up of strong solutions.

Keywords: potential wells, space-fractional wave equations, global solutions, fractional Sobolev spaces.

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1 Introduction

There exist several natural phenomena that cannot be modeled by partial differential equations based on ordinary calculus, since they depend on the so-called memory effect. In order to take account of this dependence, we may use fractional differential calculus. Fractional differential equations have gained considerable importance due to their applications in various sciences, such as physics, mechanics, chemistry, engineering, etc. In recent years, there has been a significant development in fractional differential equations which may be ordinary or partial, see for examples [8–10, 15, 16, 18, 21, 34, 35] and the references therein.

A space–time fractional diffusion-wave equation is obtained from the classical diffusion or wave equation by replacing the first or second order time derivatives and second order space derivatives by fractional derivatives, see for examples [12, 17]. We can describe space–time fractional diffusion-wave equations with three space variables as

$$ {}_0^C D_t^{\alpha} u = \chi \left(\frac{\partial^{\beta} u}{\partial |x|^{\beta}} + \frac{\partial^{\gamma} u}{\partial |y|^{\gamma}} + \frac{\partial^{\delta} u}{\partial |z|^{\delta}} \right), \tag{1.1} $$

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where χ is a positive constant, ${}_0^C D_t^\alpha$ is the Caputo derivative of order α and $\partial^\beta/\partial|x|^\beta$, $\partial^\gamma/\partial|y|^\gamma$ and $\partial^\delta u/\partial|z|^\delta$ are symmetric Riesz derivatives of orders β, γ and δ, respectively. If $\beta = \gamma = \delta$, symmetric Riesz derivatives can be treated as fractional Laplace operators. Equation (1.1) yields different diffusion-wave equations for various values of the parameters α, β, γ and δ. Precisely,

(1) Classical diffusion equation $\alpha = 1$, $\beta = \gamma = \delta = 2$.

(2) Time-fractional diffusion equation $0 < \alpha < 1$, $\beta = \gamma = \delta = 2$, see for examples [23,24,36].

(3) Space-fractional diffusion equation $\alpha = 1$, either $0 < \beta, \gamma \leq 2$ $0 < \delta < 2$, or $0 < \beta < 2$, $0 < \gamma, \delta \leq 2$, or $0 < \beta, \delta \leq 2$, $0 < \gamma < 2$, see for examples [6].

(4) Space–time fractional diffusion equation $0 < \alpha < 1$, either $0 < \beta, \gamma \leq 2$, $0 < \delta < 2$, or $0 < \beta < 2$, $0 < \gamma, \delta \leq 2$ or $0 < \beta, \delta \leq 2$, $0 < \gamma < 2$, see for examples [7,19,29].

(5) Classical wave equation $\alpha = \beta = \gamma = \delta = 2$.

(6) Time-fractional wave equation $1 < \alpha < 2$, $\beta = \gamma = \delta = 2$, see for examples [22,27,33].

(7) Space-fractional wave equation $\alpha = 2$, either $0 < \beta, \gamma \leq 2$,$0 < \delta < 2$, or $0 < \beta < 2$, $0 < \gamma, \delta \leq 2$, or $0 < \beta, \delta \leq 2$, $0 < \gamma < 2$, see for examples [2].

(8) Space–time fractional wave equation $1 < \alpha < 2$, either $0 < \beta, \gamma \leq 2$, $0 < \delta < 2$, or $0 < \beta < 2$, $0 < \gamma, \delta \leq 2$, or $0 < \beta, \delta \leq 2$, $0 < \gamma < 2$, see for examples [5,14].

In this paper, we study the space-fractional diffusion problem:

$$\begin{cases} u_t + (-\triangle)^s u = |u|^{p-1}u, & x \in \Omega,\ t > 0, \qquad (1.2) \\ u(x,0) = u_0(x), & x \in \Omega, \qquad (1.3) \\ u(x,t) = 0, & x \in \mathbb{R}^N \backslash \Omega,\ t \geq 0, \qquad (1.4) \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $N > 2s$, and p satisfies $1 < p \leq 2_s^* - 1 = (N+2s)/(N-2s)$.

A suitable stationary fractional Sobolev space for (1.2)–(1.4) is $X_0(\Omega)$ which consists of all functions $u \in H^s(\mathbb{R}^N)$ with $u = 0$ a.e. in $\mathbb{R}^N \setminus \Omega$. We refer to Section 2 for further details and recall that the use of the space $X_0(\Omega)$ to find solutions of nonlinear fractional elliptic problems was begun in [30].

Let T be the existence time of the solution u for the problem (1.2)–(1.4), where T may be ∞. We say that $u \in L^\infty(0,T;X_0(\Omega))$, with $u_t \in L^2(0,T;L^2(\Omega))$, is a *weak solution* of (1.2)–(1.4) if

$$\int_0^t \int_\Omega u_\tau(x,\tau)\varphi(x,\tau)dxdt + C(N,s)\int_0^t \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(u(x,\tau)-u(y,\tau))(\varphi(x,\tau)-\varphi(y,\tau))}{|x-y|^{N+2s}}dxdyd\tau$$
$$= \int_0^t \int_\Omega |u(x,\tau)|^{p-1}u(x,\tau)\varphi(x,\tau)dxd\tau$$

for any $\varphi \in L^1(0,\infty;X_0(\Omega))$ and any $t \in [0,T)$, and

$$u(x,0) = u_0(x) \in X_0(\Omega).$$

Here and in the following

$$\frac{1}{C(N,s)} = \int_{\mathbb{R}^N} \frac{1-\cos\xi_1}{|\xi|^{N+2s}} d\xi.$$

If a weak solution u belongs to $C(0,T;X_0(\Omega))$, we call u a *strong solution*.

In order to find solutions of (1.2)–(1.4), we use the potential well theory, see for examples [11,13,20,25,28] and the references therein. All the results obtained for the problem (1.2)–(1.4) are still valid if we replace the equation (1.2) by

$$u_t + (-\triangle)^s u = f(u),$$

provided that f satisfies the following conditions first introduced in [25]:

(f_1) *$f \in C^1(\mathbb{R})$ and $f(0) = f'(0) = 0$;*

(f_2) *f is monotone increasing in $\mathbb{R}$, and is convex $\mathbb{R}^+$, concave $\mathbb{R}^-$;*

(f_3) *$(p+1)F(u) \le uf(u)$, $|uf(u)| \le \gamma|F(u)|$, where $2 < p+1 \le \gamma < \infty$ if $N = 1,2$, and $2 < p+1 \le \gamma \le 2N/(N-2s) = 2^*_s$ if $N \ge 3$, and $F(u) = \int_0^u f(s)ds$.*

For example, concerning a global existence theorem, i.e. Theorem 4.1 in Section 4, the key functionals associated to problem (1.2)–(1.4) are

$$J(u) = \frac{C(N,s)}{2}\|u\|^2_{X_0(\Omega)} - \frac{1}{p+1}\|u\|^{p+1}_{L^{p+1}(\Omega)}, \qquad I(u) = C(N,s)\|u\|^2_{X_0(\Omega)} - \|u\|^{p+1}_{L^{p+1}(\Omega)}.$$

If we replace $|u|^{p-1}u$ by $f(u)$ which satisfies (f_1)–(f_3), then we should replace the key functionals by

$$J(u) = \frac{C(N,s)}{2}\|u\|^2_{X_0(\Omega)} - \int_\Omega F(u)dx, \qquad I(u) = C(N,s)\|u\|^2_{X_0(\Omega)} - \int_\Omega f(u)u dx.$$

After the replacement, Theorem 4.1 in Section 4 is still valid.

The paper is organized as follows. In Section 2, we provide notations and some facts concerning fractional Sobolev spaces which shall be used later. In Section 3, we introduce a family of potential wells in order to study the space-fractional diffusion equations. In Section 4, we obtain the existence of global weak solutions. In Section 5, we establish the phenomenon of vacuum isolating and blow up for strong solutions.

2 Preliminaries

Let $s \in (0,1)$ and $2s < N$. The fractional Laplace operator for a function $\varphi \in C_0^\infty(\mathbb{R}^N)$ is defined pointwise by

$$(-\triangle)^s\varphi(x) = \frac{C(N,s)}{2}\int_{\mathbb{R}^N} \frac{\varphi(x)-\varphi(x+y)-\varphi(x-y)}{|y|^{N+2s}}dy, \qquad \frac{1}{C(N,s)} = \int_{\mathbb{R}^N}\frac{1-\cos\xi_1}{|\xi|^{N+2s}}d\xi,$$

for all $x \in \mathbb{R}^N$.

The fractional Sobolev space $H^s(\mathbb{R}^N)$ is set as

$$H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \frac{|u(x)-u(y)|}{|x-y|^{\frac{N}{2}+s}} \in L^2(\mathbb{R}^N \times \mathbb{R}^N) \right\},$$

endowed with the norm

$$\|u\|_{H^s(\mathbb{R}^N)} = \left(\|u\|^2_{L^2(\mathbb{R}^N)} + \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} dxdy \right)^{1/2}.$$

Let

$$X_0(\Omega) = \left\{ u \in H^s(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \right\}.$$

In the sequel we take

$$\|u\|_{X_0(\Omega)} = \left(\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} dxdy \right)^{1/2}$$

as norm on $X_0(\Omega)$. It is easily seen that $X_0(\Omega) = \left(X_0(\Omega), \|\cdot\|_{X_0(\Omega)}\right)$ is a Hilbert space with inner product

$$\langle u, v \rangle_{X_0(\Omega)} = \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+2s}} dxdy.$$

Since $u \in X_0(\Omega)$, we know that the norm and inner product can be extended to all $\mathbb{R}^N \times \mathbb{R}^N$.

Denote by

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$$

the distinct eigenvalues and e_k the eigenfunction corresponding to λ_k of the elliptic eigenvalue problem:

$$\begin{cases} (-\triangle)^s u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{2.1}$$

Concerning the eigenvalue of the problem (2.1), by [31] we have for $k \in \mathbb{N}$

$$\lambda_k = \frac{C(N,s)}{2} \min_{u \in \mathscr{P}_k \setminus \{0\}} \frac{\|u\|^2_{X_0(\Omega)}}{\|u\|^2_{L^2(\Omega)}},$$

where

$$\mathscr{P}_1 = X_0(\Omega)$$

and for all $k \geq 2$

$$\mathscr{P}_k = \left\{ u \in X_0(\Omega) \,:\, \langle u, e_j \rangle_{X_0(\Omega)} = 0 \text{ for all } j = 1, 2, \ldots, k-1 \right\}.$$

For the readers' convenience, we recall the main embedding results for the fractional Sobolev spaces, see [3] for details.

Lemma 2.1. *Let Ω be bounded domain. Then*

(1) *the embedding $X_0(\Omega) \to L^p(\mathbb{R}^N)$ is compact for any $p \in [1, 2^*_s)$;*
(2) *the embedding $X_0(\Omega) \hookrightarrow L^{2^*_s}(\mathbb{R}^N)$ is continuous.*

Let $1 \leq p < \infty$ and let X be a Banach space. The space $L^p(0,T;X)$ denotes the space of L^p-integrable functions from $[0,T)$ into X with the norm

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(\cdot,t)\|^p_X dt \right)^{1/p}.$$

If $p = \infty$, the space $L^\infty(0, T; X)$ is the space of essentially bounded functions from $[0, \infty)$ into X with norm

$$\|u\|_{L^\infty(0,\infty;X)} = \sup_{t\in[0,T)} \|u(\cdot, t)\|_X.$$

The space $C(0, T; X)$ consists of all functions u from $[0, T)$ into X such that $\|u\|_X$ is continuous on $[0, T)$. See for example [32] for facts concerning this kind of spaces. In this paper we take either $X = L^2(\Omega)$, or $X = L^{p+1}(\Omega)$, or $X = X_0(\Omega)$.

3 Potential wells in variational stationary setting

For simplicity, in this section we consider the problem (1.2)–(1.4) in stationary case. In fact, if we replace u in this section by $u(t)$ for any $t \in [0, T)$, all the facts are still valid.

We define

$$J(u) = \frac{C(N,s)}{2}\|u\|^2_{X_0(\Omega)} - \frac{1}{p+1}\|u\|^{p+1}_{L^{p+1}(\Omega)}, \qquad I(u) = C(N,s)\|u\|^2_{X_0(\Omega)} - \|u\|^{p+1}_{L^{p+1}(\Omega)}$$

and the potential well

$$W = \{u \in X_0(\Omega) \,:\, I(u) > 0,\ J(u) < d\} \cup \{0\},$$

where

$$d = \inf_{\substack{u\in X_0(\Omega)\\ u\neq 0}} \sup_{\lambda\geq 0} J(\lambda u).$$

It is easy to see that $J(\lambda u)$ attains its maximum, with respect to λ, at

$$\lambda^* = \left(\frac{C(N,s)\|u\|^2_{X_0(\Omega)}}{\|u\|^{p+1}_{L^{p+1}(\Omega)}}\right)^{1/(p-1)}.$$

Normalizing u so that $\lambda^* = 1$, i.e. $C(N,s)\|u\|^2_{X_0} = \|u\|^{p+1}_{L^{p+1}(\Omega)}$, we get

$$d = \inf J(u)$$

subject to $u \in X_0(\Omega)$, $\|u\|_{X_0(\Omega)} \neq 0$, $I(u) = 0$.

From [31], we know that the problem

$$\begin{cases} (-\triangle)^s w = |w|^{p-1}w, & \text{in } \Omega,\\ w = 0, & \text{in } \mathbb{R}^N\backslash\Omega. \end{cases} \tag{3.1}$$

admits a nontrivial solution. Then for all $\lambda > 0$, the function $v = \lambda^{\frac{1}{1-p}} w$ is a solution of

$$\begin{cases} (-\triangle)^s v = \lambda|v|^{p-1}v, & \text{in } \Omega,\\ v = 0, & \text{in } \mathbb{R}^N\backslash\Omega. \end{cases}$$

Define

$$S_{p+1} = \inf_{\substack{u\in X_0(\Omega)\\ u\neq 0}} \frac{C(N,s)\|u\|_{X_0(\Omega)}}{\|u\|_{L^{p+1}(\Omega)}}.$$

The Euler equation for this homogeneous variational problem is

$$(-\triangle)^s u = \lambda |u|^{p-1} u,$$

where λ is a Lagrange multiplier. Therefore, the nontrivial solution w of (3.1) attains the infimum, that is

$$S_{p+1} = \frac{C(N,s)\|w\|_{X_0(\Omega)}}{\|w\|_{L^{p+1}(\Omega)}}.$$

On the other hand, by the definition of solution for (3.1),

$$C(N,s)\|w\|^2_{X_0(\Omega)} = \|w\|^{p+1}_{L^{p+1}(\Omega)}.$$

Thus

$$d = \|w\|^{p+1}_{L^{p+1}} \frac{p-1}{2(p+1)}.$$

Therefore

$$S_{p+1} = C(N,s)^{\frac{1}{2}} \|w\|^{\frac{p-1}{2}}_{L^{p+1}(\Omega)} = C(N,s)^{\frac{1}{2}} \left(\frac{2(p+1)}{p-1} d \right)^{\frac{p-1}{2(p+1)}}$$

and

$$d = \frac{p-1}{2(p+1)} C(N,s)^{\frac{p+1}{1-p}} S_{p+1}^{\frac{2(p+1)}{p-1}}.$$

Furthermore, for problem (1.2)–(1.4) and $\delta \in (0,1)$ we define

$$J_\delta(u) = \frac{\delta}{2} C(N,s)\|u\|^2_{X_0} - \frac{1}{p+1}\|u\|^{p+1}_{L^{p+1}(\Omega)}, \tag{3.2}$$

$$d(\delta) = (1-\delta)[(p+1)\delta]^{\frac{2}{p-1}} \left(\frac{S^2_{p+1}}{2C(N,s)} \right)^{\frac{p+1}{p-1}}. \tag{3.3}$$

From the definition of S_{p+1}, it is easy to get the following lemmas.

Lemma 3.1. *If $J(u) \le d(\delta)$, then*

(i) $J_\delta(u) > 0$ if and only if

$$0 < \|u\|_{X_0(\Omega)} < \left(\frac{(p+1)\delta S^{p+1}_{p+1}}{2C(N,s)^p} \right)^{1/(p-1)};$$

(ii) $J_\delta(u) < 0$ if and only if

$$\|u\|_{X_0(\Omega)} > \left(\frac{(p+1)\delta S^{p+1}_{p+1}}{2C(N,s)^p} \right)^{1/(p-1)}.$$

Lemma 3.2. *If $J(u) = d(\delta)$, then $J_\delta(u) = 0$ if and only if*

$$\|u\|_{X_0(\Omega)} = \left(\frac{(p+1)\delta S^{p+1}_{p+1}}{2C(N,s)^p} \right)^{1/(p-1)}.$$

Lemma 3.3. *The function $d = d(\delta)$ has the following properties on the interval $[0, 1]$.*

(i) $d(0) = d(1) = 0$;

(ii) *d takes the maximum value at* $\delta_0 = 2/(p+1)$ *and* $d(\delta_0) = d$;

(iii) *d is increasing on* $[0, \delta_0]$ *and decreasing on* $[\delta_0, 1]$;

(iv) *for any given* $e \in (0, d)$, *the equation* $d(\delta) = e$ *has exactly two solutions* $\delta_1 \in (0, \delta_0)$ *and* $\delta_2 \in (\delta_0, 1)$.

Theorem 3.4. $d(\delta) = \min J(u)$ *subject to* $u \in X_0(\Omega)$, $\|u\|_{X_0(\Omega)} \neq 0$, $J_\delta(u) = 0$.

First, it is easy to show that $J(u) \geq d(\delta)$, when $u \in X_0(\Omega)$, $\|u\|_{X_0(\Omega)} \neq 0$, $J_\delta(u) = 0$, and, in view of the definition of S_{p+1}, this concludes the proof of Theorem 3.4.

Corollary 3.5. $d = d(\delta_0) = \min J(u)$ *subject to* $u \in X_0(\Omega)$, $\|u\|_{X_0(\Omega)} \neq 0$, $I(u) = 0$.

The proof of Corollary 4.2 is an immediate consequence of Theorem 3.4 and the fact that $J_{\delta_0}(u) = 0$ is equivalent to $I(u) = 0$.

Let us define the following *family of potential wells* for all $\delta \in (0, 1)$

$$W_\delta = \{u \in X_0(\Omega) \,:\, J_\delta(u) > 0,\ J(u) < d(\delta)\} \cup \{0\},$$
$$\overline{W}_\delta = W_\delta \cap \partial W_\delta = \{u \in X_0(\Omega) \,:\, J_\delta(u) \geq 0,\ J(u) \leq d(\delta)\} \cup \{0\}.$$

Clearly $W_{\delta_0} = W$. In addition, let us introduce for all $\delta \in (0, 1)$

$$V_\delta = \{u \in X_0(\Omega) \,:\, J_\delta(u) < 0,\ J(u) < d(\delta)\},$$
$$\overline{V_\delta} = V_\delta \cup \partial V_\delta = \{u \in X_0(\Omega) \,:\, J_\delta(u) \leq 0,\ J(u) \leq d(\delta)\},$$
$$V = \{u \in X_0(\Omega) : I(u) < 0, J(u) < d\},$$
$$B_\delta = \left\{ u \in X_0(\Omega) \,:\, \|u\|_{X_0(\Omega)} < \left(\frac{(p+1)\delta S_{p+1}^{p+1}}{2C(N,s)^p} \right)^{1/(p-1)} \right\},$$
$$\overline{B_\delta} = B_\delta \cup \partial B_\delta = \left\{ u \in X_0(\Omega) \,:\, \|u\|_{X_0(\Omega)} \leq \left(\frac{(p+1)\delta S_{p+1}^{p+1}}{2C(N,s)^p} \right)^{1/(p-1)} \right\},$$
$$B_\delta^c = \left\{ u \in X_0(\Omega) \,:\, \|u\|_{X_0(\Omega)} > \left(\frac{(p+1)\delta S_{p+1}^{p+1}}{2C(N,s)^p} \right)^{1/(p-1)} \right\}.$$

Furthermore, $V_{\delta_0} = V$.

Let $u \in X_0(\Omega) \setminus \{0\}$, with $J(u) \leq C(N,s)\|u\|_{X_0}^2/2$. Then for any given $\delta \in (0, 1)$ such that

$$0 < \|u\|_{X_0} < (1-\delta)^{\frac{1}{2}} \left(\frac{(p+1)\delta S_{p+1}^{p+1}}{2C(N,s)^p} \right)^{1/(p-1)},$$

we get $J(u) < d(\delta)$ and $J_\delta(u) > 0$. Consequently,

$$B_{\overline{\delta}} \subset W_\delta, \quad \text{with } \overline{\delta} = (1-\delta)^{\frac{p-1}{2}}\delta.$$

From this and Lemma 3.1, it is immediate to get the following theorem.

Theorem 3.6. $B_{\overline{\delta}} \subset W_\delta \subset B_\delta$, $V_\delta \subset B^c_\delta$.

Corollary 3.7. $B_{\overline{\delta_0}} \subset W \subset B_{\delta_0}$, $V \subset B^c_{\delta_0}$, *where*

$$B_{\delta_0} = \left\{ u \in X_0(\Omega) : \|u\|_{X_0(\Omega)} < C(N,s)^{\frac{p}{1-p}} S_{p+1}^{\frac{p+1}{p-1}} \right\}, \qquad \overline{\delta_0} = \left(\frac{p-1}{p+1}\right)^{\frac{p-1}{2}} \frac{2}{p+1}$$

and $\delta_0 = 2/(p+1)$ *by Lemma 3.3.*

In view of Lemma 3.3, we obtain the following lemma.

Lemma 3.8. *(i) If* $0 < \delta' < \delta'' \le \delta_0$, *then* $W_{\delta'} \subset W_{\delta''}$.
(ii) If $\delta_0 \le \delta' < \delta'' < 1$, *then* $V_{\delta''} \subset V_{\delta'}$.

It is easy to prove the following lemma by contradiction.

Lemma 3.9. *Assume that* $0 < J(u) < d$ *for some* $u \in X_0(\Omega)$, *and that* $\delta_1 < \delta_2$ *are the two solutions of the equation* $d(\delta) = J(u)$. *Then* $J_\delta(u)$ *does not change sign for* $\delta \in (\delta_1, \delta_2)$.

4 Existence of global weak solutions

In this section we study the global existence of weak solutions for the problem (1.2)–(1.4). Via the results on eigenfunctions of fractional Laplace operators established in [31], we are able to apply the Galërkin method and we construct finite-dimensional Galërkin approximations for the problem (1.2)–(1.4). In particular, we present a priori estimates, which allow us to pass to the limit and to obtain the desired weak solution u of (1.2)–(1.4). Indeed, u verifies the conditions of initial data and belongs to the family of potential wells.

Theorem 4.1. *Let* $u_0 \in X_0(\Omega)$. *Suppose that* $0 < J(u_0) < d$, $\delta_1 < \delta_2$ *are the two solutions of equation* $d(\delta) = J(u_0)$ *and* $J_{\delta_2}(u_0) > 0$. *Then problem (1.2)–(1.4) admits a global weak solution* $u \in L^\infty(0,\infty; X_0(\Omega))$, *with* $u_t \in L^2(0,\infty; L^2(\Omega))$ *and* $u(\cdot,t) \in W_\delta$ *for* $\delta \in (\delta_1,\delta_2)$ *and* $t \in \mathbb{R}^+_0$.

Proof. Fix $u_0 \in X_0(\Omega)$ such that $0 < J(u_0) < d$, $d(\delta_i) = J(u_0)$, $i = 1, 2$, and $J_{\delta_2}(u_0) > 0$.

By [31], the sequence $\{e_k\}_k$ of eigenfunctions corresponding to the sequence $\{\lambda_k\}_k$ of eigenvalues of the fractional Laplace operator $(-\triangle)^s$ is an orthonormal basis of $L^2(\Omega)$ and an orthogonal basis of $X_0(\Omega)$. Let

$$u_m(x,t) = \sum_{j=1}^m g_{jm}(t) e_j(x), \qquad m = 1, 2, \dots,$$

be the Galërkin approximate solutions of the problem (1.2)–(1.4) satisfying

$$(u_{mt}, e_j)_{L^2(\Omega)} + C(N,s)\langle u_m, e_j\rangle_{X_0(\Omega)} = \int_\Omega |u_m|^{p-1} u_m e_j dx, \tag{4.1}$$

$$u_m(\cdot,0) = \sum_{j=1}^m a_j e_j \to u_0 \quad \text{in } X_0(\Omega). \tag{4.2}$$

Substituting u_m into (4.1)–(4.2), we get

$$g'_{jm} + \lambda_j g_{jm} = \sum_{l=1}^m |g_{jl}|^{p-1} g_{jl} \int_\Omega |e_l|^{p-1} e_l e_j dx, \tag{4.3}$$

$$g_{jm}(0) = a_j, \qquad j = 1, \dots, m. \tag{4.4}$$

According to standard ordinary differential equations theory (for example Wintner's theorem), problem (4.3)–(4.4) admits a solution g_{jm} of class $C^1([0,T])$ for each m. Multiplying (4.1) by $g'_{jm}(t)$, summing for j and integrating with respect to t, we have

$$\int_0^t \|u_{m\tau}(\cdot,\tau)\|_{L^2(\Omega)} d\tau + J(u_m(\cdot,t)) = J(u_m(\cdot,0)). \tag{4.5}$$

Since $J_{\delta_2}(u_0) > 0$ implies $\|u_0\|_{X_0(\Omega)} \neq 0$, an argument similar to the proof of Lemma 3.9 gives $J_\delta(u_0) > 0$ for all $\delta \in (\delta_1, \delta_2)$. Furthermore, $J(u_0) = d(\delta_1)$ implies that the initial value u_0 is in W_δ for all $\delta \in (\delta_1, \delta_2)$. Hence, for any fixed $\delta \in (\delta_1, \delta_2)$, the inequality $J_{\delta_2}(u_0) > 0$ implies that $J_\delta(u_m(\cdot,0)) > 0$ and $J(u_m(\cdot,0)) < d(\delta)$, provided that m is sufficiently large. This happens for all $\delta \in (\delta_1, \delta_2)$ by virtue of Lemma 3.8 (i). Therefore, we may assume without loss of generality, that $u_m(\cdot,0) \in W_\delta$ for all all $\delta \in (\delta_1, \delta_2)$ and m.

We claim that $u_m(\cdot,t) \in W_\delta$ for all $\delta \in (\delta_1, \delta_2)$, all m and all $t > 0$. Otherwise there exist $\delta \in (\delta_1, \delta_2)$, m and $t_0 > 0$ such that $u_m(\cdot,t_0) \in \partial W_\delta$, i.e. either (i) $J(u_m(\cdot,t_0)) = d(\delta)$, or (ii) $J_\delta(u_m(\cdot,t_0)) = 0$ and $\|u_m(\cdot,t_0)\|_{X_0(\Omega)} \neq 0$. From (4.5), we get

$$J(u_m(\cdot,t)) \le J(u_m(\cdot,0)) < d(\delta) \quad \text{for all } t > 0.$$

Hence the case (i) is impossible. If (ii) occurs, then by Theorem 3.4 we get $J(u_m(\cdot,t_0)) \ge d(\delta)$, which is also impossible. This completes the proof of the claim.

Thus, Lemmas 3.1 and 3.3 (ii) imply that for all $t > 0$ and m

$$\|u_m(\cdot,t)\|_{X_0(\Omega)} < \left(\frac{(p+1)\delta S_{p+1}^{p+1}}{2C(N,s)^p}\right)^{1/(p-1)} < \left(\frac{(p+1)\delta_2 S_{p+1}^{p+1}}{2C(N,s)^p}\right)^{1/(p-1)},$$

$$\|u_m(\cdot,t)\|_{L^{p+1}(\Omega)} \le \frac{C(N,s)}{S_{p+1}}\|u_m(\cdot,t)\|_{X_0(\Omega)} < \left(\frac{(p+1)\delta_2 S_{p+1}^2}{2C(N,s)}\right)^{1/(p-1)},$$

$$\int_0^t \|u_{m\tau}(\cdot,\tau)\|^2_{L^2(\Omega)} d\tau < 2d(\delta) \le 2d(\delta_0),$$

being $\delta \in (\delta_1, \delta_2)$. Then by the weak* compactness of bounded sets in $L^\infty(0,\infty; L^{p+1}(\Omega))$ and in $L^\infty(0,\infty; X_0(\Omega))$ and the weak compactness of bounded sets in $L^2(0,\infty; L^2(\Omega))$, we conclude that there exists a subsequence of $(u_m)_m$ – still denoted by $(u_m)_m$ – such that

$$u_m \to u \text{ weakly}^* \text{ in } L^\infty(0,\infty; L^{p+1}(\Omega)) \text{ and in } L^\infty(0,\infty; X_0(\Omega)),$$

and

$$u_{mt} \to u_t \text{ weakly in } L^2(0,\infty; L^2(\Omega)).$$

From (4.2) we have that $u(\cdot,0) = u_0$ in $X_0(\Omega)$.

Integrating (4.1) with respect to t and letting $m \to \infty$, we obtain that for each w_j,

$$\int_0^t \int_\Omega u_\tau(x,\tau) w_j(x) dx d\tau + C(N,s) \int_0^t \langle u(\cdot,\tau), w_j \rangle_{X_0(\Omega)} d\tau = \int_0^t \int_\Omega |u(x,\tau)|^{p-1} u(x,\tau) w_j(x) dx d\tau$$

and furthermore for any $v \in X_0(\Omega)$,

$$\int_0^t \int_\Omega u_\tau(x,\tau) v(x) dx d\tau + C(N,s) \int_0^t \langle u(\cdot,\tau), v \rangle_{X_0(\Omega)} d\tau = \int_0^t \int_\Omega |u(x,\tau)|^{p-1} u(x,\tau) v(x) dx d\tau.$$

Differentiating with respect to t, we have

$$\int_\Omega u_t(x,t)v(x)dx + C(N,s)\langle u(\cdot,t), v\rangle_{X_0(\Omega)} = \int_\Omega |u(x,t)|^{p-1}u(x,t)v(x)dx.$$

For any $\varphi \in L^1(0,\infty; X_0(\Omega))$, letting $v(x) = \varphi(x,t)$, with t fixed, and integrating with respect to t, we conclude that u is a weak solution of problem (1.2)–(1.4).

Since $u_m(\cdot,t) \in W_\delta$ for all $\delta \in (\delta_1, \delta_2)$, all m and all $t \in \mathbb{R}_0^+$, we get that $u(\cdot,t) \in W_\delta$ for all $\delta \in (\delta_1,\delta_2)$ and all $t \in \mathbb{R}_0^+$. This completes the proof. □

In view of the facts that $I(u_0) > 0$ implies $J_\delta(u_0) > 0$ and that $J_{\delta_2}(u_0) \geq J_{\delta_0}(u_0)$ thanks to their definitions and Lemma 3.3 (iv), we get at once

Corollary 4.2. *If in Theorem 4.1 the assumption $J_{\delta_2}(u_0) > 0$ is replaced by $I(u_0) > 0$, i.e. $u_0 \in W$, then the result of Theorem 4.1 continues to hold.*

Next we consider the problem (1.2)–(1.4), under the critical conditions $I(u_0) \geq 0$ and $J(u_0) = d$.

Theorem 4.3. *Let $u_0 \in X_0(\Omega)$. Suppose that $J(u_0) = d$ and $I(u_0) \geq 0$, then problem (1.2)–(1.4) admits a global weak solution $u \in L^\infty(0,\infty; X_0(\Omega))$, with $u_t \in L^2(0,\infty; L^2(\Omega))$ and $u(\cdot,t) \in \overline{W}$ for all $t \in \mathbb{R}_0^+$.*

Proof. Fix $u_0 \in X_0(\Omega)$, with $J(u_0) = d$ and $I(u_0) \geq 0$.

Let $\lambda_m = 1 - 1/m$ and $u_{0m} = \lambda_m u_0$, $m \in \mathbb{N}$. Consider the problem (1.2), (1.4), with the initial condition

$$u(\cdot,0) = u_{0m}. \tag{4.6}$$

From $I(u_0) \geq 0$ we have

$$\begin{aligned}
I(u_{0m}) &= \lambda_m^2 C(N,s)\|u_0\|^2_{X_0(\Omega)} - \lambda_m^{p+1}\|u_0\|^{p+1}_{L^{p+1}(\Omega)} \\
&= \lambda_m^2 I(u_0) + (\lambda_m^2 - \lambda_m^{p+1})\|u_0\|^{p+1}_{L^{p+1}(\Omega)} > 0, \\
J(u_{0m}) &= \frac{C(N,s)}{2}\|u_{0m}\|^2_{X_0(\Omega)} - \frac{1}{p+1}\|u_{0m}\|^{p+1}_{L^{p+1}(\Omega)} \\
&= \left(\frac{1}{2} - \frac{1}{p+1}\right)C(N,s)\|u_{0m}\|^2_{X_0(\Omega)} + \frac{1}{p+1}I(u_{0m}) > 0, \\
J(u_{0m}) &= J(\lambda_m u_0) < J(u_0) = d.
\end{aligned}$$

By Theorem 4.1 for each $m \in \mathbb{N}$, problem (1.2), (1.4), under (4.6), admits a global weak solution $u_m \in L^\infty(0,\infty; X_0(\Omega))$, with $u_{mt} \in L^2(0,\infty; L^2(\Omega))$ and $u_m(\cdot,t) \in \overline{W}$ for all $t \in \mathbb{R}_0^+$.

The fact that $u_m(\cdot,t) \in \overline{W}$ for all $t \in \mathbb{R}_0^+$ implies that for all $t \in \mathbb{R}_0^+$

$$\begin{aligned}
J(u_m(\cdot,t)) &\geq \left(\frac{1}{2} - \frac{1}{p+1}\right)C(N,s)\|u_m(\cdot,t)\|^2_{X_0(\Omega)} + \frac{1}{p+1}I(u_m(\cdot,t)) \\
&\geq \frac{p-1}{2(p+1)}C(N,s)\|u_m(\cdot,t)\|^2_{X_0(\Omega)}
\end{aligned}$$

and, furthermore, for all $t \in \mathbb{R}_0^+$,

$$\|u_m(\cdot,t)\|_{X_0(\Omega)} \leq \left(2d\frac{p+1}{(p-1)C(N,s)}\right)^{1/2},$$

$$\|u_m(\cdot,t)\|_{L^{p+1}(\Omega)} \leq \frac{C(N,s)}{S_{p+1}}\|u_m(\cdot,t)\|_{X_0(\Omega)} \leq \frac{1}{S_{p+1}}\left(2d\frac{(p+1)C(N,s)}{p-1}\right)^{1/2}$$

and

$$\int_0^t \|u_{m\tau}(\cdot,\tau)\|^2_{L^2(\Omega)} d\tau < 2d.$$

Then by the weak* compactness of bounded sets in $L^\infty(0,\infty;L^{p+1}(\Omega))$ and $L^\infty(0,\infty;X_0(\Omega))$ and by the weak compactness of bounded sets in $L^2(0,\infty;L^2(\Omega))$, we conclude that problem (1.2)–(1.4) admits a global weak solution $u \in L^\infty(0,\infty;X_0(\Omega))$, with $u_t \in L^2(0,\infty;L^2(\Omega))$ and $u(\cdot,t) \in \overline{W}$ for all $t \in \mathbb{R}_0^+$. This completes the proof. □

5 Vacuum isolating and blow up of strong solutions

Let T be the existence time of any solution u of the problem (1.2)–(1.4). In the following, similar to (4.5), we assume that

$$\int_0^t \|u_\tau(\cdot,\tau)\|_{L^2(\Omega)} d\tau + J(u(\cdot,t)) \le J(u_0) \tag{5.1}$$

for all $t \in [0,T)$.

Theorem 5.1. *Let $u_0 \in X_0(\Omega)$. Fix $e \in (0,d)$ and let $\delta_1 < \delta_2$ be the two solutions of the equation $d(\delta) = e$. Then for any strong solution u of the problem (1.2)–(1.4), with initial energy $J(u_0) = e$,*

(i) $u(\cdot,t)$ belongs to W_δ for all $\delta \in (\delta_1,\delta_2)$ and $t \in [0,T)$, provided that $I(u_0) > 0$.

(ii) $u(\cdot,t)$ belongs to V_δ for all $\delta \in (\delta_1,\delta_2)$ and $t \in [0,T)$, provided that $I(u_0) < 0$.

Proof. Fix $u_0 \in X_0(\Omega)$, $e \in (0,d)$, let $\delta_1 < \delta_2$ be the two solutions of the equation $d(\delta) = e$, and fix a strong solution u of the problem (1.2)–(1.4), with initial energy $J(u_0) = e$.

(i) Corollary 4.2 and the proof of Theorem 4.1 give that $u_0 \in W_\delta$ for any $\delta \in (\delta_1,\delta_2)$. Assume by contradiction that there exists some $t_0 \in (0,T)$ such that $u(\cdot,t_0) \in \partial W_\delta$ for some $\delta \in (\delta_1,\delta_2)$, i.e. either $J(u(t_0)) = d(\delta)$ or $J_\delta(u(t_0)) = 0$ and $\|u(\cdot,t_0)\|_{X_0(\Omega)} \neq 0$. By (5.1),

$$J(u(t)) \le J(u_0) < d(\delta) \quad \text{for } t \in (0,T), \tag{5.2}$$

so that $J(u(\cdot,t_0)) = d(\delta)$ is impossible. While, if $J_\delta(u(t_0)) = 0$ and $\|u(\cdot,t_0)\|_{X_0(\Omega)} \neq 0$ occur, then Theorem 3.4 gives that $J(u(t_0)) \ge d(\delta)$, which contradicts (5.2). This completes the proof of case (i).

(ii) The assumption $I(u_0) < 0$ implies that

$$J_\delta(u_0) < \left(\frac{\delta}{2} - \frac{1}{p+1}\right) \|u\|^{p+1}_{L^{p+1}(\Omega)} < 0$$

for all $\delta \in (\delta_1,\delta_0)$ by Lemma 3.3 (ii) and (iv). Since the sign of $J_\delta(u_0)$ does not change for $\delta \in (\delta_1,\delta_2)$ by Lemma 3.9, then $J_\delta(u_0) < 0$ for all $\delta \in (\delta_1,\delta_2)$. This fact and $J(u_0) < d(\delta)$ for all $\delta \in (\delta_1,\delta_2)$ give $u_0 \in V_\delta$ for $\delta \in (\delta_1,\delta_2)$.

Assume now by contradiction that there exist some $t_0 \in (0,T)$ and $\delta \in (\delta_1,\delta_2)$ such that $u(\cdot,t_0) \in \partial V_\delta$, i.e. either $J(u(\cdot,t_0)) = d(\delta)$ or $J_\delta(u(\cdot,t_0)) = 0$. From (5.2), the case $J(u(\cdot,t_0)) = d(\delta)$ is impossible. Suppose next that t_0 is the smallest t such that $J_\delta(u(\cdot,t_0)) = 0$, then $J_\delta(u(\cdot,t)) < 0$ for all $t \in [0,t_0)$. From (5.2) and Lemma 3.1 we have

$$\|u(\cdot,t)\|_{X_0(\Omega)} > \left(\frac{(p+1)\delta S_{p+1}^{p+1}}{2C(N,s)^p}\right)^{1/(p-1)} = \kappa_\delta \quad \text{for all } t \in [0,t_0)$$

and furthermore $\|u(\cdot,t_0)\|_{X_0(\Omega)} \ge \kappa_\delta$. Thus Theorem 3.4 implies that $J(u(\cdot,t_0)) \ge d(\delta)$, which contradicts (5.2) and completes the proof of (ii). □

From Theorem 4.3 and Lemma 3.9 we obtain the following theorem.

Theorem 5.2. *Let u_0, e and δ_i, $i = 1, 2$, be as stated in Theorem 5.1. Then for any strong solution u of problem (1.2)–(1.4), with initial energy $J(u_0)$ satisfying $0 < J(u_0) \le e$,*
(i) $u(\cdot, t)$ belongs to W_δ for all $\delta \in (\delta_1, \delta_2)$ and $t \in [0, T)$, provided that $I(u_0) > 0$;
(ii) $u(\cdot, t)$ belongs to V_δ for all $\delta \in (\delta_1, \delta_2)$ and $t \in [0, T)$, provided that $I(u_0) < 0$.

Corollary 5.3. *Let u_0, e and δ_i, $i = 1, 2$, be as stated in Theorem 5.1. Then for any strong solution u of problem (1.2)–(1.4), with initial energy $J(u_0)$ satisfying $0 < J(u_0) \le e$,*
(i) $u(\cdot, t)$ belongs to $\overline{W_{\delta_1}}$ for all $t \in [0, T)$, provided that $I(u_0) > 0$;
(ii) $u(\cdot, t)$ belongs to $\overline{V_{\delta_2}}$ for all $t \in [0, T)$, provided that $I(u_0) < 0$.

Proof. From (5.1)

$$J(u(t)) \le d(\delta_1) \quad (\text{or } d(\delta_2)) \quad \text{for all } t \in [0, T).$$

Fix $t \in [0, T)$. Letting $\delta \to \delta_1$ (or $\delta \to \delta_2$) in $J_\delta(u(t)) > 0$ (or $J_\delta(u(t)) < 0$) for the case (i) (or case (ii)), we have $J_{\delta_1}(u(t)) \ge 0$ (or $J_{\delta_2}(u(t)) \le 0$) for all $t \in [0, T)$. This completes the proof. □

From Corollary 5.3 and Lemma 3.1 we get the following theorem.

Theorem 5.4. *Let u_0, e and δ_i, $i = 1, 2$, be as stated in Theorem 5.1. Then for any strong solution u of problem (1.2)–(1.4), with initial energy $J(u_0)$ satisfying $0 < J(u_0) \le e$,*
(i) $u(\cdot, t)$ lies inside the ball $\overline{B_{\delta_1}}$ for all $t \in [0, T)$, provided $u_0 \in B_{\delta_0}$;
(ii) $u(\cdot, t)$ lies outside the ball B_{δ_2} for all $t \in [0, T)$, provided $u_0 \in B^c_{\delta_0}$;
where B_δ is the open ball of $X_0(\Omega)$, with special radius, defined in Section 3.

The result of Theorem 5.4 shows that for any given $e \in (0, d)$, there exists a corresponding *vacuum region*

$$V_e = \left\{ w \in X_0(\Omega) : \left(\frac{(p+1)S_{p+1}^{p+1}}{2C(N,s)^p} \delta_1 \right)^{1/(p-1)} < \|w\|_{X_0(\Omega)} < \left(\frac{(p+1)S_{p+1}^{p+1}}{2C(N,s)^p} \delta_2 \right)^{1/(p-1)} \right\}$$

for the set of strong solutions of the problem (1.2)–(1.4), with initial energy $J(u_0)$ satisfying $0 < J(u_0) \le e$, i.e. there are no strong solutions u such that $u(\cdot, t) \in V_e$ for all $t \in [0, T)$. The vacuum region V_e becomes bigger when e decreases to 0.

Let us next consider the limit case $e = 0$.

Theorem 5.5. *Let $u_0 \in X_0(\Omega)$. Then any nontrivial strong solution u of (1.2)–(1.4), with initial energy $J(u_0) \le 0$, is such that $u(\cdot, t)$ lies outside the ball B_1 for all $t \in [0, T)$, where*

$$B_1 = \left\{ u \in X_0(\Omega) \ : \ \|u\|_{X_0(\Omega)} < \left(\frac{(p+1)S_{p+1}^{p+1}}{2C(N,s)^p} \right)^{1/(p-1)} \right\},$$

introduced in Section 3.

Proof. Fix a nontrivial strong solution u of (1.2)–(1.4), with initial energy $J(u_0) \le 0$.

Inequality (5.1) gives $J(u(\cdot, t)) \le 0$ for $t \in [0, T)$. Thus by

$$\frac{(p+1)C(N,s)}{2} \|u(\cdot, t)\|^2_{X_0(\Omega)} \le \|u(\cdot, t)\|^{p+1}_{L^{p+1}(\Omega)} \le \left(\frac{C(N,s)}{S_{p+1}} \|u(\cdot, t)\|_{X_0} \right)^{p+1},$$

we conclude that for any $t \in [0, T)$ either $\|u(\cdot, t)\|_{X_0(\Omega)} = 0$ or

$$\|u(\cdot, t)\|_{X_0(\Omega)} \geq \left(\frac{(p+1)S_{p+1}^{p+1}}{2C(N,s)^p} \right)^{1/(p-1)} = \kappa.$$

Next, we claim that either (i) $\|u(\cdot, t)\|_{X_0(\Omega)} = 0$ for all $t \in [0, T)$ or (ii) $\|u(\cdot, t)\|_{X_0(\Omega)} \geq \kappa$ for all $t \in [0, T)$. Otherwise, there are $t \in [0, T)$ such that $\|u(\cdot, t)\|_{X_0(\Omega)} = 0$ while there are $\tau \in [0, T)$ such that $\|u(\cdot, \tau)\|_{X_0(\Omega)} \geq \kappa$, since $\|u(\cdot, t)\|_{X_0(\Omega)}$ cannot take the values in $(0, \kappa)$. This contradicts the intermediate value theorem in view of the fact that the strong solution u belongs to $C(0, T; X_0(\Omega))$. As u is nontrivial, (i) is impossible. Thus we conclude that (ii) is true. This completes the proof in view of the definition of B_1. □

Next, we study blow up of strong solutions. Concerning the analysis of blow up of solutions, we refer to the papers [1], [25] and [26], where several different methods are applied.

Theorem 5.6. *Let $u_0 \in X_0(\Omega)$. Assume that $J(u_0) < d$ and $I(u_0) < 0$. Then the maximal existence time T of any strong solution u for the problem* (1.2)–(1.4) *is finite and actually the strong solution u blows up at T, i.e.*

$$\lim_{t \to T} \|u(\cdot, t)\|_{L^2(\Omega)} = \infty \quad \text{and so} \quad \lim_{t \to T} \|u(\cdot, t)\|_{X_0(\Omega)} = \infty.$$

Proof. Let u be any strong solution of (1.2)–(1.4), with $J(u_0) < d$ and $I(u_0) < 0$. Define Φu by

$$\Phi u(t) = \int_0^t \|u(\cdot, \tau)\|_{L^2(\Omega)}^2 d\tau.$$

Then

$$(\Phi u)'(t) = \|u(\cdot, t)\|_{L^2(\Omega)}^2.$$

By Theorem 2.1 in [4], we have that $u \in C(0, \infty; L^2(\Omega))$ and

$$2 \int_0^t (u(\cdot, \tau), u_\tau \cdot, \tau))_{L^2(\Omega)} d\tau = \|u(\cdot, t)\|_{L^2(\Omega)}^2 - \|u_0\|_{L^2(\Omega)}^2.$$

Therefore, differentiating with respect to t, we get

$$\begin{aligned}(\Phi u)''(t) &= 2(u(\cdot, t), u_t(\cdot, t))_{L^2(\Omega)} = -2C(N,s)\|u(\cdot, t)\|_{X_0(\Omega)}^2 + 2\|u(\cdot, t)\|_{L^{p+1}(\Omega)}^{p+1} \\ &= -2I(u(\cdot, t)).\end{aligned} \tag{5.3}$$

Consequently,

$$(\Phi u)''(t) \geq 2(p+1) \int_0^t \|u(\cdot, \tau))\|_{L^2(\Omega)}^2 d\tau + (p-1)C(N,s)(\Phi u)'(t) - 2(p+1)J(u_0)$$

and

$$\begin{aligned}\Phi u(\Phi u)'' - \frac{p+1}{2}((\Phi u)')^2 \geq 2(p+1) &\left[\int_0^t \|u\|_{L^2(\Omega)}^2 d\tau \int_0^t \|u_\tau\|_{L^2(\Omega)}^2 d\tau - \left(\int_0^t (u, u_\tau)_{L^2(\Omega)} d\tau \right)^2 \right] \\ &+ (p-1)C(N,s)\Phi u(\Phi u)' - (p+1)\|u_0\|_{L^2(\Omega)}^2 (\Phi u)' \\ &- 2(p+1)J(u_0)\Phi u + \frac{p+1}{2}\|u_0\|_{L^2(\Omega)}^4.\end{aligned}$$

Furthermore, by the Hölder inequality, we have

$$\begin{aligned}\Phi u(\Phi u)'' - \frac{p+1}{2}((\Phi u)')^2 \geq (p-1)C(N,s)\Phi u(\Phi u)' - (p+1)\|u_0\|^2_{L^2(\Omega)}(\Phi u)' \\ - 2(p+1)J(u_0)\Phi u + \frac{p+1}{2}\|u_0\|^4_{L^2(\Omega)}.\end{aligned} \tag{5.4}$$

Next we consider the following two cases.

(i) If $J(u_0) \leq 0$, then

$$\Phi u(\Phi u)'' - \frac{p+1}{2}((\Phi u)')^2 \geq (p-1)C(N,s)\Phi u(\Phi u)' - (p+1)\|u_0\|^2_{L^2(\Omega)}(\Phi u)'.$$

We claim that $I(u(\cdot,t)) < 0$ for all $t > 0$. Otherwise, there exists $t_0 > 0$ such that $I(u(\cdot,t_0)) = 0$. Let t_0 be the first time such that $I(u(\cdot,t)) = 0$, then $I(u(\cdot,t)) < 0$ for all $t \in [0,t_0)$. Similar to the proof of Lemma 3.1, we have

$$\|u(\cdot,t)\|_{X_0(\Omega)} > \left(\frac{S^{p+1}_{p+1}}{C(N,s)^p}\right)^{1/(p-1)} = \kappa$$

for all $t \in [0,t_0)$. Hence $\|u(\cdot,t_0)\|_{X_0(\Omega)} \geq \kappa$ and $J(u(t_0)) \geq d$. This contradicts (5.1) and proves the claim. Thus by (5.3) we get $(\Phi u)'' > 0$ in $\mathbb{R}^+$. From this and the fact that $(\Phi u)'(0) = \|u_0\|^2_{L^2(\Omega)} > 0$, it follows that there exists $t_0 \geq 0$ such that $(\Phi u)'(t_0) > 0$ and for all $t \geq t_0$

$$\Phi u(t) \geq (\Phi u)'(t_0)(t-t_0) + \Phi u(t_0) \geq (\Phi u)'(t_0)(t-t_0).$$

Hence, for sufficiently large t, we have

$$(p-1)C(N,s)\Phi u - (p+1)\|u_0\|^2_{L^2(\Omega)} > 0$$

and

$$\Phi u(\Phi u)'' - \frac{p+1}{2}((\Phi u)')^2 > 0. \tag{5.5}$$

(ii) If $0 < J(u_0) < d$, then Theorems 4.3, 5.1 and Corollary 5.3 imply that $u(\cdot,t) \in V_\delta$ for all $\delta \in (\delta_1,\delta_2)$ and all $t > 0$, where $\delta_1 < \delta_2$ are the two roots of the equation $d(\delta) = J(u_0)$. Hence, $I_\delta(u(\cdot,t)) < 0$ and

$$\|u(\cdot,t)\|_{X_0(\Omega)} > \left(\frac{\delta S^{p+1}_{p+1}}{C(N,s)^p}\right)^{1/(p-1)}$$

for all $\delta \in (\delta_1,\delta_2)$ and all $t > 0$. Thus, $I_{\delta_2}(u(\cdot,t)) \leq 0$ and

$$\|u(\cdot,t)\|_{X_0(\Omega)} \geq \left(\frac{\delta_2 S^{p+1}_{p+1}}{C(N,s)^p}\right)^{1/(p-1)}$$

for all $t > 0$. By (5.3) for all $t \geq 0$,

$$\begin{aligned}(\Phi u)''(t) &= 2(\delta_2 - 1)C(N,s)\|u(\cdot,t)\|^2_{X_0(\Omega)} - 2I_{\delta_2}(u(\cdot,t)) \\ &\geq 2(\delta_2 - 1)\left(\delta_2 S^{2(p+1)}_{p+1} C(N,s)^{p-3}\right)^{1/(p-1)} > 0, \\ (\Phi u)'(t) &\geq 2(\delta_2 - 1)\left(\delta_2 S^{2(p+1)}_{p+1} C(N,s)^{p-3}\right)^{1/(p-1)} t, \\ \Phi u(t) &= (\delta_2 - 1)\left(\delta_2 S^{2(p+1)}_{p+1} C(N,s)^{p-3}\right)^{1/(p-1)} t^2.\end{aligned}$$

Hence, for all t sufficiently large, we have

$$(p-1)C(N,s)\Phi u(t) > 2(p+1)\|u_0\|^2_{L^2(\Omega)}, \qquad (p-1)C(N,s)(\Phi u)'(t) > 4(p+1)J(u_0).$$

By (5.4) we again obtain (5.5) for all t sufficiently large.

By (5.5)

$$M'' = -\frac{p-1}{2(\Phi u)^{\frac{p+3}{2}}}\left[\Phi u(\Phi u)'' - \frac{p+1}{2}((\Phi u)')^2\right] \leq 0$$

where $M = (\Phi u)^{(1-p)/2}$. Therefore, M is concave for sufficiently large t, and there exists a finite T for which $\lim_{t\to T} M(t) = 0$, i.e. $\lim_{t\to T}\|u(\cdot,t)\|_{L^2(\Omega)} = \infty$. This completes the proof. □

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